This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TFUZZ.2014.2315298, IEEE Transactions on Fuzzy Systems



# Reliable State Feedback Control of T-S Fuzzy Systems with Sensor Faults

Jiuxiang Dong *Associate Member, IEEE* and Guang-Hong Yang, *Senior Member, IEEE*

***Abstract*—This paper is concerned with the reliable state feedback control synthesis for T-S fuzzy systems with sensor multiplicative faults. By considering the influences of sensor faults on both the system states and premise variables of fuzzy controllers, a class of new convex reliable stabilization conditions are proposed for T-S fuzzy systems through using the properties of fuzzy product inference engines. Furthermore, the obtained result is extended to the $H_\infty$ reliable control case. The resulting controllers are reliable in that they provide guaranteed asymptotic stability and $H_\infty$ performance when all sensors are operational as well as when some sensor experiences failures. Different from the proposed approach, the influence of sensor faults on premise variables is not considered in the existing results, then it might not guarantee the stability and control performance for T-S fuzzy systems with premise variables dependent on the system states. A numerical example is given to illustrate the effectiveness of the proposed method.**

***Index Terms*—T-S fuzzy control systems, reliable control, sensor fault, linear matrix inequalities (LMIs).**

## I. Introduction

In many engineering control systems, the control system is often required with high reliability, especially for safety-critical systems, such as aircraft systems and medical systems. In general, control of any plant depends on the availability and quality of sensor measurements, then the performance of the system relies heavily on the quality of the sensor for feedback [1]. In some feedback control applications, sudden environmental disturbances, broken or bad communication, or malfunction of some hardware or software often corrupt the measurements of the sensors, then sensor characteristics may change over time, so that it may be partial or complete failure [2], [3], which can degrade performance or even destroy the stability of the overall systems. Therefore, to increase control system reliability, reliable control for sensor failures is of both theoretical and practical importance. Thus, various reliable control techniques are developed [4], [5], [6], [7], [8], [9], [10], [11] such as by modelling sensor characteristics as parametrizable uncertain functions, an adaptive compensator is proposed for overcome the effects of sensor uncertainties in [1]. A multisensor switching control scheme is presented in [12]. Design of tolerant sensor networks is studied by the aid of decomposition technique in [13]. A multisensor fusion fault tolerant control system with fault detection and identification via set separation is presented in [14].

The authors are with the College of Information Science and Engineering, Northeastern University, Shenyang, 110819, China. They are also with State Key Laboratory of Synthetical Automation of Process Industries(Northeastern University), Shenyang, 110819, China(e-mail: dongjiuxiang@ise.neu.edu.cn;yangguanghong@ise.neu.edu.cn). Corresponding author: Prof. Jiuxiang Dong.

The above mentioned work mainly focuses on the reliable control problems of linear systems. But many practical engineering systems are nonlinear, the resulted controllers for linear operation points often might not guarantee the performance, even stability of the original nonlinear systems. Therefore, some reliable control approaches for nonlinear systems are proposed in past several decades, see [15], [16], [17], [18], [19], [20] and the references therein. In nonlinear control theory, an important approach is to model the considered nonlinear systems as Takagi and Sugeno (T-S) fuzzy systems, which are locally linear time-invariant systems connected by IF-THEN rules [21]. As a result, the conventional linear system theory can be applied for analysis and synthesis of the nonlinear control systems [22], [23], [24], [25], [26], [27], [28], [29], [30]. In particular, reliable control synthesis for nonlinear systems based on T-S fuzzy models received considerable attentions in recent years[17], [31], [32], such as reliable mixed $\mathcal{L}_2/H_\infty$ fuzzy static output feedback control is studied for T-S fuzzy systems with sensor faults by using multiple Lyapunov functions in [33]. For stochastic fuzzy systems, a new descriptor fuzzy sliding mode observer approach is proposed against simultaneous sensor and actuator faults in [3]. On the other hand, based on T-S fuzzy system models, reliable control methods of wind energy conversion systems and automatic control system of aircraft during landing are respectively proposed in [34] and [35]. Moreover, $H_\infty$ tracking control and fault detection are respectively considered in [36] and [37], [38]. The above mentioned results have given many effective methods for designing fuzzy reliable controllers, but the influences of sensor faults on the premise variables of fuzzy controllers aren't considered. Note that the premise variables are often dependent on states in many T-S fuzzy systems. If the sensor for measuring some state occur fault, then the premise variables dependent on the state in fuzzy controllers also become unprecise, which might degrade performance or even destroy the stability of the overall systems. Motivated by this, for the T-S fuzzy systems with the premise variables dependent on the system states, the paper will develop a type of new reliable control conditions by using the properties of the fuzzy product inference engine and considering the influences of sensor faults on the system states and premise variables, such that the resulting controllers are reliable in that they provide guaranteed asymptotic stability and $H_\infty$ performance when all sensors are operational as well as when some sensor experiences failures. Different from the new approach, the influence of sensor faults on premise variables is not considered in the existing results, then it might not guarantee the stability and control performance for T-S fuzzy

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systems with premise variables dependent on the system states and a numerical example will be given to illustrate the fact.

The paper is organized as follows. Section II presents system description and some notations. A class of new conditions for designing reliable controllers are proposed and extended to $H_\infty$ reliable control in Section III. A numerical example is given to illustrate the effectiveness of the new proposed methods in Section IV. Concluding remarks are given in Section V.

## II. System Description

### A. T-S fuzzy control system and fuzzy controller

The nonlinear system under consideration is described by the following fuzzy system model:

**Plant Rule** $(i_1 i_2 \cdots i_p)$**:**
IF $v_1(t)$ is $M_{1i_1}$ and $v_2(t)$ is $M_{2i_2}, \cdots, v_p(t)$ is $M_{pi_p}$
THEN

$$\begin{aligned}\dot{x}(t) &= A_{i_1 i_2 \cdots i_p} x(t) + B_{wi_1 i_2 \cdots i_p} w(t) + B_{ui_1 i_2 \cdots i_p} u(t)\\ z(t) &= C_{i_1 i_2 \cdots i_p} x(t) + D_{i_1 i_2 \cdots i_p} u(t)\end{aligned} \tag{1}$$

$x(t) \in \mathbb{R}^{n_x}$ is the state vector, $u(t) \in \mathbb{R}^{n_u}$ is the control input vector, $w(t) \in \mathbb{R}^{n_w}$ is the disturbance input, $z(t) \in \mathbb{R}^{n_z}$ is the controlled output, $v(t) = [v_1(t)\ v_2(t) \cdots v_p(t)]^T \in \mathbb{R}^p$, $v_i(t)$, $i = 1, \cdots, p$ are the premise variables, $M_{ji_j}$, $j = 1, \cdots, p$, $i_j = 1, \cdots, r_j$ denotes an $v_j(t)$-based fuzzy set and they are linguistic terms characterized by fuzzy membership functions $M_{ji_j}(v_j(t))$, where $r_j$ be the number of $v_j(t)$-based fuzzy sets. Then, the fuzzy rule base consists of $r = \prod_{i=1}^{p} r_i$ IF-THEN rules.

By using the fuzzy inference method with a singleton fuzzifier, product inference, and center average defuzzifiers, then the T-S fuzzy model is obtained as (2).

Let

$$\mu_{ji_j}(v_j(t)) = \frac{M_{ji_j}(v_j(t))}{\sum_{l_j=1}^{r_j} M_{jl_j}(v_j(t))}, \text{for } 1 \le j \le p, 1 \le i_j \le r_j \tag{3}$$

The fuzzy system from (2) and (3) can be written as follows:

$$\begin{aligned}\dot{x}(t) &= \sum_{i_1=1}^{r_1}\sum_{i_2=1}^{r_2}\cdots\sum_{i_p=1}^{r_p}\left(\prod_{j=1}^{p}\mu_{ji_j}(v_j(t))\right)\times\\ &\quad \Big(A_{i_1 i_2 \cdots i_p} x(t) + B_{wi_1 i_2 \cdots i_p} w(t) + B_{ui_1 i_2 \cdots i_p} u(t)\Big)\\ z(t) &= \sum_{i_1=1}^{r_1}\sum_{i_2=1}^{r_2}\cdots\sum_{i_p=1}^{r_p}\left(\prod_{j=1}^{p}\mu_{ji_j}(v_j(t))\right)\times\\ &\quad \Big(C_{i_1 i_2 \cdots i_p} x(t) + D_{i_1 i_2 \cdots i_p} u(t)\Big)\end{aligned} \tag{4}$$

From (3), it is resulted that

$$\sum_{i_j=1}^{r_j} \mu_{ji_j}(v_j(t)) = 1, \quad \text{for } 1 \le j \le p \tag{5}$$

In the existing literature, there are many fuzzy control schemes for T-S fuzzy systems, for example, the parallel distributed compensation (PDC) control scheme [21], non-PDC control scheme [39], switched constant gain control scheme [23], switched PDC control scheme [40], dominant dependent fuzzy control scheme [41] and so on, where the PDC control scheme is widely used and it is also adopted in this paper as follows:

**Control Rule** $(i_1 i_2 \cdots i_p)$**:**
IF $v_1(t)$ is $M_{1i_1}$ and $v_2(t)$ is $M_{2i_2}, \cdots, v_p(t)$ is $M_{pi_p}$
THEN $u(t) = K_{i_1 i_2 \cdots i_p} x(t)$ (6)

Then the final output of fuzzy controller is obtained as:

$$u(t) = \sum_{i_1=1}^{r_1}\sum_{i_2=1}^{r_2}\cdots\sum_{i_p=1}^{r_p}\prod_{j=1}^{p}\mu_{ji_j}(v_j(t)) K_{i_1 i_2 \cdots i_p} x(t) \tag{7}$$

Combining (7) and (4), then the closed-loop system is obtained as follows:

$$\begin{aligned}&\dot{x}(t) =\\ &\sum_{i_1=1}^{r_1}\sum_{i_2=1}^{r_2}\cdots\sum_{i_p=1}^{r_p}\sum_{l_1=1}^{r_1}\cdots\sum_{l_p=1}^{r_p}\prod_{j=1}^{p}\mu_{ji_j}(v_j(t))\prod_{j=1}^{p}\mu_{jl_j}(v_j(t))\\ &\times\Big((A_{i_1 i_2 \cdots i_p} + B_{ui_1 i_2 \cdots i_p} K_{l_1 l_2 \cdots l_p}) x(t) + B_{wi_1 i_2 \cdots i_p} w(t)\Big)\\ &= \sum_{i_1=1}^{r_1}\sum_{i_2=1}^{r_2}\cdots\sum_{i_p=1}^{r_p}\sum_{l_1=1}^{r_1}\cdots\sum_{l_p=1}^{r_p}\prod_{j=1}^{p}\mu_{ji_j}(v_j(t))\mu_{jl_j}(v_j(t))\\ &\times\Big((A_{i_1 i_2 \cdots i_p} + B_{ui_1 i_2 \cdots i_p} K_{l_1 l_2 \cdots l_p}) x(t) + B_{wi_1 i_2 \cdots i_p} w(t)\Big)\\ &z(t) =\\ &\sum_{i_1=1}^{r_1}\sum_{i_2=1}^{r_2}\cdots\sum_{i_p=1}^{r_p}\sum_{l_1=1}^{r_1}\sum_{l_2=1}^{r_2}\cdots\sum_{l_p=1}^{r_p}\prod_{j=1}^{p}\mu_{ji_j}(v_j(t))\mu_{jl_j}(v_j(t))\\ &\times (C_{i_1 i_2 \cdots i_p} + D_{i_1 i_2 \cdots i_p} K_{l_1 l_2 \cdots l_p}) x(t)\end{aligned} \tag{8}$$

In order to give the conditions for designing reliable controllers, a class of new descriptions based on set theory for T-S fuzzy systems are proposed.

Let sets

$$\mathbb{S}_i = \{1, 2, \cdots, r_i\} \qquad i = 1, 2 \cdots, p \tag{9}$$

The product of the sets $\mathbb{S}_i$, $i = 1, 2, \cdots, p$ is described as

$$\begin{aligned}&\mathbb{S}_1 \times \mathbb{S}_2 \times \cdots \times \mathbb{S}_p\\ &= \prod_{i=1}^{p} \mathbb{S}_i = \{i_1 i_2 \cdots i_p :\ i_1 \in \mathbb{S}_1, i_2 \in \mathbb{S}_2, \cdots, i_p \in \mathbb{S}_p\}\end{aligned}$$

Then the closed-loop system (8) can be rewritten as the following compact form:

$$\begin{aligned}&\dot{x}(t)\\ &= \sum_{\tau \in \prod_{i=1}^{p}\mathbb{S}_i}\ \sum_{\sigma \in \prod_{i=1}^{p}\mathbb{S}_i} \mu_\tau \mu_\sigma \Big((A_\tau + B_{u\tau} K_\sigma) x(t) + B_{w\tau} w(t)\Big)\\ &z(t)\\ &= \sum_{\tau \in \prod_{i=1}^{p}\mathbb{S}_i}\ \sum_{\sigma \in \prod_{i=1}^{p}\mathbb{S}_i} \mu_\tau \mu_\sigma (C_\tau + D_\tau K_\sigma) x(t)\end{aligned} \tag{10}$$

$$\dot{x}(t) = \frac{\sum_{i_1=1}^{r_1}\sum_{i_2=1}^{r_2}\cdots\sum_{i_p=1}^{r_p}\left(\prod_{j=1}^{p} M_{ji_j}(v_j(t))\right)\left(A_{i_1i_2\cdots i_p}x(t) + B_{wi_1i_2\cdots i_p}w(t) + B_{ui_1i_2\cdots i_p}u(t)\right)}{\sum_{i_1=1}^{r_1}\sum_{i_2=1}^{r_2}\cdots\sum_{i_p=1}^{r_p}\prod_{j=1}^{p} M_{ji_j}(v_j(t))}$$

$$z(t) = \frac{\sum_{i_1=1}^{r_1}\sum_{i_2=1}^{r_2}\cdots\sum_{i_p=1}^{r_p}\left(\prod_{j=1}^{p} M_{ji_j}(v_j(t))\right)\left(C_{i_1i_2\cdots i_p}x(t) + D_{i_1i_2\cdots i_p}u(t)\right)}{\sum_{i_1=1}^{r_1}\sum_{i_2=1}^{r_2}\cdots\sum_{i_p=1}^{r_p}\prod_{j=1}^{p} M_{ji_j}(v_j(t))} \tag{2}$$

where

$$\mu_\tau = \prod_{j=1}^{p}\mu_{j\tau_{\langle j\rangle}}(v_j(t)), \tau = \tau_{\langle 1\rangle}\tau_{\langle 2\rangle}\cdots\tau_{\langle p\rangle}, \tau \in \prod_{i=1}^{p}\mathbb{S}_i$$
$$\mu_\sigma = \prod_{j=1}^{p}\mu_{j\sigma_{\langle j\rangle}}(v_j(t)), \sigma = \sigma_{\langle 1\rangle}\sigma_{\langle 2\rangle}\cdots\sigma_{\langle p\rangle}, \sigma \in \prod_{i=1}^{p}\mathbb{S}_i \tag{11}$$

**Remark 1:** Note that the new description is based on the index set of membership functions and it is equivalent to the conversional description, which is shown in Appendix. In fuzzy control systems, one premise variable might be dependent on several system states, and several premise variables might be dependent on the same system state. In order to apply the relations between premise variables and system states for designing reliable controllers, the description for T-S fuzzy systems in (4) is used in this paper.

## B. Sensor fault

In this paper, multiplicative sensor faults are considered, the definition of which is given as follows:

**Definition 1:** [33] (Sensor multiplicative fault) The sensor for measuring system variable $\xi(t) \in R$ is said to have fault at time $T_f > 0$, if the output of the sensor

$$\xi^F(t) = f(t)\xi(t), \quad 0 \le f(t) < 1, \quad \forall t > T_f \tag{12}$$

## C. Fuzzy controller under sensor failures

For the fuzzy controller (7), its input can artificially be divided into two parts, they are respectively the local feedback states and the premise variables, then

- Case I: All premise variables in the controller (7) are independent on the states. If one sensor for measuring some state fails, then the disabled sensor only results in an unprecise measurement state. Other feedback states and all premise variables of the fuzzy controller are not corrupted due to the sensor fault. For example, the sensor for measuring state $x_m$ occur fault, then the fuzzy controller with the sensor fault is

$$u(t) = \sum_{i_1=1}^{r_1}\sum_{i_2=1}^{r_2}\cdots\sum_{i_p=1}^{r_p}\prod_{j=1}^{p}\mu_{ji_j}(v_j(t))K_{i_1i_2\cdots i_p}F_m x(t) \tag{13}$$

where

$$F_m = diag\left[1,\cdots,1,f_m,1,\cdots,1\right]_{n_x\times n_x}, 0 \le f_m \le 1 \tag{14}$$

- Case II: All premise variables in the controller (7) are independent on the state variables, then the measurement values of the premise variables are used in the fuzzy controller. If one sensor for measuring some premise variable fails, then the disabled sensor will result in an unprecise measurement of the premise variable. Other premise variables and the states aren't corrupted due to the sensor fault. For example, the sensor for measuring $v_m(t)$ fails, the fuzzy controller with the sensor fault is

$$u(t) = \sum_{i_1=1}^{r_1}\sum_{i_2=1}^{r_2}\cdots\sum_{i_p=1}^{r_p}\left(\prod_{j=1,j\neq m}^{p}\mu_{ji_j}(v_j(t))\right)\times \mu_{mi_m}(v_m^F(t))K_{i_1i_2\cdots i_p}x(t) \tag{15}$$

where $v_m^F(t)$ denotes the corrupted premise variable in the fuzzy controller.

- Case (III): Some premise variables in the controller (7) are dependent on the states. When one sensor for measuring some state is failed, the premise variables dependent on the state and the state itself, which are used in the fuzzy controller, are both unprecise. For example, if some sensor fault leads to the feedback state $x_q$ and the premise variables $v_{m_1}$, $v_{m_2}$, $\cdots$, $v_{m_s}$ of the fuzzy controller being unprecise (These premise variables are all dependent on the state $x_q$), then the fuzzy controller with the sensor fault is

$$u(t) = \sum_{i_1=1}^{r_1}\sum_{i_2=1}^{r_2}\cdots\sum_{i_p=1}^{r_p}\left(\prod_{j=1,j\neq m_1,\cdots,m_s}^{p}\mu_{ji_j}(v_j(t))\right)\times \left(\mu_{m_1i_{m_1}}(v_{m_1}^F(t))\cdots\mu_{m_si_{m_s}}(v_{m_s}^F(t))\right)K_{i_1i_2\cdots i_p}F_q x(t) \tag{16}$$

where $F_q$ is the same as in (14), $v_{m_1}^F(t)$, $\cdots$, $v_{m_s}^F(t)$ denote the corrupted premise variables in the fuzzy controller.

## D. Disadvantages of the existing approaches

For the case (I), many effective methods for fuzzy reliable control have been proposed and the controllers obtained by these approaches achieve good control effects when sensors occur faults, see the references [33], [42], [3], [34] and the reference therein. To our knowledge, the cases (II) and (III) are scarcely considered in the existing literature. However, the faults in cases (II) and (III) might occur in many T-S fuzzy control systems, especially, those models are derived from given nonlinear system equations based on the idea of using sector nonlinearity in [21], where premise variables are dependent on system states. For the two cases, the existing results might become invalid. Motivated by this, the approach for designing reliable controllers for the case (III) (the obtained approach can also be applied to the cases (I) and (II)) is exploited.

## III. Main result

In this section, a class of new reliable control conditions are proposed for T-S fuzzy systems with sensor faults. First, the relations between premise variables and states are characterized by some sets. Second, the reliable control condition is proposed by using these relations and the properties of the structure of product inference engine. Last, an $H_\infty$ reliable control condition is given by extending the acquired results.

### A. Description of the relations between premise variables and states

Define the following sets about the subcripts of all premise variables.

$\Lambda$: The subcripts of all premise variables are collected as set $\Lambda = \{1, 2, \cdots, p\}$;

$\Lambda_i$: The subcripts of the premise variables dependent on $x_i(t)$ are collected as set $\Lambda_i$, $i = 1, \cdots, n_x$.

For example, a T-S fuzzy system with 3 premise variables is given as follows:

**Plant Rule** $(i_1 i_2 i_3)$**:**

IF $v_1(t)$ is $M_{1i_1}$ and $v_2(t)$ is $M_{2i_2}$ and $v_3(t)$ is $M_{3i_3}$

THEN $\dot{x}(t) = A_{i_1i_2i_3}x(t) + B_{ui_1i_2i_3}u(t) + B_{wi_1i_2i_3}w(t)$ (17)

where $x(t) = [x_1(t), x_2(t), x_3(t), x_4(t)]$, $v_1(t) = x_1(t)$, $v_2(t) = x_2(t)$, $v_3(t) = \sin(x_3(t) - x_2(t))$. Then $\Lambda = \{1, 2, 3\}$, $\Lambda_1 = \{1\}$, $\Lambda_2 = \{2, 3\}$, $\Lambda_3 = \{3\}$. $\Lambda_4 = \{\}$. If the sensor for measuring $x_1(t)$ occurs fault, the corrupted measurement is denoted as $x_1^F(t) = f_1 x_1(t)$, $0 \le f_1 \le 1$. Because $v_1(t) = x_1(t)$, then the premise variable $v_1(t)$ used in the controller is also corrupted as $v_1^F(t) = x_1^F(t) = f_1 x_1(t)$. Therefore the fuzzy controller with the sensor fault is

$$
\begin{aligned}
u(t) = &\sum_{i_1=1}^{r_1}\sum_{i_2=1}^{r_2}\sum_{i_3=1}^{r_3} \mu_{1i_1}(v_1^F(t))\mu_{2i_2}(v_2(t))\mu_{3i_3}(v_3(t)) \\
&\times K_{i_1i_2i_3}F_1 x(t)
\end{aligned}
$$

where $F_1 = diag\left[f_1, 1, 1\right]$, $\quad 0 \le f_1 \le 1$.

For more general fuzzy systems (1) with $p$ premise variables, if only the sensor for measuring $x_q$ occurs fault, then all precise premise variables in the fuzzy controller are $v_i(t)$, $i \in \Lambda - \Lambda_q$, those unprecise premise variables are $v_i(t)$, $i \in \Lambda_q$. Therefore, the fuzzy controller with the sensor variable fault is given as follows:

$$
\begin{aligned}
u(t) = &\sum_{i_1=1}^{r_1}\sum_{i_2=1}^{r_2}\cdots\sum_{i_p=1}^{r_p}\left(\prod_{j\in\Lambda-\Lambda_q}\mu_{ji_j}(v_j(t))\right)\times \\
&\left(\prod_{j\in\Lambda_q}\mu_{ji_j}(v_j^F(t))\right)K_{i_1i_2\cdots i_p}F_q x(t) \\
= &\sum_{\sigma\in\prod_{i=1}^{p}\mathbb{S}_i}\left(\prod_{j\in\Lambda-\Lambda_q}\mu_{j\sigma_{\langle j]}}(v_j(t))\right)\times \\
&\left(\prod_{j\in\Lambda_q}\mu_{j\sigma_{\langle j]}}(v_j^F(t))\right)K_\sigma F_q x(t)
\end{aligned}
$$

Moreover, the following binary relation on $\mathbb{S}_i^2$ (where $\mathbb{S}_i$ is the same as in (9)) is useful.

$$
\begin{aligned}
\mathbb{R}_i = &\{(i_1i_2, j_1j_2) : (i_1 = j_1 \text{ and } i_2 = j_2), \\
&\text{or } (i_1 = j_2 \text{ and } i_2 = j_1), i_1i_2, j_1j_2 \in \mathbb{S}_i^2\}
\end{aligned} \tag{18}
$$

It is easily shown that the relation $\mathbb{R}_i$ is an equivalence relation on the set $\mathbb{S}_i^2$, then by set theory, the set $\mathbb{S}_i^2/\mathbb{R}_i = \{[\![\mathrm{x}]\!]_{\mathbb{R}} : \mathrm{x} \in \mathbb{S}_i^2\}$ is a partition of $\mathbb{S}_i^2$.

### B. Reliable stabilization

In this subsection, the relations between premise variables and the states, the properties of the structure of product inference engine and the equivalence class of set are applied for designing reliable controllers in the special description (10) based on set. Assume $w(t) \equiv 0$, then a stabilization condition for T-S fuzzy systems with a sensor fault is proposed as follows:

**Theorem 1:** If there exist symmetric matrices $Q_q > 0$, $1 \le q \le n_x$ and matrices $L_\tau$, $\tau \in \mathbb{S}$, a scalar $\epsilon > 0$, such that the following inequalities hold

$$
\begin{aligned}
&\sum_{\substack{\tau_{\langle i]}\sigma_{\langle i]}\in\mathbb{X}_i \\ i\in\Lambda-\Lambda_q}} \underline{\Phi}_{\tau\sigma q} < 0, \text{ for } \tau, \sigma \in \prod_{j=1}^{p}\mathbb{S}_j, \mathbb{X}_1 \in \mathbb{S}_1^2/\mathbb{R}_1, \cdots, \\
&\qquad \mathbb{X}_p \in \mathbb{S}_p^2/\mathbb{R}_p, 1 \le q \le n_x
\end{aligned} \tag{19}
$$

$$
\begin{aligned}
&\sum_{\substack{\tau_{\langle i]}\sigma_{\langle i]}\in\mathbb{X}_i \\ i\in\Lambda-\Lambda_q}} \bar{\Phi}_{\tau\sigma q} < 0, \quad \text{for } \tau, \sigma \in \prod_{j=1}^{p}\mathbb{S}_j, \mathbb{X}_1 \in \mathbb{S}_1^2/\mathbb{R}_1, \cdots, \\
&\qquad \mathbb{X}_p \in \mathbb{S}_p^2/\mathbb{R}_p, 1 \le q \le n_x
\end{aligned} \tag{20}
$$

where

$$
\begin{aligned}
\underline{\Phi}_{\tau\sigma q} &= \begin{bmatrix} \mathrm{He}(A_\tau Q_q + B_{u\tau}L_\sigma) & * \\ Q_q - G + \epsilon L_\sigma^T B_{u\tau}^T & -\epsilon G - \epsilon G^T \end{bmatrix}, \\
\bar{\Phi}_{\tau\sigma q} &= \begin{bmatrix} \mathrm{He}(A_\tau Q_q + B_{u\tau}L_\sigma) & * \\ \bar{F}_q Q_q - G + \epsilon L_\sigma^T B_{u\tau}^T & -\epsilon G - \epsilon G^T \end{bmatrix}, \\
\bar{F}_q &= \left[1, \cdots, 1, 0, 1, \cdots, 1\right],
\end{aligned}
$$

then the state feedback controller of the form (6) with the gain $K_\tau = L_\tau G^{-1}$, $\tau \in \mathbb{S}$ renders the system (4) in the normal case and only one sensor failure cases asymptotically stable.

*Proof:* If only the sensor for measuring the state $x_q(t)$ is failed, then

- The feedback state $x_q(t)$ is unprecise, which is denoted as $x_q^F(t)$, and others states are precise.
- The premise variables $v_i$, $i \in \Lambda_q$ as the input of the controller (6) are also unprecise due to the sensor fault,

which are denoted as $v_i^F$, $i \in \Lambda_q$ and the other premise variables of the controller (6) $v_i$, $i \in \Lambda - \Lambda_q$ are precise.

Therefore, the output of the fuzzy controller (7) with the sensor fault is the following form.

$$u(t) = \sum_{i_1=1}^{r_1}\sum_{i_2=1}^{r_2}\cdots\sum_{i_p=1}^{r_p}\left(\prod_{j\in\Lambda-\Lambda_q}\mu_{ji_j}(v_j(t))\right)\times \left(\prod_{j\in\Lambda_q}\mu_{ji_j}(v_j^F(t))\right)K_{i_1i_2\cdots i_p}F_q x(t)$$

i.e.,

$$u(t) = \sum_{\sigma\in\mathbb{S}}\left(\prod_{j\in\Lambda-\Lambda_q}\mu_{j\sigma_{\langle j\rangle}}(v_j(t))\right)\times \left(\prod_{j\in\Lambda_q}\mu_{j\sigma_{\langle j\rangle}}(v_j^F(t))\right)K_\sigma F_q x(t)$$

where the definitions of $\sigma_{\langle\cdot]}$ and $F_q$ are given in (11) and (14), respectively.

Combining it and the fuzzy system (4), then the closed-loop system with the sensor fault is obtained as follows:

$$\dot{x}(t) = \sum_{\tau\in\prod_{i=1}^p\mathbb{S}_i}\sum_{\sigma\in\prod_{i=1}^p\mathbb{S}_i}\left(\prod_{j\in\Lambda}\mu_{j\tau_{\langle j]}}(v_j(t))\right)\times \left(\prod_{j\in\Lambda-\Lambda_q}\mu_{j\sigma_{\langle j]}}(v_j(t))\right)\left(\prod_{j\in\Lambda_q}\mu_{j\sigma_{\langle j]}}(v_j^F(t))\right)\times \Big(A_\tau + B_{u\tau}K_\sigma F_q\Big)x(t) \tag{21}$$

Let $P_q = Q_q^{-1}$, $q = 1,\cdots,p$, then $P_q > 0$, $q = 1,\cdots,p$, choose Lyapunov function $V(t) = x^T(t)P_q x(t)$, then

$$\begin{aligned}\dot{V}(t) =& 2x^T(t)\Bigg[\sum_{\tau\in\prod_{i=1}^p\mathbb{S}_i}\sum_{\sigma\in\prod_{i=1}^p\mathbb{S}_i}\left(\prod_{j\in\Lambda}\mu_{j\tau_{\langle j]}}(v_j(t))\right)\times \left(\prod_{j\in\Lambda-\Lambda_q}\mu_{j\sigma_{\langle j]}}(v_j(t))\right)\left(\prod_{j\in\Lambda_q}\mu_{j\sigma_{\langle j]}}(v_j^F(t))\right)\times \Big(P_qA_\tau + P_qB_{u\tau}K_\sigma F_q\Big)\Bigg]x(t)\\ =& 2x^T(t)\Bigg[\sum_{\tau\in\prod_{i=1}^p\mathbb{S}_i}\sum_{\sigma\in\prod_{i=1}^p\mathbb{S}_i}\left(\prod_{j\in\Lambda-\Lambda_q}\mu_{j\tau_{\langle j]}}(v_j(t))\mu_{j\sigma_{\langle j]}}(v_j(t))\right)\times \left(\prod_{j\in\Lambda_q}\mu_{j\tau_{\langle j]}}(v_j(t))\mu_{j\sigma_{\langle j]}}(v_j^F(t))\right)\times \Big(P_qA_\tau + P_qB_{u\tau}K_\sigma F_q\Big)\Bigg]x(t)\end{aligned}$$

In order to give a compact description, $\mu_{j\tau_{\langle j]}}(v_j(t))$ and $\mu_{j\sigma_{\langle j]}}(v_j^F(t))$ are respectively denoted as $\mu_{j\tau_{\langle j]}}$ and $\mu^F_{j\sigma_{\langle j]}}$, then it follows from the above inequality that

$$\begin{aligned}\dot{V}(t) =& 2x^T(t)\Bigg[\sum_{\tau\in\prod_{i=1}^p\mathbb{S}_i}\sum_{\sigma\in\prod_{i=1}^p\mathbb{S}_i}\left(\prod_{j\in\Lambda-\Lambda_q}\mu_{j\tau_{\langle j]}}\mu_{j\sigma_{\langle j]}}\right)\times\left(\prod_{j\in\Lambda_q}\mu_{j\tau_{\langle j]}}\mu^F_{j\sigma_{\langle j]}}\right)\Big(P_qA_\tau + P_qB_{u\tau}K_\sigma F_q\Big)\Bigg]x(t)\\ =& 2x^T(t)\Big(P_qA(\mu) + P_qB_u(\mu)K(\mu,\mu^F)F_q\Big)x(t)\end{aligned} \tag{22}$$

where

$$\begin{aligned} A(\mu) &= \sum_{\tau\in\prod_{i=1}^p\mathbb{S}_i}\prod_{j\in\Lambda}\mu_{j\tau_{\langle j]}}A_\tau = \sum_{\tau\in\prod_{i=1}^p\mathbb{S}_i}\prod_{j=1}^p\mu_{j\tau_{\langle j]}}A_\tau\\ &= \sum_{\tau\in\prod_{i=1}^p\mathbb{S}_i}\mu_\tau A_\tau\\ B_u(\mu) &= \sum_{\tau\in\prod_{i=1}^p\mathbb{S}_i}\prod_{j\in\Lambda}\mu_{j\tau_{\langle j]}}B_{u\tau} = \sum_{\tau\in\prod_{i=1}^p\mathbb{S}_i}\prod_{j=1}^p\mu_{j\tau_{\langle j]}}B_{u\tau}\\ &= \sum_{\tau\in\prod_{i=1}^p\mathbb{S}_i}\mu_\tau B_{u\tau},\\ K(\mu,\mu^F) &= \sum_{\sigma\in\prod_{i=1}^p\mathbb{S}_i}\prod_{j\in\Lambda-\Lambda_q}\mu_{j\sigma_{\langle j]}}\prod_{j\in\Lambda_q}\mu^F_{j\sigma_{\langle j]}}K_\sigma \end{aligned} \tag{23}$$

On the other hand, from (19) and (20), we have that

$$\sum_{\substack{\tau_{\langle i]}\sigma_{\langle i]}\in\mathbb{X}_i\\ i\in\Lambda-\Lambda_q}}\Phi_{\tau\sigma q} < 0, \quad \text{for } \tau,\sigma\in\prod_{j=1}^p\mathbb{S}_j, \mathbb{X}_1\in\mathbb{S}_1^2/\mathbb{R}_1,\cdots, \mathbb{X}_p\in\mathbb{S}_p^2/\mathbb{R}_p, 1\le q\le n_x \tag{24}$$

where $\Phi_{\tau\sigma q} = \begin{bmatrix}\mathtt{He}(A_\tau Q_q + B_{u\tau}L_\sigma) & * \\ F_qQ_q - G + \epsilon L_\sigma^T B_{u\tau}^T & -\epsilon G - \epsilon G^T\end{bmatrix}$, $F_q$ is the same as in (14).

For $\mathbb{X}_i\in\mathbb{S}_i^2/\mathbb{R}_i$, $i\in\Lambda-\Lambda_q$, let $\bar{\tau}_{\langle i]}\bar{\sigma}_{\langle i]}\in\mathbb{X}_i$, and satisfying $\tau_{\langle i]}\le\bar{\tau}_{\langle i]}$ and $\sigma_{\langle i]}\le\bar{\sigma}_{\langle i]}$ for all $\tau_{\langle i]}\sigma_{\langle i]}\in\mathbb{X}_i$. According to the definition (18) of the binary relation $\mathbb{R}_i$, it follows that $\mu_{\tau_{\langle i]}}\mu_{\sigma_{\langle i]}} = \mu_{\bar{\tau}_{\langle i]}}\mu_{\bar{\sigma}_{\langle i]}}$ for all $\tau_{\langle i]}\sigma_{\langle i]}\in\mathbb{X}_i$, $i\in\Lambda-\Lambda_q$. Therefore,

$$\begin{aligned}&\left(\prod_{j\in\Lambda-\Lambda_q}\mu_{j\bar{\tau}_{\langle j]}}\mu_{j\bar{\sigma}_{\langle j]}}\right)\sum_{\substack{\tau_{\langle i]}\sigma_{\langle i]}\in\mathbb{X}_i\\ i\in\Lambda-\Lambda_q}}\Phi_{\tau\sigma q}\\ &= \sum_{\substack{\tau_{\langle i]}\sigma_{\langle i]}\in\mathbb{X}_i\\ i\in\Lambda-\Lambda_q}}\left(\prod_{j\in\Lambda-\Lambda_q}\mu_{j\bar{\tau}_{\langle j]}}\mu_{j\bar{\sigma}_{\langle j]}}\right)\Phi_{\tau\sigma q}\\ &= \sum_{\substack{\tau_{\langle i]}\sigma_{\langle i]}\in\mathbb{X}_i\\ i\in\Lambda-\Lambda_q}}\left(\prod_{j\in\Lambda-\Lambda_q}\mu_{j\tau_{\langle j]}}\mu_{j\sigma_{\langle j]}}\right)\Phi_{\tau\sigma q}\end{aligned}$$

Combining it and (24), we have that

$$\sum_{\substack{\tau_{\langle i]}\sigma_{\langle i]}\in\mathbb{X}_i\\ i\in\Lambda-\Lambda_q}}\left(\prod_{j\in\Lambda-\Lambda_q}\mu_{j\tau_{\langle j]}}\mu_{j\sigma_{\langle j]}}\right)\Phi_{\tau\sigma q} < 0, \quad \text{for}$$

$$\tau,\sigma\in\prod_{j=1}^{p}\mathbb{S}_j, \mathbb{X}_1\in\mathbb{S}_1^2/\mathbb{R}_1,\cdots,\mathbb{X}_p\in\mathbb{S}_p^2/\mathbb{R}_p, 1\le q\le n_x$$

Then

$$\left(\prod_{j\in\Lambda_q}\mu_{j\tau_{\langle j\rangle}}\mu^F_{j\sigma_{\langle j\rangle}}\right)\sum_{\substack{\tau_{\langle i\rangle}\sigma_{\langle i\rangle}\in\mathbb{X}_i\\ i\in\Lambda-\Lambda_q}}\left(\prod_{j\in\Lambda-\Lambda_q}\mu_{j\tau_{\langle j\rangle}}\mu_{j\sigma_{\langle j\rangle}}\right)\Phi_{\tau\sigma q}$$
$$=\sum_{\substack{\tau_{\langle i\rangle}\sigma_{\langle i\rangle}\in\mathbb{X}_i\\ i\in\Lambda-\Lambda_q}}\left(\prod_{j\in\Lambda-\Lambda_q}\mu_{j\tau_{\langle j\rangle}}\mu_{j\sigma_{\langle j\rangle}}\right)\left(\prod_{j\in\Lambda_q}\mu_{j\tau_{\langle j\rangle}}\mu^F_{j\sigma_{\langle j\rangle}}\right)\times$$
$$\Phi_{\tau\sigma q}<0,$$

for $\tau,\sigma\in\prod_{j=1}^{p}\mathbb{S}_j, \mathbb{X}_1\in\mathbb{S}_1^2/\mathbb{R}_1,\cdots,\mathbb{X}_p\in\mathbb{S}_p^2/\mathbb{R}_p, 1\le q\le n_x$

Further, we have

$$\sum_{\substack{\tau_{\langle i\rangle}\sigma_{\langle i\rangle}\in\mathbb{S}_i^2\\ i\in\Lambda_q}}\sum_{\substack{\mathbb{X}_i\in\mathbb{S}_i^2/\mathbb{R}_i\\ i\in\Lambda-\Lambda_q}}\sum_{\substack{\tau_{\langle i\rangle}\sigma_{\langle i\rangle}\in\mathbb{X}_i\\ i\in\Lambda-\Lambda_q}}\left(\prod_{j\in\Lambda-\Lambda_q}\mu_{j\tau_{\langle j\rangle}}\mu_{j\sigma_{\langle j\rangle}}\times\right.$$
$$\left.\prod_{j\in\Lambda_q}\mu_{j\tau_{\langle j\rangle}}\mu^F_{j\sigma_{\langle j\rangle}}\right)\Phi_{\tau\sigma q}$$
$$=\sum_{\substack{\tau_{\langle i\rangle}\sigma_{\langle i\rangle}\in\mathbb{S}_i^2\\ i\in\Lambda_q}}\sum_{\substack{\tau_{\langle i\rangle}\sigma_{\langle i\rangle}\in\mathbb{S}_i^2\\ i\in\Lambda-\Lambda_q}}\left(\prod_{j\in\Lambda-\Lambda_q}\mu_{j\tau_{\langle j\rangle}}\mu_{j\sigma_{\langle j\rangle}}\prod_{j\in\Lambda_q}\mu_{j\tau_{\langle j\rangle}}\mu^F_{j\sigma_{\langle j\rangle}}\right)$$
$$\times\Phi_{\tau\sigma q}$$
$$=\sum_{\substack{\tau_{\langle i\rangle}\sigma_{\langle i\rangle}\in\mathbb{S}_i^2\\ i\in\Lambda}}\left(\prod_{j\in\Lambda-\Lambda_q}\mu_{j\tau_{\langle j\rangle}}\mu_{j\sigma_{\langle j\rangle}}\prod_{j\in\Lambda_q}\mu_{j\tau_{\langle j\rangle}}\mu^F_{j\sigma_{\langle j\rangle}}\right)\Phi_{\tau\sigma q}$$
$$=\sum_{\substack{\tau_{\langle i\rangle}\in\mathbb{S}_i\\ i\in\Lambda}}\sum_{\substack{\sigma_{\langle i\rangle}\in\mathbb{S}_i\\ i\in\Lambda}}\left(\prod_{j\in\Lambda-\Lambda_q}\mu_{j\tau_{\langle j\rangle}}\mu_{j\sigma_{\langle j\rangle}}\prod_{j\in\Lambda_q}\mu_{j\tau_{\langle j\rangle}}\mu^F_{j\sigma_{\langle j\rangle}}\right)\Phi_{\tau\sigma q}$$
$$=\sum_{\tau\in\prod_{i=1}^{p}\mathbb{S}_i}\sum_{\sigma\in\prod_{i=1}^{p}\mathbb{S}_i}\left(\prod_{j\in\Lambda-\Lambda_q}\mu_{j\tau_{\langle j\rangle}}\mu_{j\sigma_{\langle j\rangle}}\prod_{j\in\Lambda_q}\mu_{j\tau_{\langle j\rangle}}\mu^F_{j\sigma_{\langle j\rangle}}\right)$$
$$\times\Phi_{\tau\sigma q}$$
$$=\sum_{\tau\in\prod_{i=1}^{p}\mathbb{S}_i}\sum_{\sigma\in\prod_{i=1}^{p}\mathbb{S}_i}\left(\prod_{j\in\Lambda-\Lambda_q}\mu_{j\tau_{\langle j\rangle}}\mu_{j\sigma_{\langle j\rangle}}\prod_{j\in\Lambda_q}\mu_{j\tau_{\langle j\rangle}}\mu^F_{j\sigma_{\langle j\rangle}}\right)$$
$$\times\begin{bmatrix}\mathrm{He}(A_\tau Q_q+B_{u\tau}L_\sigma) & *\\ F_qQ_q-G+\epsilon L_\sigma^TB_{u\tau}^T & -\epsilon G-\epsilon G^T\end{bmatrix}$$
$$<0$$

Consider the gain $K_\sigma=L_\sigma G^{-1}$, $\sigma\in\mathbb{S}$, then substituting $L_\sigma$ by $K_\sigma G$ in the above inequality, it yields that

$$\begin{bmatrix}\mathrm{He}(A(\mu)Q_q+B_u(\mu)K(\mu,\mu^F)G) & *\\ F_qQ_q-G+\epsilon G^TK^T(\mu,\mu^F)B_u^T(\mu) & -\epsilon G-\epsilon G^T\end{bmatrix}<0 \tag{25}$$

where $A(\mu)$, $B_u(\mu)$, $K(\mu,\mu^F)$ are the same as in (23).

Let vectors $x(t)$ and $\bar{x}(t)$ satisfy

$$\bar{x}(t)=Q_q^{-1}x(t)=P_qx(t) \tag{26}$$

For $x(t)\neq 0$, it follows that $\bar{x}(t)\neq 0$. Pre- and post-multiplying (25) with

$$\begin{bmatrix}\bar{x}^T(t) & \bar{x}^T(t)B_u(\mu)K(\mu,\mu^F)\end{bmatrix}$$

and its transpose, then it follows that

$$2\bar{x}^T(t)\Big(A(\mu)Q_q+B_u(\mu)K(\mu,\mu^F)F_qQ_q\Big)\bar{x}(t)<0$$

i.e.,

$$2x^T(t)\Big(P_qA(\mu)+P_qB_u(\mu)K(\mu,\mu^F)F_q\Big)x(t)<0$$

which implies that $\dot{V}(t)<0$ from (22). Therefore, the system is asymptotically stable with the controller (6) for the error measurement state vector $x^F(t)=F_qx(t)$, $F_q=diag\left[1,\cdots,1,f_q,1,\cdots,1\right]_{n_x\times n_x}$, $0\le f_q\le 1$. For all $q\in\{1,2,\cdots,n_x\}$, it is easily proved from the above discuss that the closed-loop system is asymptotically stable with the error measurement of the state $x_q(t)$, if the conditions (19) and (20) hold. Therefore, the state feedback controller of the form (6) renders the system (4) in the normal case and only one sensor failure cases asymptotically stable. Thus, the proof is complete. ■

**Remark 2:** A new reliable control synthesis condition is obtained in Theorem 1 by using the properties of fuzzy product inference engine and the equivalence class in set theory. The new methods are applicable for the fuzzy systems, the premise variables of which are dependent on the states, therefore, it can guarantee the stability for T-S fuzzy systems with fault case (III) (see Definition 1). However, the existing approaches only consider the fault case (I), i.e., the premise variables are independent on the states, then they might be ineffective for T-S fuzzy systems with fault case (III). In next section, a numerical example will be given to show the advantage of the new method.

**Remark 3:** Note that the condition of Theorem 1 is a set of LMIs with a line search over a scalar variable $\epsilon$, then Theorem 1 is no longer convex. Because $\epsilon$ is a scalar variable, a constructive numerical procedure can be given. The procedure always achieves a reasonable solution provided $\epsilon$ is initialized with a sufficiently large value and the search is carefully performed (for instance with small enough steps near the optimum). Some methods for a line search can be found in [43], [44].

### C. $H_\infty$ reliable control

Consider the T-S control system (8) with sensor fault (12) and affected by unknown disturbance $w(t)$. In order to guarantee a good disturbance attenuation property, when the sensors occur faults, $H_\infty$ reliable control methods will be proposed in this subsection. Firstly, $H_\infty$ performance definition is given as follows:

**Definition 2:** [45], [46] Let $\gamma>0$ be a constant. If (8) is asymptotically stable, and for any $w(t)\in L^2[0,\infty)$ (the space of square integrable functions) and $x(0)=0$, the following inequality holds:

$$\int_0^\infty z^T(t)z(t)dt\le\gamma^2\int_0^\infty w^T(t)w(t)dt \tag{27}$$

then the system (8) is said to be with an $H_\infty$-norm less than or equal to $\gamma$.

Based on the description (10) and adopting the similar technique in the above subsection, a new $H_\infty$ reliable control synthesis condition is given as follows:

**Theorem 2:** If there exist symmetric matrices $Q_q > 0, 1 \le q \le n_x$ and matrices $G$, $L_\tau$, $\tau \in \mathbb{S}$, a scalar $\epsilon > 0$ satisfying

$$\sum_{\substack{\tau_{\langle i\rangle}\sigma_{\langle i\rangle}\in\mathbb{X}_i\\ i\in\Lambda-\Lambda_q}} \underline{\Psi}_{\tau\sigma q} < 0, \quad \text{for } \tau,\sigma \in \prod_{j=1}^{p}\mathbb{S}_j, \mathbb{X}_1 \in \mathbb{S}_1^2/\mathbb{R}_1, \cdots, \mathbb{X}_p \in \mathbb{S}_p^2/\mathbb{R}_p, 1 \le q \le n_x \tag{28}$$

$$\sum_{\substack{\tau_{\langle i\rangle}\sigma_{\langle i\rangle}\in\mathbb{X}_i\\ i\in\Lambda-\Lambda_q}} \bar{\Psi}_{\tau\sigma q} < 0, \quad \text{for } \tau,\sigma \in \prod_{j=1}^{p}\mathbb{S}_j, \mathbb{X}_1 \in \mathbb{S}_1^2/\mathbb{R}_1, \cdots, \mathbb{X}_p \in \mathbb{S}_p^2/\mathbb{R}_p, 1 \le q \le n_x \tag{29}$$

where

$$\underline{\Psi}_{\tau\sigma q} = \begin{bmatrix} \mathrm{He}\left(\bar{A}_\tau\bar{Q}_q + \bar{B}_{u\tau}L_\sigma[I\ 0]\right) & * & * & * \\ [I\ 0]\bar{Q}_q - G[I\ 0] + \epsilon L_\sigma^T\bar{B}_{u\tau}^T & -\epsilon G - \epsilon G^T & * & * \\ \bar{B}_{w\tau}^T & 0 & -\gamma^2 I & * \\ \bar{C}_\tau\bar{Q}_q & 0 & 0 & -I \end{bmatrix}$$

$$\bar{\Psi}_{\tau\sigma q} = \begin{bmatrix} \mathrm{He}\left(\bar{A}_\tau\bar{Q}_q + \bar{B}_{u\tau}L_\sigma[I\ 0]\right) & * & * & * \\ [\bar{F}_q\ 0]\bar{Q}_q - G[I\ 0] + \epsilon L_\sigma^T\bar{B}_{u\tau}^T & -\epsilon G - \epsilon G^T & * & * \\ \bar{B}_{w\tau}^T & 0 & -\gamma^2 I & * \\ \bar{C}_\tau\bar{Q}_q & 0 & 0 & -I \end{bmatrix}$$

$$\bar{A}_\tau = \begin{bmatrix} A_\tau & 0 \\ 0 & -\frac{1}{2}I \end{bmatrix}, \bar{B}_{u\tau} = \begin{bmatrix} B_{u\tau} \\ D_\tau \end{bmatrix}, \bar{C}_\tau = \begin{bmatrix} C_\tau & 0 \end{bmatrix},$$
$$\bar{B}_{w\tau} = \begin{bmatrix} B_{w\tau} \\ 0 \end{bmatrix}, \bar{Q}_q = \begin{bmatrix} Q_q & 0 \\ 0 & I \end{bmatrix}$$

then the state feedback controller of the form (6) with the gain $K_\tau = L_\tau G^{-1}$, $\tau \in \mathbb{S}$ renders the system (4) in the normal case and only one sensor failure cases asymptotically stable, and $H_\infty$-norm less than or equal to $\gamma$.

*Proof:* From (14) and (29), we have that

$$\sum_{\substack{\tau_{\langle i\rangle}\sigma_{\langle i\rangle}\in\mathbb{X}_i\\ i\in\Lambda-\Lambda_q}} \Psi_{\tau\sigma q} < 0, \quad \text{for } \tau,\sigma \in \prod_{j=1}^{p}\mathbb{S}_j, \mathbb{X}_1 \in \mathbb{S}_1^2/\mathbb{R}_1, \cdots, \mathbb{X}_p \in \mathbb{S}_p^2/\mathbb{R}_p, 1 \le q \le n_x \tag{30}$$

where

$$\Psi_{\tau\sigma q} = \begin{bmatrix} \mathrm{He}\left(\bar{A}_\tau\bar{Q}_q + \bar{B}_{u\tau}L_\sigma[I\ 0]\right) & * & * & * \\ [F_q\ 0]\bar{Q}_q - G[I\ 0] + \epsilon L_\sigma^T\bar{B}_{u\tau}^T & -\epsilon G - \epsilon G^T & * & * \\ \bar{B}_{w\tau}^T & 0 & -\gamma^2 I & * \\ \bar{C}_\tau\bar{Q}_q & 0 & 0 & -I \end{bmatrix} \tag{31}$$

and $F_q$ is the same as in (14).

Adopting the same technique from (24) to (25), then it follows from (31) that

$$\begin{bmatrix} \mathrm{He}\left(\bar{A}(\mu)\bar{Q}_q + \bar{B}_u(\mu)K(\mu,\mu^F)G[I\ 0]\right) & * & * & * \\ [F_q\ 0]\bar{Q}_q - G[I\ 0] + \epsilon G^T K^T(\mu,\mu^F)\bar{B}_u^T(\mu) & -\epsilon G - \epsilon G^T & * & * \\ \bar{B}_w^T(\mu) & 0 & -\gamma^2 I & * \\ \bar{C}(\mu)\bar{Q}_q & 0 & 0 & -I \end{bmatrix} < 0$$

where

$$\bar{A}(\mu) = \sum_{\tau\in\prod_{i=1}^p\mathbb{S}_i}\mu_\tau\bar{A}_\tau, \quad \bar{B}_u(\mu) = \sum_{\tau\in\prod_{i=1}^p\mathbb{S}_i}\mu_\tau\bar{B}_{u\tau},$$
$$\bar{B}_w(\mu) = \sum_{\tau\in\prod_{i=1}^p\mathbb{S}_i}\mu_\tau\bar{B}_{w\tau}, \quad \bar{C}(\mu) = \sum_{\tau\in\prod_{i=1}^p\mathbb{S}_i}\mu_\tau\bar{C}_\tau,$$
$$K(\mu,\mu^F) = \sum_{\sigma\in\in\prod_{i=1}^p\mathbb{S}_i}\left(\prod_{j\in\Lambda-\Lambda_q}\mu_{j\sigma_{\langle j\rangle}}\right)\left(\prod_{j\in\Lambda_q}\mu^F_{j\sigma_{\langle j\rangle}}\right)K_\sigma$$

Applying Schur complement lemma to the above inequality, then we have

$$\begin{bmatrix} \mathrm{He}\left(\bar{A}(\mu)\bar{Q}_q + \bar{B}_u(\mu)K(\mu,\mu^F)G[I\ 0]\right) + \bar{Q}_q\bar{C}^T(\mu)\bar{C}(\mu)\bar{Q}_q & * & * \\ [F_q\ 0]\bar{Q}_q - G[I\ 0] + \epsilon G^T K^T(\mu,\mu^F)\bar{B}_u^T(\mu) & -\epsilon G - \epsilon G^T & * \\ \bar{B}_w^T(\mu) & 0 & -\gamma^2 I \end{bmatrix} < 0 \tag{32}$$

For $\bar{x}(t) \neq 0$, pre- and post-multiplying the above inequality with $\begin{bmatrix}\bar{x}^T(t) & \bar{x}^T(t)\bar{B}_u(\mu)K(\mu,\mu^F) & w^T(t)\end{bmatrix}$ and its transpose, it yields that

$$\begin{aligned}
&2\bar{x}^T\left(\bar{A}(\mu)\bar{Q}_q + \bar{B}_u(\mu)K^T(\mu,\mu^F)G[I\ 0]\right)\bar{x} \\
&+ \bar{x}^T\bar{Q}_q\bar{C}^T(\mu)\bar{C}(\mu)\bar{Q}_q\bar{x} + 2\bar{x}^T\bar{B}_u(\mu)K(\mu,\mu^F)[F_q\ 0]\bar{Q}_q\bar{x} \\
&- 2\bar{x}^T\bar{B}_u(\mu)K(\mu,\mu^F)G[I\ 0]\bar{x} \\
&+ 2\epsilon\bar{x}^T\bar{B}_u(\mu)K(\mu,\mu^F)G^TK^T(\mu,\mu^F)\bar{B}_u^T(\mu)\bar{x} \\
&- 2\epsilon\bar{x}^T\bar{B}_u(\mu)K(\mu,\mu^F)G^TK^T(\mu,\mu^F)\bar{B}_u^T(\mu)\bar{x} \\
&+ 2w^T\bar{B}_w^T(\mu)\bar{x} - \gamma^2 w^T w \\
=&2\bar{x}^T\bar{A}(\mu)\bar{Q}_q\bar{x} + \bar{x}^T\bar{Q}_q\bar{C}^T(\mu)\bar{C}(\mu)\bar{Q}_q\bar{x} + 2\bar{x}^T\bar{B}_w(\mu)w \\
&+ 2\bar{x}^T\bar{B}_u(\mu)K(\mu,\mu^F)[F_q\ 0]\bar{Q}_q\bar{x} - \gamma^2 w^T w \\
<&0
\end{aligned}$$

Since $Q_q > 0$, it is invertible, let

$$\bar{P}_q = (\bar{Q}_q)^{-1}$$

Therefore, the above inequality can be rewritten as follows:

$$\tilde{x}^T(t)\left(\bar{P}_q\left(\bar{A}(\mu) + \bar{B}_u(\mu)K(\mu,\mu^F)[F_q\ 0]\right) + \left(\bar{A}(\mu) + \bar{B}_u(\mu)K(\mu,\mu^F)[F_q\ 0]\right)^T\bar{P}_q + \bar{C}^T(\mu)\bar{C}(\mu)\right)\tilde{x}(t) + \tilde{x}^T\bar{P}_q\bar{B}_w(\mu)w + w^T\bar{B}_w^T(\mu)\bar{P}_q\tilde{x} - \gamma^2 w^T w < 0 \tag{33}$$

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TFUZZ.2014.2315298, IEEE Transactions on Fuzzy Systems



where

$$\tilde{x}(t) = \bar{Q}_q \bar{x}(t) \tag{34}$$

(33) is equivalent to

$$\begin{bmatrix} \mathtt{He}\Big(\bar{P}_q\big(\bar{A}(\mu) + \bar{B}_u(\mu)K(\mu,\mu^F)[F_q\ 0]\big)\Big) + \bar{C}^T(\mu)\bar{C}(\mu) & * \\ \bar{B}_w^T(\mu)\bar{P}_q & -\gamma^2 I \end{bmatrix} < 0$$

which can be rewritten as follows:

$$\begin{bmatrix} \mathtt{He}\big(P_q A(\mu) + P_q B_u(\mu)K(\mu,\mu^F)F_q\big) + C^T(\mu)C(\mu) & F_q K^T(\mu,\mu^F)D^T(\mu) & P_q B_w(\mu) \\ D(\mu)K(\mu,\mu^F)F_q & -I & 0 \\ B_w^T(\mu)P_q & 0 & -\gamma^2 I \end{bmatrix} < 0$$

Applying Schur complement to the above inequality, then we can obtain

$$\begin{bmatrix} \Xi_{11} & P_q B_w(\mu) \\ B_w^T(\mu)P_q & -\gamma^2 I \end{bmatrix} < 0$$

where $\Xi_{11} = \mathtt{He}\big(P_q A(\mu) + P_q B_u(\mu)K(\mu,\mu^F)F_q\big) + C^T(\mu)C(\mu) + F_q K^T(\mu,\mu^F)D^T(\mu)D(\mu)K(\mu,\mu^F)F_q$. Combining the above inequality and $C^T(\mu)D(\mu) = 0$, yields

$$\begin{bmatrix} \bar{\Xi}_{11} & P_q B_w(\mu) \\ B_w^T(\mu)P_q & -\gamma^2 I \end{bmatrix} < 0$$

where $\bar{\Xi}_{11} = \mathtt{He}\big(P_q A(\mu) + P_q B_u(\mu)K(\mu,\mu^F)F_q\big) + \Big(C(\mu) + D(\mu)K(\mu,\mu^F)F_q\Big)^T \Big(C(\mu) + D(\mu)K(\mu,\mu^F)F_q\Big)$. Pre- and post-multiplying the above inequality with $\begin{bmatrix} x^T(t) & w^T(t) \end{bmatrix}$ and its transpose, then we have

$$\begin{aligned} &2x^T(t)P_q\big(A(\mu) + B_u(\mu)K(\mu,\mu^F)F_q\big)x(t) + x^T(t)\times \\ &\big(C(\mu) + D(\mu)K(\mu,\mu^F)F_q\big)^T\big(C(\mu) + D(\mu)K(\mu,\mu^F)F_q\big)\times \\ &x(t) + 2x^T(t)P_q B_w(\mu)w(t) - \gamma^2 w^T(t)w(t) < 0 \end{aligned} \tag{35}$$

Choose Lyapunov function $V(t) = x^T(t)P_q x(t)$ for the case that only the sensor for measuring the state $x_q(t)$ is failed, then (35) can be rewritten as follows:

$$\dot{V}(t) + z^T(t)z(t) - \gamma^2 w^T(t)w(t) < 0 \tag{36}$$

Integrating both sides of this inequality yields

$$\begin{aligned} &\int_0^\infty \dot{V}(t) + \int_0^\infty z^T(t)z(t) - \gamma^2 \int_0^\infty w^T(t)w(t) \\ =&V(\infty) - V(0) + \int_0^\infty z^T(t)z(t) - \gamma^2 \int_0^\infty w^T(t)w(t) \\ <&0 \end{aligned}$$

Using the fact that $x(0) = 0$ and $V(\infty) \geq 0$, we obtain

$$\int_0^\infty z^T(t)z(t)dt \leq \gamma^2 \int_0^\infty w^T(t)w(t)dt$$

Hence, (27) holds and the $H_\infty$ performance is fulfilled. If the disturbance $w(t) = 0$, then from (36), we have $\dot{V}(t) < 0$. Hence, the system (1) with sensor faults by the controller (16) is asymptotically stable. Thus, the proof is complete. ■

## IV. EXAMPLE

In this section, an example is given to illustrate the effectiveness of the proposed method. Consider the following nonlinear system.

$$\begin{aligned} \dot{x}_1(t) =& x_2(t) - 3x_3(t) + w(t) \\ \dot{x}_2(t) =& -x_1(t) - 0.2x_2(t) - x_3(t) + 0.5\sin(x_2(t))x_2(t) \\ & - u(t) \\ \dot{x}_3(t) =& 2x_1(t) + x_2(t) - 3x_3(t) + 1.6\sin(x_1(t))x_3(t) + u(t) \\ z_1(t) =& x_3(t) \\ z_2(t) =& u(t) \end{aligned}$$

Choose premise variables $x_1(t)$, $x_2(t)$ and use the modelling method in [21], then we have that the nonlinear system can be exactly represented by the following the T-S fuzzy model:

**Plant Rule** $(i_1 i_2)$**:**

IF $v_1(t)$ is $M_{1i_1}$ and $v_2(t)$ is $M_{2i_2}$

THEN $\dot{x}(t) = A_{i_1 i_2}x(t) + B_{wi_1 i_2}w(t) + B_{ui_1 i_2}u(t)$

$z(t) = C_{i_1 i_2}x(t) + D_{i_1 i_2}u(t)$

where

$$A_{11} = \begin{bmatrix} 0 & 1 & -3 \\ -1 & -0.7 & -1 \\ 2 & 1 & -4.6 \end{bmatrix}, A_{12} = \begin{bmatrix} 0 & 1 & -3 \\ -1 & 0.3 & -1 \\ 2 & 1 & -4.6 \end{bmatrix},$$

$$A_{21} = \begin{bmatrix} 0 & 1 & -3 \\ -1 & -0.7 & -1 \\ 2 & 1 & -1.4 \end{bmatrix}, A_{22} = \begin{bmatrix} 0 & 1 & -3 \\ -1 & 0.3 & -1 \\ 2 & 1 & -1.4 \end{bmatrix}$$

$$B_{w11} = B_{w12} = B_{w21} = B_{w22} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T,$$

$$B_{u11} = B_{u12} = B_{u21} = B_{u22} = \begin{bmatrix} 0 & -1 & 1 \end{bmatrix}^T$$

$$C_{11} = C_{12} = C_{21} = C_{22} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

$$D_{11} = D_{12} = D_{21} = D_{22} = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}$$

and the fuzzy membership functions are $\mu_{11}(x_1(t)) = \frac{1-\sin(x_1(t))}{2}$, $\mu_{12}(x_1(t)) = \frac{1+\sin(x_1(t))}{2}$, $\mu_{21}(x_2(t)) = \frac{1-\sin(x_2(t))}{2}$, $\mu_{22}(x_2(t)) = \frac{1+\sin(x_2(t))}{2}$. By using the product inference engine [21], the weights of the fuzzy rules (11), (12), (21), (22) are obtained as $\mu_{11}(x_1)\mu_{21}(x_2)$, $\mu_{11}(x_1)\mu_{22}(x_2)$, $\mu_{12}(x_1)\mu_{21}(x_2)$ and $\mu_{12}(x_1)\mu_{22}(x_2)$. Moreover, by the mapping in Appendix, the fuzzy model can also be rewritten as follows:

**Plant Rule** $i$**:**

IF $v_1(t)$ is $M_{1i}$ and $v_2(t)$ is $M_{2i}$

THEN $\dot{x}(t) = A_i x(t) + B_{wi}w(t) + B_{ui}u(t)$

$z(t) = C_i x(t) + D_i u(t)$

where

$$\begin{aligned} &A_1 = A_{11}, A_2 = A_{12}, A_3 = A_{21}, A_4 = A_{22}, B_{u1} = B_{u11}, \\ &B_{u2} = B_{u12}, B_{u3} = B_{u21}, B_{u4} = B_{u22}, B_{w1} = B_{w11}, \\ &B_{w2} = B_{w12}, B_{w3} = B_{w21}, B_{w4} = B_{w22}, C_1 = C_{11}, \\ &C_2 = C_{12}, C_3 = C_{21}, C_4 = C_{22}, D_1 = D_{11}, D_2 = D_{12}, \end{aligned}$$

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TFUZZ.2014.2315298, IEEE Transactions on Fuzzy Systems



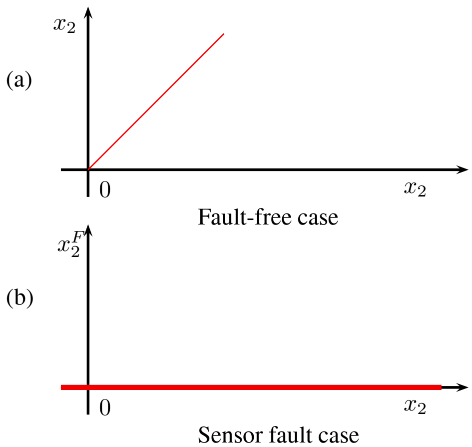


Fig. 1: The relation of $x_2^F$ and $x_2$ of the second state for the fault-free and fault cases

$$D_3 = D_{21}, D_4 = D_{22},$$

and the fuzzy rule weights of the corresponding conventional fuzzy model are $\alpha_1(v(t)) = \mu_{11}(x_1)\mu_{21}(x_2)$, $\alpha_2(v(t)) = \mu_{11}(x_1)\mu_{22}(x_2)$, $\alpha_3(v(t)) = \mu_{12}(x_1)\mu_{21}(x_2)$, $\alpha_4(v(t)) = \mu_{12}(x_1)\mu_{22}(x_2)$. Note that each one of the weights $\alpha_i(v(t))$, $i = 1, \cdot, 4$ is dependent on $x_1$ and $x_2$, then there are influences on all fuzzy rule weights if one sensor for measuring some state is failure.

First, assume the disturbance $w(t) \equiv 0$ and use different methods to design fuzzy controllers for guaranteeing the stability of the system with sensor faults. The methods in [33], [47] and Theorem 1 are adopted inhere and the computational results are given in Table I. Assume that the sensor for measuring state $x_2$ is outage and no fault in the other sensors, then the second state, which is used in the fuzzy controller, is 0, i.e., $x_2^F = 0$. The membership functions $\mu_{21}(x_2^F) = \frac{1-\sin(x_2^F)}{2} = 0.5$, $\mu_{22}(x_2^F) = \frac{1+\sin(x_2^F)}{2} = 0.5$ in the fuzzy controller (16) are different from $\mu_{21}(x_2) = \frac{1-\sin(x_2)}{2}$, $\mu_{22}(x_2) = \frac{1+\sin(x_2)}{2}$ in the fuzzy system (1), and the fuzzy controller with the sensor fault can be written as follows:

$$\begin{aligned} & u(t) \\ =& (\mu_{11}(x_1)\mu_{21}(x_2^F)K_{11} + \mu_{11}(x_1)\mu_{22}(x_2^F)K_{12} + \mu_{12}(x_1) \\ & \times \mu_{21}(x_2^F)K_{21} + \mu_{12}(x_1)\mu_{22}(x_2^F)K_{22}) \begin{bmatrix} x_1 & x_2^F & x_3 \end{bmatrix}^T \\ =& (\mu_{11}(x_1)\mu_{21}(x_2^F)K_{11} + \mu_{11}(x_1)\mu_{22}(x_2^F)K_{12} \\ & + \mu_{12}(x_1)\mu_{21}(x_2^F)K_{21} + \mu_{12}(x_1)\mu_{22}(x_2^F)K_{22})F_2 x(t) \end{aligned}$$

In order to illustrate the influences of the sensor fault on the states and membership functions, the relations of $x_2^F$, $\mu_{21}(x_2^F)$, $\mu_{21}(x_2^F)$ $\mu_{22}(x_2)$, $\mu_{22}(x_2^F)$ and $x_2$ are given in Figs. 1-3, from which, it can be seen that the fault has much of impact on the membership functions of the fuzzy controller (16). The simulations are done and the corresponding state responds are shown in Figs. 4-6.

From Figs. 4-6, it can be seen that the existing methods in [47] and [33] cannot guarantee the stability of the resulted closed-loop system and the new proposed method (Theorem 1) presents an effective reliable controller. In particular, the

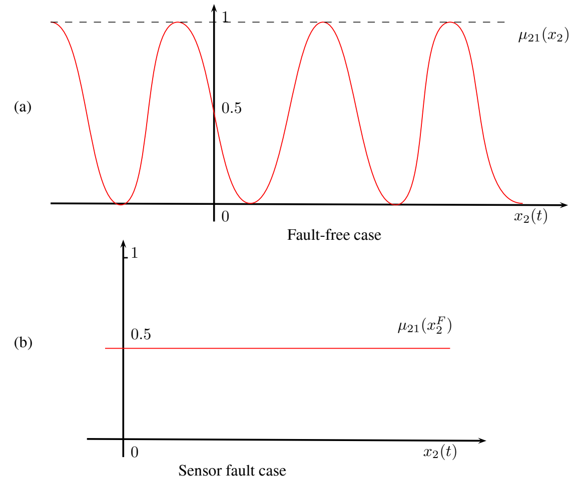


Fig. 2: The relations of $\mu_{21}(x_2)$, $\mu_{21}(x_2^F)$ and $x_2$ for the fault-free and fault cases

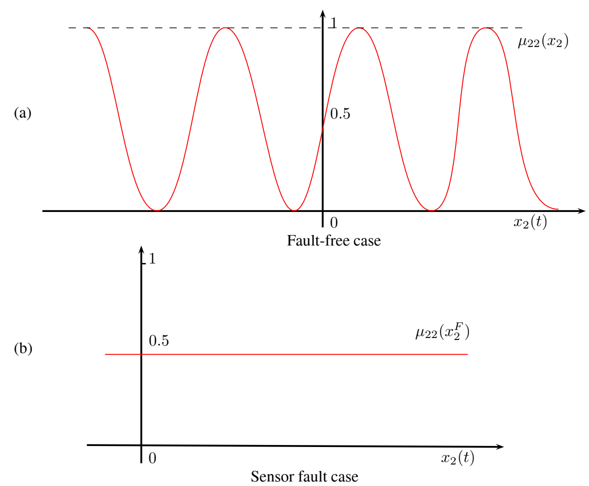


Fig. 3: The relations of $\mu_{22}(x_2)$, $\mu_{22}(x_2^F)$ and $x_2$ for the fault-free and fault cases

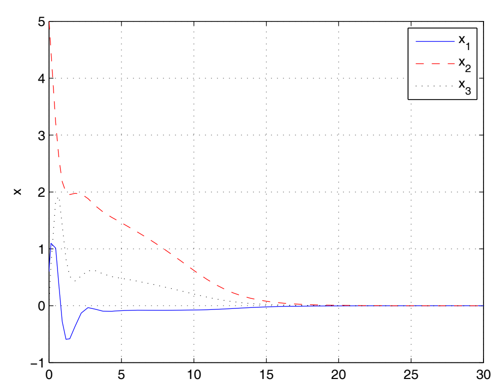


Fig. 4: The state responds by using the controller obtained based on Theorem 1

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TABLE I: Controller gains

| Theorem 1 with $\epsilon = 0.6$ | The method in [33] with $\lambda = 50$ | The method in [47] |
|---|---|---|
| $K_{11} = [-0.8144\ 1.7659\ 0.2007]$ | $K_1 = [-0.8149\ 10.8548\ -2.4226]$ | $K_1 = [1.4306\ 6.5104\ 7.3707]$ |
| $K_{12} = [-0.8359\ 2.0636\ 0.1525]$ | $K_2 = [-0.9587\ 6.5834\ 0.8549]$ | $K_2 = [1.5732\ 8.0527\ 8.3183]$ |
| $K_{21} = [-0.8240\ 1.8171\ 0.3148]$ | $K_3 = [-0.7797\ 4.2596\ -2.3550]$ | $K_3 = [0.9513\ 3.0071\ 2.5858]$ |
| $K_{22} = [-0.8347\ 1.9621\ 0.2789]$ | $K_4 = [-0.9486\ 1.7885\ 0.1064]$ | $K_4 = [1.0939\ 4.5494\ 3.5334]$ |

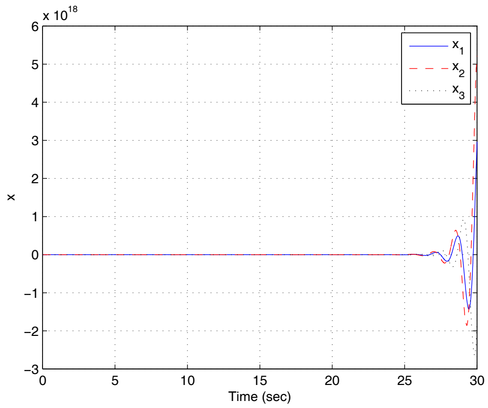


Fig. 5: The state responds by using the controller obtained based on the method in [47]

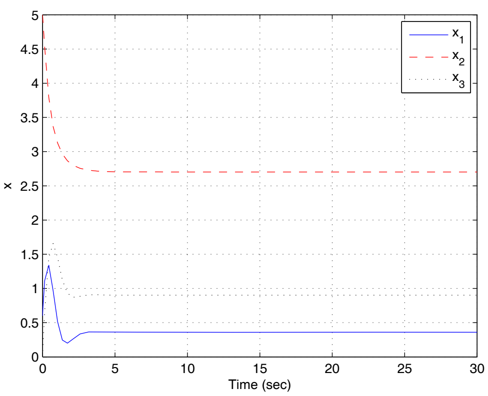


Fig. 6: The state responds by using the controller obtained based on the method in [33]

method in [33] is for designing reliable controllers, but it is only applicable for the fuzzy system with the premise variables independent on the states, then it is failed for the example.

Assume the disturbance

$$w(t) = \begin{cases} 1, & 6 \le t \le 15 \\ 0, & \text{others} \end{cases}$$

and the initial condition $x(0) = 0$. Theorem 2 and the method in [33] are used for designing $H_\infty$ controllers, the obtained $H_\infty$ performance indices are respectively 9.98 and 2.80. Note that the obtained index by the method in [33] is smaller. However, it is not the actual bound of the $H_\infty$ performance, because the all premise variables are assumed to be reliable in the method of [33]. Moreover, the obtained gains by these methods are given in the following Table II.

The simulations are done with the sensor for measuring state

TABLE II: Controller gains

| Theorem 2 with $\epsilon = 0.4$ | The method in [33] with $\lambda = 1.7$ |
|---|---|
| $K_{11} = [-0.9006\ 2.5372\ 0.0802]$ | $K_1 = [-0.9362\ 7.0083\ -2.1257]$ |
| $K_{12} = [-0.9007\ 2.5374\ 0.0803]$ | $K_2 = [-0.9931\ 4.0159\ 0.4995]$ |
| $K_{21} = [-0.9154\ 2.6312\ 0.1725]$ | $K_3 = [-0.9463\ 2.5456\ -1.9976]$ |
| $K_{22} = [-0.9154\ 2.6313\ 0.1725]$ | $K_4 = [-1.0026\ 0.9888\ 0.0933]$ |

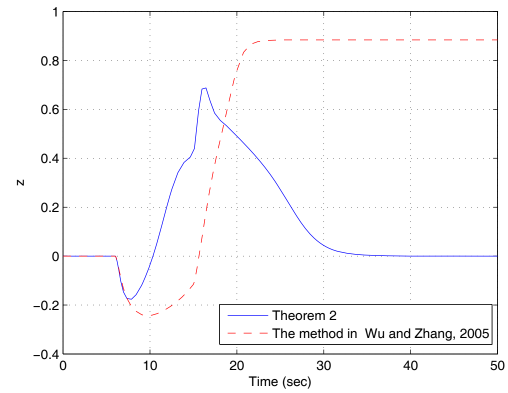


Fig. 7: The responds of the controlled output $z_1(t)$

$x_2$ being outage. The responds of the controlled output $z(t)$ are given in Fig. 7. It can be seen that the controller obtained by Theorem 2 achieve a better $H_\infty$ performance, which illustrates the effectiveness of the new method.

## V. Conclusion

The reliable control problem for T-S fuzzy control systems with sensor multiplicative faults has been investigated in this paper. By using the properties of fuzzy product inference engine, a class of new reliable fuzzy control techniques are proposed for T-S fuzzy systems with sensor faults. In the new conditions, we consider the influences of sensor faults on both the system states and premise variables of fuzzy controllers, then the proposed controllers can maintain the stability and the control performance when all sensors are operational as well as when some sensor experiences failures. A numerical example has been given to illustrate the effectiveness of the new approach. The influences of sensor faults on both the premise variables and system states have been considered in this paper, then the proposed methods are valuable in practical use for guaranteeing the performance of T-S fuzzy control systems against sensor faults. Planned future work by the authors will be directed at reliable control problems via dynamic output feedback for T-S fuzzy systems.

## ACKNOWLEDGMENT

This work was supported by the Funds of National Science of China (Grant Nos. 60904010 and 61273148), the Program for New Century Excellent Talents in University (Grant No. NCET-11-0072), the Fundamental Research Funds for the Central Universities (Grant Nos. N120504004 and N110804001), the Foundation for the Author of National Excellent Doctoral Dissertation of P.R. China (Grant No. 201157), the IAPI Fundamental Research Funds (Grant No. 2013ZCX01-01), China Postdoctoral Science Foundation (Grant No. 20100470074), China Postdoctoral Science Foundation Special Funded Project (Grant No. 201104608), the Nature Science of Foundation of Liaoning Province (Grant No. 201202063), the 985 fund and Postdoctoral Science Foundation of Northeastern University, China.

## APPENDIX

Relations between the new description and the existing descriptions of T-S fuzzy systems:

Note that the set $\prod_{i=1}^{p}\mathbb{S}_i$ is with $r=\prod_{i=1}^{p}r_i$ elements, then a $1-1$ mapping can be defined as follows:

$$q:\prod_{i=1}^{p}\mathbb{S}_i \longrightarrow \{1,2,\cdots,r\} \tag{37}$$

where

$$\begin{aligned}q(\tau)=&\tau_{\langle p]}+(\tau_{\langle p-1]}-1)r_p+(\tau_{\langle p-2]}-1)r_pr_{p-1}+\\ &(\tau_{\langle p-3]}-1)r_pr_{p-1}r_{p-2}+\cdots+(\tau_{\langle 1]}-1)\prod_{j=1}^{p-1}r_{p+1-j}\\ =&\tau_{\langle p]}+\sum_{i=2}^{p}\prod_{j=1}^{i-1}r_{p+1-j}(\tau_{\langle i]}-1)\end{aligned}$$

i.e.,

$$q:\tau_{\langle 1]}\tau_{\langle 2]}\cdots\tau_{\langle p]}\longmapsto \tau_{\langle p]}+\sum_{i=2}^{p}\prod_{j=1}^{i-1}r_{p+1-j}(\tau_{\langle i]}-1) \tag{38}$$

Let

$$\begin{aligned}&\alpha_{q(\tau)}(v(t))=\mu_\tau=\prod_{j=1}^{p}\mu_{j\tau_{\langle j]}}(v_j(t)),\quad \bar{A}_{q(\tau)}=A_\tau,\\ &\bar{B}_{q(\tau)}=B_{u\tau},\quad \bar{K}_{q(\tau)}=K_\tau\end{aligned} \tag{39}$$

where $v(t)=[v_1(t)\ v_2(t)\ \cdots\ v_p(t)]^T$.

Then (8) can be rewritten as follows:

$$\dot{x}(t)=\sum_{\tau\in\prod_{i=1}^{p}\mathbb{S}_i}\alpha_{q(\tau)}(v(t))(\bar{A}_{q(\tau)}x(t)+\bar{B}_{q(\tau)}u(t))$$

which is equivalent to

$$\dot{x}(t)=\sum_{i=1}^{r}\alpha_i(v(t))(\bar{A}_ix(t)+\bar{B}_iu(t)) \tag{40}$$

Further, the fuzzy controller (7) can be rewritten as follows:

$$u(t)=\sum_{i=1}^{r}\alpha_i(v(t))\bar{K}_iy(t) \tag{41}$$

Moreover, we can easily obtain $\sum_{i=1}^{r}\alpha_i(v(t))=1$. Then the fuzzy system description (40) with (41) is widely used in the existing literature [21], [23], [25], [29], [32], [39].

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**Jiuxiang Dong** received the B.S. degree in mathematics and applied mathematics, the M.S. degree in applied mathematics from Liaoning Normal University, China, in 2001 and 2004, respectively. He received the Ph.D. degree in navigation guidance and control from Northeastern University, China, in 2009. He is currently a Professor at the College of Information Science and Engineering, Northeastern University. His research interests include fuzzy control, robust control and reliable control. Dr. Dong is an Associate Editor for the International Journal of Control, Automation, and Systems (IJCAS).

**Guang-Hong Yang** (SM'04) received the B.S. and M.S. degrees from Northeast University of Technology, Liaoning, China, in 1983 and 1986, respectively, and the Ph.D. degree in Control Engineering from Northeastern University, China (formerly, Northeast University of Technology), in 1994. He was a Lecturer/Associate Professor with Northeastern University from 1986 to 1995. He joined the Nanyang Technological University in 1996 as a Postdoctoral Fellow. From 2001 to 2005, he was a Research Scientist/Senior Research Scientist with the National University of Singapore. He is currently a Professor at the College of Information Science and Engineering, Northeastern University. His current research interests include fault-tolerant control, fault detection and isolation, non-fragile control systems design, and robust control. Dr. Yang is an Associate Editor for the International Journal of Control, Automation, and Systems (IJCAS), the International Journal of Systems Science (IJSS), the IET Control Theory & Applications, and the IEEE Transactions on Fuzzy Systems.