Inverse-scattering theory within the Rytov approximation

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A method for determining the internal structure of a localized scattering potential from field measurements performed outside the scattering volume is developed by using the Rytov approximation. The theory is compared with the inverse-scattering method within the Born and eikonal approximations and found to reduce to these methods in the weak-scattering (Born) and very-short-wavelength (eikonal) limits.

There are two seemingly distinct approaches employed to determine the internal structure of objects from scattered-field data. On the one hand there are radiographic imaging techniques that, when used in conjunction with computerized tomography (CT) give good results at x-ray wavelengths.[1] These techniques rely on a geometrical-optics description[2] of the imaging process that is adequate at x-ray wavelengths but that has questionable validity in optical and acoustic applications in which diffraction and scattering effects can become important. The second technique is the method of inverse scattering.[3] In this method, the object's structure (as represented by a scattering potential) is approximately reconstructed from measurements of the scattered field produced in a sequence of experiments employing monochromatic plane waves having varying angles of illumination to the object. The method of inverse scattering is customarily based on the first Born approximation to the scattered field and thus partially accounts for diffraction and scattering but is applicable only to weakly scattering objects (semitransparent objects).

In this Letter we reformulate the inverse-scattering problem by using the Rytov approximation[4] in place of the first Born approximation. In most optical applications the wavelength of the illuminating plane wave can be expected to be much smaller than the finest-structure detail of the object. For such cases the Rytov approximation will be superior to both the Born approximation[5] and the eikonal approximation[6] of geometrical optics. Consequently, the reformulated inverse-scattering method can be expected to be superior to both of the currently employed inverse techniques discussed above. This claim is substantiated by the fact that the method is shown to reduce in the very-short-wavelength limit to the radiographic CT method and in the weakly scattering limit to the usual inverse-scattering method.

The idea of using the Rytov approximation in the inverse-scattering problem is not new. Iwata and Nagata[7] have discussed its use in optical applications, and Mueller *et al.*[8] have investigated its application in ultrasonic computerized tomography. The work reported here differs from these earlier studies in that our analysis is a straightforward modification of the method of inverse scattering within the Born approximation. As a result the development is both conceptually and mathematically simpler than the treatments presented in the earlier work. Another advantage of the present formulation is that reconstruction algorithms developed for semitransparent objects[3] can be readily modified to be applicable within the reformulated theory.

Let $U(\mathbf{r})$ be the total field (incident plus scattered) generated by an object having an index of refraction $n(\mathbf{r}) = 1 + \delta n(\mathbf{r})$ and illuminated by a monochromatic plane wave $\exp(ik\mathbf{s}_0 \cdot \mathbf{r})$. $U(\mathbf{r})$ is then that solution to the equation

$$\{\nabla^2 + k^2[1 + \delta n(\mathbf{r})]^2\}\, U(\mathbf{r}) = 0 \tag{1}$$

that reduces to the sum of the incident plane wave and an outgoing spherical wave as $kr \to \infty$. Here, $k = 2\pi/\lambda$ is the wave number of free space, and we shall assume that the fluctuation in the index $\delta n(\mathbf{r})$ is small compared to unity and is localized to within some finite volume τ centered at the origin.

The first Born approximation to $U(\mathbf{r})$ is given by[9]

$$U_B(\mathbf{r}) = \exp(ik\mathbf{s}_0 \cdot \mathbf{r}) + \frac{k^2}{2\pi} \int_\tau \mathrm{d}^3 r' \delta n(\mathbf{r}') \times \exp(ik\mathbf{s}_0 \cdot \mathbf{r}') \frac{\exp(ik|\mathbf{r} - \mathbf{r}'|)}{|\mathbf{r} - \mathbf{r}'|}. \tag{2}$$

The method of inverse scattering is based on the fact that the twofold Fourier transform of the scattered field within the Born approximation $U_B^{(s)}(\mathbf{r})$ [second term on the right-hand side of Eq. (2)] over any plane of constant z that does not intersect the scattering volume τ reduces to[10]

$$\begin{aligned}\tilde{U}_B^{(s)}(K_x, K_y, z) &= \int \mathrm{d}x\mathrm{d}y U_B^{(s)}(\mathbf{r}) \times \exp[-i(K_x x + K_y y)] \\ &= i\frac{k}{s_z} \exp(iks_z z)\delta\tilde{n}[k(\mathbf{s} - \mathbf{s}_0)],\end{aligned} \tag{3}$$

where

$$\delta\tilde{n}(\mathbf{K}) = \int_\tau \mathrm{d}^3 r \delta n(\mathbf{r}) e^{-i\mathbf{K}\cdot\mathbf{r}} \tag{4}$$

is the threefold Fourier transform of the index-of-refraction fluctuation $\delta n(\mathbf{r})$ and $k\mathbf{s} = [K_x, K_y, (k^2 - K_x^2 - K_y^2)^{1/2}]$. It is seen from Eq. (3) that $\tilde{U}_B^{(s)}(K_x, K_y,$

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z) allows one to determine the transform $\delta\tilde{n}(\mathbf{K})$ on the surface of a sphere (called the Ewald sphere[11]) whose radius is k and that is centered at $k\mathbf{s}_0$. By performing a number of experiments using different values of $\mathbf{s}_0$ (i.e., different incident plane waves), it is possible to determine $\delta\tilde{n}(\mathbf{K})$ at a sufficient number of points within the region $|\mathbf{K}| < 2k$ (the so-called *limiting* Ewald sphere[11]) to obtain an approximate reconstruction of $\delta n(\mathbf{r})$.[3] This approximation will be accurate if the object [i.e., $\delta n(\mathbf{r})$] is effectively band limited to within the limiting Ewald sphere and if the scattered field is accurately represented by the first Born approximation, Eq. (2).

We can express the total field $U(\mathbf{r})$ (incident plus scattered) in the form

$$U(\mathbf{r}) = \exp[ik\mathbf{s}_0 \cdot \mathbf{r} + \psi(\mathbf{r})]. \qquad (5)$$

The real part of the complex phase function $\psi(\mathbf{r})$ is the log amplitude of $U(\mathbf{r})$, and the imaginary part is the fluctuation in the phase of $U(\mathbf{r})$ from the unperturbed value $ik\mathbf{s}_0 \cdot \mathbf{r}$ [obtained in the limit when $\delta n(\mathbf{r}) = 0$]. Within the Rytov approximation the complex phase $\psi(\mathbf{r})$ is found to be[4]

$$\psi_R(r) = \exp(-ik\mathbf{s}_0 \cdot \mathbf{r}) \times U_B{}^{(s)}(\mathbf{r}). \qquad (6)$$

Although the phase function $\psi(\mathbf{r})$ in the Rytov approximation is simply related to the Born approximation to the scattered field $U_B{}^{(s)}(\mathbf{r})$, the limits of validity of the two approximations are quite different. In particular, the Rytov approximation requires that $|\nabla\psi| \ll 2\pi/\lambda$. As was discussed by Chernov and by Tatarski,[4] this condition will be met as long as the scale at which $\delta n(\mathbf{r})$ fluctuates is large compared to the wavelength λ. On the other hand, the Born approximation requires that the total scattered field be small. Since the scattered field can be expected to increase with increasing distance of propagation, it is clear that the Born approximation can easily break down for large extended objects even though $|\delta n(\mathbf{r})| \ll 1$.[12]

The inverse-scattering method within the Rytov approximation follows at once from Eqs. (3) and (6). We find that

$$\begin{aligned} &\int dx dy [\exp(ik\mathbf{s}_0 \cdot \mathbf{r})\psi(\mathbf{r})]\exp[-i(K_x x + K_y y)] \\ &\quad\approx \int dx dy [\exp(ik\mathbf{s}_0 \cdot \mathbf{r})\psi_R(\mathbf{r})]\exp[-i(K_x x + K_y y)] \\ &\quad= i\frac{k}{s_z} e^{iks_z z}\delta\tilde{n}[k(\mathbf{s} - \mathbf{s}_0)], \qquad (7) \end{aligned}$$

where

$$\psi(\mathbf{r}) = \log U(\mathbf{r}) - ik\mathbf{s}_0 \cdot \mathbf{r} \qquad (8)$$

is the fluctuation of the complex phase of the total field over the (x, y) plane. The procedure is seen to differ from that used when the Born approximation is employed only in the replacement of the Fourier transform of the scattered field over the (x, y) plane by the Fourier transform of $\exp(ik\mathbf{s}_0 \cdot \mathbf{r})\psi(\mathbf{r})$ over this plane. All the earlier remarks concerning the inverse-scattering method continue to apply with the exception that Eq. (7) will be valid in certain cases in which the first Born approximation, and hence Eq. (3), will break down.

It is clear that Eq. (7) reduces to Eq. (3) when the Born approximation holds [i.e., when $|U_B{}^{(s)}(\mathbf{r})| \ll 1$]. A more interesting limiting case occurs when the wavelength is significantly smaller than the smallest object structure. In this case $\delta\tilde{n}(\mathbf{K}) \approx 0$ when $K_x{}^2 + K_y{}^2 \gg K_0{}^2 \ll k^2$. Taking the direction $\mathbf{s}_0$ of the incident plane wave to be along the z axis, we then find that

$$k\frac{\exp(iks_z z)}{s_z}\delta\tilde{n}[k(\mathbf{s} - \mathbf{s}_0)] \approx ke^{ikz}\,\delta\tilde{n}(K_x, K_y, 0). \qquad (9)$$

On substituting Eq. (9) into Eq. (7) and Fourier transforming both sides of the resulting equation, we obtain

$$ik\int_{-\infty}^{z} dz'\delta n(x, y, z') = (2\pi)^2\psi(x, y, z). \qquad (10)$$

Equation (10) states that the *projection* of the index fluctuation $\delta n(\mathbf{r})$ onto the (x, y) plane is proportional to the fluctuation in the complex phase of the total field over this plane. This result can be derived directly from the eikonal approximation to the field over this plane and forms the basis for the radiographic CT methods mentioned earlier.[1] It is seen to be a limiting case of the inverse-scattering method developed here, which is valid in the very-short-wavelength limit.

References

1. For a survey of these techniques, see A. C. Kak, Proc. IEEE **67**, 1245 (1979).
2. More specifically, they employ the *eikonal* approximation (see Ref. 6) to obtain a relationship between the complex phase of the field and so-called *projections* (see Ref. 1) of the object's internal structure. This relationship is derived as a limiting case of the inverse-scattering method later in this Letter.
3. See, for example, E. Wolf, Opt. Commun. **1**, 153 (1969); W. H. Carter, J. Opt. Soc. Am. **60**, 306 (1970); W. H. Carter and P.-C. Ho, Appl. Opt. **13**, 162 (1974); A. F. Fercher *et al.*, Appl Opt. **18**, 2427 (1979); A. J. Devaney, J. Math. Phys. **19**, 1526 (1978). An overall review of the method of inverse scattering is given by A. J. Devaney in *Optics in Four Dimensions*, M. A. Machado and L. M. Narducci, eds., Conference Proceedings #65 (American Institute of Physics, New York, 1981). See also the collection of articles in *Inverse Scattering Problems in Optics*, H. P. Baltes, ed. (Springer-Verlag, Heidelberg, 1980).
4. See, for example, V. I. Tatarski, *Wave Propagation in a Turbulent Medium* (McGraw-Hill, New York, 1961), Chap. 7; L. A. Chernov, *Wave Propagation in a Random Medium* (Dover, New York, 1967). Chap. 5.
5. The relative merits of the Rytov approximation over the Born approximation are discussed in Ref. 4 and by J. B. Keller, J. Opt. Soc. Am. **59**, 1003 (1969). It should be emphasized that the Rytov approximation may lose its advantage over the Born approximation in cases in which the wavelength is not significantly shorter than the finest object detail. (In this connection, see Ref. 8).
6. For a discussion of the eikonal approximation, see L. Schiff, *Quantum Mechanics*, 3rd ed. (McGraw-Hill, New York, 1968), p. 339. The Rytov approximation reduces to the eikonal approximation in the limit of very short wavelengths (see discussions in Ref. 4).

7. K. Iwata and R. Nagata, Jpn. J. Appl. Phys. **14**, Suppl. 14-1, 379 (1975).
8. R. K. Mueller, M. Kaveh, and G. Wade, Proc. IEEE **67**, 567 (1979).
9. See Ref. 4. In deriving Eq. (2) the quantity $[\delta n(\mathbf{r})]^2$ has been assumed to be negligible in comparison to $\delta n(\mathbf{r})$.
10. E. Wolf, Opt. Commun. **1**, 153 (1969).
11. R. Hosemann and S. H. Baghi, *Direct Analysis of Diffraction of Matter* (North-Holland, Amsterdam, 1962), Chap. I, Sec. 6.
12. See Ref. 5 and also the discussion in K. Gottfried, *Quantum Mechanics* (Benjamin, New York, 1966), Sec. 13.