Design of Optimized Radar Codes with a Peak to Average Power Ratio Constraint

A. De Maio, Y. Huang, M. Piezzo, S. Zhang, A. Farina

Abstract

This paper considers the problem of radar waveform design in the presence of colored Gaussian disturbance under a Peak to Average power Ratio (PAR) and an energy constraint. First of all, we focus on the selection of the radar signal optimizing the Signal to Noise Ratio (SNR) in correspondence of a given expected target Doppler frequency (Algorithm 1). Then, through a max-min approach, we make robust the technique with respect to the received Doppler (Algorithm 2), namely we optimize the worst case SNR under the same constraints as in the previous problem. Since Algorithms 1 and 2 do not impose any condition on the waveform phase, we also devise their phase quantized versions (Algorithms 3 and 4 respectively), which force the waveform phase to lie within a finite alphabet. All the problems are formulated in terms of non-convex quadratic optimization programs with either a finite or an infinite number of quadratic constraints. We prove that these problems are NP-hard and, hence, introduce design techniques, relying on Semidefinite Programming (SDP) relaxation and randomization as well as on the theory of trigonometric polynomials, providing high quality sub-optimal solutions with a polynomial time computational complexity. Finally, we analyze the performance of the new waveform design algorithms in terms of detection performance and robustness with respect to Doppler shifts.

Index Terms

Radar Waveform Design, Waveform Diversity, Non-Convex Quadratic Optimization, Semidefinite Programming Relaxation, Randomized Algorithm, Approximation Bound, Nonnegative Trigonometric Polynomials.

A. De Maio and M. Piezzo are with the Università degli Studi di Napoli "Federico II", Dipartimento di Ingegneria Biomedica, Elettronica e delle Telecomunicazioni, Via Claudio 21, I-80125 Napoli, Italy. E-mail: ademaio@unina.it, marpiez@libero.it.

Y. Huang and S. Zhang are with the Department of Systems Engineering and Engineering Management, The Chinese University of Hong Kong, Shatin, Hong Kong. E-mail: ywhuang@se.cuhk.edu.hk, zhang@se.cuhk.edu.hk.

A. Farina is with SELEX Sistemi Integrati, via Tiburtina Km.12.4, I-00131 Roma, Italy. E-mail: afarina@selex-si.com.

I. Introduction

Modern digital technology and adaptive transmitters now give the ability to generate high-accuracy, sophisticated, broad-bandwidth radar waveforms, dynamically adaptable to and optimized for a range of different tasks (detection, tracking, target recognition, etc.) potentially on a pulse-by-pulse and channel-by-channel basis. For instance, a modern multifunction phased array radar can adapt the waveform, dwell time, and update interval according to the nature of the surrounding clutter environment, the Signal to Noise Ratio (SNR), and the particular target (the most likely type of target, the threat that it may represent, and the degree to which it is manoeuvering, etc.). This is essentially the subject of waveform diversity [1], [2], [3], [4], [5], namely a new flexibility and dynamic adaptation which demands new ways of characterizing waveform properties and optimizing waveform design.

The possibility of modulating adaptively the radar signal depending on the surrounding environment and on the expected target characteristics has lead to the concept of matched-illumination [6], [7], [8], which determines the optimized transmission waveform and the corresponding receiver response through the maximization of SNR. This concept is also thoroughly investigated in [9], with reference to a Gaussian point-target and stationary Gaussian clutter, showing that the optimum allocation procedure places the signal energy in the noise band having minimum power. Recent studies concerning waveform optimization in the presence of colored disturbance can be found in [10], where a signal design approach relying on the maximization of the SNR under a similarity constraint with a given waveform is proposed and assessed. In [11], focusing on the class of linearly coded pulse trains (both in amplitude and in phase), the authors introduce a code selection algorithm which maximizes the detection performance but, at the same time, is capable of controlling both the region of achievable values for the Doppler estimation accuracy and the degree of similarity with a pre-fixed radar code. In [12] and [13], the approach is extended to account for a Space-Time Adaptive Processing and an unknown target Doppler frequency respectively. However, since in several practical situations, the radar amplifiers might work in saturation conditions and hence an amplitude modulation might be difficult to perform, in [14], the authors also consider the synthesis of constant modulus (unimodular) phase coding schemes for radar coherent pulse trains.

In this paper, we introduce a new waveform design approach relying on the maximization of the detection performance under a more general constraint than unimodularity. Specifically, we design waveforms with a bounded transmitted Peak to Average power Ratio (PAR). This constraint is very reasonable for radar applications and includes, as a special case, the phase only condition. Indeed, it has also been imposed in [15] for the synthesis of waveforms with stopband and correlation constraints. Actually, controlling the PAR permits to constrain the excursions of the squared code elements around their mean value. This also allows to keep under control the dynamic range of the transmitted waveform which is an important practical issue (for the current technology) because high PAR values necessitate a linear amplifier having a large dynamic range and this is difficult to accommodate. Finally, the PAR control is also a crucial task in OFDM systems and the interested reader might refer to [16] and references therein where this issue is addressed.

First of all, we focus on the selection of the radar waveform optimizing the SNR in correspondence of a given expected target Doppler frequency, under a PAR and an energy constraint (Algorithm 1). Notice that this problem is of practical importance when it is required a confirmation of an initial detection in a certain Doppler bin, namely when some knowledge about the Doppler frequency is available. Besides, when the Doppler parameter is unknown, the practical application of Algorithm 1 can be obtained either tuning the design Doppler to a challenging condition, dictated by the clutter Power Spectral Density (PSD) shape, or optimizing the waveform to an average scenario. This is tantamount to considering as objective function the average SNR over the possible target Doppler shifts.

Afterword, we make robust the technique with respect to the received target Doppler frequency resorting to a max-min approach (Algorithm 2). Otherwise stated, we optimize the worst case (over the target Doppler) SNR under the same constraints as in the previous problem. Since Algorithms 1 and 2 do not impose any condition on the waveform phase (i.e. the waveform phase can range within the continuous interval $[0, 2\pi)$), we also devise their phase quantized versions (Algorithms 3 and 4 respectively) which force the waveform phase to belong to a finite alphabet.

All the problems are formulated in terms of non-convex quadratic optimization problems with a finite (cases of Algorithms 1 and 3) or an infinite (cases of Algorithms 2 and 4) number of quadratic constraints. We prove that these problems are NP-hard and, hence, introduce design techniques, relying on Semidefinite Programming (SDP) relaxation and randomization[1] as well as on the theory of trigonometric polynomials [19], which approximate the optimal solution with a polynomial time computational complexity. For Algorithms 1 and 3, we also provide an analytical expression of the approximation bound which quantifies the quality of the obtained waveforms.

At the analysis stage, we assess the performance of the new technique in terms of detection probability achievable by the Neyman-Pearson receiver and robust behavior of the detection performance with respect

[1]SDP relaxation and randomization techniques have also been used in other signal processing fields. For instance, in maximum likelihood multiuser detection [17] and transmit beamforming [18].

to the Doppler frequency. The results show that the new algorithms trade off detection performance and SNR robustness with small desirable values of the PAR as well as (Algorithms 3 and 4) with the number of quantization levels used to represent the waveform phase.

The paper is organized as follows. In Section II, we present the system model and the formulation of the waveform design problems; in Sections III-VI, we devise solution algorithms for the considered problems; in Section VII, we analyze the performance of the new waveform design techniques providing numerical results aimed at assessing their quality. Finally, conclusions are given in Section VIII.

A. Notation

We adopt the notation of using boldface for vectors $\boldsymbol{a}$ and matrices $\boldsymbol{A}$. The i-th element of $\boldsymbol{a}$ and the (i,j)-th entry of $\boldsymbol{A}$ are respectively denoted by a_i and $\boldsymbol{A}_{ij}$. The transpose operator and the conjugate transpose operator are denoted by the symbols $(\cdot)^T$ and $(\cdot)^H$ respectively. $\mathrm{tr}(\cdot)$ is the trace of the square matrix argument, $\boldsymbol{I}$ and $\boldsymbol{0}$ denote respectively the identity matrix and the matrix with zero entries, while $\boldsymbol{e}_k$ is the vector with all zeros except 1 in the k-th position (their size is determined from the context). The letter j represents the imaginary unit (i.e. $j=\sqrt{-1}$), while the letter i often serves as index in this paper. $\mathbb{R}$ and $\mathbb{C}$ are respectively the set of real and complex numbers. For any complex number x, we use $\Re(x)$ and $\Im(x)$ to denote respectively the real and the imaginary parts of x, $|x|$ and $\arg(x)$ represent the modulus and the argument of x, and x^* stands for the conjugate of x. The Euclidean norm of the vector $\boldsymbol{x}$ is denoted by $\|\boldsymbol{x}\|$. The symbols $\odot$ represents the Hadamard element-wise product [20], while $E[\cdot]$ stands for the expected value operator. The curled inequality symbol $\succeq$ (and its strict form $\succ$) is used to denote generalized inequality: $\boldsymbol{A}\succeq\boldsymbol{B}$ means that $\boldsymbol{A}-\boldsymbol{B}$ is an Hermitian positive semidefinite matrix ($\boldsymbol{A}\succ\boldsymbol{B}$ for positive definiteness). $\mathbf{diag}\,(\cdot)$ denotes the vector formed by the diagonal elements of matrix argument whereas $\mathbf{Diag}\,(\cdot)$ indicates the diagonal matrix formed by the components of vector argument.

II. System Model and Formulation of the Problems

Let us focus on a monostatic radar transmitting a linearly encoded pulse train and consider the signal model of [11], where the N-dimensional column vector $\boldsymbol{v}=[v(t_0),v(t_1),\ldots,v(t_{N-1})]^T$ of the observations is expressed as

$$\boldsymbol{v}=\alpha\boldsymbol{c}\odot\boldsymbol{p}+\boldsymbol{w}\,, \tag{1}$$

with α a parameter accounting for channel propagation and target backscattering effects, $\boldsymbol{c}$ the N-dimensional column vector containing the code elements, $\boldsymbol{p}=[1,e^{j2\pi\nu_d},\ldots,e^{j2\pi(N-1)\nu_d}]^T$ the temporal steering vector, ν_d the normalized Doppler frequency, and $\boldsymbol{w} = [w(t_0), w(t_1), \ldots, w(t_{N-1})]^T$ the vector of the disturbance samples.

We are looking for codes optimizing the SNR (either in the matched case, namely in correspondence of a given normalized target Doppler, or in the worst normalized Doppler case), under a constraint on the transmitted energy, namely $\|\boldsymbol{c}\|^2 = N$, and forcing an upper bound to the PAR, i.e.

$$\text{PAR} = \frac{\max_{i=1,\ldots,N} |c_i|^2}{\frac{1}{N}\|\boldsymbol{c}\|^2} = \max_{i=1,\ldots,N} |c_i|^2, \tag{2}$$

where $\boldsymbol{c} = [c_1, \ldots, c_N]^T \in \mathbb{C}^N$. Evidently, a bound on the PAR is tantamount to imposing a more general constraint than the phase-only condition, which can be obtained letting PAR=1.

In the following, we formulate mathematically the waveform design problems, showing how the matched or worst case SNR can be optimized and the constraints can be enforced, under the assumption that $\boldsymbol{w}$ is a zero-mean complex circular Gaussian vector with known positive definite covariance matrix $E[\boldsymbol{w}\boldsymbol{w}^H] = \boldsymbol{M}$. First of all, we remind that the SNR is defined as (see [11] for the derivation of this expression as well as of the decision rule to which it refers)

$$\text{SNR} = |\alpha|^2(\boldsymbol{c}\odot\boldsymbol{p})^H\boldsymbol{M}^{-1}(\boldsymbol{c}\odot\boldsymbol{p}) = |\alpha|^2\boldsymbol{c}^H\left(\boldsymbol{M}^{-1}\odot(\boldsymbol{p}\boldsymbol{p}^\dagger)^*\right)\boldsymbol{c} = |\alpha|^2\boldsymbol{c}^H\boldsymbol{R}\boldsymbol{c}\,, \tag{3}$$

where $\boldsymbol{R} = \boldsymbol{M}^{-1}\odot(\boldsymbol{p}\boldsymbol{p}^H)^*$. Note that $\boldsymbol{R}$ is positive definite since $\boldsymbol{x}^H\boldsymbol{R}\boldsymbol{x} = (\boldsymbol{x}\odot\boldsymbol{p})^H\boldsymbol{M}^{-1}(\boldsymbol{x}\odot\boldsymbol{p}) > 0$ for any $\boldsymbol{x} \neq \boldsymbol{0}$ (which is equivalent to $\boldsymbol{x}\odot\boldsymbol{p} \neq \boldsymbol{0}$), and that $\boldsymbol{M}^{-1} = (\boldsymbol{M}^{-1}\odot(\boldsymbol{p}\boldsymbol{p}^H))\odot(\boldsymbol{p}\boldsymbol{p}^H)^*$. Hence, for a given normalized target Doppler ν_d, we can formulate the Waveform Design Problem (WDP) in terms of the following complex quadratic optimization program

$$\begin{array}{ll} \max_{\boldsymbol{c}} & \boldsymbol{c}^H\boldsymbol{R}\boldsymbol{c} \\ \text{s.t.} & \text{PAR} = \max_{i=1,\ldots,N}|c_i|^2 \leq \gamma \\ & \|\boldsymbol{c}\|^2 = N \end{array} \tag{4}$$

(PAR constrained WDP) where $1 \leq \gamma \leq N$ rules the maximum allowable PAR.

If the target Doppler is not a-priori known, it makes sense to consider the waveform optimizing the worst case SNR. This criterion leads to the following Robust PAR constrained WDP

$$\begin{array}{ll} \max_{\boldsymbol{c}} & \min_{\nu_d\in[0,1]}\boldsymbol{c}^H\boldsymbol{R}\boldsymbol{c} \\ \text{s.t.} & \text{PAR} = \max_{i=1,\ldots,N}|c_i|^2 \leq \gamma, \\ & \|\boldsymbol{c}\|^2 = N. \end{array} \tag{5}$$

Since problems (4) and (5) do not impose any condition on the waveform phase (i.e. the waveform phase can range within the continuous interval $[0, 2\pi)$), it is of interest to consider also their phase quantized versions, forcing the waveform phase to belong to a finite set. This observation leads to PAR constrained and phase quantized WDP

$$\begin{array}{ll} \max_{\boldsymbol{c}} & \boldsymbol{c}^H \boldsymbol{R} \boldsymbol{c} \\ \text{s.t.} & \text{PAR} = \max_{i=1,\ldots,N} |c_i|^2 \leq \gamma \\ & \arg c_i \in \{0, \frac{1}{M}2\pi, \ldots, \frac{M-1}{M}2\pi\},\ i=1,\ldots,N \\ & \|\boldsymbol{c}\|^2 = N \end{array} \tag{6}$$

(where the number of quantization levels M is an integer such that $M \geq 2$) and robust PAR constrained and phased quantized WDP:

$$\begin{array}{ll} \max_{\boldsymbol{c}} & \min_{\nu_d \in [0,1]} \boldsymbol{c}^H \boldsymbol{R} \boldsymbol{c} \\ \text{s.t.} & \text{PAR} = \max_{i=1,\ldots,N} |c_i|^2 \leq \gamma, \\ & \arg c_i \in \{0, \frac{1}{M}2\pi, \ldots, \frac{M-1}{M}2\pi\},\ i=1,\ldots,N \\ & \|\boldsymbol{c}\|^2 = N \end{array} \tag{7}$$

which respectively refer to the case of known and and unknown normalized target Doppler.

Before proceeding with the design of solution techniques for (4), (5), (6), and (7), we address the differences between them and the optimization problems formulated and solved in our previous papers. To this end, let us highlight what we have done in previous works:

1) the problem in [11] is a non-convex homogeneous Quadratically Constrained Quadratic Programming (QCQP) with three constraints, the strong duality holds for the problem, and a polynomial-time algorithm is established based on a suitable rank-one decomposition;
2) the problem in [12] is a non-convex homogeneous QCQP with four constraints for which strong duality does not hold in general. Nevertheless, we have shown how to construct an optimal solution in polynomial-time, provided only that the SDP relaxation of the original problem gives an optimal solution with rank not equal to two;
3) the problem in [14] is an NP-hard QCQP optimization problem due to the phase-only and the possibly finite alphabet constraint, whose optimal solution is approximated using the relaxation and randomization approach typical of the boolean Quadratic Programming (QP) problems;
4) the problem in [13] is a QCQP with infinitely many constraints, for which we establish a *deterministic* approximation procedure, with polynomial time computational complexity, to output a solution leading to high-quality radar waveforms.

In this work, we will establish new randomized approximation algorithms for the WDP (4) and its phase-quantized version (6) respectively. Due to the PAR constraint considered in (4) quite different in nature from the constraint (the similarity constraint under the infinite norm) in the optimization problem considered in [14], the approximation procedures for (4) and (6) must be re-designed and the mathematical analysis for the approximation bounds has to be re-assessed. For the robust PAR constrained WDPs (5) and (7), we will propose respective randomized approximation algorithms, in contrast to the deterministic approximation algorithm built in [13], according to some convex optimization techniques and the new randomization procedures.

III. PAR CONSTRAINED WDP

Problem (4) can be equivalently reformulated as

$$\begin{array}{ll} \max_{\boldsymbol{c}} & \boldsymbol{c}^H \boldsymbol{R} \boldsymbol{c} \\ \text{s.t.} & |c_i|^2 \le \gamma,\ i = 1, \dots, N \\ & \|\boldsymbol{c}\|^2 = N. \end{array} \tag{8}$$

Notice that when $\gamma = 1$, a feasible point for (8) has the property that $|c_i| = 1\ \forall i$, and thus the norm constraint $\|\boldsymbol{c}\|^2 = N$ is redundant, i.e., (8) reduces to

$$\begin{array}{ll} \max_{\boldsymbol{c}} & \boldsymbol{c}^H \boldsymbol{R} \boldsymbol{c} \\ \text{s.t.} & |c_i|^2 \le 1,\ i = 1, \dots, N. \end{array} \tag{9}$$

Problem (9) has been proven NP-hard in [21][2] (see relating works [22], [23], [24]) and approximation algorithms for (9) are established in [21] (see [25] also). An interesting application for (9) with all parameters and design variable being real-valued can be found with reference to blind Maximum-Likelihood (ML) detection of Orthogonal Space-Time Block Codes (OSTBCs) with unknown Channel State Information (CSI) in Multiple-Input-Multiple-Output (MIMO) transmissions [26].

In this section, we consider (8) with $\gamma > 1$, which means that the norm constraint does not vanish. Clearly, problem (8) is a non-convex QCQP with multiple constraints[3]. We claim that problem (8) with γ greater than one is NP-hard by a reduction from an even partition problem which is known to be NP-complete.

[2]Indeed, problem (9) is equivalent to (9) with all the inequality constraints becoming equality constraints, due to the fact that the maximal value of a convex function is attained only the boundary of a convex region. In other words, replacing the inequality constraints in (9) into equality ones, neither the optimal value nor the optimal solution set of problem (9) would be changed. It has been shown in [21] that the problem (9) with all equality constraints is NP-hard, thus problem (9) is NP-hard, as it stands now.

[3]For a QCQP, non-convexity does not imply that it is hard to solve; it turns out that, if the number of constraints is not too high, the QCQP can be solved efficiently; in other words, the SDP relaxation of it is tight. See [27], [28].

Proposition 3.1: The radar code design problem (8) is NP-hard with parameters $\boldsymbol{R} \succeq \boldsymbol{0}$ and $\gamma > 1$.

Proof: See Appendix A. ■

Thanks to Proposition 3.1, the radar code design problem (8) is unlikely to admit a polynomial time solution method (which means (8) is computational intractable in general). Thus, we will make efforts toward the design of an approximation algorithm for (8).

A. Approximation algorithm via semidefinite programming relaxation and randomization

To get an approximate solution (alternatively termed as a suboptimal solution) of (8), we consider its SDP relaxation:

$$\begin{array}{ll} \max_{\boldsymbol{C}} & \operatorname{tr}(\boldsymbol{RC}) \\ \text{s.t.} & C_{ii} \leq \gamma,\, i = 1, \ldots, N \\ & \operatorname{tr}(\boldsymbol{C}) = N \\ & \boldsymbol{C} \succeq \boldsymbol{0}. \end{array} \tag{10}$$

Evidently, problem (10) with the additional rank constraint $\operatorname{Rank}(\boldsymbol{C}) = 1$ is equivalent to (8). It follows from the strong duality theorem [29, Theorem 1.7.1] of SDP that (10) is solvable[4], since the SDP (10) is feasible (for example, $\boldsymbol{I}$ is a feasible point) and its dual is strictly feasible:

$$\begin{array}{ll} \min_{t_i} & t_0 N + \gamma \sum_{i=1}^{N} t_i \\ \text{s.t.} & \boldsymbol{R} - \sum_{i=1}^{N} t_i \boldsymbol{E}_i - t_0 \boldsymbol{I} \preceq \boldsymbol{0} \\ & t_i \geq 0,\, i = 1, \ldots, N \end{array} \tag{11}$$

where $\boldsymbol{E}_i$ stands for the $N \times N$ matrix with the ii-th entry being one and all other entries being zero. In practice, an optimal solution of (10) can be obtained using public solvers (such as cvx [30] and SeDuMi [31]).

Let $\boldsymbol{C}^\star$ be an optimal solution of (10). We intend to extract a rank-one feasible solution of (10) with mathematically provable quality from $\boldsymbol{C}^\star$, which may or may not be of rank-one. We remark that if $\operatorname{Rank} \boldsymbol{C}^\star$ happens to be one, then the radar code design problem (8) is solved and the SDP relaxation is tight.

However, often, it is not the case that Rank $\boldsymbol{C}^\star$ is one, which means that the SDP relaxation (10) is not tight for (8). Therefore, the design of a suitable procedure to construct in polynomial time a *suboptimal* solution of problem (8) is a compromising must. The idea of a Gaussian randomization procedure to

[4]By saying "solvable", we mean the problem is feasible, bounded above (for maximization problem), and the optimal value is attained [29, page 13].

produce an approximate solution to an NP-hard optimization problem comes from the seminal work [32] by Goemans and Williamson where the authors proposed a randomized approximation algorithm for the NP-hard max-cut problem, with the approximation bound 0.87856, via the SDP relaxation technique. Since then, a large number of NP-hard optimization problems have been solved by the approximation method of SDP-relaxation-plus-randomization, importantly with improved/tight approximation bound provable mathematically. For an overview of it from a perspective of signal processing, we suggest the reader to refer to the magazine paper [28]. Using the idea (mainly from [32] and [33] and references therein), we are going to present a Gaussian randomization procedure to obtain an approximate solution of problem (8), based on the optimal solution $\boldsymbol{C}^\star$ of the SDP relaxation problem (10). The quoted procedure requires the definition of a suitable "ad hoc" covariance matrix of the Gaussian distribution to be adopted in the randomization step. The basic criterion for selecting such a covariance matrix is that the entire randomization procedure has to lead to a feasible solution of the original problem with probability one and it has also to provide mathematical tractability in assessing the quality of the resulting solution. According to this guideline, denote by

$$\boldsymbol{d} = \sqrt{\textbf{diag}\,(\boldsymbol{C}^\star)}, \tag{12}$$

and by $\boldsymbol{d}^-$

$$(\boldsymbol{d}^-)_i = \begin{cases} 1/d_i, & \text{if } d_i > 0 \\ 1, & \text{if } d_i = 0 \end{cases} \quad i = 1, \ldots, N. \tag{13}$$

Additionally, let

$$\boldsymbol{D} = \textbf{Diag}\,(\boldsymbol{d}),\ \boldsymbol{D}^- = \textbf{Diag}\,(\boldsymbol{d}^-), \tag{14}$$

and observe that, from (12)-(14),

$$(\boldsymbol{D}^-\boldsymbol{D})_{ii} = \begin{cases} 1, & \text{if } d_i > 0 \\ 0, & \text{if } d_i = 0 \end{cases} \quad i = 1, \ldots, N. \tag{15}$$

Hence, the entries of the matrix

$$\tilde{\boldsymbol{C}}^\star = \boldsymbol{C}^\star + (\boldsymbol{I} - \boldsymbol{D}^-\boldsymbol{D}) \tag{16}$$

comply with

$$(\tilde{\boldsymbol{C}}^\star)_{ik} = \begin{cases} (\boldsymbol{C}^\star)_{ik}, & \text{if } i \neq k \\ (\boldsymbol{C}^\star)_{ii}, & \text{if } (\boldsymbol{C}^\star)_{ii} > 0 \\ 1, & \text{if } (\boldsymbol{C}^\star)_{ii} = 0 \end{cases} . \tag{17}$$

By the construction of $\tilde{\boldsymbol{C}}^\star$, we see that the diagonal elements $\tilde{\boldsymbol{C}}^\star$ are positive and and that $\tilde{C}^\star_{ii} = 1$ provided that $C^\star_{ii}$ vanishes. Exploiting the above definitions and observations, we have further important properties about $\tilde{\boldsymbol{C}}^\star$:

Proposition 3.2: Let $\boldsymbol{C}^\star$ be a positive semidefinite matrix and $\boldsymbol{d}$, $\boldsymbol{d}^-$, $\boldsymbol{D}$, $\boldsymbol{D}^-$, $\tilde{\boldsymbol{C}}^\star$ be defined as (12)-(14), (16), respectively. Then, the matrix $\boldsymbol{D}^-\tilde{\boldsymbol{C}}^\star\boldsymbol{D}^-$ enjoys the following properties:

(i) $\boldsymbol{D}^-\tilde{\boldsymbol{C}}^\star\boldsymbol{D}^- \succeq \boldsymbol{0}$;

(ii) the diagonal elements of $\boldsymbol{D}^-\tilde{\boldsymbol{C}}^\star\boldsymbol{D}^-$ are one.

Proof: See Appendix B. ■

This proposition indicates that $\boldsymbol{D}^-\tilde{\boldsymbol{C}}^\star\boldsymbol{D}^-$ can be a suitable choice for the covariance matrix of a Gaussian distribution to be adopted in our randomized approximation algorithm. Indeed, suppose that we take a Gaussian random vector $\boldsymbol{\xi}$ from the distribution $\mathcal{N}_{\mathbb{C}}(\boldsymbol{0}, \boldsymbol{D}^-\tilde{\boldsymbol{C}}^\star\boldsymbol{D}^-)$; then each component of $\boldsymbol{\xi}$ is with zero mean and unit variance (according to (ii) of Proposition 3.2), i.e., the vector $\boldsymbol{\xi}$ enjoys dependent standard complex Gaussian random components. It can be seen that with probability one, $(\sqrt{C^\star_{11}}\frac{\xi_1}{|\xi_1|}, \ldots, \sqrt{C^\star_{NN}}\frac{\xi_N}{|\xi_N|})$ is feasible for the PAR constrained WDP (4). Additionally, such a construction of the covariance $\boldsymbol{D}^-\tilde{\boldsymbol{C}}^\star\boldsymbol{D}^-$ shares some advantages in mathematically assessing the quality of a randomized approximation algorithm (as can be seen in the next sub-section). Based on these observations, in order to produce an approximate solution (i.e., a suboptimal solution, or a feasible solution) of (8), we propose the following randomization procedure (in Algorithm 1).

Algorithm 1 Gaussian randomization procedure for radar code design problem (8)

`Input:` $\boldsymbol{R}$, γ;

`Output:` a randomized approximate solution $\boldsymbol{c}$ of (8);

1: solve the SDP (10) finding $\boldsymbol{C}^\star$;

2: define $\boldsymbol{d}$, $\boldsymbol{d}^-$, $\boldsymbol{D}$, $\boldsymbol{D}^-$ according to (12)-(14);

3: draw a random vector $\boldsymbol{\xi} \in \mathbb{C}^N$ from the complex normal distribution $\mathcal{N}_{\mathbb{C}}(\boldsymbol{0}, \boldsymbol{D}^-(\boldsymbol{C}^\star + (\boldsymbol{I} - \boldsymbol{D}^-\boldsymbol{D}))\boldsymbol{D}^-)$;

4: let $c_i = \sqrt{C^\star_{ii}}e^{j\arg\xi_i}$, $i = 1, \ldots, N$.

We remark that in practice the randomization steps 3 and 4 can be repeated many times, in order to obtain a solution with better quality. As it can be directly seen, the computational cost of Algorithm 1 is dominated by solving SDP (10) which has a complexity of $O(N^{3.5}\log(1/\epsilon))$ [28], given a solution accuracy $\epsilon > 0$.

B. Approximation bound

The approximation bound of an approximation algorithm is a measure characterizing the approximate quality of the algorithm. For a randomized approximation algorithm solving a maximization (minimization) problem, an approximation bound[5] $R \in (0, 1]$ ($R \in [1, +\infty)$) means that for all instances of the problem, the algorithm always delivers a feasible solution whose expected objective functional value is at least (at most) R times the optimal value. Such an algorithm is usually called randomized R-approximation algorithm. More precisely, let $v(\cdot)$ be the optimal value of an instance of a given maximization (minimization) problem $(\cdot)$, then a feasible solution z produced by a randomized R-approximation algorithm, complies with

$$E[\text{the objective function evaluated at } z] \geq Rv(\cdot)$$

($E[\text{the objective function evaluated at } z] \leq Rv(\cdot)$ for minimization problem). It is clear that an algorithm produces a *better* approximation (for either maximization problem or minimization problem), if the approximation bound is *closer* to 1. In this subsection, we aim at establishing an approximation bound for Algorithm 1. Toward this end, let us revoke a result proved in Section 3.3, page 884 of [21]:

Lemma 3.3: Let $\boldsymbol{Z}$ be a positive semidefinite matrix with all one diagonal elements and $\boldsymbol{z}$ be a randomized vector generated setting $z_i = e^{j \arg \xi_i}$, $i = 1, \ldots, N$, where $\boldsymbol{\xi} \sim \mathcal{N}_{\mathbb{C}}(\boldsymbol{0}, \boldsymbol{Z})$. Then,

$$E[\boldsymbol{z}\boldsymbol{z}^H] = F(\boldsymbol{Z}) = \frac{\pi}{4}\boldsymbol{Z} + \frac{\pi}{2}\sum_{k=1}^{\infty} \frac{((2k)!)^2}{2^{4k+1}(k!)^4(k+1)} (\boldsymbol{Z}^T \odot \boldsymbol{Z})^{(k)} \odot \boldsymbol{Z} \succeq \frac{\pi}{4}\boldsymbol{Z} \tag{18}$$

where $(\boldsymbol{A})^{(k)}$ denotes the Hadamard product of k copies of $\boldsymbol{A}$.

Besides, from Proposition 3.2, we get

Proposition 3.4: Let $\boldsymbol{C}^\star$ be a positive semidefinite matrix and $\boldsymbol{d}$, $\boldsymbol{d}^-$, $\boldsymbol{D}$, $\boldsymbol{D}^-$, $\tilde{\boldsymbol{C}}^\star$ be defined as (12)-(14), (16), respectively. Then,

$$\boldsymbol{D}(\boldsymbol{D}^- \tilde{\boldsymbol{C}}^\star \boldsymbol{D}^-)\boldsymbol{D} = \boldsymbol{C}^\star.$$

Proof: See Appendix C. ■

Capitalizing Lemma 3.3 and Proposition 3.4, we obtain the proposition below showing that the randomized Algorithm 1 has the approximation abound $\frac{\pi}{4}$.

Proposition 3.5: Let $\boldsymbol{c}$ be the randomized solution output by Algorithm 1. Then,

$$E[\boldsymbol{c}^H \boldsymbol{R} \boldsymbol{c}] = \operatorname{tr}(\boldsymbol{R}(\boldsymbol{D}F(\boldsymbol{D}^- \tilde{\boldsymbol{C}}^\star \boldsymbol{D}^-)\boldsymbol{D})) \geq \frac{\pi}{4}\operatorname{tr}(\boldsymbol{R}\boldsymbol{C}^\star) \geq \frac{\pi}{4} v((8)) \tag{19}$$

[5]It also is termed as performance guarantee, or worst case ratio in the open literature.

where $\tilde{\boldsymbol{C}}^\star$ is defined in (16) and the function $F(\cdot)$ is defined in (18).

Proof: See Appendix D. ■

Before concluding, we remark that problem (8) is equivalent to the real-valued quadratic program:

$$\begin{array}{ll} \max_{\boldsymbol{u},\boldsymbol{v}} & [\boldsymbol{u}^T \ \boldsymbol{v}^T] \begin{bmatrix} \Re(\boldsymbol{R}) & -\Im(\boldsymbol{R}) \\ \Im(\boldsymbol{R}) & \Re(\boldsymbol{R}) \end{bmatrix} \begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{v} \end{bmatrix} \\ \text{s.t.} & u_i^2 + v_i^2 \leq \gamma, \, i = 1, \ldots, N \\ & \sum_{i=1}^N (u_i^2 + v_i^2) = N \end{array} \tag{20}$$

where $\boldsymbol{u} = \Re(\boldsymbol{c})$ and $\boldsymbol{v} = \Im(\boldsymbol{c})$. The approximation bound for the approximation algorithm solving a real-valued quadratic program like in (20) but without any special structure of the positive semidefinite matrix appearing in the objective function, obtained in [33], is $\frac{2}{\pi}$ (≈ 0.6366), instead of $\frac{\pi}{4}$ (≈ 0.7854). We see that complex quadratic program (8) is a structured real quadratic program (20); in other words, the matrix appearing in the objective function of (20) has the structure

$$\begin{bmatrix} \Re(\boldsymbol{R}) & -\Im(\boldsymbol{R}) \\ \Im(\boldsymbol{R}) & \Re(\boldsymbol{R}) \end{bmatrix},$$

rather than a general $(2N) \times (2N)$ positive semidefinite matrix. As a consequence, the complex quadratic program (8) is equivalent to a subclass of real quadratic programs, and it is reasonable that it shares a tighter approximation bound. Indeed, this phenomenon happens also in related literature as for instance in [21], [22] and [27].

IV. Robust PAR Constrained WDP

Problem (5) can be equivalently expressed as

$$\begin{array}{ll} \max_{\boldsymbol{c},t} & t \\ \text{s.t.} & t \leq \boldsymbol{p}^H (\boldsymbol{M}^{-1} \odot (\boldsymbol{c}\boldsymbol{c}^H)^*) \boldsymbol{p}, \, \forall \nu_d \in [0, 1] \\ & |c_i|^2 \leq \gamma, \, i = 1, \ldots, N \\ & \|\boldsymbol{c}\|^2 = N. \end{array} \tag{21}$$

The conventional SDP relaxation of (21) is

$$\begin{array}{ll} \max_{\boldsymbol{C},t} & t \\ \text{s.t.} & t \leq \boldsymbol{p}^H (\boldsymbol{M}^{-1} \odot (\boldsymbol{C})^*) \boldsymbol{p}, \, \forall \nu_d \in [0, 1] \\ & C_{ii} \leq \gamma, \, i = 1, \ldots, N \\ & \text{tr}\,(\boldsymbol{C}) = N \\ & \boldsymbol{C} \succeq \boldsymbol{0}. \end{array} \tag{22}$$

Problem (22) includes the infinitely many quadratic constraints $t \leq \boldsymbol{p}^H(\boldsymbol{M}^{-1} \odot (\boldsymbol{C})^*)\boldsymbol{p}$, $\forall \nu_d \in [0,1]$. However, we prove that they can be transformed into a finite number convex constraints, resorting to the SDP representation of nonnegative trigonometric polynomials [19]. To this end, we first observe that

$$\boldsymbol{p}^H(\boldsymbol{M}^{-1} \odot (\boldsymbol{C})^*)\boldsymbol{p} - t = x_0 - t + 2\Re\left(\sum_{k=1}^{N-1} x_k e^{-jk\omega}\right),$$

where $\omega = 2\pi\nu_d$ and

$$x_k = \sum_{i=1}^{N-k} (\boldsymbol{M} \odot (\boldsymbol{C})^*)_{i+k,i}, \qquad k = 0, 1, \ldots, N-1. \tag{23}$$

Hence, we exploit the following theorem, proved in [19, Theorem 3.1] and quoted here as a lemma.

Lemma 4.1: The trigonometric polynomial $f(\omega) = x_0 + 2\Re\left(\sum_{k=1}^{N-1} x_k e^{-jk\omega}\right)$ is nonnegative over $[0, 2\pi]$, if and only if there exists an $N \times N$ Hermitian matrix $\boldsymbol{X} \succeq \boldsymbol{0}$ such that

$$\boldsymbol{x} = \boldsymbol{W}^H \textbf{diag}\,(\boldsymbol{W}\boldsymbol{X}\boldsymbol{W}^H), \tag{24}$$

where $\boldsymbol{x} = [x_0, \ldots, x_{N-1}]^T$, $\boldsymbol{W} = [\boldsymbol{w}_0, \ldots, \boldsymbol{w}_{N-1}] \in \mathbb{C}^{L \times N}$, $\boldsymbol{w}_k = [1, e^{-jk\theta}, \ldots, e^{-j(L-1)k\theta}]^T$, $k = 0, \ldots, N-1$, $\theta = 2\pi/L$, $L \geq 2N-1$.

The above Lemma implies that (22) can be recast equivalently as the following SDP:

$$\begin{array}{ll} \max_{\boldsymbol{C}, \boldsymbol{X}, t} & t \\ \text{s.t.} & \boldsymbol{W}^H \textbf{diag}\,(\boldsymbol{W}\boldsymbol{X}\boldsymbol{W}^H) + t\boldsymbol{e}_1 = \boldsymbol{x} \\ & C_{ii} \leq \gamma,\ i = 1, \ldots, N \\ & \text{tr}\,(\boldsymbol{C}) = N \\ & \boldsymbol{C} \succeq \boldsymbol{0},\ \boldsymbol{X} \succeq \boldsymbol{0} \end{array} \tag{25}$$

where $\boldsymbol{x}$ is defined by (23), $\boldsymbol{e}_1 = [1, 0, \ldots, 0]^T$, and $\boldsymbol{W}$ is the same as the one defined in Lemma 4.1 by taking $L = 2N-1$.

Proposition 4.2: It holds that SDP problem (25) is solvable.

Proof: See Appendix E. ■

Let $(\boldsymbol{C}^\star, \boldsymbol{X}^\star, t^\star)$ be an optimal solution of (22). We generate feasible solutions $\boldsymbol{c}_k$, $k = 1, \ldots, K$ (K will be referred to as the number of randomizations), of (5) using $\boldsymbol{C}^\star$ in a way similar to Algorithm 1. Then we pick $\boldsymbol{c}_k$, say $\boldsymbol{c}_1$, such that the objective function value t_1 is maximal over all

$$t_k = \min_{\nu_d \in [0,1]} \boldsymbol{p}^H(\boldsymbol{M} \odot (\boldsymbol{c}_k\boldsymbol{c}_k^H)^*)\boldsymbol{p},\ k = 1, \ldots, K. \tag{26}$$

The minimization problems (26) are one dimensional optimization problem. It is seen that each problem in (26) is equivalent to an SDP. In fact, for each k, we have

$$t_k = \max_{s}\ s \ \ \text{s.t.}\ \boldsymbol{p}^H(\boldsymbol{M} \odot (\boldsymbol{c}_k\boldsymbol{c}_k^H)^*)\boldsymbol{p} \geq s,\ \forall \nu_d \in [0,1]. \tag{27}$$

It follows from Lemma 4.1 that problem (27) is equivalent to

$$\begin{aligned} t_k = \max_{\boldsymbol{X}_1, s} \quad & s \\ \text{s.t.} \quad & \boldsymbol{W}^H \mathbf{diag}\,(\boldsymbol{W}\boldsymbol{X}_1\boldsymbol{W}^H) + s\boldsymbol{e}_1 = \boldsymbol{x}_k \\ & \boldsymbol{X}_1 \succeq \mathbf{0},\ s \in \mathbb{R} \end{aligned} \tag{28}$$

where the l-th element of $\boldsymbol{x}_k$ is similar that defined in (23), i.e.,

$$(\boldsymbol{x}_k)_l = \sum_{i=1}^{N-l} (\boldsymbol{M} \odot (\boldsymbol{c}_k \boldsymbol{c}_k^H)^*)_{i+l,i}, \qquad l = 0, 1, \ldots, N-1. \tag{29}$$

Algorithm 2 summarizes the procedure to generate an approximate solution of (5).

Algorithm 2 Gaussian randomization procedure for the code design problem (5)

`Input:` $\boldsymbol{M}$, γ;

`Output:` a randomized approximate solution $\boldsymbol{c}$ of (5);

1: solve the SDP (25) finding $\boldsymbol{C}^\star$;

2: define $\boldsymbol{d}$, $\boldsymbol{d}^-$, $\boldsymbol{D}$, $\boldsymbol{D}^-$ according to (12)-(14);

3: draw random vectors $\boldsymbol{\xi}_k \in \mathbb{C}^N$ from the complex normal distribution $\mathcal{N}_{\mathbb{C}}(\mathbf{0}, \boldsymbol{D}^-(\boldsymbol{C}^\star + (\boldsymbol{I} - \boldsymbol{D}^-\boldsymbol{D}))\boldsymbol{D}^-)$, $k = 1, \ldots, K$;

4: let $(\boldsymbol{c}_k)_i = \sqrt{C^\star_{ii}} e^{j \arg(\xi_k)_i}$, $i = 1, \ldots, N$, $k = 1, \ldots, K$;

5: compute

$$t_k = \min_{\nu_d \in [0,1]} \boldsymbol{p}^H (\boldsymbol{M} \odot (\boldsymbol{c}_k \boldsymbol{c}_k^H)^*) \boldsymbol{p},\ k = 1, \ldots, K,$$

by solving the SDPs (28);

6: pick the maximal value over $\{t_1, \ldots, t_K\}$, say t_1, and output $\boldsymbol{c}_1$.

We remark that the complexity of the algorithm is dominated by the computation required for solving SDPs (25) and (28). Lastly, we point out that an alternative way to numerically solve the one dimensional problems is to perform one dimension search since each of the problems has sufficiently smooth objective function and compact feasible interval. In the numerical simulation, we shall use the Matlab ©command `fminbnd` to perform it.

V. PAR Constrained and Phase Quantized WDP

In this section, we consider the synthesis of an approximation algorithm for (6), equivalently reformulated as:

$$\begin{array}{ll} \max_{\boldsymbol{c}} & \boldsymbol{c}^H \boldsymbol{R} \boldsymbol{c} \\ \text{s.t.} & |c_i|^2 \leq \gamma \\ & \arg c_i \in \{0, \frac{1}{M}2\pi, \ldots, \frac{M-1}{M}2\pi\},\ i = 1, \ldots, N \\ & \|\boldsymbol{c}\|^2 = N. \end{array} \tag{30}$$

Clearly, when M goes to infinity, (30) becomes (8). We claim that problem (30) is also NP-hard, as shown below.

Proposition 5.1: The phase quantized code design problem (30) is NP-hard with parameters $\boldsymbol{R} \succeq \boldsymbol{0}$ and $\gamma > 1$.

Proof: See Appendix F. ■

Due to the hardness of problem (30), similar to Algorithm 1, we propose a randomized approximation algorithm based on the SDP relaxation technique (as explained in Algorithm 3). Notice that the SDP relaxation problem for (30) is (10) as well.

Algorithm 3 Gaussian randomization procedure for radar code design problem (30)

`Input:` $\boldsymbol{R}$, γ, M;

`Output:` a randomized approximate solution $\boldsymbol{c}$ of (8);

1: solve the SDP (10) finding $\boldsymbol{C}^\star$;

2: define $\boldsymbol{d}$, $\boldsymbol{d}^-$, $\boldsymbol{D}$, $\boldsymbol{D}^-$ according to (12)-(14);

3: draw a random vector $\boldsymbol{\xi} \in \mathbb{C}^N$ from the complex normal distribution $\mathcal{N}_{\mathbb{C}}(\boldsymbol{0}, \boldsymbol{D}^-(\boldsymbol{C}^\star + (\boldsymbol{I} - \boldsymbol{D}^-\boldsymbol{D}))\boldsymbol{D}^-)$;

4: let $c_i = \sqrt{C^\star_{ii}}\mu(\xi_i)$, $i = 1, \ldots, N$. where $\mu(x)$ is defined as

$$\mu(x) = \begin{cases} 1, & \text{if } \arg x \in [0, 2\pi\frac{1}{M}) \\ e^{j2\pi\frac{1}{M}}, & \text{if } \arg x \in [2\pi\frac{1}{M}, 2\pi\frac{2}{M}) \\ \vdots & \\ e^{j2\pi\frac{M-1}{M}}, & \text{if } \arg x \in [2\pi\frac{M-1}{M}, 2\pi) \end{cases}. \tag{31}$$

We remark that, using the related idea in [33], the approximation algorithm is applicable to the following quadratic program:

$$\begin{array}{ll} \max_{\boldsymbol{c}} & \boldsymbol{c}^H \boldsymbol{R}\boldsymbol{c} \\ \text{s.t.} & \arg c_i \in \{0, \frac{1}{M}2\pi, \ldots, \frac{M-1}{M}2\pi\},\ i = 1, \ldots, N \\ & [|c_1|^2, \ldots, |c_N|^2]^T \in \mathcal{F} \end{array} \tag{32}$$

where $\mathcal{F} \subseteq \mathbb{R}_+^N$ is a closed convex set. In this case, the convex relaxation of (32) is

$$\begin{array}{ll} \max_{\boldsymbol{C}} & \text{tr}\,(\boldsymbol{R}\boldsymbol{C}) \\ \text{s.t.} & \textbf{diag}\,(\boldsymbol{C}) \in \mathcal{F} \\ & \boldsymbol{C} \succeq \boldsymbol{0} \end{array} \tag{33}$$

which can be solved efficiently due to the convexity of the problem. As to the approximation bound for Algorithm 3, let us quote Lemma 3.3 of [21] as the following lemma.

Lemma 5.2: Let $\boldsymbol{Z}$ be a positive semidefinite matrix with all diagonal elements being one, $\boldsymbol{z}$ be a randomized vector generated setting $z_i = \mu(\xi_i)$, $i = 1, \ldots, N$, where $\boldsymbol{\xi} \sim \mathcal{N}_{\mathbb{C}}(\boldsymbol{0}, \boldsymbol{Z})$, and the rounding function $\mu(x)$ is defined according to (31). Then,

$$E[\boldsymbol{z}\boldsymbol{z}^H] \succeq \frac{2}{\pi}\Re(\boldsymbol{Z}) \text{ for } M = 2, \text{ and } E[\boldsymbol{z}\boldsymbol{z}^H] \succeq \frac{M^2 \sin^2 \frac{\pi}{M}}{4\pi}\boldsymbol{Z} \text{ for } M \geq 3. \tag{34}$$

Resorting to the above lemma, we have the following result concerning the approximation bound.

Proposition 5.3: Let $\boldsymbol{c}$ be the randomized solution obtained through Algorithm 3. Then,

$$E[\boldsymbol{c}^H \boldsymbol{R}\boldsymbol{c}] \geq R(M) \times \text{tr}\,(\boldsymbol{R}\boldsymbol{C}^\star) \geq R(M) \times v((8)) \tag{35}$$

where

$$R(M) = \begin{cases} \frac{2}{\pi}, & \text{if } M = 2 \\ \frac{M^2 \sin^2 \frac{\pi}{M}}{4\pi}, & \text{if } M \geq 3 \end{cases}. \tag{36}$$

Proof: The proof is based on Propositions 3.2, 3.4, and Lemma 5.2. It is completely similar to the proof of Proposition 3.5 and, thus, it is omitted here. ■

In words, Algorithm 3 is a randomized $R(M)$-approximation algorithm for (30), where some examples of $R(M)$ are $R(4) = 0.6366$, $R(8) = 0.7458$, $R(16) = 0.7754$, $R(32) = 0.7829$, $R(64) = 0.7848$, $R(128) = 0.7852$.

VI. Robust PAR Constrained and Phase Quantized WDP

In this section, we aim at solving problem (7), which can be equivalently written as

$$\begin{array}{ll} \max_{\boldsymbol{c},\, t} & t \\ \text{s.t.} & t \leq \boldsymbol{p}^H(\boldsymbol{M}^{-1} \odot (\boldsymbol{c}\boldsymbol{c}^H)^*)\boldsymbol{p},\ \forall \nu_d \in [0,1] \\ & |c_i|^2 \leq \gamma,\ i = 1, \ldots, N \\ & \arg c_i \in \{0, \frac{1}{M}2\pi, \ldots, \frac{M-1}{M}2\pi\},\ i = 1, \ldots, N \\ & \|\boldsymbol{c}\|^2 = N. \end{array} \tag{37}$$

It is verified that (22) is an SDP relaxation of (37). Let $(\boldsymbol{C}^\star, \boldsymbol{X}^\star, t^\star)$ be an optimal solution of (22). Based on $\boldsymbol{C}^\star$, we construct approximate solutions of (7), and then select the one with the best performance. Algorithm 4 summarizes the procedure to generate an approximate solution of (7).

Algorithm 4 Gaussian randomization procedure for radar code design problem (7)

`Input:` $\boldsymbol{M}$, γ, M;

`Output:` a randomized approximate solution $\boldsymbol{c}$ of (7);

1: solve the SDP (25) finding $\boldsymbol{C}^\star$;
2: define $\boldsymbol{d}$, $\boldsymbol{d}^-$, $\boldsymbol{D}$, $\boldsymbol{D}^-$ according to (12)-(14);
3: draw random vectors $\boldsymbol{\xi}_k \in \mathbb{C}^N$ from the complex normal distribution $\mathcal{N}_{\mathbb{C}}(\boldsymbol{0}, \boldsymbol{D}^-(\boldsymbol{C}^\star + (\boldsymbol{I} - \boldsymbol{D}^-\boldsymbol{D}))\boldsymbol{D}^-)$, $k = 1, \ldots, K$;
4: let $(\boldsymbol{c}_k)_i = \sqrt{C^\star_{ii}}\mu((\xi_k)_i)$, $i = 1, \ldots, N$, $k = 1, \ldots, K$, where $\mu(x)$ is defined in (31);
5: compute

$$t_k = \min_{\nu_d \in [0,1]} \boldsymbol{p}^H(\boldsymbol{M} \odot (\boldsymbol{c}_k\boldsymbol{c}_k^H)^*)\boldsymbol{p},\ k = 1, \ldots, K$$

by solving the SDPs (28);
6: pick the maximal value over $\{t_1, \ldots, t_K\}$, say t_1, and output $\boldsymbol{c}_1$.

We point out that, although we do not have an analytical approximation bound, our numerical simulations indicate that such an approximate scheme leads to high quality radar waveforms, also with a moderate sample size K. This point will be better elicited in the section addressing numerical results.

VII. Performance Analysis

This section is devoted to the performance analysis of the proposed waveform design techniques in correspondence of different values for the design parameters (namely, the PAR constraint γ, the number of randomizations K, the number of phase quantization levels M, etc.). To this end, we assume a disturbance covariance matrix $\boldsymbol{M}$, accounting for both clutter and thermal noise, with the following structure:

$$\boldsymbol{M} = \sum_{i=1}^{N_c} \beta_i \boldsymbol{p}(\nu_{d,i}) \boldsymbol{p}(\nu_{d,i})^H + \beta_n \boldsymbol{I}$$

where the number of discrete clutter scatterers $N_c = 10$, their strength $\beta_i = \beta = 10^3$, $\nu_{d,i} = (i-1)/2$, $i = 1, \ldots, 10$, and $\beta_n = 10^{-2}$.

The analysis is conducted in terms of P_d of the GLRT receiver [11] (or equivalently the standard matched flter with pre-whitening, followed by squared modulus operation and threshold comparison) for a fixed target normalized Doppler frequency $\bar{\nu}_d$ (design parameter for Algorithms 1 and 3), and robustness of the detection capabilities with respect to Doppler shifts for a fixed $\bar{\alpha}$:

$$P_d(\alpha, \bar{\nu}_d) = Q\left(\sqrt{2|\alpha|^2 \boldsymbol{c}^H \boldsymbol{R}(\bar{\nu}_d)\boldsymbol{c}}, \sqrt{-2\ln P_{fa}}\right),$$
$$P_{d,rob} = P_d(\bar{\alpha}, \nu_d), \qquad \nu_d = -\frac{1}{2}, \ldots, \frac{1}{2},\ \alpha = \bar{\alpha},$$

where $Q(\cdot, \cdot)$ is the Marcum Q function [34], assuming a false alarm probability $P_{fa} = 10^{-6}$. Additionally, due to the randomization procedures involved into Algorithms 1-4, the aforementioned performance metrics have been averaged over 500 independent trials. We explicitly highlight that, for Algorithms 1 and 3, $P_{d,rob} = P_d(\bar{\alpha}, \nu_d)$ is the detection performance obtained when the code is designed for the given $\bar{\nu}_d$, while the actual target and the receiver steering vectors are matched to the same Doppler ν_d.

In Figure 1, we plot P_d, achieved using the code devised according to Algorithm 1, versus $|\alpha|^2$, for $N = 10$, some values of γ (precisely, $\gamma \in \{1, 1.3, 1.6, 1.9, 2.2, 2.5\}$), and $\bar{\nu}_d = 0.1$. The curves highlight that greater and greater PAR parameters lead to better and better P_d values. Such behaviour was indeed expected, because increasing γ (namely, imposing a less restrictive PAR constraint on the devised code) is tantamount to increasing the size of the feasible set. However, it is also evident that, after a threshold value for γ, depending on the maximum eigenvalue of the covariance matrix $\boldsymbol{M}$, no additional performance improvements can be observed. This phenomenon has a clear analytical interpretation. In fact, for γ greater than the threshold value, the PAR constraint becomes inactive and an optimal solution to (4) coincides with an optimal solution to

$$\begin{array}{ll} \max_{\boldsymbol{c}} & \boldsymbol{c}^H \boldsymbol{R} \boldsymbol{c} \\ \text{s.t.} & \|\boldsymbol{c}\|^2 = N. \end{array} \tag{38}$$

In other words, the optimal waveform is proportional to the eigenvector of $\boldsymbol{R}$ corresponding to the maximum eigenvalue.

The robustness of Algorithms 1 and 2 with respect to target Doppler shifts is studied in Figure 2. Therein, we plot $P_{d,rob}$ versus the actual ν_d for the PAR constrained (Algorithm 1) and the Robust PAR constrained (Algorithm 2) codes, assuming $N = 10$, $K = 10$, $|\bar{\alpha}|^2 = 0$ dB, and $\gamma = \{1, 1.3, 1.6, 1.9, 2.2\}$. The nominal target Doppler for Algorithm 1 is set to $\bar{\nu}_d = 0.1$, while Algorithm 2 does not require this information. Inspection of the curves shows that Algorithm 1 outperforms Algorithm 2 when the actual target Doppler is sufficiently close to the nominal one. However, in the presence of significant Doppler mismatches, $P_{d,rob}$ of Algorithm 1 exhibits a significant deterioration, approaching values very close to zero. Besides, the transition from the Doppler interval with close to 1 detection rates to the undetectability region is quite sharp. On the contrary, the performance curves of Algorithm 2 show a quite flat shape with respect to Doppler variations, outperforming Algorithm 1 for a wide range of Doppler shifts. This feature is far more evident as γ increases, leading (for the considered values of the parameters) to codes with greater and greater detection capabilities, due to the less restrictive constraints forced into the optimization problem.

In Figure 3, we analyze the impact of the number of randomizations K on the detection performance of Algorithm 2. Specifically, we plot the worst case P_d versus $|\alpha|^2$ for $N = 10$, $\gamma = 1.3$, and several values of K ($K \in \{1, 5, 10, 25\}$). We can notice a performance improvement as K increases. This behavior can be explained based on Step 6 of Algorithm 2, which selects the code ensuring the best performance among all the K randomization experiments. It is also worth pointing out that, for a quite moderate number of randomizations, $K = 5, 10$, the performance can be considered satisfactory, in the sense that a further increase in K does not lead to additional sensible improvements in P_d.

In Figures 4 and 5, we conduct the same analysis developed in Figures 1 and 2 (for Algorithms 1 and 2) with reference to the performance of Algorithms 3 and 4. Precisely, in Figure 4, we plot P_d of the code designed according to Algorithm 3 versus $|\alpha|^2$ for $N = 10$, $\bar{\nu}_d = 0.1$, some values of the PAR parameter $\gamma \in \{1, 1.3, 1.6, 1.9, 2.2\}$, and $M = 4$ levels for the phase quantization. As in Figure 1, increasing γ leads to better and better detection levels. In Figure 5, we plot $P_{d,rob}$ versus the actual ν_d for the PAR constrained Phase quantized (Algorithm 3) and the Robust PAR constrained Phase quantized (Algorithm 4) codes, assuming $N = 10$, $K = 10$, $|\bar{\alpha}|^2 = 0$ dB, $M = 4$ and $\gamma \in \{1, 1.3, 1.6, 1.9, 2.2\}$. The nominal target Doppler for Algorithm 1 is set to $\bar{\nu}_d = 0.1$, while Algorithm 2 does not require this information. Analyzing the curves, we can repeat the same considerations as in Figure 2.

Let us now focus on Algorithms 1 and 3 and the corresponding approximation bounds. In Figure 6, we assume $N = 10$, $\bar{\nu}_d = 0.1$, $K = 10$, $M = 4$ and compare the performance of Algorithms 1 and 3 with the P_d curves obtained exploiting their approximation bounds defined by (19) and (35) respectively (i.e.

TABLE I: Average CPU time in seconds required to solve problems (10) and (25).

γ	1	1.3	1.6	1.9	2.5
SDP (10)	0.083	0.104	0.097	0.085	0.086
SDP (25)	0.097	0.143	0.158	0.128	0.112

using (19) or (35) in the first argument of the Marcum Q function in place of the respective quadratic form). Each subplot refers to a specific value of the PAR parameter γ. The plots highlight that Algorithm 1 performs better than Algorithm 3, which quantizes the phase of the transmitted waveform on four different levels. The performance loss of the latter with respect to the former is kept within 1 dB, for $P_d = 0.9$, and is quite acceptable considering also the easiest hardware implementation of a phase quantized waveform. It is also interesting to observe that the P_d curves obtained using the approximation bound provide a quite good approximation of the actual detection performance, for all the considered values of the parameter γ and for both the considered algorithms. As a matter of fact, the lower bound approximation is at most 2 dB far from the true P_d curve.

In the last part of this section, we investigate the effects of the number of quantization levels. Specifically, in Figure 7, we plot P_d versus $|\alpha|^2$ for $\bar{\nu}_d = 0.1$, $K = 10$, $\gamma = 1.3$, and several values of M ($M \in \{2, 4, 8, 16\}$). As expected, increasing the number of quantization levels, leads to better and better performances until $M \leq 8$. Then, a saturation effect is experienced and the performance obtained by the phase quantized Algorithm 3 ends up coincident with that provided by Algorithm 1, which, as already pointed out, assumes code elements with phases ranging in a continuous interval.

Finally, before concluding this section, we provide in Table I the average CPU time required to solve the SDP problem (10) (and (25)) which is the most computational expensive step of Algorithms 1 and 3 (Algorithms 2 and 4). All the experiments were conducted on a desktop computer equipped with a Intel Core 2 Quad Q9400 CPU (2.66 GHz). The results highlight that the computational time is quite modest and acceptable for all the considered values of γ. Nevertheless, it is also worth pointing out that the waveform design must not necessary be performed on-line. It can be also implemented off-line producing a waveform library [5] and then during the operation a waveform from the library is selected for that particular scenario.

VIII. Conclusions

In this paper, we have considered radar waveform design in the presence of colored Gaussian disturbance under a PAR and an energy constraint. First of all, we have focused on the selection of the radar signal optimizing the SNR in correspondence of a given expected target Doppler frequency (Algorithm 1). Then, through a max-min approach, we have devised a robust version (with respect to the received Doppler) of the aforementioned technique (Algorithm 2), optimizing the worst case SNR under the same constraints as in the previous problem. Since Algorithms 1 and 2 do not impose any condition on the waveform phase, we have also introduced their phase quantized versions (Algorithms 3 and 4 respectively), forcing the waveform phase to belong to a finite alphabet. Actually, this is a quite nice feature for a practical implementation of the techniques. All the problems have been formulated in terms of non-convex quadratic optimization programs with a finite (Algorithm 1 and 3) or an infinite (Algorithm 2 and 4) number of quadratic constraints. We have proved the NP-hard nature of the problems and, hence, have introduced design techniques, relying on Semidefinite Programming (SDP) relaxation and randomization as well as on the theory of trigonometric polynomials, which provide high quality sub-optimal solutions with a polynomial time computational complexity.

At the analysis stage, we have evaluated the performance of the devised algorithms, considering both the detection probability achieved by the Neyman-Pearson detector, as well as the robustness with respect to target Doppler shifts. Additionally, we have studied the effects of the possible phase quantization showing the trade off existing between the number of quantization levels and some simplicity in circuitry implementation.

Possible future research tracks might concern the generalization of the waveform design problem so as to account for an additional similarity constraint with a known code sequence. This new approach will pave the way to a joint control of both the PAR and the waveform ambiguity function. Unfortunately, the additional constraint cannot be easily handled and the design of a solution method to the resulting optimization problems is still an open issue.

Appendix

A. Proof of Proposition 3.1

Proof: It is clear that problem (8) is equivalent to the problem:

$$\begin{array}{ll} \max_{\boldsymbol{z}} & \boldsymbol{z}^H \boldsymbol{R} \boldsymbol{z} \\ \text{s.t.} & |z_i|^2 \leq 1,\, i = 1, \ldots, N \\ & \|\boldsymbol{z}\|^2 = N/\gamma. \end{array} \tag{39}$$

Let $N = 3P+1$, $\gamma = 1 + \frac{P}{2P+1}$, $\boldsymbol{z} = [\boldsymbol{x}^T, \boldsymbol{y}^T]^T$, where $\boldsymbol{x} = [z_0, z_1, \ldots, z_P, z_{P+1}, \ldots, z_{2P}]^T$ and $\boldsymbol{y} = [z_{2P+1}, \ldots, z_{3P}]^T$; let $\boldsymbol{b}_0 = [\frac{-j}{2}\boldsymbol{e}^T\boldsymbol{a}, \boldsymbol{a}^T, \boldsymbol{0}_P^T, \boldsymbol{0}_P^T]^T$, $\boldsymbol{b}_i = [-j, \boldsymbol{e}_i^T, -\boldsymbol{e}_i^T, \boldsymbol{0}_P^T]^T$, $i = 1, \ldots, P$, where $\boldsymbol{a} \in \mathbb{R}^P$ is a given vector with integer-valued components and $\boldsymbol{e} \in \mathbb{R}^P$ is the all-one vector. Let λ be any number not less than the maximal eigenvalue of $\sum_{i=0}^P \boldsymbol{b}_i\boldsymbol{b}_i^H$, and $\boldsymbol{R}$ be

$$\begin{bmatrix} \lambda \boldsymbol{I}_{2P+1} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0}_{P\times P} \end{bmatrix} - \sum_{i=0}^P \boldsymbol{b}_i\boldsymbol{b}_i^H. \tag{40}$$

This previous assumption ensures $\boldsymbol{R} \succeq \boldsymbol{0}$. Therefore, we have

$$\boldsymbol{z}^H\boldsymbol{R}\boldsymbol{z} = \lambda\|\boldsymbol{x}\|^2 - \sum_{i=0}^P |\boldsymbol{z}^H\boldsymbol{b}_i|^2 \leq \lambda N/\gamma = \lambda(2P+1) \tag{41}$$

and the equality holds for any feasible point $\boldsymbol{z}$ for (39), if and only if $|z_i| = 1$, $i = 0, \ldots, 2P$, and $\boldsymbol{b}_i^H\boldsymbol{z} = 0$, $i = 0, \ldots, P$. That is, all z_i, $i = 0, \ldots, 2P$, are of unit modulus and

$$\frac{j}{2}\boldsymbol{e}^T\boldsymbol{a}z_0 + \sum_{i=1}^P a_i z_i = 0,\ jz_0 + z_i - z_{P+i} = 0,\ k = 1, \ldots, P,$$

which, due to nonzero z_0, are equivalent to

$$\frac{j}{2}\boldsymbol{e}^T\boldsymbol{a} + \sum_{i=1}^P a_i(z_i/z_0) = 0,\ j + z_i/(z_0) - z_{P+i}/(z_0) = 0,\ i = 1, \ldots, P, \tag{42}$$

Set $z_i/z_0 = e^{j\theta_i}$, $i = 1, \ldots, 2P$, and the last P equations of (42) become

$$\cos\theta_i - \cos\theta_{P+i} = 0,\ \ 1 + \sin\theta_i - \sin\theta_{P+i} = 0,\ i = 1, \ldots, P,$$

which imply that $\theta_i = -\theta_{P+i} \in \{-\frac{\pi}{6}, -\frac{5}{6}\pi\}$, and the first equation of (42) becomes

$$\frac{1}{2}\boldsymbol{e}^T\boldsymbol{a} + \sum_{i=1}^P a_i \sin\theta_i = 0,\ \sum_{i=1}^P a_i\cos\theta_i = 0,$$

which further amounts to

$$\sum_{i=1}^P a_i\cos\theta_i = 0,\ \theta_i \in \{-\frac{\pi}{6}, -\frac{5}{6}\pi\},\ i = 1, \ldots, P.$$

This is clearly equivalent to the partition problem described in [35, pp 47 - 60], namely finding a binary vector $\boldsymbol{x}$ such that

$$\sum_{i=1}^P a_i x_i = 0,\ x_i \in \{\pm 1\},\ i = 1, \ldots, P. \tag{43}$$

Summarizing, we arrive at the conclusion that finding a feasible solution such that (41) is valid with equality is equivalent to finding a solution $\boldsymbol{x} \in \mathbb{R}^P$ of (43). ∎

B. Proof of Proposition 3.2

Proof: (i) It follows from (15) that $\boldsymbol{I} - \boldsymbol{D}^{-}\boldsymbol{D} \succeq \boldsymbol{0}$. Thus $\tilde{\boldsymbol{C}}^{\star} = \boldsymbol{C}^{\star} + (\boldsymbol{I} - \boldsymbol{D}^{-}\boldsymbol{D}) \succeq \boldsymbol{0}$, which implies $\boldsymbol{D}^{-}\tilde{\boldsymbol{C}}^{\star}\boldsymbol{D}^{-} \succeq \boldsymbol{0}$.

(ii) It is seen immediately from (12)-(14) and (17). ■

C. Proof of Proposition 3.4

Proof: Notice that $\boldsymbol{D}^{-}\boldsymbol{D} = \boldsymbol{D}\boldsymbol{D}^{-}$, namely $\boldsymbol{D}$ and $\boldsymbol{D}^{-}$ commute. Since $\boldsymbol{C}^{\star}$ is positive semidefinite, then

$$\boldsymbol{D}\boldsymbol{D}^{-}\boldsymbol{C}^{\star}\boldsymbol{D}^{-}\boldsymbol{D} = \boldsymbol{C}^{\star},$$

where we use the fact that if a positive semidefinite matrix has a diagonal element 0, then the corresponding row and column contains all zero elements. Observe that $(\boldsymbol{I} - \boldsymbol{D}^{-}\boldsymbol{D})\boldsymbol{D}^{-}\boldsymbol{D} = \boldsymbol{0}$. Then, it follows that

$$\boldsymbol{D}\boldsymbol{D}^{-}\tilde{\boldsymbol{C}}^{\star}\boldsymbol{D}^{-}\boldsymbol{D} = \boldsymbol{D}\boldsymbol{D}^{-}(\boldsymbol{C}^{\star} + (\boldsymbol{I} - \boldsymbol{D}^{-}\boldsymbol{D}))\boldsymbol{D}^{-}\boldsymbol{D} = \boldsymbol{C}^{\star}.$$

■

D. Proof of Proposition 3.5

Proof: Let $y_i = e^{j \arg \xi_i}$, $i = 1, \ldots, N$, where ξ_i is generated by step 3 of Algorithm 1. Thus $\boldsymbol{c} = \boldsymbol{D}\boldsymbol{y}$. It follows from Lemma 3.3 that the expectation of $\boldsymbol{y}\boldsymbol{y}^H$ is

$$E[\boldsymbol{y}\boldsymbol{y}^H] = F(\boldsymbol{D}^{-}\tilde{\boldsymbol{C}}^{\star}\boldsymbol{D}^{-}) \succeq \frac{\pi}{4}\boldsymbol{D}^{-}\tilde{\boldsymbol{C}}^{\star}\boldsymbol{D}^{-}.$$

Therefore, we have

$$\begin{aligned}
E[\boldsymbol{c}^H\boldsymbol{R}\boldsymbol{c}] &= E[\boldsymbol{y}^H\boldsymbol{D}\boldsymbol{R}\boldsymbol{D}\boldsymbol{y}] \\
&= \operatorname{tr}(\boldsymbol{D}\boldsymbol{R}\boldsymbol{D}E[\boldsymbol{y}\boldsymbol{y}^H]) \\
&\geq \frac{\pi}{4}\operatorname{tr}(\boldsymbol{D}\boldsymbol{R}\boldsymbol{D}\boldsymbol{D}^{-}\tilde{\boldsymbol{C}}^{\star}\boldsymbol{D}^{-}) \\
&= \frac{\pi}{4}\operatorname{tr}(\boldsymbol{R}\boldsymbol{D}\boldsymbol{D}^{-}\tilde{\boldsymbol{C}}^{\star}\boldsymbol{D}^{-}\boldsymbol{D}) \\
&= \frac{\pi}{4}\operatorname{tr}(\boldsymbol{R}\boldsymbol{C}^{\star}) \\
&\geq \frac{\pi}{4}v((8))
\end{aligned}$$

where the first inequality is due to the fact that $\boldsymbol{D}\boldsymbol{R}\boldsymbol{D} \succeq \boldsymbol{0}$ and, in the last equality, we apply Proposition 3.4. ■

E. Proof of Proposition 4.2

Proof: Let us work on the dual problem of (25), and show that it is strictly feasible and bounded above, which by the strong duality [29, Theorem 1.7.1], means that (25) is solvable.

Recall that $\boldsymbol{W} = [\boldsymbol{w}_0, \ldots, \boldsymbol{w}_{N-1}] \in \mathbb{C}^{L\times N}$, $\boldsymbol{w}_k = [1, e^{-jk\theta}, \ldots, e^{-j(L-1)k\theta}]^T$, $k = 0, \ldots, N-1$, $\theta = 2\pi/L$, $L = 2N-1$. Then, we can rewrite $\boldsymbol{W}$ as

$$\boldsymbol{W} = \begin{bmatrix} \boldsymbol{v}_0^H \\ \boldsymbol{v}_1^H \\ \vdots \\ \boldsymbol{v}_{L-1}^H \end{bmatrix}, \ \boldsymbol{v}_m = \begin{bmatrix} 1 \\ e^{jm\theta} \\ \vdots \\ e^{j(N-1)m\theta} \end{bmatrix}, \ m = 0, \ldots, L-1. \tag{44}$$

Thus, $\boldsymbol{W}^H \mathbf{diag}\,(\boldsymbol{W}\boldsymbol{X}\boldsymbol{W}^H) = \sum_{m=0}^{L-1} (\boldsymbol{v}_m^H \boldsymbol{X} \boldsymbol{v}_m)\boldsymbol{v}_m$. From the equality constraint $t\boldsymbol{e}_1 = \boldsymbol{x} - \boldsymbol{W}^H \mathbf{diag}\,(\boldsymbol{W}\boldsymbol{X}\boldsymbol{W}^H)$, we have

$$t = \sum_{i=1}^{N} (\boldsymbol{M} \odot \boldsymbol{C}^*)_{ii} - \sum_{m=0}^{L-1} \boldsymbol{v}_m^H \boldsymbol{X} \boldsymbol{v}_m, \tag{45}$$

and

$$\sum_{i=1}^{N-k} (\boldsymbol{M} \odot \boldsymbol{C}^*)_{i+k,i} = \sum_{m=0}^{L-1} \boldsymbol{v}_m^H \boldsymbol{X} \boldsymbol{v}_m e^{jkm\theta}, \ k = 1, \ldots, N-1. \tag{46}$$

It is clear that (45) can be further rewritten into

$$t = \mathrm{tr}\,(\boldsymbol{A}_0 \boldsymbol{C}) - \mathrm{tr}\,(\boldsymbol{B}_0 \boldsymbol{X}), \tag{47}$$

where

$$\boldsymbol{A}_0 = \boldsymbol{I} \odot \boldsymbol{M}, \qquad \boldsymbol{B}_0 = \sum_{m=0}^{L-1} \boldsymbol{v}_m \boldsymbol{v}_m^H, \tag{48}$$

Observe that (46) has $2(N-1)$ equalities (counting the real part and imaginary part):

$$\mathrm{tr}\,(\boldsymbol{A}_{k,1}\boldsymbol{C}) = \mathrm{tr}\,(\boldsymbol{B}_{k,1}\boldsymbol{X}), \ \ \mathrm{tr}\,(\boldsymbol{A}_{k,2}\boldsymbol{C}) = \mathrm{tr}\,(\boldsymbol{B}_{k,2}\boldsymbol{X}), \ k = 1, \ldots, N-1 \tag{49}$$

where

$$\boldsymbol{B}_{k,1} = \sum_{m=0}^{L-1} \boldsymbol{v}_m \boldsymbol{v}_m^H \cos(km\theta), \ \ \boldsymbol{B}_{k,2} = \sum_{m=0}^{L-1} \boldsymbol{v}_m \boldsymbol{v}_m^H \sin(km\theta), \ k = 1, \ldots, N-1, \tag{50}$$

and

$$\boldsymbol{A}_{k,1} = \frac{1}{2}\boldsymbol{M}_k, \qquad \boldsymbol{A}_{k,2} = \frac{1}{2}(\boldsymbol{M}_k \odot \boldsymbol{E}), \qquad k = 1, \ldots, N-1. \tag{51}$$

The $N \times N$ Hermitian matrices $\boldsymbol{M}_k$, $k = 1, \ldots, N-1$, are defined by

$$(\boldsymbol{M}_k)_{i+k,i} = (\boldsymbol{M})_{i+k,i}, \qquad i = 1, \ldots, N-k \tag{52}$$

and the diagonal elements and the other lower triangular elements of $\boldsymbol{M}_k$ are zero. The $N \times N$ Hermitian matrix $\boldsymbol{E}$ is defined by

$$\begin{cases} (\boldsymbol{E})_{ii} = 1, \ \ i = 1, \ldots, N, \\ (\boldsymbol{E})_{il} = -j, \ \ \forall i > l. \end{cases} \tag{53}$$

By considering (45)-(47), (49), we can rewrite problem (25) equivalently into the following form

$$\begin{array}{ll} \max_{\boldsymbol{X}, \boldsymbol{C}} & \text{tr}\,(\boldsymbol{A}_0 \boldsymbol{C}) - \text{tr}\,(\boldsymbol{B}_0 \boldsymbol{X}) \\ \text{s.t.} & \text{tr}\,(\boldsymbol{A}_{k,1} \boldsymbol{C}) - \text{tr}\,(\boldsymbol{B}_{k,1} \boldsymbol{X}) = 0, \ \ k = 1, \ldots, N-1 \\ & \text{tr}\,(\boldsymbol{A}_{k,2} \boldsymbol{C}) - \text{tr}\,(\boldsymbol{B}_{k,2} \boldsymbol{X}) = 0, \ \ k = 1, \ldots, N-1 \\ & \text{tr}\,(\boldsymbol{E}_i \boldsymbol{C}) \leq \gamma, \, i = 1, \ldots, N \\ & \text{tr}\,(\boldsymbol{C}) = N \\ & \boldsymbol{C} \succeq \boldsymbol{0}, \, \boldsymbol{X} \succeq \boldsymbol{0} \end{array} \tag{54}$$

where $\boldsymbol{E}_i$ are the same as those in problem (11). Therefore, the dual problem of (54) is

$$\begin{array}{ll} \min_{\{y_i\}, z, \{x_{k,1}\}, \{x_{k,2}\}} & \gamma \sum_{i=1}^N y_i + Nz \\ \text{s.t.} & z\boldsymbol{I} + \sum_{i=1}^N y_i \boldsymbol{E}_i + \sum_{k=1}^{N-1} (x_{k,1} \boldsymbol{A}_{k,1} + x_{k,2} \boldsymbol{A}_{k,2}) \succeq \boldsymbol{A}_0, \\ & \sum_{k=1}^{N-1} (x_{k,1} \boldsymbol{B}_{k,1} + x_{k,2} \boldsymbol{B}_{k,2}) \preceq \boldsymbol{B}_0, \\ & y_i \geq 0, \, i = 1, \ldots, N, \, z \in \mathbb{R}, \, x_{k,1} \in \mathbb{R}, \, x_{k,2} \in \mathbb{R}, \, k = 1, \ldots, N-1. \end{array} \tag{55}$$

We take a point $\boldsymbol{c}$ satisfying $|c_i| \leq \gamma$ for $i = 1, \ldots, N$ and $\|\boldsymbol{c}\| = N$, and set

$$t = \min_{\nu_d \in [0,1]} \boldsymbol{p}^\dagger (\boldsymbol{M} \odot (\boldsymbol{c}_0 \boldsymbol{c}_0^\dagger)^*) \boldsymbol{p},$$

which is a one-dimensional optimization. It follows from (28) that solving the one-dimensional optimization is equivalent to solving an SDP. Thus $(\boldsymbol{c}, t)$ is feasible for (21) and $(\boldsymbol{c}\boldsymbol{c}^H, t)$ is feasible for (22), and thus (25) is feasible. It follows by the weak duality theorem that the dual SDP (55) is bounded below.

It is further seen that problem (55) is strictly feasible. In fact, let z be a sufficiently large positive number, y_i positive numbers sufficiently close to zero, $x_{k,1}$, $x_{k,2}$ equal to zero; then $(z, y_1, \ldots, y_N, x_{1,1}, x_{1,2}, \ldots, x_{N-1,1}, x_{N-1,2})$ is a strictly feasible solution of (55). Here, we note that $\boldsymbol{B}_0 = \boldsymbol{W}^H \boldsymbol{W}$ is the diagonal matrix with each diagonal element being L. Therefore, we conclude that problem (25) is solvable, because the dual is bounded below and strictly feasible. ■

F. Proof of Proposition 5.1

Proof: We are showing that problem (30) includes the max-cut problem and the max-3-cut problem which are known to be NP-hard [32], [36], and [37]. In fact, problem (30) is equivalent to

$$\begin{aligned} \max_{\boldsymbol{c}} \quad & \boldsymbol{c}^H \boldsymbol{R} \boldsymbol{c} \\ \text{s.t.} \quad & |c_i|^2 \leq 1 \\ & \arg c_i \in \{0, \tfrac{1}{M}2\pi, \dots, \tfrac{M-1}{M}2\pi\},\ i = 1, \dots, N \\ & \|\boldsymbol{c}\|^2 = N/\gamma. \end{aligned} \tag{56}$$

The max-cut problem for a given undirected weighted graph (E, V) with P nodes, is cast as

$$\begin{aligned} \max_{\boldsymbol{x}} \quad & \textstyle\sum_{k<l} w_{kl}(1 - x_k x_l)/2 \\ \text{s.t.} \quad & x_k \in \{\pm 1\},\ k = 1, \dots, P \end{aligned} \tag{57}$$

where $w_{kl} \geq 0$ is the weight on the edge between nodes k and l[6]. Let $\boldsymbol{Q}$ be the Laplacian matrix of the graph, i.e., $Q_{kl} = -w_{kl}$ for $k \neq l$ and $Q_{kk} = \sum_{l \neq k,\, l=1}^{P} w_{kl}$. Thus, $\boldsymbol{Q} \succeq \boldsymbol{0}$ and the objective function of max-cut problem (57) is equal to $\frac{1}{4}\boldsymbol{x}^T \boldsymbol{Q} \boldsymbol{x}$. Now, in (56), setting $M = 2$ (this means that $\arg c_i \in \{0, \pi\}$, $\forall i$, i.e., any c_i is real-valued), $N = 2P$, $\gamma = 2$ (this implies that $\|\boldsymbol{c}\|^2 = P$), and

$$\boldsymbol{R} = \begin{bmatrix} \frac{1}{4}\boldsymbol{Q} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0}_{P \times P} \end{bmatrix}$$

(the so-defined $\boldsymbol{R}$, together with $\|\boldsymbol{c}\|^2 = P$ and $|c_i| \leq 1\ \forall i$, implies that an optimal solution $\boldsymbol{c}^\star$ of the maximization problem (56), has $|c_i^\star| = 1$, $i = 1, \dots, P$, and $|c_i^\star| = 0$, $i = P+1, \dots, 2P$), we reduce (56) into the max-cut problem (57).

■

References

[1] A. Farina, "Waveform Diversity: Past, Present, and Future", *Third International Waveform Diversity & Design Conference*, Plenary Talk, Pisa, June 2007.

[2] M. Wicks, "A Brief History of Waveform Diversity", *Proceedings of the IEEE 2009 Radar Conference*, Pasadena, May 2009.

[3] Special issue on "Adaptive Waveform Design for Agile Sensing and Communications", Edited by A. Nehorai, F. Gini, M. S. Greco, A. Papandreou-Suppappola, and M. Rangaswamy, *IEEE Journal of Selected Topics in Signal Processing*, Vol. 1, no. 1, pp. 2-213, June 2007.

[4] R. Calderbank, S. Howard and B. Moran, "Waveform Diversity in Radar Signal Processing", *IEEE Signal Processing Magazine*, Vol. 26, no. 1, pp. 32-41, February 2009.

[6] When there is no edge between k and l, one sets $w_{kl} = 0$.

[5] D. Cochran, S. Suvorova, S. Howard and B. Moran, “Waveform Libraries”, *IEEE Signal Processing Magazine*, Vol. 26, no. 1, pp. 12-21, February 2009.

[6] D. T. Gjessing, *Target Adaptive Matched Illumination Radar: Principles and Applications*, Peter Peregrinus: IEE Electromagnetic Waves Series, July 1986.

[7] A. Farina and F. A. Studer, “Detection with High Resolution Radar: Great Promise, Big Challenge”, *Microwave Journal*, pp. 263-273, May 1991.

[8] S. U. Pillai, H. S. Oh, D. C. Youla, and J. R. Guerci, “Optimum Transmit-Receiver Design in the Presence of Signal-Dependent Interference and Channel Noise”, *IEEE Transactions on Information Theory*, Vol. 46, no. 2, pp. 577-584, March 2000.

[9] S. Kay, “Optimal Signal Design for Detection of Point Targets in Stationary Gaussian Clutter/Reverberation”, *IEEE Journal on Selected Topics in Signal Processing*, Vol. 1, no. 1, pp. 31-41, June 2007.

[10] J. Li, J. R. Guerci, and L. Xu, “Signal Waveform’s Optimal-under-Restriction Design for Active Sensing”, *IEEE Signal Processing Letters*, Vol. 13, no. 9, pp. 565-568, September 2006.

[11] A. De Maio, S. De Nicola, Y. Huang, S. Zhang, and A. Farina, “Code Design to Optimize Radar Detection Performance under Accuracy and Similarity Constraints”, *IEEE Transactions on Signal Processing*, Vol. 56, no. 11, pp. 5618-5629, November 2008.

[12] A. De Maio, S. De Nicola, Y. Huang, D. Palomar, S. Zhang, and A. Farina, “Code Design for Radar STAP via Optimization Theory”, *IEEE Transactions on Signal Processing*, vol. 58, no. 2, pp. 679-694, February 2010.

[13] A. De Maio, Y. Huang, and M. Piezzo, “A Doppler Robust Max-Min Approach to Radar Code Design”, ”, *IEEE Transactions on Signal Processing*, vol. 58, no. 9, pp. 4943-4947, September 2010.

[14] A. De Maio, S. De Nicola, Y. Huang, Z. Q. Luo, and S. Zhang, “Design of Phase Codes for Radar Performance Optimization with a Similarity Constraint”, *IEEE Transactions on Signal Processing*, Vol. 57, no. 2, pp. 610-621, February 2009.

[15] H. He, P. Stoica, and J. Li, “Waveform Design with StopBand and Correlation Constraints for Cognitive Radar,” *The 2nd International Workshop on Cognitive Information Processing*, Elba Island, Italy, June 2010.

[16] T. Jiang and Y. Wu, “An Overview: Peak-to-Average Power Ratio Reduction Techniques for OFDM Signals,” *IEEE Trans. on Broadcasting*, vol. 54, no. 2, pp. 257-268, February 2008.

[17] W. K. Ma, T. N. Davidson, K. M. Wong, Z. Q. Luo, and P. C. Ching, “Quasi-Maximum-Likelihood Multiuser Detection Using Semi-Definite Relaxation with Application to Synchronous CDMA”, *IEEE Transactions on Signal Processing*, Vol. 50, no. 4, pp. 912 - 922, April 2002.

[18] E. Karipidis, N. D. Sidiropoulos, Z. Q. Luo, “Convex Transmit Beamforming for Downlink Multicasting to Multiple Co-Channel Groups”, *Proceedings of the 2006 IEEE International Conference on Acoustics, Speech and Signal Processing*, Vol. 5, Toulouse, France, 14-19 May 2006.

[19] T. Roh and L. Vandenberghe, “Discrete Transforms, Semidefinite Programming, and Sum-of-Squares Representations of Nonnegative Polynomials,” *SIAM Journal on Optimization*, Vol. 16, no. 4, pp. 939 - 964, 2006.

[20] R. A. Horn and C. R. Johnson, *Matrix Analysis*, Cambridge University Press, 1985.

[21] S. Zhang and Y. Huang, “Complex Quadratic Optimization and Semidefinite Programming,” *SIAM Journal on Optimization*, Vol. 16, no. 3, pp. 871 - 890, 2006.

[22] Z. Q. Luo, N. D. Sidiropoulos, P. Tseng, and S. Zhang, ”Approximation Bounds for Quadratic Optimization with Homogeneous Quadratic Constraints,” *SIAM Journal on Optimization*, Vol. 18, pp. 1-28, February 2007.

[23] A. Nemirovski, C. Roos, and T. Terlaky, ”On Maximization of Quadratic Form over Intersection of Ellipsoids with Common Center,” Mathematcial Programming, Vol. 86, pp. 463-473, 1999.

[24] N. Sidiropoulos, T. Davidson, and Z.-Q. Luo, “Transmit Beamforming for Physical-Layer Multicasting,” *IEEE Transactions on Signal Processing*, Vol. 54, no. 6, pp. 2239 - 2251, June 2006.

[25] A. So, J. Zhang, and Y. Ye, “On Approximating Complex Quadratic Optimization Problems Via Semidefinite Programming Relaxations,” *Mathematical Programming*, Series B, Vol. 110, no. 1, pp. 93 - 110, June 2007.

[26] W.-K. Ma, B.-N. Vo, T. N. Davidson, and P.-C. Ching, “Blind ML Detection of Orthogonal Space-Time Block Codes: Efficient High-Performance Implementations”, *IEEE Transactions on Signal Processing*, Vol. 54, no. 2, pp. 738 - 751, November 2008.

[27] Y. Huang and S. Zhang, “Complex Matrix Decomposition and Quadratic Programming”, *Mathematics of Operations Research*, Vol. 32, no. 3, pp. 758-768, Aug. 2007.

[28] Z.-Q. Luo, W.-K. Ma, A.M.-C. So, Y. Ye, and S. Zhang, “Semidefinite Relaxation of Quadratic Optimization Problems,” *IEEE Signal Processing Magazine*, Vol. 27, no. 3, pp. 20 - 34, May 2010.

[29] A. Nemirovski, “Lectures on Modern Convex Optimization”, *Class Notes, Georgia Institute of Technology*, Fall 2005.

[30] M. Grant and S. Boyd, “CVX: Matlab Software for Disciplined Convex Programming (web page and software),” http://stanford.edu/∼boyd/cvx, December 2008.

[31] J. F. Sturm, “ Using SeDuMi 1.02, a MATLAB Toolbox for Optimization over Symmetric Cones,” *Optimization Methods and Software*, Vol. 11-12, pp. 625-653, August 1999.

[32] M. X. Goemans and D. P. Williamson, “Improved Approximation Algorithms for Maximum Cut and Satisfiability Problem using Semi-Definite Programming,” *Journal of the ACM*, Vol. 42, pp. 1115 - 1145, 1995.

[33] S. Zhang, “Quadratic Maximization and Semidefinite Relaxation,” *Mathematical Programming*, Ser. A, Vol. 87, no. 3, pp. 453 - 465, 2000.

[34] S. M. Kay, *Fundamentals of Statistical Signal Processing: Detection Theory*, Englewood Cliffs, NJ: Prentice-Hall, 1998, vol. II.

[35] M. R. Garey and D. S. Johnson, *Computers and Intractability: A Guide to the Theory of NP-Completeness*. San Francisco, CA: Freeman, 1979.

[36] M. X. Goemans and D. P. Williamson, “Approximation Algorithms for MAX-3-CUT and Other Problems Via Complex Semidefinite Programming”, *Journal of Computer and System Sciences*, Vol. 68, no. 2, pp. 442 - 470, March 2004.

[37] A. Frieze and M. Jerrum, “Improved Approximation Algorithms for MAX k-CUT and MAX BISECTION”, *Algorithmica*, Vol. 18, no. 1, pp. 67 - 81, May 1997.

Figure 1: P_d versus $|\alpha|^2$ for $P_{fa} = 10^{-6}$, $\bar{\nu}_d = 0.1$, $N = 10$ and $\gamma \in \{1, 1.3, 1.6, 1.9, 2.2, 2.5\}$. Algorithm 1 - PAR constrained code.

Figure 2: P_d versus ν_d for $P_{fa} = 10^{-6}$, $|\bar{\alpha}|^2 = 0$ dB, $\bar{\nu}_d = 0.1$, $K = 10$, $N = 10$, and

Figure 3: Worst case P_d versus $|\alpha|^2$ for $P_{fa} = 10^{-6}$, $\gamma = 1.3$, $N = 10$, and $K \in \{1, 5, 10, 25\}$ randomizations. Algorithm 2 - Robust PAR constrained code.

Figure 4: P_d versus $|\alpha|^2$ for $P_{fa} = 10^{-6}$, $\bar{\nu}_d = 0.1$, $M = 4$, $N = 10$, and $\gamma \in \{1, 1.3, 1.6, 1.9, 2.2\}$. Algorithm 3 - PAR constrained Phase quantized code.

Figure 5: P_d versus ν_d for $P_{fa} = 10^{-6}$, $|\bar{\alpha}|^2 = 0$ dB, $\bar{\nu}_d = 0.1$,

Figure 6a: P_d versus $|\alpha|^2$ for $P_{fa} = 10^{-6}$, $\bar{\nu}_d = 0.1$, $M = 4$, $K = 10$, $N = 10$ and $\gamma = 1$. Algorithm 1 - PAR constrained code (solid line). Approximation Bound of Algorithm 1 (dashed o-marked

Figure 6b: P_d versus $|\alpha|^2$ for $P_{fa} = 10^{-6}$, $\bar{\nu}_d = 0.1$, $M = 4$, $K = 10$, $N = 10$ and $\gamma = 1.3$. Algorithm 1 - PAR constrained code (solid line). Approximation Bound of Algorithm 1 (dashed o-marked

Figure 6c: P_d versus $|\alpha|^2$ for $P_{fa} = 10^{-6}$, $\bar{\nu}_d = 0.1$, $M = 4$, $K = 10$, $N = 10$ and $\gamma = 1.9$. Algorithm 1 - PAR constrained code (solid line). Approximation Bound of Algorithm 1 (dashed o-marked

Figure 6d: P_d versus $|\alpha|^2$ for $P_{fa} = 10^{-6}$, $\bar{\nu}_d = 0.1$, $M = 4$, $K = 10$, $N = 10$ and $\gamma = 2.5$. Algorithm 1 - PAR constrained code (solid line). Approximation Bound of Algorithm 1 (dashed o-marked

Figure 7: P_d versus $|\alpha|^2$ for $P_{fa} = 10^{-6}$, $\bar{\nu}_d = 0.1$, $\gamma = 1.3$, $K = 10$, and $M \in \{2, 4, 8, 16\}$. Algorithm 3 - PAR constrained Phase quantized code (dashed-dotted lines). Algorithm 1 - PAR constrained code (o-marked curve). Notice that the curve of Algorithm 1 overlaps with that referring to Algorithm 3 for $M = 8$ and $M = 16$.