Capacity of the Gaussian Relay Channel with Correlated Noises to Within a Constant Gap

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***Abstract*—This paper studies the relaying strategies and the approximate capacity of the classic three-node Gaussian relay channel, but where the noises at the relay and at the destination are correlated. It is shown that the capacity of such a relay channel can be achieved to within a constant gap of $\frac{1}{2}\log_2 3 = 0.7925$ bits using a modified version of the noisy network coding strategy, where the quantization level at the relay is set in a correlation dependent way. As a corollary, this result establishes that the conventional compress-and-forward scheme also achieves to within a constant gap to the capacity. In contrast, the decode-and-forward and the single-tap amplify-and-forward relaying strategies can have an infinite gap to capacity in the regime where the noises at the relay and at the destination are highly correlated, and the gain of the relay-to-destination link goes to infinity.**

***Index Terms*—Relay channel, approximate capacity, noise correlation, noisy network coding.**

I. Introduction

The relay channel models a communication scenario where an intermediate relay is deployed to assist the direct communication between a source and the destination. Although the capacity of the relay channel is still not known exactly even for the Gaussian case, much progress has been made recently in the characterization of its approximate capacity [1]–[3].

In the classic Gaussian relay channel, the noises at the relay and at the destination are independent. In many practical systems, however, the noises at the relay and at the destination may be *correlated*. This may arise, for example, due to the presence of a common interference, which in a practical system is often treated as a part of the background noise, but nevertheless contributes to the correlation between the noises.

The Gaussian relay channel with correlated noises has been studied in [4], where relaying strategies such as the decode-and-forward and the compress-and-forward schemes are studied in full-duplex or half-duplex modes. Likewise, the effect of noise correlation for the single-tap amplify-and-forward scheme has been studied for the diamond network and the two-hop parallel relay network in [5]. In both papers, noise correlation has been found to be beneficial. Neither [4] nor [5], however, addresses the question of whether the classic relaying strategies are able to achieve to within constant bits of the capacity for the relay channel with correlated noises.

Inspired by the recent work [1] and [3], where the quantize-map-and-forward and the noisy network coding strategies with fixed quantization level at the relays are shown to achieve the capacity of arbitrary Gaussian relay networks with uncorrelated noises to within a constant gap, this paper shows that such strategies are also capable of approximating the capacity of the three-node Gaussian relay channel with correlated noises. However, unlike the existing schemes of [1] and [3], this paper shows that the relay quantization level needs to be modified to be noise-correlation dependent in the correlated-noise case. As a corollary, this paper also establishes that the conventional compress-and-forward scheme [6] achieves to within constant bits of the capacity for the Gaussian relay channel in the correlated-noise case as well. Finally, in contrast to the case with uncorrelated noises, the decode-and-forward and the single-tap amplify-and-forward strategies can have an infinite gap to capacity, when the noise correlation goes to ± 1 and the gain of the relay-to-destination link goes to infinity.

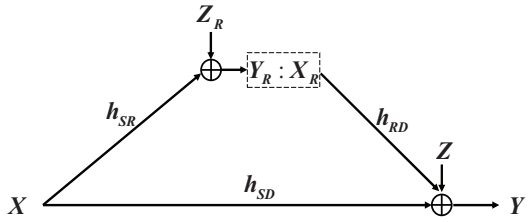

Fig. 1. Three-node Gaussian relay channel with correlated noises

II. Channel Model

This paper considers a real-valued discrete-time three-node Gaussian relay channel as depicted in Fig. 1, which consists of a source X, a destination Y, and a relay. The relay observes a noise-corrupted version of the source signal, denoted by Y_R, and transmits X_R to the destination. The source-to-destination channel is denoted h_{SD}, the relay-to-destination channel h_{RD}, and the source-to-relay channel h_{SR}. The additive Gaussian noises at the relay and at the destination are denoted as Z_R and Z respectively. Mathematically, the channel model is:

$$Y_R = h_{SR}X + Z_R, \quad (1)$$

$$Y = h_{SD}X + h_{RD}X_R + Z. \quad (2)$$

Without loss of generality, the power constraints at the source and at the relay can both be normalized to one, i.e., $\mathbb{E}[X^2] \leq 1$ and $\mathbb{E}[X_R^2] \leq 1$, and so can the noise variances, i.e., $Z_R \sim \mathcal{N}(0,1)$ and $Z \sim \mathcal{N}(0,1)$. Different from most of the literature that assumes independence between Z_R and Z, this paper introduces a correlation between the two noises

$$\rho_z \triangleq \frac{\mathbb{E}\left[Z_R Z\right]}{\sqrt{\mathbb{E}[|Z_R|^2]\mathbb{E}[|Z|^2]}}. \quad (3)$$

Note that Z and Z_R are both i.i.d. in time. Further, the relay operation is causal.

III. Within Constant Bits of the Capacity

To approach capacity, the relaying strategy must take advantage of the noise correlation. Consider the limiting scenario of $\rho_z \to \pm 1$. The relay's observation becomes more and more useful to the destination in this case, thus an increasingly fine quantization resolution at the relay is required — the fixed quantization strategy of [1] and [3] would result in significant inefficiency. The main contribution of this paper is to introduce a *correlation-aware* quantization strategy at the relay, which better exploits the noise correlation and achieves to within $\frac{1}{2}\log_2 3$ bits of the capacity of the Gaussian relay channel with correlated noises.

Theorem 1. *The capacity of the three-node Gaussian relay channel with correlated noises, as shown in Fig. 1, can be achieved to within $\frac{1}{2}\log_2 3$ bits to capacity using a noisy network coding strategy with independent Gaussian inputs $X \sim \mathcal{N}(0,1)$, $X_R \sim \mathcal{N}(0,1)$ and Gaussian quantization at the relay with quantization variance $\mathsf{q}^* = 2(1-\rho_z^2)$.*

Proof: First, the capacity of the relay channel is upper bounded by the cut-set bound, i.e.,

$$\begin{aligned}
\overline{C} &= \max_{p(x,x_R)} \min\{I(X,X_R;Y), I(X;Y,Y_R|X_R)\} \\
&= \max_{\rho_x} \min\left\{\frac{1}{2}\log(1+h_{SD}^2+h_{RD}^2+2\rho_x h_{SD}h_{RD}),\right. \\
&\qquad \left.\frac{1}{2}\log\left(1+\frac{(1-\rho_x^2)(h_{SD}^2+h_{SR}^2-2\rho_z h_{SD}h_{SR})}{1-\rho_z^2}\right)\right\} \\
&\le \min\left\{\frac{1}{2}\log(1+h_{SD}^2+h_{RD}^2+2h_{SD}h_{RD}),\right. \\
&\qquad \left.\frac{1}{2}\log\left(1+\frac{h_{SD}^2+h_{SR}^2-2\rho_z h_{SD}h_{SR}}{1-\rho_z^2}\right)\right\} \\
&= \min\{R_{UB1}, R_{UB2}\}, \qquad (4)
\end{aligned}$$

where ρ_x is the correlation between X and X_R.

The achievable rate by noisy network coding or compress-and-forward with joint decoding can be readily obtained from [7, Proposition 2] and [3, Theorem 1]:

$$\begin{aligned}
R &= \min\{I(X,X_R;Y) - I(Y_R;\hat{Y}_R|X,X_R,Y), \\
&\qquad I(X;Y,\hat{Y}_R|X_R)\} \\
&= \min\{R_1, R_2\} \qquad (5)
\end{aligned}$$

for any distribution

$$p(x,x_R,y_R,\hat{y}_R) = p(x)p(x_R)p(y_R|x,x_R)p(\hat{y}_R|x_R,y_R).$$

Substitute independent Gaussian distributions $X \sim \mathcal{N}(0,1)$ and $X_R \sim \mathcal{N}(0,1)$ into (5), and set $\hat{Y}_R = Y_R + e$, where the quantization noise $e \sim \mathcal{N}(0,\mathsf{q})$ is independent with everything else, we have

$$\begin{aligned}
R_1 &= I(X,X_R;Y) - I(Y_R;\hat{Y}_R|X,X_R,Y) \\
&= \frac{1}{2}\log(1+h_{SD}^2+h_{RD}^2) - \frac{1}{2}\log\left(1+\frac{1-\rho_z^2}{\mathsf{q}}\right), \qquad (6)
\end{aligned}$$

and

$$\begin{aligned}
R_2 &= I(X;Y,\hat{Y}_R|X_R) \\
&\overset{(a)}{=} \frac{1}{2}\log(1+h_{SD}^2) \\
&\quad + \frac{1}{2}\log\left(\frac{\mathsf{q}+\sigma^2_{h_{SR}X+Z_R|h_{SD}X+Z}}{\mathsf{q}+1-\rho_z^2}\right) \\
&= \frac{1}{2}\log\left(1+\frac{\mathsf{q}+(\mathsf{q}+1)h_{SD}^2+h_{SR}^2-2\rho_z h_{SD}h_{SR}}{1-\rho_z^2}\right) \\
&\quad - \frac{1}{2}\log\left(1+\frac{\mathsf{q}}{1-\rho_z^2}\right), \qquad (7)
\end{aligned}$$

where in (a) the conditional variance of $h_{SR}X + Z_R$ given $h_{SD}X + Z$ is calculated as

$$\begin{aligned}
&\sigma^2_{h_{SR}X+Z_R|h_{SD}X+Z} \\
&= \mathbb{E}[|h_{SR}X+Z_R|^2] - \frac{|\mathbb{E}[(h_{SR}X+Z_R)(h_{SD}X+Z)]|^2}{\mathbb{E}[|h_{SD}X+Z|^2]} \\
&= \frac{1-\rho_z^2+h_{SR}^2+h_{SD}^2-2\rho_z h_{SR}h_{SD}}{1+h_{SD}^2}. \qquad (8)
\end{aligned}$$

Comparing R_1 and the upper bound R_{UB1}, we have

$$\begin{aligned}
&R_{UB1} - R_1 \\
&= \frac{1}{2}\log(1+h_{SD}^2+h_{RD}^2+2h_{SD}h_{RD}) \\
&\quad - \frac{1}{2}\log(1+h_{SD}^2+h_{RD}^2) + \frac{1}{2}\log\left(1+\frac{1-\rho_z^2}{\mathsf{q}}\right) \\
&= \frac{1}{2}\log\left(\frac{1+h_{SD}^2+h_{RD}^2+2h_{SD}h_{RD}}{2+2h_{SD}^2+2h_{RD}^2}\right) \\
&\quad + \frac{1}{2}\log\left(1+\frac{1-\rho_z^2}{\mathsf{q}}\right) + \frac{1}{2} \\
&< \frac{1}{2}\log\left(1+\frac{1-\rho_z^2}{\mathsf{q}}\right) + \frac{1}{2}. \qquad (9)
\end{aligned}$$

Comparing R_2 and the upper bound R_{UB2}, we have

$$\begin{aligned}
&R_{UB2} - R_2 \\
&= \frac{1}{2}\log\left(1+\frac{h_{SD}^2+h_{SR}^2-2\rho_z h_{SD}h_{SR}}{1-\rho_z^2}\right) \\
&\quad - \frac{1}{2}\log\left(1+\frac{\mathsf{q}+(\mathsf{q}+1)h_{SD}^2+h_{SR}^2-2\rho_z h_{SD}h_{SR}}{1-\rho_z^2}\right) \\
&\quad + \frac{1}{2}\log\left(1+\frac{\mathsf{q}}{1-\rho_z^2}\right) \\
&< \frac{1}{2}\log\left(1+\frac{\mathsf{q}}{1-\rho_z^2}\right). \qquad (10)
\end{aligned}$$

The gap between the cut-set bound $\overline{C}$ and the achievable rate R is then upper bounded by the maximum of (9) and (10), i.e.

$$\begin{aligned}
\overline{C} - R &\le \min\{R_{UB1}, R_{UB2}\} - \min\{R_1, R_2\} \\
&\le \max\{R_{UB1} - R_1, R_{UB2} - R_2\} \\
&< \max\left\{\frac{1}{2}\log\left(1+\frac{1-\rho_z^2}{\mathsf{q}}\right) + \frac{1}{2},\right. \\
&\qquad \left.\frac{1}{2}\log\left(1+\frac{\mathsf{q}}{1-\rho_z^2}\right)\right\}. \qquad (11)
\end{aligned}$$

The first term above monotonically decreases with q, while the second term monotonically increases with q. To minimize the maximum of the two terms, we set

$$\frac{1}{2}\log\left(1+\frac{1-\rho_z^2}{\mathsf{q}^*}\right)+\frac{1}{2}=\frac{1}{2}\log\left(1+\frac{\mathsf{q}^*}{1-\rho_z^2}\right), \quad (12)$$

which results in $\mathsf{q}^*=2(1-\rho_z^2)$. Substituting q^* into (11), we have $\overline{C}-R<\frac{1}{2}\log_2 3=0.7925$. ∎

In addition, it can be shown that the conventional compress-and-forward rate is also within the same constant gap to capacity. To prove this directly would have been quite involved (see [2] for the computation of the gap for the case of $\rho_z=0$). Instead, we obtain the result as a direct consequence of Theorem 1.

Corollary 1. *The following rate, which is achieved by the classic compress-and-forward strategy on the three-node Gaussian relay channel with correlated noises shown in Fig. 1:*

$$R_{CF}=\frac{1}{2}\log\left(1+h_{SD}^2+\frac{(h_{SR}-\rho_z h_{SD})^2}{1-\rho_z^2+\mathsf{q}_c}\right), \quad (13)$$

where

$$\mathsf{q}_c=\frac{(1-\rho_z^2)(1+h_{SD}^2)+(h_{SR}-\rho_z h_{SD})^2}{h_{RD}^2} \quad (14)$$

is within $\frac{1}{2}\log_2 3$ bits to the capacity.

Proof: The rate expression R_{CF} for the correlated-noise Gaussian relay channel has been obtained in [4, Proposition 5]. The derivation is based on the classic compress-and-forward rate for the relay channel by Cover and El Gamal [6, Theorem 6], which is $R_{CF}=I(X;\hat{Y}_R,Y|X_R)$ subject to $I(X_R;Y)\geq I(Y_R;\hat{Y}_R|X_R,Y)$ for some joint distribution $p(x)p(x_R)p(y_R|x,x_R)p(\hat{y}_R|x_R,y_R)$. Using the same signaling scheme as in Theorem 1, i.e., $X\sim\mathcal{N}(0,1)$ and $X_R\sim\mathcal{N}(0,1)$ are independent, and $\hat{Y}_R=Y_R+e$, where $e\sim\mathcal{N}(0,\mathsf{q}_c)$ is chosen to satisfy the relay-destination rate constraint, we obtain (13).

In the following, we prove the constant gap result for the compress-and-forward rate by showing that R_{CF} in (13) is greater than the noisy network coding rate, i.e., $R_{CF}\geq\min(R_1,R_2)$, where R_1 and R_2 are as in (6) and (7) respectively. Substituting q_c in (14) as q in R_1 and R_2, it is easy to verify that $R_1(\mathsf{q}_c)=R_2(\mathsf{q}_c)=R_{CF}$. Since R_1 increases with q and R_2 decreases with q, we have $R_{CF}=\min\{R_1(\mathsf{q}_c),R_2(\mathsf{q}_c)\}=\max_{\mathsf{q}}\min\{R_1(\mathsf{q}),R_2(\mathsf{q})\}\geq\min\{R_1(\mathsf{q}^*),R_2(\mathsf{q}^*)\}$ for any $\mathsf{q}*$ and in particular for $\mathsf{q}^*=2(1-\rho_z^2)$. Since it has been show in Theorem 1 that $\min\{R_1(\mathsf{q}^*),R_2(\mathsf{q}^*)\}$ is within $\frac{1}{2}\log 3$ bits of the cut-set upper bound, so is R_{CF}. ∎

IV. Suboptimality of Decode-and-Forward and Single-Tap Amplify-and-Forward

The decode-and-forward and the single-tap amplify-and-forward strategies have been shown to achieve to within a constant gap to the capacity of the Gaussian relay channel with uncorrelated noises [1], [2]. In this section, we show that this is no longer the case when noises are correlated.

A. Decode-and-Forward

Consider a decode-and-forward strategy as described in [1, Appendix A], in which when the source-to-relay link is weaker than the source-to-destination link, i.e., $h_{SR}\leq h_{SD}$, the relay is simply ignored, otherwise the relay decodes and forwards a bin index to the destination as in the original scheme of [6]. The following rate is achievable:

$$R_{DF}=\max\left\{\frac{1}{2}\log(1+h_{SD}^2),\right.$$
$$\left.\min\left\{\frac{1}{2}\log(1+h_{SR}^2),\frac{1}{2}\log(1+h_{SD}^2+h_{RD}^2)\right\}\right\} \quad (15)$$

In the extreme scenario where $\rho_z=1$ and

$$h_{RD}^2\gg h_{SR}^2\gg h_{SD}^2\gg 1, \quad (16)$$

the above decode-and-forward rate (15) becomes

$$R_{DF}=\frac{1}{2}\log(1+h_{SR}^2). \quad (17)$$

Meanwhile, when $\rho_z=1$, the cut-set bound (4) becomes

$$\overline{C}=\frac{1}{2}\log(1+h_{SD}^2+h_{RD}^2+2h_{SD}h_{RD}). \quad (18)$$

Comparing (17) with (18), we observe that

$$\begin{aligned}\overline{C}-R_{DF} &= \frac{1}{2}\log\left(\frac{1+h_{SD}^2+h_{RD}^2+2h_{SD}h_{RD}}{1+h_{SR}^2}\right)\\ &\to \frac{1}{2}\log\left(\frac{h_{RD}^2}{h_{SR}^2}\right), \end{aligned} \quad (19)$$

which is unbounded in the asymptotic regime (16). This is not unexpected, because the decoding at the relay eliminates the noise. Therefore, noise correlation is not exploited.

B. Single-Tap Amplify-and-Forward

In the single-tap amplify-and-forward, the relay scales the current observation and forwards to the destination in the next time instance, i.e.,

$$X_R[i]=\alpha(h_{SR}X[i-1]+Z_R[i-1]), \quad (20)$$

where $\alpha\leq\frac{1}{\sqrt{1+h_{SR}^2}}$ is chosen to satisfy the power constraint at the relay. Since $Y[i]=h_{SD}X[i]+h_{RD}X_R[i]+Z[i]$, the relay channel is now converted into a single-tap inter-symbol-interference (ISI) channel:

$$Y[i]=h_{SD}X[i]+\alpha h_{RD}h_{SR}X[i-1]+Z[i]+\alpha h_{RD}Z_R[i-1]. \quad (21)$$

The capacity of the Gaussian ISI channel is given by

$$R_{AF}=\max_{S(\omega)}\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{2}\log\left(1+S(\omega)\frac{|H(\omega)|^2}{N(\omega)}\right)d\omega, \quad (22)$$

subject to

$$\frac{1}{2\pi}\int_0^{2\pi}S(\omega)d\omega\leq 1,\ \text{ and }\ S(\omega)\geq 0,\quad 0\leq\omega\leq 2\pi, \quad (23)$$

where $N(\omega)=1+\alpha^2h_{RD}^2+2\rho_z\alpha h_{RD}\cos(\omega)$ is the power spectrum density of the noise, and $H(\omega)=h_{SD}+$

$\alpha h_{RD}h_{SR}e^{j\omega}$ is the Fourier transform of the channel coefficients, and $S(\omega) = \left(\lambda - \frac{N(\omega)}{|H(\omega)|^2}\right)^+$ is the water-filling power allocation over the frequencies.

Consider again the case of $\rho_z = 1$ and the asymptotic regime of (16), i.e. $h_{RD}^2 \gg h_{SR}^2 \gg h_{SD}^2 \gg 1$. In this high signal-to-noise ratio regime, it is easy to verify that the water-filling power spectrum converges to an equal power allocation, i.e., $S(\omega) = 1$, $0 \le \omega \le 2\pi$. Substituting $N(\omega)$, $H(\omega)$ and $S(\omega) = 1$ into (22) and using table of integrals, after some algebra, it is possible to show that

$$R_{AF} \le \frac{1}{2}\log(2 + h_{SR}^2 + h_{SD}^2).$$

Comparing the above with the cut-set bound, we see that

$$\begin{aligned} \overline{C} - R_{AF} &\ge \frac{1}{2}\log\left(\frac{1 + h_{SD}^2 + h_{RD}^2 + 2h_{SD}h_{RD}}{2 + h_{SR}^2 + h_{SD}^2}\right) \\ &\to \frac{1}{2}\log\left(\frac{h_{RD}^2}{h_{SR}^2}\right) \end{aligned} \quad (24)$$

in the asymptotic regime of (16), which is unbounded.

V. Numerical Simulation

This section numerically compares the cut-set upper bound and the achievable rates of different relaying schemes. Here, the noisy network coding rate is computed with $\mathsf{q}^* = 2(1-\rho_z^2)$. We consider two examples: Fig. 2 shows the case for $h_{SD}^2 =$ 20dB, $h_{SR}^2 =$ 40dB and $h_{RD}^2 =$ 60dB, corresponding to an extreme scenario of $h_{RD}^2 \gg h_{SR}^2 \gg h_{SD}^2 \gg 1$. Fig. 3 shows the case for $h_{SD}^2 =$ 5dB, $h_{SR}^2 =$ 10dB, and $h_{RD}^2 =$ 10dB. It is clear that in both cases, compress-and-forward is always better than the noisy network coding scheme with the specific q^*, and both are within a constant gap to the cut-set upper bound for all values of ρ_z.

The decode-and-forward rate is always independent of ρ_z. In the asymptotic regime as shown in Fig. 2, the single-tap amplify-and-forward rate is almost independent of ρ_z as well, and it coincides with the decode-and-forward rate. Both can have an unbounded gap to the cut-set bound as $h_{RD}^2 \to \infty$ and $\rho_z \to \pm 1$. Compress-and-forward, on the other hand, closely tracks the cut-set bound. (Note that the above observations are not true in the non-asymptotic SNR regime as shown in Fig. 3.) The noisy-network-coding scheme, although not as good as compress-and-forward, nevertheless is always within a constant gap to the cut-set bound.

It is interesting to see that the noisy-network-coding rate resembles the shape of the cut-set upper bound as shown in both Fig. 2 and Fig. 3. It is also interesting to note that the decode-and-forward curve touches the cut-set bound at a particular value of ρ_z. This is because at this value of ρ_z, the relay channel becomes degraded [4, Theorem 1].

VI. Conclusion

This paper investigates different relaying strategies for the three-node Gaussian relay channel with correlated noises. It is shown that both the proposed correlation-aware noisy network coding scheme and the conventional compress-and-forward relaying scheme can achieve to within a constant gap to the capacity, while the decode-and-forward scheme and the single-tap amplify-and-forward scheme cannot.

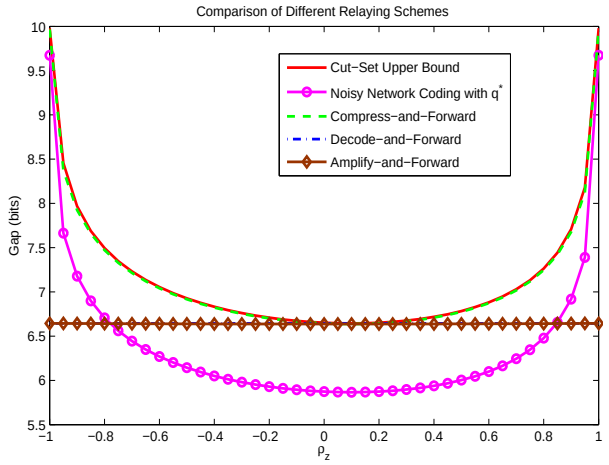

Fig. 2. Comparison of different relaying schemes for $h_{SD}^2 =$ 20dB, $h_{SR}^2 =$ 40dB and $h_{RD}^2 =$ 60dB

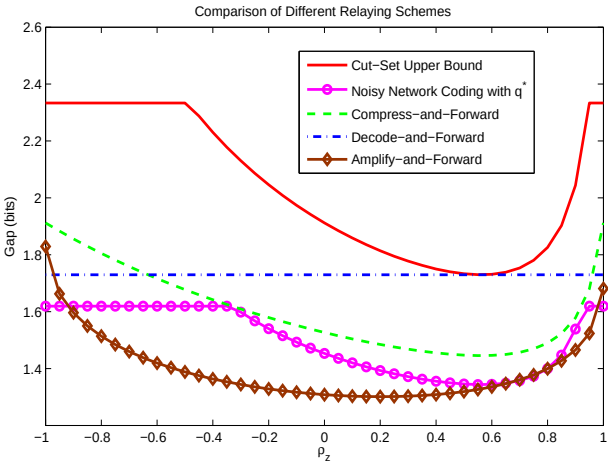

Fig. 3. Comparison of different relaying schemes for $h_{SD}^2 =$ 5dB, $h_{SR}^2 =$ 10dB, and $h_{RD}^2 =$ 10dB

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