Elementary duality and local duality for modules

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Given a ring Λ (associative with identity) we denote by $\mathrm{Mod}(\Lambda)$ the category of (right) Λ-modules and by $\mathrm{mod}(\Lambda)$ the full subcategory of all finitely presented Λ-modules. Our aim in this note is to point out a relation between elementary duality for modules as introduced by Herzog [6] and a classical construction in module theory. We call an indecomposable pure-injective Λ-module *simply reflexive* provided that there exists a map $X \to Y$ in $\mathrm{mod}(\Lambda)$ such that the cokernel of the induced map $\mathrm{Hom}(Y, M) \to \mathrm{Hom}(X, M)$ is simple when it is regarded in the natural way as an $\mathrm{End}_\Lambda(M)^{\mathrm{op}}$-module. Given such a module M it has been shown in [8] that there exists, up to isomorphism, a unique indecomposable pure-injective Λ^{op}-module DM such that any map φ in $\mathrm{mod}(\Lambda)$ induces an epimorphism $\mathrm{Hom}_\Lambda(\varphi, M)$ if and only if it induces a monomorphism $\varphi \otimes_\Lambda DM$. Moreover, DM is again simply reflexive and satisfies $DDM = M$. Note that M is simply reflexive if and only if it is reflexive in the sense of Herzog [6] and that DM coincides with the dual of M in the sense of Herzog [6]. In this note we shall prove the following result.

Theorem *Let M be an indecomposable pure injective Λ-module and suppose that M is simply reflexive. Suppose also that M is a Λ-Γ-bimodule and that I is an injective cogenerator for* $\mathrm{Mod}(\Gamma)$. *Then DM is isomorphic to a direct summand of the Λ^{op}-module* $\mathrm{Hom}_\Gamma(M, I)$.

Remark (1) The proof of the theorem shows how to construct DM.

(2) For examples of simply reflexive modules we refer to [6]. For instance, any Σ-pure-injective module is simply reflexive.

(3) Let $\Gamma = \mathrm{End}_\Lambda(M)^{\mathrm{op}}$ and regard M in the natural way as a Γ-module. Taking a minimal injective cogenerator I the Λ^{op}-module $\mathrm{Hom}_\Gamma(M, I)$ might be called the *local dual* of M. It is well-known that $\mathrm{Hom}_\Gamma(M, I)$ is indecomposable if M is finitely presented. If M is of finite length as a Γ-module, then it has been shown by Crawley-Boevey [3] that $\mathrm{Hom}_\Gamma(M, I)$ is a coproduct of copies of DM.

(4) Recall from [8] that the *endocategory* $\mathcal{E}_M$ of M is the smallest abelian subcategory of $\mathrm{Mod}(\mathrm{End}_\Lambda(M)^{\mathrm{op}})$ containing M and all the endomorphisms of M induced by multiplication with an element from Λ. The modules M, DM and $\mathrm{Hom}_\Gamma(M, I)$ are related by a duality between $\mathcal{E}_M$ and $\mathcal{E}_{DM}$ and an equivalence between $\mathcal{E}_{DM}$ and $\mathcal{E}_{\mathrm{Hom}_\Gamma(M,I)}$.

To give a proof of the theorem we need to recall some background material. Let $\mathcal{C}_\Lambda = \operatorname{mod}(\Lambda^{\mathrm{op}})^{\mathrm{op}}$. We denote by $\operatorname{Mod}(\mathcal{C}_\Lambda)$ the category of all additive functors $(\mathcal{C}_\Lambda)^{\mathrm{op}} \to \mathrm{Ab}$ into the category of abelian groups and by $\operatorname{mod}(\mathcal{C}_\Lambda)$ the full subcategory of all finitely presented functors which is abelian. The fully faithful functor

$$\operatorname{Mod}(\Lambda) \to \operatorname{Mod}(\mathcal{C}_\Lambda), \quad M \mapsto M \otimes_\Lambda -$$

will play an important role in our considerations. An exact sequence $0 \to L \to M \to N \to 0$ in $\operatorname{Mod}(\Lambda)$ is said to be *pure-exact* if its image

$$0 \longrightarrow L \otimes_\Lambda - \longrightarrow M \otimes_\Lambda - \longrightarrow M \otimes_\Lambda - \longrightarrow 0$$

under this functor is exact and $M \in \operatorname{Mod}(\Lambda)$ is *pure-injective* if $M \otimes_\Lambda -$ is injective. Given a module M we denote by $\mathcal{S} = \mathcal{S}_M$ the kernel of the functor

$$\operatorname{mod}(\mathcal{C}_\Lambda) \to \operatorname{Mod}(\Gamma), \quad X \mapsto \operatorname{Hom}(X, M \otimes_\Lambda -)$$

where $\Gamma = \operatorname{End}_\Lambda(M)^{\mathrm{op}}$ is identified with $\operatorname{End}(M \otimes_\Lambda -)^{\mathrm{op}}$. Furthermore, denote by $\vec{\mathcal{S}}$ the full subcategory of $\operatorname{Mod}(\mathcal{C}_\Lambda)$ which consists of all direct limits $\varinjlim X_i$ with $X_i \in \mathcal{S}$ for all i.

Recall that a full subcategory $\mathcal{T}$ of any module category $\operatorname{Mod}(\mathcal{C})$ is *localizing* if it is closed under subobjects, quotients, extensions and coproducts. For any localizing subcategory $\mathcal{T}$ one can form the *quotient category* $\operatorname{Mod}(\mathcal{C})/\mathcal{T}$ which is abelian, has injective envelopes and admits an exact *quotient functor* $q\colon \operatorname{Mod}(\mathcal{C}) \to \operatorname{Mod}(\mathcal{C})/\mathcal{T}$ with $\operatorname{Ker}(q) = \mathcal{T}$ [4].

Lemma 1 *The subcategory $\vec{\mathcal{S}}$ is localizing. The quotient functor $q\colon \operatorname{Mod}(\mathcal{C}_\Lambda) \to \operatorname{Mod}(\mathcal{C}_\Lambda)/\vec{\mathcal{S}}$ sends any injective object N satisfying $\operatorname{Hom}(\mathcal{S}, N) = 0$ to an injective object and q induces an isomorphism $\operatorname{Hom}(X, N) \to \operatorname{Hom}(q(X), q(N))$ for every $X \in \operatorname{Mod}(\mathcal{C}_\Lambda)$.*

Proof: The first statement is proved in [7]. The properties of q are well-known facts which may be found in [4].

Lemma 2 *Let $\varphi\colon X \to Y$ be a map in $\operatorname{mod}(\Lambda)$ and let $U = \operatorname{Ker}(\varphi \otimes_\Lambda -)$ in $\operatorname{Mod}(\mathcal{C}_\Lambda)$. Then $\operatorname{Hom}(U, M \otimes_\Lambda -) \cong \operatorname{Coker}(\operatorname{Hom}_\Lambda(\varphi, M))$ for any $M \in \operatorname{Mod}(\Lambda)$.*

Proof: The exact sequence $0 \to U \to X \otimes_\Lambda - \to Y \otimes_\Lambda -$ induces an exact sequence

$$\operatorname{Hom}(Y \otimes_\Lambda -, M \otimes_\Lambda -) \to \operatorname{Hom}(X \otimes_\Lambda -, M \otimes_\Lambda -) \to \operatorname{Hom}(U, M \otimes_\Lambda -) \to 0.$$

Lemma 3 *Let M be indecomposable pure-injective and suppose that the cokernel of* $\mathrm{Hom}(\varphi, M)$ *is a simple* $\mathrm{End}_\Lambda(M)^{\mathrm{op}}$*-module for some φ in* $\mathrm{mod}(\Lambda)$*. Then the quotient functor* $q\colon \mathrm{Mod}(\mathcal{C}_\Lambda) \to \mathrm{Mod}(\mathcal{C}_\Lambda)/\vec{\mathcal{S}}_M$ *sends* $U = \mathrm{Ker}(\varphi \otimes_\Lambda -)$ *to a simple object and* $M \otimes_\Lambda -$ *to an injective envelope of* $q(U)$.

Proof: The object $q(U)$ is simple precisely if for any exact sequence $0 \to U' \to U \to U'' \to 0$ either $U' \in \vec{\mathcal{S}}_M$ or $U'' \in \vec{\mathcal{S}}_M$. Writing $U' = \varinjlim U_i$ as a direct limit of all its finitely generated submodules we obtain induced sequences $0 \to U_i \to U \to U/U_i \to 0$ in $\mathrm{mod}(\mathcal{C}_\Lambda)$ since $\mathrm{mod}(\mathcal{C}_\Lambda)$ is abelian. Now, for any i either $U_i \in \mathcal{S}_M$ or $U/U_i \in \mathcal{S}_M$ since there is an induced exact sequence

$$0 \to \mathrm{Hom}(U/U_i, M \otimes_\Lambda -) \to \mathrm{Hom}(U, M \otimes_\Lambda -) \to \mathrm{Hom}(U_i, M \otimes_\Lambda -) \to 0$$

of $\mathrm{End}_\Lambda(M)^{\mathrm{op}}$-modules and $\mathrm{Hom}(U, M \otimes_\Lambda -) \cong \mathrm{Coker}(\mathrm{Hom}_\Lambda(\varphi, M))$ is simple by assumption and Lemma 2. If $U/U_i \in \mathcal{S}_M$ for some i, then $U'' \in \vec{\mathcal{S}}_M$ since U'' is a quotient of U/U_i. Otherwise all $U_i \in \mathcal{S}_M$ and therefore $U' \in \vec{\mathcal{S}}_M$. Thus $q(U)$ is simple. Using again Lemma 2 there is a non-zero morphism $U \to M \otimes_\Lambda -$ and this is taken to a non-zero morphism $q(U) \to q(M \otimes_\Lambda -)$ by Lemma 1. Thus $q(M \otimes_\Lambda -)$ is an injective envelope of $q(U)$ since $q(M \otimes_\Lambda -)$ is indecomposable injective.

Lemma 4 *If M is a Λ-Γ-bimodule and $I \in \mathrm{Mod}(\Gamma)$ is injective, then* $\mathrm{Hom}_\Gamma(M, I)$ *is a pure injective Λ^{op}-module.*

Proof: See [1, I, Proposition 10.1].

Recall from [8] that a pair of modules $M \in \mathrm{Mod}(\Lambda)$ and $N \in \mathrm{Mod}(\Lambda^{\mathrm{op}})$ is *purely opposed* provided that any map φ in $\mathrm{mod}(\Lambda)$ induces an epi $\mathrm{Hom}_\Lambda(\varphi, M)$ iff it induces a mono $\varphi \otimes_\Lambda N$, equivalently if any map ψ in $\mathrm{mod}(\Lambda^{\mathrm{op}})$ induces an epi $\mathrm{Hom}_{\Lambda^{\mathrm{op}}}(\psi, N)$ iff it induces a mono $\psi \otimes_{\Lambda^{\mathrm{op}}} M$.

Lemma 5 *If M is a Λ-Γ-bimodule and $I \in \mathrm{Mod}(\Gamma)$ is an injective cogenerator, then M and* $\mathrm{Hom}_\Gamma(M, I)$ *are purely opposed.*

Proof: If I is any injective Γ-module, then there is a well-known isomorphism

$$X \otimes_\Lambda \mathrm{Hom}_\Gamma(M, I) \longrightarrow \mathrm{Hom}_\Gamma(\mathrm{Hom}_\Lambda(X, M), I)$$

for all $X \in \mathrm{mod}(\Lambda)$ which is functorial in X [2, VI, Proposition 5.2]. Taking a map φ in $\mathrm{mod}(\Lambda)$ it follows that $\varphi \otimes_\Lambda \mathrm{Hom}_\Gamma(M, I)$ is a mono iff $\mathrm{Hom}(\varphi, M)$ is an epi provided that I cogenerates $\mathrm{Mod}(\Gamma)$. Thus M and $\mathrm{Hom}_\Gamma(M, I)$ are purely opposed.

Recall from [8] that a pair of Λ-modules M and N is *purely equivalent* provided that $\mathcal{S}_M = \mathcal{S}_N$, equivalently if any map φ in $\mathrm{mod}(\Lambda)$ induces an epi $\mathrm{Hom}_\Lambda(\varphi, M)$ iff it induces an epi $\mathrm{Hom}_\Lambda(\varphi, N)$.

Lemma 6 *Let M and N be a pair of purely equivalent pure-injective Λ-modules. If M is indecomposable and simply reflexive, then M is isomorphic to a direct summand of N.*

Proof: We use again the quotient functor $q: \mathrm{Mod}(\mathcal{C}_\Lambda) \to \mathrm{Mod}(\mathcal{C}_\Lambda)/\vec{\mathcal{S}_M}$. By assumption and Lemma 1 the objects $q(M \otimes_\Lambda -)$ and $q(N \otimes_\Lambda -)$ are both injective and there are non-zero morphisms $q(U) \to q(M \otimes_\Lambda -)$ and $q(U) \to q(N \otimes_\Lambda -)$ where $q(U)$ is a simple object as defined in Lemma 3. It follows that there exists a split mono $q(M \otimes_\Lambda -) \to q(N \otimes_\Lambda -)$ which is the image of a split mono $M \otimes_\Lambda - \to N \otimes_\Lambda -$ by Lemma 1. Thus M is isomorphic to a direct summand of N.

Proof of the theorem: Let M be simply reflexive and suppose that M is a Λ-Γ-bimodule. Also let I be an injective cogenerator for $\mathrm{Mod}(\Gamma)$. The module DM is purely opposed to M and therefore purely equivalent to $\mathrm{Hom}_\Gamma(M, I)$ by Lemma 5. It follows from Lemma 6 that DM is isomorphic to a direct summand of $\mathrm{Hom}_\Gamma(M, I)$. Thus the proof is complete.

Our argument in the preceding proof shows how to construct DM for any simply reflexive Λ-module M. One uses the well-known duality $d: \mathrm{mod}(\mathcal{C}_\Lambda) \to \mathrm{mod}(\mathcal{C}_{\Lambda^{\mathrm{op}}})$ [5]. Let $U = \mathrm{Ker}(\varphi \otimes_\Lambda -) \in \mathrm{mod}(\mathcal{C}_\Lambda)$ be as in Lemma 3 and put $\mathcal{T} = d(\mathcal{S}_M)$. Adapting the argument of Lemma 3 one shows that the quotient functor $q: \mathrm{Mod}(\mathcal{C}_{\Lambda^{\mathrm{op}}}) \to \mathrm{Mod}(\mathcal{C}_{\Lambda^{\mathrm{op}}})/\vec{\mathcal{T}}$ sends $d(U)$ to a simple object. Now one uses the section functor $s: \mathrm{Mod}(\mathcal{C}_{\Lambda^{\mathrm{op}}})/\vec{\mathcal{T}} \to \mathrm{Mod}(\mathcal{C}_{\Lambda^{\mathrm{op}}})$ to find an indecomposable pure-injective Λ^{op}-module N such that $\mathrm{Hom}(\mathcal{T}, N \otimes_{\Lambda^{\mathrm{op}}} -) = 0$ and $q(N \otimes_{\Lambda^{\mathrm{op}}} -)$ is an injective envelope of $q(d(U))$. It is not hard to check that M and N are purely opposed. Finally, the uniqueness of N follows from the fact that an injective envelope of $q(d(U))$ is unique up to isomorphism. Thus $DM = N$.

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