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Mean, median and mode filtering of images

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If a two-dimensional image is simplified by repeatedly replacing its values with the mean in an infinitesimal neighbourhood, it evolves according to the diffusion equation $L_t = L_{xx} + L_{yy}$ (subscripts denote differentiation). This equation can alternatively be written in gauge coordinates as $L_t = L_{vv} + L_{ww}$, where the v-direction is tangent to the isophote and the w-direction is along the gradient. The alternative evolution scheme of $L_t = L_{vv}$ has also attracted attention. Guichard & Morel showed, in 1996, that this equation describes the operation of repeated infinitesimal median filtering. In this paper it is proved that repeated infinitesimal mode filtering is described by $L_t = L_{vv} - 2L_{ww}$ at regular points, and $L_t = 0$ at critical points. Other new results are (i) an approximate equation for median filtering at critical points, and (ii) a derivation of the equation for median filtering at regular points, which generalizes to mean and mode filtering. Finally, the results of numerical implementation of all three filtering schemes are briefly presented.

Keywords: scale space; anisotropic diffusion; measures of central location; partial differential equations; mean curvature flow

1. Introduction

The concept of progressively simplifying an image is well established (Marr 1982; ter Haar Romeny 1994; Griffin 1995). The obvious technique is to replace simultaneously the value at each point of the image by the average of the values within an aperture around the point. Iterating this procedure progressively simplifies the image. The effect of such filtering will depend upon the size and shape of the aperture. To remove this dependency it is natural to consider the limiting process as smaller apertures are used and the number of iterations is increased. In the limit, the apertures become infinitesimal and the iteration number is replaced by a continuous time parameter. At the limit, the effect of filtering can only be dependent on differential measures of the image, and so infinitesimal filtering is always describable by a partial differential equation (PDE). PDE schemes for processing images are becoming increasingly important (Weickert 1998).

Different definitions of average result in different filtering schemes (Davies 1988; Evans & Nixon 1995; Torroba *et al.* 1994). Using the mean as the averaging operator results in a filtering scheme equivalent to linear diffusion (Koenderink 1984). Linear diffusion is described (in the case of two-dimensional images) by the equation $L_{t_2} = L_{xx} + L_{yy}$, where x and y are image coordinates and t_2 is the time parameter.

Given that mean filtering is described by a simple equation and has found wide applicability in image processing, it seems productive to consider the effect of using

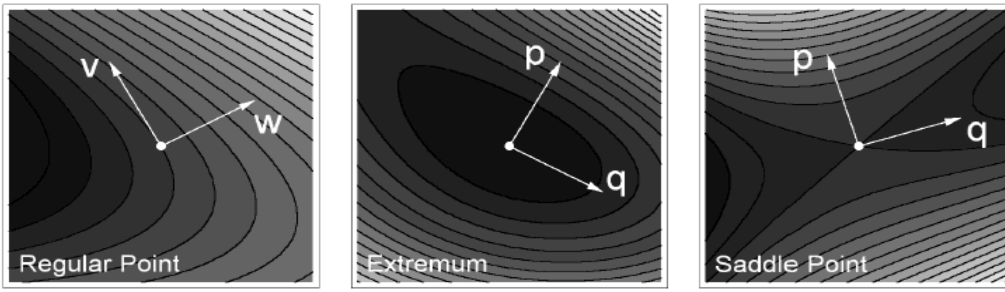

Figure 1. The gauge coordinate systems used in the paper. The w, v-gauge is used at regular points: the w-direction is along the gradient, and the v-direction is tangent to the isophote. The p, q-gauge is used at critical points (extremum and saddles); it is chosen such that the mixed derivative, L_{pq}, disappears; and the magnitude of the second derivative in the p-direction exceeds that in the q-direction.

averaging operators other than the mean, in particular the median and mode. The case of median filtering has been analysed by Guichard & Morel (1995), who showed that in the limiting case, at points with non-zero gradient (regular points), the image evolves according to

$$L_{t_1} = \frac{L_y^2 L_{xx} - 2L_x L_y L_{xy} + L_x^2 L_{yy}}{L_x^2 + L_y^2}.$$

This equation may be written more simply in gauge coordinates. In particular, in w, v-gauge coordinates, where the w-direction is along the gradient and the v-direction is tangent to the isophote (figure 1*a*), the equation becomes $L_{t_1} = L_{vv}$. Mean filtering is described, in such coordinates, by $L_{t_2} = L_{vv} + L_{ww}$.

The main result proved in this paper (conjectured in Griffin (1998)) is that mode filtering is described, at regular points, by $L_{t_0} = L_{vv} - 2L_{ww}$. Exact and approximate equations for mode and median filtering, respectively, at critical points are also derived. Finally, results of numerical implementations of mean, median and mode filtering are briefly presented.

2. Mathematical preliminaries

The equivalence between mean filtering and linear diffusion is almost independent of the shape of the apertures considered when deriving the action of mean filtering. This is because

(i) mean filtering is equivalent to convolution with the filter aperture;

(ii) n-times application of a convolution kernel is equal to the single application of the n-times convolution product of the kernel; and

(iii) the limiting n-times product of most symmetrical kernels is the same, a two-dimensional Gaussian.

Given (iii) (which is the central limit theorem), the most natural apertures to consider are Gaussian apertures of non-zero width. Additional arguments for the naturalness of Gaussian apertures have been made (Koenderink 1984; Alvarez *et al.* 1993).

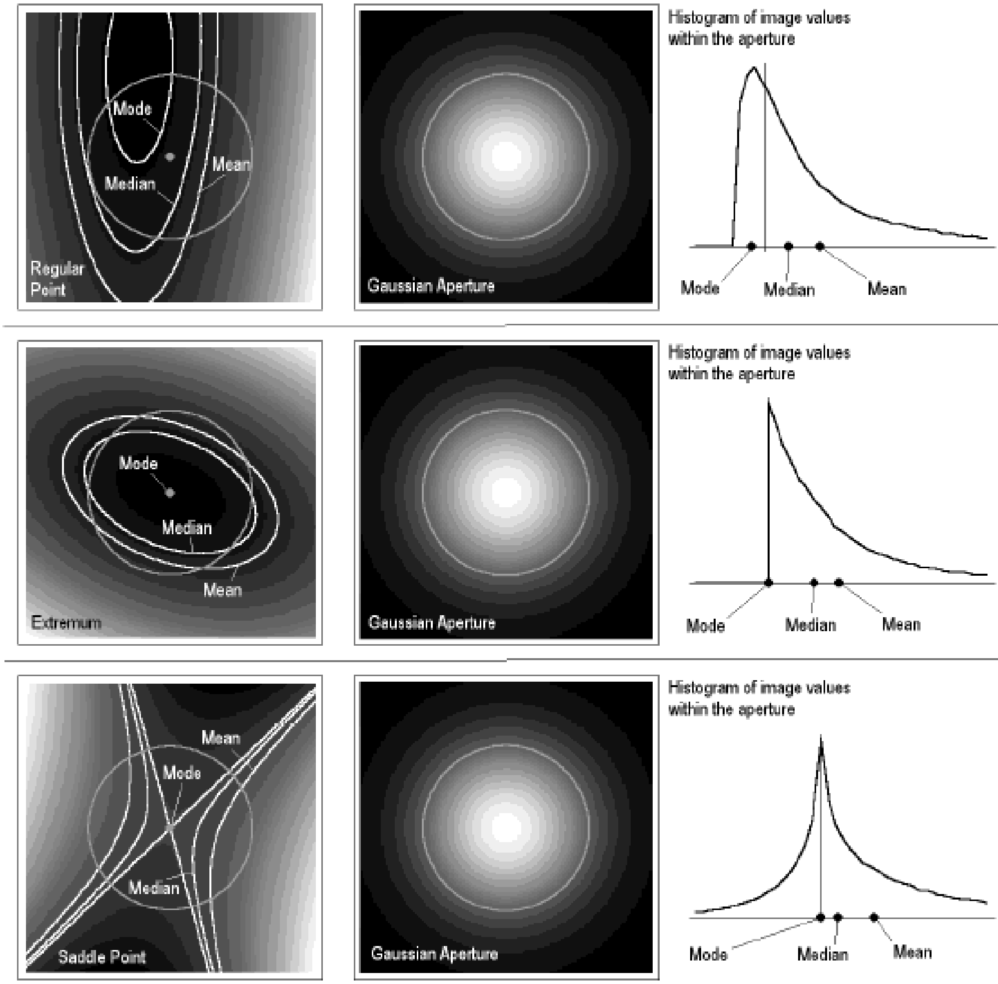

Figure 2. Example image regions (left column, density plot) centred on regular and critical points. The central column shows a Gaussian aperture. The right column shows the histograms of the images on the left within the aperture shown in the centre. The mean, median and mode of these histograms are marked, as are the corresponding isophotes in the underlying images (left). The median isophotes divide the image into two regions with equal integral of the aperture weighting.

To define Gaussian apertures I first define the one-dimensional Gaussian function with scale parameter t:

$$G_t(x) := \frac{1}{\sqrt{4\pi t}} \mathrm{e}^{-x^2/4t}.$$

A two-dimensional origin-centred aperture may then be defined as

$$G_t\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) := G_t(x) G_t(y).$$

The effect of filtering an image $L : \mathbb{R}^2 \to \mathbb{R}$ with non-infinitesimal apertures is most easily defined in two steps (see Griffin (1997) and Koenderink & van Doorn (1999); see also figure 2). The first step is to form the histogram of the image values visible within the aperture; at the origin this is given by

$$H_t[L](p) := \int_{\boldsymbol{r}\in\mathbb{R}^2} G_t(\boldsymbol{r})\delta(L(\boldsymbol{r}) - p),$$

where p ranges over image values and δ is the Dirac delta function (Richards & Youn 1990). The second step is to operate on the histogram to produce an average value. The three averaging operators considered here are

(i) the mean operator,

$$\mu_2(D) := \int_{p\in\mathbb{R}} pD(p) \Big/ \int_{p\in\mathbb{R}} D(p)$$

(where D is a histogram);

(ii) the median operator,

$$\mu_1(D) := m \qquad \text{s.t. } \int_{-\infty}^{m} D(p)\,\mathrm{d}p = \frac{1}{2}\int_{-\infty}^{\infty} D(p)\,\mathrm{d}p;$$

(iii) and the mode operator

$$\mu_0(D) := m \qquad \text{s.t. } D'(m) = 0, \quad D''(m) < 0.$$

I denote the combined effect of the two steps, when performed at the origin, by $a_r(t) := \mu_r(H_t[L])$, where $r = 0, 1, 2$. Here, as elsewhere, the subscripts 0, 1 and 2 are used to indicate mode, median and mean schemes, respectively. This labelling is not arbitrary and follows from the definition of central locations as values where $|p|^r \otimes D(p)$ is a local minimum (Fréchet 1948).

It will be necessary to consider μ'_r the derivatives of the averaging operators; defined by

$$\mu'_r[D](p) := \lim_{\alpha\to 0}(1/\alpha)(\mu_r(D + \alpha\delta_p) - \mu_r(D)),$$

where δ_p stands for a delta function at image value p. The derivative of the mean operator is easily shown to be $\mu'_2[D](p) = p$, which is independent of D because of the linearity of the mean operator. In contrast, the nonlinearity of the median and mode operators means that their derivatives do depend on D, so calculation will be deferred until later sections when relevant histograms have been identified.

The action of filtering with infinitesimal apertures is equal to

$$\lim_{t\to 0}(1/t)a_r(t) = a'_r(0).$$

To calculate this, I start from the easy result that

$$a_r(t) = t^{1/2}\mu_r(H_1[t^{-1/2}L(t^{1/2}\cdot_{-})]).$$

Taking the derivative with respect to t, one obtains

$$a_r'(t) = \tfrac{1}{2}t^{-1/2}\mu_r(H_1[t^{-1/2}L(t^{1/2}._{-})]) \\ + \int_{p\in\mathbb{R}} \left\{ t^{1/2}\frac{\partial}{\partial t}H_1[t^{-1/2}L(t^{1/2}._{-})](p) \right\} \mu_r'[H_1[t^{-1/2}L(t^{1/2}._{-})]](p). \quad (2.1)$$

The expression in (2.1) within curly braces can be expanded as follows:

$$t^{1/2}\frac{\partial}{\partial t}H_1[t^{-1/2}L(t^{1/2}._{-})](p) \\ = \tfrac{1}{2}t^{1/2}\int_{\boldsymbol{r}\in\mathbb{R}^2} G_1(\boldsymbol{r})\delta'(t^{-1/2}L(t^{1/2}\boldsymbol{r}) - p)(-t^{-3/2}L(t^{1/2}\boldsymbol{r}) + t^{-1}\boldsymbol{r}\cdot L'(t^{1/2}\boldsymbol{r})). \quad (2.2)$$

Further steps in deriving the action of mean, median and mode filtering depend on whether or not the image is regular at the point being considered, and will be taken in the following two sections.

3. Filtering at regular points

At a regular point one may always choose orthogonal directions w and v such that the image gradient vanishes along the v-direction and is positive in the w-direction (figure 1, left). Thus, since addition of a constant image has a trivial effect on all filtering schemes, the image in the neighbourhood of the origin can be described by

$$L\left(\begin{pmatrix} w \\ v \end{pmatrix}\right) = wL_w + \tfrac{1}{2}w^2L_{ww} + wvL_{wv} + \tfrac{1}{2}v^2L_{vv} + \cdots,$$

where $L_w > 0$. Given that one is at a regular point, the derivation started in (2.2) can be continued and then evaluated at $t = 0$:

$$\left\{ t^{1/2}\frac{\partial}{\partial t}H_1[t^{-1/2}L(t^{1/2}._{-})](p) \right\}_{t=0} \\ = \frac{1}{2}\iint_{w,v\in\mathbb{R}} G_1(w,v)\delta'(wL_w - p)(\tfrac{1}{2}w^2L_{ww} + wvL_{wv} + \tfrac{1}{2}v^2L_{vv}) \\ = \frac{-1}{4L_w^2}(2L_w^2(L_{vv} - 2L_{ww}) + p^2L_{ww})G'_{L_w^2}(p). \quad (3.1)$$

It is easily confirmed that, with appropriate scaling, the histogram of image values within the aperture tends to Gaussian form as the aperture shrinks to a point, and the width of the distribution is a function of the gradient, L_w, at the aperture centre. Symbolically,

$$\lim_{t\to 0} H_1[t^{-1/2}L(t^{1/2}._{-})] = G_{L_w^2}. \quad (3.2)$$

Equations (3.1) and (3.2) can now be used to evaluate equation (2.1) at $t = 0$:

$$a_r'(0) = \tfrac{1}{2}a_r'(0) - \frac{1}{4L_w^2}\int_{p\in\mathbb{R}} \mu_r'[G_{L_w^2}](p)\, G'_{L_w^2}(p)\,(2L_w^2(L_{vv} - 2L_{ww}) + p^2L_{ww}).$$

Hence

$$a'_r(0) = \frac{-1}{2L_w^2} \int_{p\in\mathbb{R}} \mu'_r[G_{L_w^2}](p)\, G'_{L_w^2}(p)\, (2L_w^2(L_{vv} - 2L_{ww}) + p^2 L_{ww}). \tag{3.3}$$

Knowing that the limiting shape of the histogram is Gaussian, the appropriate derivatives of the averaging operators μ_r may now be calculated. They are

$$\begin{aligned} &\mu'_2[\_](p) = p, && \text{as already noted,} \\ &\mu'_1[G_{L_w^2}] = \frac{-\Delta}{2\sqrt{\pi}L_w}, && \text{where } \Delta \text{ is the unit step function } \Delta := \int \delta, \\ &\mu'_0[G_{L_w^2}] = -\frac{L_w}{\sqrt{\pi}}\delta'. && \end{aligned}$$

Inserting these into (3.3) and evaluating leads to

$$a'_2(0) = L_{vv} + L_{ww}, \qquad a'_1(0) = L_{vv}, \qquad a'_0(0) = L_{vv} - 2L_{ww},$$

and thus the PDEs for mean, median and mode filtering, at regular points, are:

$$\begin{aligned} L_{t_2} &= L_{vv} + L_{ww}, && \text{mean filtering,} \\ L_{t_1} &= L_{vv}, && \text{median filtering,} \\ L_{t_0} &= L_{vv} - 2L_{ww}, && \text{mode filtering.} \end{aligned}$$

4. Filtering at critical points

At a critical point one may always choose orthogonal directions p and q such that the mixed second derivative L_{pq} vanishes at the point. To resolve the ambiguity as to which direction is p and which is q, the constraint $|L_{pp}| > |L_{qq}|$ is used. Examples of such a gauge are shown in figure 1 (centre and right). Using this gauge, the image in the neighbourhood of a critical point is described by

$$L\left(\begin{pmatrix} p \\ q \end{pmatrix}\right) = \tfrac{1}{2}p^2 L_{pp} + \tfrac{1}{2}q^2 L_{qq} + \cdots .$$

Taking the limit to infinitesimal apertures is easier at critical points than at regular ones, because the irrelevant first-order structure does not need to be finessed by algebraic manipulation. In particular, it can be shown that

$$\lim_{t\to 0}(1/t)\mu_r(H_t[\tfrac{1}{2}p^2 L_{pp} + \tfrac{1}{2}q^2 L_{qq} + \cdots]) = \mu_r(H_1[\tfrac{1}{2}p^2 L_{pp} + \tfrac{1}{2}q^2 L_{qq}]).$$

(*a*) *Mean filtering*

Mean filtering does not show any special behaviour at critical points. The diffusion equation can be written as

$$L_{t_2} = L_{pp} + L_{qq} = L_{xx} + L_{yy} = \nabla^2 L,$$

where ∇^2 is the coordinate-independent Laplacian operator. This is well defined at critical points just as it is at regular ones.

(*b*) *Mode filtering*

Mode filtering does not affect the image value at critical points, i.e. $L_{t_0} = 0$. This is apparent from considering the forms of the local histograms at such points (figure 2, middle and bottom). For example, at a minimum (L_{pp}, $L_{qq} \geqslant 0$) the histogram is given by

$$H_1[\tfrac{1}{2}p^2 L_{pp} + \tfrac{1}{2}q^2 L_{qq}](v) = \begin{cases} \dfrac{1}{4\pi}\displaystyle\int_{-\pi}^{\pi} \dfrac{\exp\{-v/[2(\cos(\theta)^2 L_{pp} + \sin(\theta)^2 L_{qq})]\}}{\cos(\theta)^2 L_{pp} + \sin(\theta)^2 L_{qq}}\,\mathrm{d}\theta & \text{if } v \geqslant 0, \\ 0 & \text{if } v < 0. \end{cases} \tag{4.1}$$

By taking the derivative of (4.1) with respect to v, the histogram at a minimum can be shown to be decreasing for positive v and, thus, has a single maximum at $v = 0$. An equation similar to (4.1) can be deduced for the histogram at a saddle point; and again by taking derivatives it can be shown that the only maximum is at $v = 0$.

(*c*) *Median filtering*

Calculating the effect of median filtering at critical points is easy for the following special cases:

$$\begin{aligned} L_{pp} = -L_{qq} \quad &\Rightarrow \quad L_{t_1} = 0, \qquad \text{a balanced saddle (Griffin \& Colchester 1995)}, \\ L_{qq} = 0 \quad &\Rightarrow \quad L_{t_1} = 2L_{pp}\,\mathrm{erf}^{-1}(\tfrac{1}{2})^2 \approx 0.455(L_{pp} + L_{qq}), \\ &\qquad\qquad \text{effectively a one-dimensional extremum}, \\ L_{pp} = L_{qq} \quad &\Rightarrow \quad L_{t_1} = 2\ln(2)L_{pp} \approx 0.693(L_{pp} + L_{qq}), \qquad \text{a circular extremum.} \end{aligned}$$

To derive an exact result for median filtering at general critical points it is necessary to integrate equation (4.1) with respect to v between limits 0 and c; and then find the value of c that produces an integral of value $\frac{1}{2}$. I have failed to find an exact solution to this procedure. Instead, by numerical methods, I have fitted the following approximate equation, which

(i) satisfies the property

$$\mu_1(H_1[\tfrac{1}{2}p^2(\alpha L_{pp}) + \tfrac{1}{2}q^2(\alpha L_{qq})]) = \alpha\mu_1(H_1[\tfrac{1}{2}p^2 L_{pp} + \tfrac{1}{2}q^2 L_{qq}]);$$

(ii) is exact for the three special cases above; and

(iii) is accurate to within 5% for other cases:

$$\begin{aligned} \mu_1(H_1[\tfrac{1}{2}p^2 L_{pp} + \tfrac{1}{2}q^2 L_{qq}]) &\approx \Pi(L_{pp}, L_{qq}) \\ &= L_{pp}\left(0.455 + 0.907\frac{L_{qq}}{L_{pp}} + 0.457\left(\frac{L_{qq}}{L_{pp}}\right)^2 \right. \\ &\qquad \left. - 0.214\left(\frac{L_{qq}}{L_{pp}}\right)^3 - 0.218\left(\frac{L_{qq}}{L_{pp}}\right)^4\right). \end{aligned} \tag{4.2}$$

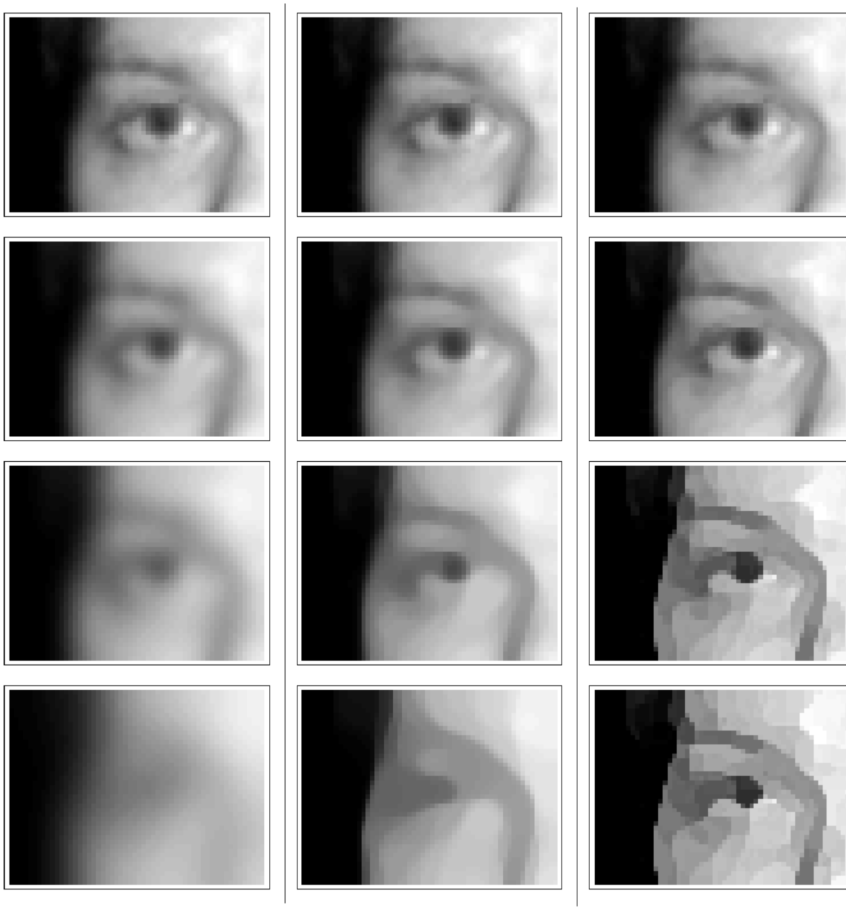

Figure 3. The effect of mean (left column), median (centre) and mode (right) filtering. The first row shows the original image; the other rows reading downwards show the progressive effect of repeated filtering after 4, 16 and 64 iterations. The mode-filtered image at bottom right is the final state for this image; further mode filtering has no effect.

5. Numerical implementation

Figure 3 shows the results of mean, median and mode filtering applied to an image of a human eye. Full details of the algorithm will be given in a further publication. In brief, Euler's method (Press *et al.* 1992) is used to compute each filtered image from the preceding one. Spatial derivatives are calculated with respect to the weighted neighbourhood

$$\frac{1}{36}\begin{pmatrix} 1 & 4 & 1 \\ 4 & 16 & 4 \\ 1 & 4 & 1 \end{pmatrix}$$

commonly used in image processing (Lindeberg 1990).

Of the three filtering methods, mean and median filtering are the most similar in effect. The difference between the two is most easily grasped when viewing an animation of the filtering process. It is then clear that in mean filtering, light and dark blobs spread, thin and overlap without interacting (because of the linearity of mean filtering). In contrast, median filtering is clearly nonlinear. The impression when viewing it is of the isophotes of the image simplifying and shrinking to points. This is a correct impression because isophote curvature can be expressed as $\kappa_{\text{iso}} = -L_{vv}/L_w$ (ter Haar Romeny *et al*. 1994), and, hence, at regular points, median filtering moves isophotes in a locally normal direction at a speed proportional to their local curvature (Guichard & Morel 1995).

Mode filtering is distinctly different in effect from mean or median. As mode filtering progresses, the $-2L_{ww}$ term has the effect of de-blurring and so enhancing edges, while the $+L_{vv}$ term stabilizes this process and prevents the developing loci of discontinuity from becoming too ragged. Away from developing edges, the image changes in value towards nearby critical points. The final result seems to be a mosaic of plateaus separated by discontinuities, though this has not been proved. At this point, the image is unaffected by further applications of the mode-filtering procedure.

Inspecting the final mode-filtered image in figure 3 one can see several features that seem artefactual, for example the dark spur extending from the top left of the iris region. This is confirmed when viewing the animation, when occasional 'leakages' can be observed. These artefacts seem to be a consequence of the way the PDE has been discretized rather than a genuine effect of mode filtering.

6. Concluding remarks

The new results in this paper are

(i) the PDEs for mode filtering at regular and critical points;

(ii) an approximate PDE for median filtering at critical points; and

(iii) a new proof of the PDE for median filtering at regular points which generalizes to mean and mode filtering.

All results have been confirmed by numerical calculations (using Monte Carlo techniques).

Mode filtering appears to have the potential to be a useful tool in image processing. The aim of finding PDEs that will 'clean up' images and generate natural segmentations has led several authors (Gabor 1965; see also Lindenbaum *et al*. 1994; Kramer & Bruckner 1975; Osher & Rudin 1990; Pollak *et al*. 1997) to propose similar PDEs, primarily on ad hoc grounds.

Finally, there is an interesting relationship that exists between the PDEs for mean, median and mode filtering at regular points, namely $3\delta_{t_1} = \delta_{t_0} + 2\delta_{t_2}$. This exact equation reminds us of the approximate relationship $3\mu_1 \approx \mu_0 + 2\mu_2$ that has been observed to be true for moderately skewed histograms (Pearson 1895). That the relationship holds exactly in this case is presumably due to the infinitesimal nature of the histograms involved.

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