Homological Methods in Commutative Algebra

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Abstract. This essay provides a short introduction to the theme of the workshop.

1. Introduction

Section 2 presents classical homological algebra in general algebraic terms. The classes of projective modules, injective modules, and flat modules are defined, and the first two are used to introduce the derived functors $\mathrm{Ext}^{\ell}(-,-)$ and $\mathrm{Tor}_{\ell}(-,-)$. Furthermore, homological characterizations of semisimple rings and of von Neumann regular rings are presented. The Gorenstein homological algebra, an important relative version of the classical homological algebra, is mentioned. So is a cotorsion theory based on vanishing of $\mathrm{Ext}^{1}(-,-)$. Finally, we present hyperhomological algebra which is a powerful extension of the classical homological algebra.

Section 3 lists homological characterizations of selected classes of commutative Noetherian rings: Dedekind rings, regular local rings, regular rings, Gorenstein local rings, Cohen–Macaulay local rings, and local complete intersections. Furthermore, classical and newer applications of these are presented. Although the rings above are introduced using classical algebraic terminology, the proofs of many of the results use the homological characterizations: Often no classical proofs are known! Betti numbers are important invariants throughout; over a polynomial ring over a field they yield the famous Castelnuovo–Mumford Regularity which is briefly recalled. When a polynomial ring over a field is equipped with a suitable ordering of the monomials, each ideal induces an initial ideal which is generated by monomials; we mention some results concerning the transition of homological conditions to an ideal from its initial ideal. Finally, Grothendieck's local cohomology modules are presented, and Hartshorne's concept of cofiniteness of a module with respect to an ideal is mentioned.

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2. Homological algebra

The title of this section is that of the September 1953 manuscript by Henri Cartan and Samuel Eilenberg. It was published as a book in 1956 by Princeton University Press, and the thirteenth printing [8] of it appeared in 1999 in the series "Princeton Landmarks in Mathematics and Physics". The book provides the foundations for homological algebra which yields a very powerful theory of functors between categories of modules over associative rings.

(2.1) **Functors.** To give examples, let R be an associative ring and let M and N be R–modules. In this essay, rings have always a multiplicative unit, and all modules are unitary left modules. The set $\mathrm{Hom}_R(M,N)$ of R–homomorphisms is an abelian group, and the assignment $N \mapsto \mathrm{Hom}_R(M,N)$ induces a covariant functor $\mathrm{Hom}_R(M,-)$ from the category of R–modules to that of abelian groups. That is, for any R–homomorphism $\varphi\colon N \to N'$ there is an induced group homomorphism

$$\varphi_* \overset{\mathrm{def}}{=} \mathrm{Hom}_R(M,\varphi)\colon\ \mathrm{Hom}_R(M,N) \to \mathrm{Hom}_R(M,N'),\ \mu \mapsto \varphi\mu\,.$$

This is such that $(\varphi'\varphi)_* = (\varphi')_*\varphi_*$, if $\varphi'\colon N' \to N''$ is also an R–homomorphism, and with $(1_N)_* = 1_{\mathrm{Hom}_R(M,N)}$ for the identity on N.

On the other hand, the assignment $M \mapsto \mathrm{Hom}_R(M,N)$ induces a contravariant functor $\mathrm{Hom}_R(-,N)$ from the category of R–modules to that of abelian groups: for any R–homomorphism $\psi\colon M \to M'$ there is an induced group homomorphism

$$\psi^* \overset{\mathrm{def}}{=} \mathrm{Hom}_R(\psi,N)\colon\ \mathrm{Hom}_R(M',N) \to \mathrm{Hom}_R(M,N),\ \mu' \mapsto \mu'\psi\,;$$

if, furthermore, $\psi'\colon M' \to M''$ is an R–homomorphism, then $(\psi'\psi)^* = \psi^*(\psi')^*$; and the equality $(1_M)^* = 1_{\mathrm{Hom}_R(M,N)}$ holds.

It turns out that $\mathrm{Hom}_R(-,-)$ is a functor in two variables, contravariant in the first and covariant in the second.

(2.2) **Projectivity.** An R–module P is said to be *projective*, if the functor $\mathrm{Hom}_R(P,-)$ takes surjective R–homomorphisms into surjective group homomorphisms. If P is a free R–module (that is, admits a basis over R) then P is projective, and thus their are sufficiently many projective modules: To every R–module M there exists a projective R–module P and a surjective R–homomorphism $\pi\colon P \to M$. As a consequence, every R–module M admits a *projective resolution*, that is, a sequence of R–homomorphisms

$$P_\bullet = \cdots \to P_{\ell+1} \xrightarrow{\partial_{\ell+1}} P_\ell \xrightarrow{\partial_\ell} P_{\ell-1} \xrightarrow{\partial_{\ell-1}} \cdots \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \to O$$

together with an R–homomorphism $\partial_0\colon P_0 \to M$ such that the all modules P_ℓ are projective, and such that the augmented sequence

$$\cdots \to P_{\ell+1} \xrightarrow{\partial_{\ell+1}} P_\ell \xrightarrow{\partial_\ell} P_{\ell-1} \xrightarrow{\partial_{\ell-1}} \cdots \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\partial_0} M \xrightarrow{\partial_{-1}} O$$

is exact (that is, $\operatorname{Im}\partial_\ell = \operatorname{Ker}\partial_{\ell-1}$ for all ℓ where $\operatorname{Im}\partial_\ell$ is the image of ∂_ℓ and $\operatorname{Ker}\partial_{\ell-1}$ is the kernel of $\partial_{\ell-1}$).

(2.3) **Injectivity.** By reversing arrows in the theory of projective modules one obtains that of *injective* modules: An R–module N is said to be injective exactly when the functor $\operatorname{Hom}_R(-,N)$ takes injective R–homomorphisms into surjective group homomorphisms. The $\mathbb{Z}$–modules $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are injective.

Let R° be the opposite ring of R, that is, it has same additive group as R, and its multiplication is given by $(r,r') \mapsto r'r$. Let P is an R°–module[1]. The abelian group $\operatorname{Hom}_{\mathbb{Z}}(P,\mathbb{Q}/\mathbb{Z})$ is then an R–module, and the following hold.

(2.3.1) P projective over $R^\circ \implies \operatorname{Hom}_{\mathbb{Z}}(P,\mathbb{Q}/\mathbb{Z})$ injective over R.

This results from the (so-called) *Swap Isomorphism:*

(2.3.2) $\operatorname{Hom}_R(-,\operatorname{Hom}_{\mathbb{Z}}(P,\mathbb{Q}/\mathbb{Z})) \xrightarrow{\cong} \operatorname{Hom}_{R^\circ}(P,\operatorname{Hom}_{\mathbb{Z}}(-,\mathbb{Q}/\mathbb{Z}))$
defined by $\varphi \mapsto \big(p \mapsto (x \mapsto \big(\varphi(x)\big)(p)\big)$.

It follows that their are sufficiently many injective modules: To every R–module M there exists an injective R–module I and an injective R–homomorphism $\iota\colon M \to I$, and this, in turn, yields that every R–module M admits an *injective resolution*, that is, a sequence

$$I^\bullet = O \to I^0 \xrightarrow{\partial^0} I^1 \to \cdots \to I^\ell \xrightarrow{\partial^\ell} I^{\ell+1} \xrightarrow{\partial^{\ell+1}} \cdots$$

of injective modules together with an R–homomorphism $\partial^{-1}\colon M \to I^0$ yielding an exact augmented sequence:

$$O \to M \xrightarrow{\partial^{-1}} I^0 \xrightarrow{\partial^0} I^1 \to \cdots \to I^\ell \xrightarrow{\partial^\ell} I^{\ell+1} \xrightarrow{\partial^{\ell+1}} \cdots .$$

[1] Some refer to this situation as P being a *right* R–module.

(2.4) **Derived functors.** Any projective resolution $P_\bullet$ of an R–module M, confer above, induces for any R–module N a sequence of homomorphisms:

$$\mathrm{Hom}_R(P_\bullet, N) \overset{\text{def}}{=}$$
$$\cdots \to \mathrm{Hom}_R(P_{\ell-1}, N) \overset{(\partial_\ell)^*}{\to} \mathrm{Hom}_R(P_\ell, N) \overset{(\partial_{\ell+1})^*}{\to} \mathrm{Hom}_R(P_{\ell+1}, N) \to \dots .$$

For each ℓ, this yields the *ℓ th right derived functor* $\mathrm{Ext}_R^\ell(-,N)$ of the functor $\mathrm{Hom}_R(-,N)$ by taking cohomology:

$$\mathrm{Ext}_R^\ell(M,N) \overset{\text{def}}{=} \mathrm{H}^\ell(\mathrm{Hom}_R(P_\bullet, N))\,;$$

(the latter is the group $\mathrm{Ker}(\partial_{\ell+1})^* / \mathrm{Im}(\partial_\ell)^*$). Up to canonical isomorphism, this does not depend upon the choice of projective resolution, and $\mathrm{Ext}_R^\ell(-,N)$ becomes a contravariant functor. For fixed M, it turns out that also the assignment $N \mapsto \mathrm{Ext}_R^\ell(M,N)$ can be turned into a covariant functor, and that $\mathrm{Ext}_R^\ell(-,-)$ is a functor in two variables, contravariant in the first and covariant in the second. Furthermore, there is an isomorphism of functors $\mathrm{Hom}_R(-,-) \cong \mathrm{Ext}_R^0(-,-)$, and the functor $\mathrm{Ext}_R^\ell(-,-)$ vanishes for all negative ℓ. There is a dual construction of $\mathrm{Ext}_R^\ell(-,-)$: For fixed R–modules M and N, an injective resolution $I^\bullet$ of N, and an integer ℓ, set $\widetilde{\mathrm{Ext}}_R^\ell(M,N) =^{\text{def}} \mathrm{H}^\ell(\mathrm{Hom}_R(M, I^\bullet))$. It turns out that also $\widetilde{\mathrm{Ext}}_R^\ell(-,-)$ is a functor in two variables and that it is actually isomorphic to the already defined $\mathrm{Ext}_R^\ell(-,-)$. The next theorems show that the functor Ext determines projectivity and injectivity of modules as well as semisimplicity of rings.

(2.5) **Projectivity Theorem.** *For any module M the next are equivalent:*
(i) *M is projective;*
(ii) $\mathrm{Ext}_R^\ell(M,-) = O$ *for all* $\ell > 0$;
(iii) $\mathrm{Ext}_R^1(M,-) = O$.

(2.6) **Injectivity Theorem.** *For any module N the next are equivalent:*
(i) *N is injective;*
(ii) $\mathrm{Ext}_R^\ell(-,N) = O$ *for all* $\ell > 0$;
(iii) $\mathrm{Ext}_R^1(-,N) = O$.

(2.7) **Semisimplicity Theorem.** *For any ring R the next are equivalent:*
(i) *R is semisimple*[2];
(ii) *every R–module is projective;*
(iii) *every R–module is injective;*
(iv) $\mathrm{Ext}_R^\ell(-,-) = O$ *for all* $\ell > 0$.

[2] In the classical sense that every R–module is a direct sum of simple modules.

(2.8) **Tensor products.** Let N be an R–module and let M be a module over the opposite ring R°. The *tensor product* $M \otimes_R N$ is then an abelian group generated by symbols $m \otimes n$ for $m \in M$ and $n \in N$ subject to the relations $(m+m') \otimes n = m \otimes n + m' \otimes n$, $m \otimes (n+n') = m \otimes n + m \otimes n'$, and $(rm) \otimes n = m \otimes (rn)$ for $m, m' \in M$, $n, n' \in N$, and $r \in R$. This becomes a functor in two variables: Whenever $\varphi\colon M \to M'$ is an R°–homomorphism there is a group homomorphism $\varphi \otimes N\colon M \otimes_R N \to M' \otimes_R N$ well-defined by $m \otimes n \mapsto \varphi(m) \otimes n$, and whenever $\psi\colon N \to N'$ is an R–homomorphism there is a group homomorphism $M \otimes \psi\colon M \otimes_R N \to M \otimes_R N'$ well-defined by $m \otimes n \mapsto m \otimes \psi(n)$. There is a natural *commutativity isomorphism:*

(2.8.1) $$M \otimes_R N \xrightarrow{\cong} N \otimes_{R^\circ} M,\ m \otimes n \mapsto n \otimes m\,,$$

and a natural *adjointness isomorphism:*

(2.8.2)
$$\operatorname{Hom}_{\mathbb{Z}}(M \otimes_R N, Q) \xrightarrow{\cong} \operatorname{Hom}_{R^\circ}(M, \operatorname{Hom}_{\mathbb{Z}}(N, Q)),\ \varphi \mapsto \big(m \mapsto (n \mapsto m \otimes n)\big)\,,$$
when Q is a $\mathbb{Z}$–module.

(2.9) **Torsion functors.** The *ℓ th left derived functor* $\operatorname{Tor}^R_\ell(M,-)$ of the functor $M \otimes_R -$ is defined using projective resolutions: $\operatorname{Tor}^R_\ell(M,N) =^{\mathrm{def}} \mathrm{H}_\ell(M \otimes_R Q_\bullet)$ whenever $Q_\bullet$ is a projective resolution of N. For fixed N this provides a functor $\operatorname{Tor}^R_\ell(-,N)$, and this induces a functor $\operatorname{Tor}^R_\ell(-,-)$ in two variables; the latter turns out to be isomorphic to the one induced by $(M,N) \mapsto \mathrm{H}_\ell(P_\bullet \otimes_R N)$ whenever $P_\bullet$ is a projective resolution of the R°–module M. Furthermore, the functors $- \otimes_R -$ and $\operatorname{Tor}^R_0(-,-)$ are isomorphic, and the functor $\operatorname{Tor}^R_\ell(-,-)$ vanishes for all negative ℓ.

An R–module N is said to be *flat* whenever the functor $- \otimes_R N$ is exact.

(2.10) **Flatness Theorem.** *The next are equivalent for any R–module N.*
(i) *N is flat;*
(i') $\operatorname{Hom}_{\mathbb{Z}}(N, \mathbb{Q}/\mathbb{Z})$ *is an injective R°–module;*
(ii) $\operatorname{Tor}^R_\ell(-,N) = O$ *for all* $\ell > 0$*;*
(iii) $\operatorname{Tor}^R_1(-,N) = O$.
Furthermore, if N is projective, then it is flat.

Concerning the equivalence between (i) and (i') the following should be noted; the points of view in (2) and (3) can be important in other categories than the one of modules.
(1) It results from the isomorphism (2.8.2) with $Q = \mathbb{Q}/\mathbb{Z}$.
(2) It yields the last assertion by (2.3.1).
(3) It provides most of the theory of flat modules directly from that of injective ones.

(2.11) **von Neumann Regularity.** The ring R is said to be *von Neumann regular* exactly when there to any $r \in R$ exists an $x \in R$ such that $rxr = r$. Fields are von Neumann regular, and so are direct products of von Neumann regular rings.

(2.12) **von Neumann Regularity Theorem.** *The following are equivalent:*

(i) *R is von Neumann regular;*
(ii) *$R/\mathfrak{a}$ is a projective R–module for all finitely generated ideals $\mathfrak{a}$;*
(iii) *every R–module is flat*
(iv) $\operatorname{Tor}_\ell^R(-,-) = O$ *for all* $\ell > 0$*;*
(v) $\operatorname{Tor}_1^R(-,-) = O$.

(2.13) **Gorenstein projective modules.** The class of, for example, projective R–modules has very nice (so-called) resolving properties, and these are essential in the usage of this class of modules in the foundation of homological algebra. There are other classes of modules with almost as nice resolving properties. One of these, the class of (so-called) *Gorenstein projective R–modules* was introduced by Enochs and Jenda in [10], see also [9] and [18]. It turns out that a non-zero finitely generated module over a Noetherian commutative ring is G–projective if and only if it has G–dimension zero in the sense of Auslander [1]. Other references concerning Gorenstein homological algebra include [4], [19], [20], and [23]. Several talks at the workshop focused on Gorenstein projective modules and/or *Gorenstein injective* modules; the latter ones are defined dually, and they are scrutinized in [12] (in these proceedings).

(2.14) **Cotorsion theories.** By saying that R–modules M and N are orthogonal exactly when $\operatorname{Ext}_R^1(M,N)$ vanishes, Enochs and Jenda in [11] (in these proceedings) consider the corresponding cotorsion pairs and study situations when also higher Ext groups vanish. They also report on results concerning covers and envelopes associated to cotorsion pairs.

(2.15) **Hyperhomological algebra.** Some of the talks at the workshop used a powerful extention of classical homological algebra called *hyperhomological algebra*. Its objects are complexes

$$M_\bullet = \cdots \to M_{\ell+1} \xrightarrow{\partial_{\ell+1}} M_\ell \xrightarrow{\partial_\ell} M_{\ell-1} \to \cdots$$

of R–modules, that is, $\operatorname{Im}\partial_{\ell+1} \subseteq \operatorname{Ker}\partial_\ell$ for all ℓ. Each R–module M is considered as a complex concentrated in degree zero, that is, $M_0 = M$ and $M_\ell = O$ for $\ell \neq 0$. The morphisms $\varphi\colon X \to Y$ are (so-called) chain maps of degree zero, that is, families $\varphi = (\varphi_\ell\colon X_\ell \to Y_\ell)_{\ell\in\mathbb{Z}}$ of R–homomorphisms such that $\partial_\ell^Y \varphi_\ell = \varphi_{\ell-1}\partial_\ell^X$. The projective resolutions in classical homological algebra are in hyperhomological algebra replaced by (so-called) semiprojective resolutions. The left column in the next table describes the formation in classical homological algebra of the left derived functor $\mathrm{L}_\ell T(M)$ of a covariant additive functor T applied to an R–module M, and the right column describes the formation in hyperhomological algebra of the left derived functor $\mathbf{L}T(M_\bullet)$ of a suitable functor T applied to a complex $M_\bullet$ of R–modules. If M is a module, then $\mathrm{L}_\ell T(M)$ is just the ℓth homology of $\mathbf{L}T(M)$.

Classical homological algebra	*Hyperhomological algebra*
(1) Take projective resolution $P_\bullet$ of M;	(1$_\bullet$) Take semiprojective resolution $P_\bullet$ of $M_\bullet$;
(2) apply T to $P_\bullet$ to get $T(P_\bullet)$;	(2$_\bullet$) apply T to $P_\bullet$ to get $\mathbf{L}T(M_\bullet)$.
(3) take ℓth homology to get $\mathrm{L}_\ell(M)$.	

Note that the procedure in classical homological algebra is one step longer than in hyperhomological algebra, and that information is lost in that step! The latter represents one the advantages of hyperhomological algebra. The appendix of [9] is a good reference.

3. Commutative Algebra

Commutative algebra emerged in the late 1800s as part of the foundation of *algebraic geometry.* Moreover, commutative rings are ubiquitous in *algebraic number theory*[3]. Let R, M, and N be as above, and assume, in addition, that the ring R is *commutative.* The abelian groups $\mathrm{Hom}_R(M,N)$ and $M \otimes_R N$ are then R–modules and so are $\mathrm{Ext}_R^\ell(M,N)$ and $\mathrm{Tor}_\ell^R(M,N)$ for all ℓ.

Assume, furthermore, that the ring R is also *Noetherian;* this means that every ideal is finitely generated, and this is tantamount to: every ascending chain of ideals is ultimately stationary. Under these assumptions, if the R–modules M and N are both finitely generated, then so are $\mathrm{Hom}_R(M,N)$ and $M \otimes_R N$ as well as $\mathrm{Ext}_R^\ell(M,N)$ and $\mathrm{Tor}_\ell^R(M,N)$ for all ℓ.

(3.1) **Assumptions.** From now on, all rings are assumed to be *commutative* and *Noetherian.*

(3.2) **Dedekind rings.** Let the ring R be an integral domain; it is then said to be *Dedekind* exactly when every non-zero ideal is the product of prime ideals. Obviously, any Principal Ideal Domain is a Dedekind ring. The latter ones also play a crucial role in algebraic number theory, partly due to the following fact: Let R be a Dedekind ring with K as field of fractions, and let

[3] Actually, it seems that the word *ring* appeared for the first time in 1897 on page 237 in David Hilbert's *Zahlbericht* [17].

K' be a finite field extension of K. The integral closure of D in K' is then a Dedekind ring. There is the following homological characterization.

(3.3) **Dedekindness Theorem.** *The following are equivalent:*

(i) *R is Dedekind;*
(ii) $\operatorname{Ext}^{\ell}_R(-,-) = O$ *for* $i \geq 2$;
(iii) $\operatorname{Ext}^2_R(-,-) = O$.

(3.4) **Krull dimension.** The *Krull dimension* is defined as $\dim R \stackrel{\text{def}}{=}$

$$\sup\{\, n \mid \text{there exists a chain } \mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_n \text{ of prime ideals in } R \,\}\,.$$

This number might be infinite. If k is a field, the $\dim k = 0$ and, for any positive integer n, $\dim k[x_1, \ldots, x_n] = n$. If R is a Dedekind ring and not a field, then $\dim R = 1$. The term "dimension" comes from classical algebraic geometry: Let V be an affine variety in complex n–space $\mathbb{C}^n$. There is then an associated ideal $\mathfrak{I}(V)$ in $\Gamma(\mathbb{C}^n) =^{\text{def}} \mathbb{C}[x_1, \ldots, x_n]$, namely the one consisting exactly of those $f \in \Gamma(\mathbb{C}^n)$ with $f(Q) = 0$ for all $Q \in V$. The ring of residues $\Gamma(V) =^{\text{def}} \Gamma(\mathbb{C}^n)/\mathfrak{I}(V)$ is the *coordinate ring* of V. It turns out that $\dim \Gamma(V)$ equals the *geometrical dimension* of V, that is, the maximal length n of a chain $U_n \subset \cdots \subset U_1 \subset U_0$ of subvarieties of V.

(3.5) **Local rings.** Let R be a ring. It is said to be *local* precisely when it has exactly one maximal ideal $\mathfrak{m}$, and in this case we let k denote the residue field $R/\mathfrak{m}$ and say that $(R, \mathfrak{m}, k)$ is a local ring. If $\mathfrak{p}$ is a prime ideal in R, then we let $R_{\mathfrak{p}}$ denote the ring $\{\, r/s \mid r \in R\,,\ s \in R \setminus \mathfrak{p} \,\}$ of fractions. This is a local ring called the localization of R; its maximal ideal is $\mathfrak{p}R_{\mathfrak{p}} = \{\, p/s \mid p \in \mathfrak{p}\,,\ s \in R \setminus \mathfrak{p} \,\}$.

The term "local" comes from classical algebraic geometry: to any point P in an affine variety V in complex n–space $\mathbb{C}^n$, there is an associated local ring $\mathcal{O}_P(V)$. Namely, consider the coordinate ring $\Gamma(V)$, cf. (3.4), and its maximal ideal $\mathfrak{m}_P(V)$ at P, that is,

$$\mathfrak{m}_P(V) \stackrel{\text{def}}{=} \{\, f \in \Gamma(\mathbb{C}^n) \mid f(P) = 0 \,\}/\mathfrak{I}(V)\,.$$

The local ring $\mathcal{O}_P(V)$ of V at P is then the ring of fractions $\Gamma(V)_{\mathfrak{m}_P(V)}$ which is local with maximal ideal $\mathfrak{m}_P(V)\mathcal{O}_P(V)$.

(3.6) **Regular local rings.** Let $(R, \mathfrak{m}, k)$ be a local ring. The (initial) *Betti number* of $\mathfrak{m}$ is defined as

(3.6.1) $$\beta_0^R(\mathfrak{m}) \stackrel{\text{def}}{=} \text{ the minimal number of generators of } \mathfrak{m}\,.$$

Krull's Principal Ideal Theorem yields the inequality $\dim R \leq \beta_0^R(\mathfrak{m})$. The ring R is said to be a *regular* exactly when there is equality $\dim R = \beta_0^R(\mathfrak{m})$.

The following hold. If $\dim R = 0$, then R is regular if and only if it is a field. If $\dim R = 1$, then R is regular if and only if it is a Principal Ideal Domain. If R is regular then so is the ring $R[[X]]$ of formal power series (and $\dim R[[X]] = \dim R + 1$). If R is regular then it is a domain. For a point P in an affine variety V as above, then the local ring $\mathcal{O}_P(V)$ is regular if and only if V is non-singular at P (cf. e.g. Theorem I.5.1 in [15]).

The next two assertions are often referred to as *Krull's Conjectures*; the second one can be found in [21].

- If $\mathfrak{p}$ is a prime ideal in a regular ring R, then the ring $R_{\mathfrak{p}}$ is regular.
- If R is regular, then R is a Unique Factorization Domain.

Concerning the first, note that regularity is defined in terms of a special property of the maximal ideal $\mathfrak{m}$, and that localization at a prime ideal $\mathfrak{p} \neq \mathfrak{m}$ seemingly ruins this property as $\mathfrak{m}R_{\mathfrak{p}} = R_{\mathfrak{p}}$. Nevertheless, it holds: it follows quite easily from the homological characterization (a) below and is restated as (b) below. The second one also follows from this homological characterization—after further preparations, cf. e.g. Theorem 2.2.19 in [7]; it is (c) below. The next results are proved by Auslander, Buchsbaum, and Serre, cf. e.g. Theorem 2.2.7 in [7].

(3.7) **Local Regularity Theorem.** *Let $(R, \mathfrak{m}, k)$ be a local ring.*

(a) *The following are equivalent for R.*
  (i) *R is regular;*
  (ii) $\operatorname{Ext}^{\ell}_R(-,-) = O$ *for* $\ell \gg 0$;
  (iii) $\operatorname{Ext}^{\ell}_R(-,-) = O$ *for* $\ell > \dim R$;
  (iv) $\operatorname{Ext}^{\ell}_R(k,-) = O$ *for* $\ell > \dim R$;
  (v) $\operatorname{Ext}^{\dim R+1}_R(k,k) = O$.

(b) *If R is regular, then so is the localization $R_{\mathfrak{p}}$ for each prime ideal $\mathfrak{p}$.*

(c) *If R is regular, then R is a Unique Factorization Domain.*

(3.8) **Regular rings.** A ring R, cf. (3.1), is said to be *regular* exactly when the localization $R_{\mathfrak{p}}$ is a regular local ring for each prime ideal $\mathfrak{p}$ in R. By (3.7.b) above this provides an extension of the class of regular local rings. For the first part of the next result consult Corollary 2.2.20 in [7]. The last part follows from the Local Regularity Theorem (3.7.a) by localization.

(3.9) **Regularity Theorem.**

(a) *Any regular ring is isomorphic to a finite direct sum of regular domains.*

(b) *The ring R is regular if and only if for every finitely generated R–module M the functors $\operatorname{Ext}^{\ell}_R(M,-)$ vanish for $\ell \gg 0$* [4].

[4] That is, every finitely generated R–module has *finite projective dimension.*

(c) *The following are equivalent for any ring R with* $\dim R < \infty$.
   (i) *R is regular;*
   (ii) $\operatorname{Ext}_R^{\ell}(-,-) = O$ *for* $\ell \gg 0$;
   (iii) $\operatorname{Ext}_R^{\ell}(-,-) = O$ *for* $\ell > \dim R$;
   (iv) $\operatorname{Ext}_R^{\dim R+1}(-,-) = O$.
   (v) $\operatorname{Tor}_{\ell}^R(-,-) = O$ *for* $\ell \gg 0$;
   (vi) $\operatorname{Tor}_{\ell}^R(-,-) = O$ *for* $\ell > \dim R$;
   (vii) $\operatorname{Tor}_{\dim R+1}^R(-,-) = O$.

If R is a domain, then it is a Dedekind ring if and only if it is a regular ring of dimension at most one.

(3.10) **Betti numbers.** If $(R, \mathfrak{m}, k)$ is a local ring, M is a finitely generated R–module, and ℓ is any integer, then the ℓ th *Betti number* $\beta_{\ell}^R(M)$ is the dimension of $\operatorname{Tor}_{\ell}^R(k, M)$ considered as a vector space over the field k, and this number is known to be finite whenever M is finitely generated. The special case in which $\ell = 0$ and $M = \mathfrak{m}$ turns out to be the one mentioned in (3.6.1). It follows from (3.7.a) that the ring R is regular if and only $\beta_{\ell}^R(M) = 0$ for all $\ell \geq \dim R$ and all finitely generated R–modules M.

(3.11) **Castelnuovo–Mumford Regularity.** Let M be a finitely generated graded module over $S = k[x_1, \ldots, x_n]$ where k is a field. The local rings $S_{\mathfrak{q}}$ are then regular for all prime ideals $\mathfrak{q}$, and the k–module $\operatorname{Tor}_{\ell}^S(k, M)$ is a graded vector space over k; the (ℓ, j) th *graded Betti number* $\beta_{\ell j}^S(M)$ of M is defined as the dimension of the j th component of $\operatorname{Tor}_{\ell}^S(k, M)$. The *Castelnuovo–Mumford regularity* $\operatorname{reg}_S M$ is defined by

$$\operatorname{reg}_S M \stackrel{\text{def}}{=} \sup \{ \sup\{ j \mid \beta_{\ell j}^S(M) \neq 0 \} - \ell \mid \ell \in \mathbb{N}_0 \}$$

In the paper [16] (in these proceedings) Herzog considers a graded ideal $\mathfrak{a}$ in S and reports on results concerning $\operatorname{reg}_S(\mathfrak{a}^n)$ as a function of n as well as its behavior for large n.

(3.12) **Gorenstein local rings.** Let $(R, \mathfrak{m}, k)$ be a local ring with $d = \dim R$. If $\mathfrak{a}$ is an ideal in R, then the Betti number $\beta_0^R(\mathfrak{a})$ turns out to be the minimal number of generators of $\mathfrak{a}$. It is a consequence of Krull's Principal Ideal Theorem that $d \leq \beta_0^R(\mathfrak{a})$ if the dimension of the ring $R/\mathfrak{a}$ is zero. The ideal $\mathfrak{a}$ is said to be a *parameter ideal* exactly when $\dim(R/\mathfrak{a}) = 0$ and $d = \beta_0^R(\mathfrak{a})$, and one can prove that any local ring admits a parameter ideal. An ideal $\mathfrak{a}$ in R is said to be *irreducible* whenever it is not the intersection of two strictly larger ideals. The ring R is said to be *Gorenstein* whenever it admits an irreducible parameter ideal. Any regular local ring is Gorenstein. Next follows, in (a) below, a homological characterization of Gorenstein local rings, and this implies that the Gorenstein property is stable under localization, this is (b) below. These results are due to Bass [5].

(3.13) **Local Gorensteinness Theorem.** *Let* $(R, \mathfrak{m}, k)$ *be a local ring.*

(a) *The following are equivalent.*
 (i) *R is Gorenstein;*
 (ii) $\mathrm{Ext}^{\ell}_R(-, R) = O$ *for* $\ell \gg 0$ [5]*;*
 (iii) $\mathrm{Ext}^{\ell}_R(-, R) = O$ *for* $\ell > \dim R$*;*
 (iv) $\mathrm{Ext}^{\ell}_R(k, R) = O$ *for* $\ell \gg 0$*;*
 (v) $\mathrm{Ext}^{\ell}_R(k, R) = O$ *for* $\ell > \dim R$*;*
 (vi) $\mathrm{Ext}^{\dim R+1}_R(k, R) = O$.

(b) *If R is a Gorenstein, then so is $R_{\mathfrak{p}}$ for each prime ideal $\mathfrak{p}$.*

(3.14) **Cohen–Macaulay local rings.** Let $(R, \mathfrak{m}, k)$ be a local ring. If $\mathfrak{a}$ is an ideal in R and x is an element in R, then x is said to be a *zero-divisor* on the residue ring $R/\mathfrak{a}$ whenever there exists $r \in R$ such that $r \notin \mathfrak{a}$ and $xr \in \mathfrak{a}$. A sequence $x_1, \dots, x_n$ of elements in the maximal ideal $\mathfrak{m}$ is said to be *regular* whenever, for all $i = 1, \dots, n$, the element x_i is not a zero-divisor on $R/(x_1, \dots, x_{i-1})$; for $i = 1$ this means that x_1 is not a zero-divisor on R. The *depth* of R is defined as follows:

$$\operatorname{depth} R \overset{\text{def}}{=} \text{ the maximal length } n \text{ of regular sequence } x_1, \dots, x_n \text{ in } \mathfrak{m}\,.$$

It turns out that $\operatorname{depth} R$ can be computed homologically as follows.

(3.14.1) $$\operatorname{depth} R = \inf\{\, \ell \mid \mathrm{Ext}^{\ell}_R(k, R) \neq 0 \,\}\,.$$

From this equality one can deduce the next inequality.

(3.14.2) $$\operatorname{depth} R \leq \operatorname{depth} R_{\mathfrak{p}} + \dim(R/\mathfrak{p})\,.$$

Here, $\operatorname{depth} R_{\mathfrak{p}}$ is the depth of the local ring $(R_{\mathfrak{p}}, \mathfrak{p}R_{\mathfrak{p}}, R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}})$, and $\dim(R/\mathfrak{p})$ is the Krull dimension of the local ring $(R/\mathfrak{p}, \mathfrak{m}/\mathfrak{p}, k)$. The last inequality yields the next one.

(3.14.3) $$\operatorname{depth} R \leq \dim R\,.$$

The *Cohen–Macaulay defect* of R is the number:

$$\operatorname{cmd} R \overset{\text{def}}{=} \dim R - \operatorname{depth} R\,;$$

this number is always non-negative by (3.14.3) above. The inequality (3.14.2) has the next easily established counterpart concerning Krull dimension.

(3.14.4) $$\dim R_{\mathfrak{p}} + \dim(R/\mathfrak{p}) \leq \dim R\,.$$

[5] That is, the R–module R has *finite injective dimension.*

Combining (3.14.2) and (3.14.4) one obtains

$$\operatorname{cmd} R_{\mathfrak{p}} \le \operatorname{cmd} R\,. \qquad (3.14.5)$$

The ring R is said to be *Cohen–Macaulay* whenever $\operatorname{depth} R = \dim R$, that is, exactly when $\operatorname{cmd} R = 0$. It follows from (3.14.5) that Cohen–Macaulayness behaves well under localization, see (b) below.

(3.15) **Local Cohen–Macaulayness Theorem.** *Let $(R, \mathfrak{m}, k)$ be a local ring.*

(a) *The following are equivalent for R.*
  (i) *R is Cohen–Macaulay;*
  (ii) *$\operatorname{Ext}_R^{\ell}(k, R) = O$ for all $\ell < \dim R$;*
  (iii) *$\operatorname{Ext}_R^{\dim R+1}(-, M) = O$ for some non-zero finitely generated R–module M.*

(b) *If R is Cohen–Macaulay, then so is the localization $R_{\mathfrak{p}}$ for each prime ideal $\mathfrak{p}$.*

The implication (iii)⇒(i) is a deep result, see Corollary 9.6.2 and Remark 9.6.4(a)(ii) in [7].

(3.16) **Local Complete Intersections.** Let $(R, \mathfrak{m}, k)$ be a local ring, and consider its (so-called) $\mathfrak{m}$–adic completion $\widehat{R}$ which also is a local ring. The ring R is said to a *complete intersection* exactly if there exists a regular local ring Q and a surjective homomorphism $\varphi\colon Q \to \widehat{R}$ of rings such that the kernel $\operatorname{Ker}\varphi$ ($\subset Q$) is generated by a regular sequence.

A finitely generated R–module M is said to be of *finite complexity* exactly when its sequence of Betti numbers $\beta_0^R(M), \beta_1^R(M), \ldots, \beta_n^R(M), \ldots$ has only polynomial growth, that is, there exists a polynomial $f \in \mathbb{Z}[x]$ such that $\beta_n^R(M) \le f(n)$ for all n. If R is a regular local ring, respectively, a regular local ring modulo a non-unit, then the Betti number sequences are all eventually zero, respectively, eventually constant.

(3.17) **Local Complete Intersection Theorem.** *Let $(R, \mathfrak{m}, k)$ be a local ring.*

(a) *The following are equivalent for R.*
  (i) *R is a complete intersection;*
  (ii) *Every finitely generated R–module has finite complexity.*

(b) *If R is a complete intersection, then so is the localization $R_{\mathfrak{p}}$ for each prime ideal $\mathfrak{p}$.*

These two results are due to Gulliksen and Avramov, respectively, cf. e.g. Theorem 8.1.2 and Corollary 7.4.6, respectively, in [2]. The next result is Theorem III in [3].

(3.18) **Ext–Tor Vanishing Theorem.** *If $(R, \mathfrak{m}, k)$ is a local complete intersection, then the following are equivalent for all finitely generated R–modules M and N.*

(i) $\mathrm{Ext}_R^{\ell}(M, N) = O$ *for* $\ell \gg 0$*;*
(ii) $\mathrm{Ext}_R^{\ell}(N, M) = O$ *for* $\ell \gg 0$*;*
(iii) $\mathrm{Tor}_{\ell}^{R}(M, N) = O$ *for* $\ell \gg 0$.

If R is a local ring such that (i), (ii), and (iii) above are equivalent for all finitely generated modules M and N, then R is necessarily a Gorenstein ring; namely, set $M = R$ and $N = k$ and appeal to (iv) in (3.13.a). For more precise relations between (i), (ii), and (iii) consult Remark 6.3 in [3].

For the next result, consult e.g. Proposition 3.5 in [7].

(3.19) **Hierarchy of Local Rings.** *The next hold for all local rings R.*

$$\begin{aligned} R \text{ is regular } &\Longrightarrow R \text{ is a complete intersection} \\ &\Longrightarrow R \text{ is Gorenstein} \\ &\Longrightarrow R \text{ is Cohen–Macaulay.} \end{aligned}$$

(3.20) **Initial ideals.** Let $S = k[x_1, \ldots, x_n]$ be a polynomial ring over a field k. Moreover, assume that each x_i is assigned a positive (integer) degree, and that the set of monomials are equipped with a suitable well-ordering. For further details, consult the paper [6] (in these proceedings) by Bruns and Conca. To each ideal $\mathfrak{a}$ in S one can associate its (so-called) *initial* ideal $\widetilde{\mathfrak{a}}$ which is generated by monomials. Consider furthermore the residue rings $R =^{\mathrm{def}} S/\mathfrak{a}$ and $\widetilde{R} =^{\mathrm{def}} S/\widetilde{\mathfrak{a}}$. Properties of $\widetilde{R}$ imply sometimes the corresponding ones for R. For example: If $\widetilde{R}$ is such that for each prime ideal $\mathfrak{q}$ in $\widetilde{R}$ the local ring $\widetilde{R}_{\mathfrak{q}}$ is Gorenstein, respectively Cohen–Macaulay, then the ring R has the corresponding property. Furthermore, if there is an integer p such that $\mathrm{Ext}_S^{\ell}(\widetilde{R}, -) = O$ for $\ell > p$, then $\mathrm{Ext}_S^{\ell}(R, -) = O$ for $\ell > p$.

(3.21) **Local cohomology.** Let $(R, \mathfrak{m}, k)$ be a local ring. Define a covariant endo-functor $\Gamma_{\mathfrak{m}}(-)$ on the category of R–modules by $\Gamma_{\mathfrak{m}}(M) =^{\mathrm{def}} \{\, m \in M \mid \mathfrak{m}^n m = 0 \text{ for } n \gg 0 \,\}$ when M is an R–module; for any R–homomorphism $\alpha\colon M \to N$ let $\Gamma_{\mathfrak{m}}(\alpha)\colon \Gamma_{\mathfrak{m}}(M) \to \Gamma_{\mathfrak{m}}(N)$ be the restriction of α. Furthermore, use injective resolutions to define its ℓth right derived functor $\mathrm{H}_{\mathfrak{m}}^{\ell}(-) =^{\mathrm{def}} R^{\ell}\,\Gamma_{\mathfrak{m}}(-)$ which is said to be the the *ℓ th local cohomology.* If R is a homomorphic image of a Gorenstein local ring Q, and E is an indecomposable[6] injective R–module such that $\mathrm{Hom}_R(k, E) \cong k$ (and such exists always and is said to be the injective envelope of k), then there are

[6] That is, it is not the direct sum of two non-zero submodules.

isomorphisms of functors

$$\mathrm{H}^{\ell}_{\mathfrak{m}}(-) \cong \mathrm{Hom}_R(\mathrm{Ext}_Q^{\dim Q-\ell}(-,Q),E)\,.$$

This is Grothendieck's *Local Duality Theorem,* cf. Theorem 6.3 in [13]. It follows that:

$$\begin{aligned}\operatorname{depth} R &= \inf\{\,\ell \mid \mathrm{H}^{\ell}_{\mathfrak{m}}(R) \neq 0\,\}\,;\\ \dim R &= \sup\{\,\ell \mid \mathrm{H}^{\ell}_{\mathfrak{m}}(R) \neq 0\,\}\end{aligned}$$

(and (3.14.3) above has been rediscovered).

(3.22) **Cofinite modules.** One can prove that the modules $\mathrm{Ext}^{\ell}_R(k,\mathrm{H}^{\ell}_{\mathfrak{m}}(M))$ are finitely generated for all ℓ and all finitely generated R–modules M. If we, in the definition of the section functor $\Gamma_{\mathfrak{m}}(-)$, replace the maximal ideal $\mathfrak{m}$ by any ideal $\mathfrak{a}$, then we obtain the section functor $\Gamma_{\mathfrak{a}}(-)$ supported at $\{\,\mathfrak{p}\text{ prime ideal} \mid \mathfrak{p} \supseteq \mathfrak{a}\,\}$ and its derived functors $\mathrm{H}^{\ell}_{\mathfrak{a}}(-)$. Hartshorne studied in [14] R–modules M such that $\mathrm{Ext}^{\ell}_R(R/\mathfrak{a},\mathrm{H}^{\ell}_{\mathfrak{a}}(M))$ is finitely generated for all ℓ. These investigations have been continued by Melkersson as reported in [22] (in these proceedings). An R–module N is said to be $\mathfrak{a}$–*cofinite* exactly when $\mathrm{Ext}^{\ell}_R(R/\mathfrak{a},N)$ is finitely generated for all ℓ and $\mathfrak{a} \supseteq \mathfrak{p}$ whenever $\mathfrak{p}$ is a prime ideal such that $M_{\mathfrak{p}} \neq O$. Melkersson proves that the former is tantamount to finite generation of $\mathrm{Tor}^R_{\ell}(R/\mathfrak{a},M)$ for all ℓ, and interesting relations between the cofiniteness of M and that of $\mathrm{H}^{\ell}_{\mathfrak{a}}(M)$ result.

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