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Vibratory Identification of Beam Boundary Conditions

In previous work, by means of stochastic and deterministic models, the authors presented a system identification theory for performing nondestructive testing of elastic systems. The procedure requires identification of the structure's boundary conditions. Herein, the mathematical models are improved, and vibration data are presented for the determination of the boundary parameters. These experimentally derived results are shown to validate the models.

Introduction

In [1],[2] by means of stochastic and deterministic models, we presented a system-identification theory for predicting buckling loads of elastic systems from vibration data; thus creating a theory for non-destructive testing. An integral part of the procedure is the determination of the structure's boundary conditions. In this paper the mathematical models presented in [1] are improved and attention is focused on estimating the system's boundary conditions by means of vibration testing. By use of the vibration data the improved models are validated. In a subsequent paper, use is made of these results to describe a nondestructive testing procedure for determining buckling criteria for structures.

Single Boundary Parameter Model

Consider a uniform cantilever beam which is partially restrained against translation and rotation at its base. Denote the torsional restraint by the lumped parameter c and the translational restraint by the lumped parameter k as shown in Fig. 1. The differential equation for the free vibration of this beam is easily shown to be

$$EIu_{xxxx} + \rho A u_{tt} = 0 \qquad (1)$$

in which $u(x, t)$ is the beam's lateral deflection, ρ its density, A its cross-sectional area, E its modulus of elasticity, and I its second moment of area. Its associated boundary conditions are

$$\begin{aligned} ku(o, t) + EIu_{xxx}(o, t) &= 0 \\ EIu_{xx}(o, t) - cu_x(o, t) &= 0 \\ u_{xxx}(L, t) = 0 &= u_{xx}(L, t) \end{aligned} \qquad (2)$$

where L is the beam's length.

The natural frequencies of this beam are the solutions of the characteristic equation

$$CKq_1(F) - KFq_2(F) - CF^3q_3(F) + F^4q_5(F) = 0 \qquad (3)$$

where the $q_i(F)$ are transcendental functions defined in the nomenclature and its eigenfunctions are given by

$$\begin{aligned} X(x) = B\{&[KC(\cos F + \cosh F) - 2CF^3 \sinh F \\ &+ F^4(\cos F - \cosh F)] \cos \lambda x + [KC(\sin F - \sinh F) \\ &- 2KF \cosh F + F^4(\sin F + \sinh F)] \sin \lambda x \\ &+ [- KC(\cos F + \cosh F) + 2CF^3 \sin F \end{aligned}$$

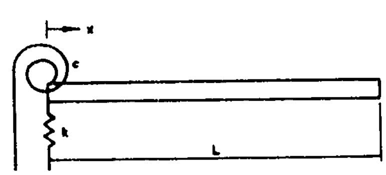

Fig. 1 Cantilever beam with rotational and translational restraints

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[2]Numbers in brackets designate References at end of paper.

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$$+ F^4(\cos F - \cosh F)] \cosh \lambda x + [KC(\sinh F - \sin F)$$
$$- 2KF \cos F + F^4(\sin F + \sinh F)] \sinh \lambda x\} \quad (4)$$

In these equations, the parameter F is a dimensionless representation of the natural frequency, ω, and is given by

$$F = \lambda L \quad \text{where } \lambda^4 = \frac{\omega^2 \rho A}{EI} \quad (5)$$

The dimensionless parameters

$$C = \frac{cL}{EI} \quad \text{and } K = \frac{kL^3}{EI} \quad (6)$$

represent the rotational and translational restraints. The arbitrary constant B in (4) is determined from the initial conditions.

Our objective is to create a mathematical model to be used to predict results obtained experimentally. Specifically our experimental setup was for a system whose support stiffness was much larger in translation than rotation. To this end let us consider the case where k is infinite, thus the characteristic equation reduces to

$$Cq_1(F) - Fq_2(F) = 0 \quad (7)$$

If the fundamental frequency of this beam was known, then equation (7) could be used to calculate the values of its rotational restraint, thus identifying its boundary condition. However, any variation associated with the measurement of the fundamental frequency will lead to variations in the values of the restraint computed in this manner. For this reason, the variation associated with the measurement of the natural frequency will be modeled, and its effect on the identification of the restraint will be discussed.

Assume that the jth measurement, $\hat{\Omega}_j^{(1)}$, of the beam's fundamental frequency parameter is a random variable which can be represented as

$$\hat{\Omega}_j^{(1)} = \bar{F}^{(1)} + Z_j^{(1)} \quad j = 1, \ldots J \quad (8)$$

where $Z_j^{(1)}, j = 1, \ldots J$ are a sequence of independent, identically distributed random variables such that

$$E\{Z_j^{(1)}\} = 0 \quad j = 1, \ldots J \quad (9)$$

$$E\{Z_{j_1}^{(1)} Z_{j_2}^{(1)}\} = \text{Var}\,\{Z^{(1)}\}\delta_{j_1 j_2} \quad j_1, j_2 = 1, \ldots J \quad (10)$$

In words, each fundamental frequency measurement is a random variable which can be expressed as the sum of a deterministic and a random part. This random part has a mean of zero, is independent from measurement to measurement, and represents the scatter which appears in a sequence of identical experiments. For the experiment described below, the random part has a standard deviation which is much smaller than the deterministic part of the measurement; that is

$$\text{Var}^{1/2}\{Z^{(1)}\} << \bar{F}^{(1)} \quad (11)$$

and the variance of the $Z_j^{(1)}$ was small compared to one. These facts make it unnecessary to assume that distribution functions for the random variables $Z_j^{(1)}$ are known.

An estimator of the beam's fundamental frequency, $\bar{F}^{(1)}$, is the sample mean of the measurements

$$\Omega^{(1)} = \frac{1}{J} \sum_{j=1}^{J} \Omega_j^{(1)} \quad (12)$$

If $\Omega^{(1)}$ is substituted for F in (7), then an estimator, C, of the support's restraint, C, can be obtained.

Now writing a Taylor series expansion for C about $\Omega^{(1)} = \bar{F}^{(1)}$ and from the assumptions stated earlier it can be shown that the expected value and variance of C are given by

$$E\{C\} = C + \frac{1}{2} \frac{d^2C}{d\bar{F}^{(1)2}} \text{Var}\,\{\Omega^{(1)}\} \quad (13)$$

$$\text{Var}\,\{C\} = \left[\frac{dC}{d\bar{F}^{(1)}}\right]^2 \text{Var}\,\{\Omega^{(1)}\} \quad (14)$$

Nomenclature

A = cross-sectional area
B = constant in mode shapes
c = rotational spring constant
C = dimensionless rotational spring parameter $= cL/EI$
E = Young's modulus
$E\{\ \}$ = expectation of a random variable
F = dimensionless frequency parameter $= (\omega^2\rho A/EI)L^{1/4}$
I = second moment of cross-sectional area; number of ensemble members
J = number of frequency measurements
k = translational spring constant
K = dimensionless translational spring parameter $= kL^3/EI$
L = length of beam
m = concentrated mass
M = dimensionless mass parameter $= m/\rho AL$; number of natural frequencies measured
$q_1(F)$ = $1 + \cos F \cosh F$
$q_2(F)$ = $\sin F \cosh F - \cos F \sinh F$
$q_3(F)$ = $\sin F \cosh F + \cos F \sinh F$
$q_4(F)$ = $\sin F \sinh F$
$q_5(F)$ = $1 - \cos F \cosh F$
$s^2\{\ \}$ = estimator for variance of a random variable
t = time
u = lateral displacement of beam neutral axis
$\text{Var}\{\ \}$ = variance of a random variable
x = coordinate along axis of beam
X_i = random component of rotational restraint
$X(x)$ = mode shapes of beam
$Z_i^{(k)}$ = random component of frequency parameter measurement
δ_{ij} = Kronecker delta
λ = beam parameter $= (\omega^2\rho A/EI)^{1/4}$
ρ = density
ω = angular frequency
Ω = measurement of dimensionless frequency parameter

Subscripts

$i = 1, \ldots I$ = ith assembly
$j = 1, \ldots J$ = jth measurement
t = partial differentiation with respect to time
x = partial differentiation with respect to spatial coordinate

Superscripts

k = kth mode
$m = 1, \ldots M$ = mth mode used in identification
$\wedge$ = estimator
$-$ = mean value
$\sim$ = value of frequency corresponding to the mean value of restraint in an ensemble

The derivatives in equations (13) and (14) are evaluated at $\bar{F}^{(1)}$ and can be obtained by implicit differentiation. Notice in (13) that C is a biased estimator of the restraint, C, but that this bias decreases as the variance of the sample's mean of the frequency measurements decreases. Equation (14) indicates that the variance of the restraint estimator also decreases under the same condition.

The experiment performed to obtain the vibration data for use in the model was a beam of 4150 steel having a measured density of 7772 kg/m^3 and a measured Young's modulus of 2.127×10^{11} N/m^2. The beam was 39.55 cm long and had a rectangular cross-section which averaged 1.270 cm $\times$ 2.567 cm. The beam was supported at its root by bolting it to steel blocks as shown in Fig. 2. The beam's first five natural frequencies were measured in the steady-state vibration test which is block-diagrammed in Fig. 3.

The beam was excited by applying the amplified signal of an oscillator to a loudspeaker which was connected to the beam through a force gage. The gage consisted of a hollow plexiglass rod on which two strain gages were mounted and wired with two identical gages on a dummy rod so as to read axial strain with compensation for strains due to bending and temperature change. The response of the beam was monitored at 3 locations with piezoelectric accelerometers. Using the response and frequency data, a particular natural frequency could be obtained. The results of twelve measurements of each of the beam's first five natural frequencies are summarized in Table 1.

Table 1 Summary of data from 12 experiments on a single cantilever beam

Mode k	Sample mean of frequency $\hat{\Omega}^{(k)}$	Sample variance of frequency $s^2\{Z^{(k)}\}$ 10^{-6}	Fractional standard deviation $\frac{s\{Z^{(k)}\}}{\hat{\Omega}^{(k)}}$
1	1.7045	1.1977	0.06%
2	4.299	3.709	0.04%
3	7.249	4.980	0.03%
4	10.122	8.626	0.03%
5	12.984	9.580	0.02%

The fundamental frequency measured in the experiment will now be used to identify the beam's boundary condition. Substitution of $\hat{\Omega}^{(1)} = 1.7045$ from Table 1 into (7) gives $\hat{C} = 8.694$ as the estimate for the beam's restraint. Recall from (13) that $\hat{C}$ is a biased estimator of the restraint C. However, an unbiased estimate of the restraint can be calculated from this equation by using the value of $\hat{C}$ for $E\{\hat{C}\}$, $s^2\{Z^{(1)}\}$ from Table 1 for the variance of $Z^{(1)}$, and evaluating the derivatives of the restraint at $\hat{\Omega}^{(1)}$. The result is 8.694 indicating that the bias is negligible due to the small value achieved for $s^2\{Z^{(1)}\}$. An estimate of the variance of $\hat{C}$ is similarly found from (14) to be $s^2\{\hat{C}\} = 4.075 \times 10^{-4}$. The accuracy of the prediction of the restraint can be judged by using the unbiased estimate of C in the characteristic equation to obtain estimates of the beam's higher frequencies by use of the equations

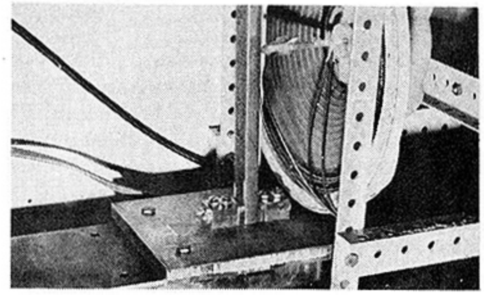

Fig. 2 Mounted cantilever beam

$$Cq_1(\hat{F}^{(k)}) - \hat{F}^{(k)}q_2(\hat{F}^{(k)}) = 0 \qquad k = 2, \ldots \tag{15}$$

The estimates of the frequencies computed from (15) and the corresponding measurements for the beam's second through fifth modes are shown in Table 2. The difference between these estimates and the corresponding measurements ranges from 1.7 percent in the second mode to 4.7 percent in the fifth mode.

Table 2 Comparison of cantilever data and the single parameter (C) model

Mode k	Estimate $\hat{F}^{(k)}$	Measurement $\hat{\Omega}^{(k)}$	$\frac{\hat{F}^{(k)} - \hat{\Omega}^{(k)}}{\hat{\Omega}^{(k)}}$ Percent
2	4.373	4.299	+1.7
3	7.423	7.249	+2.4
4	10.495	10.122	+3.7
5	13.590	12.984	+4.7

Determination of One Parameter From Multiple Frequencies

In this section, the measurements of the beam's higher natural frequencies will be used to obtain an improved estimate of the boundary parameter. The scatter in the measurements of the higher natural frequencies will be modeled by assuming that the jth measurement of the mth natural frequency is a random variable, denoted by $\hat{\Omega}_j^{(m)}$, such that

$$\hat{\Omega}_j^{(m)} = \bar{F}^{(m)} + Z_j^{(m)} \qquad j = 1, \ldots J; m = 1, \ldots M \tag{16}$$

$$E\{Z_j^{(m)}\} = 0 \qquad j = 1, \ldots J; m = 1, \ldots M \tag{17}$$

$$E\{Z_{j_1}^{(m_1)} Z_{j_2}^{(m_2)}\} = \mathrm{Var}\{Z^{(m_1)}\}\delta_{j_1 j_2}\delta_{m_1 m_2}$$

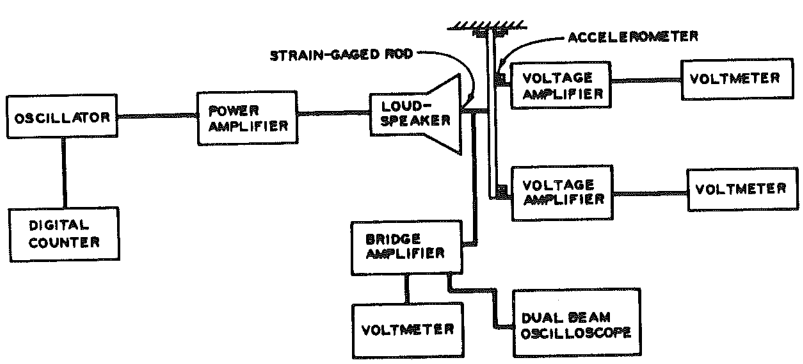

Fig. 3 Block diagram of instrumentation for vibration experiment

$$j_1, j_2 = 1, \ldots J; m_1, m_2 = 1, \ldots M \quad (18)$$

These random variables were modeled to have a different variance for each mode, $m = 1, \ldots M$, since the variance increases in the higher modes for the experimental data in Table 2. One also notices from this table that the standard deviations of the random variables are much smaller than the corresponding deterministic parts of the measurements. Thus the same observations as expressed in (11) and immediately following it can be used.

Suppose that each of the first M natural frequencies was observed a total of J times. The estimators for the beam's natural frequencies are obtained by calculating the sample means of these measurements

$$\hat{\Omega}^{(m)} = \frac{1}{J} \sum_{j=1}^{J} \hat{\Omega}_j^{(m)} \quad m = 1, \ldots M \quad (19)$$

Now, the estimator of each of the beam's natural frequencies can be used in the characteristic equation (7) to derive an estimator, $\hat{C}^{(m)}$, of the partial restraint

$$\hat{C}^{(m)} q_1(\hat{\Omega}^{(m)}) - \hat{\Omega}^{(m)} q_2(\hat{\Omega}^{(m)}) = 0 \quad m = 1, \ldots M \quad (20)$$

whose variance is found as in the preceding section to be

$$\text{Var}\{\hat{C}^{(m)}\} = \left[\frac{dC}{d\bar{F}^{(m)}}\right]^2 \text{Var}\{\hat{\Omega}^{(m)}\} \quad m = 1, \ldots M \quad (21)$$

A single estimator, $\hat{C}$, of the partial restraint, C, can be found by minimizing the quantity

$$\sum_{m=1}^{M} \frac{[\hat{C} - \hat{C}^{(m)}]^2}{\text{Var}\{\hat{C}^{(m)}\}}$$

which gives greater consideration to those estimators having smaller variances. The minimizing value of $\hat{C}$ is given by

$$\hat{C} = \sum_{m=1}^{M} a_m \hat{C}^{(m)} \quad (22)$$

in which the weighting factors are found from

$$\left[\sum_{m=1}^{M} \frac{1}{\text{Var}\{\hat{C}^{(m)}\}}\right] a_m = \frac{1}{\text{Var}\{\hat{C}^{(m)}\}} \quad m = 1, \ldots M \quad (23)$$

The expectation and variance of the restraint estimator, $\hat{C}$, are computed to be

$$E\{\hat{C}\} = C + \frac{1}{2} \sum_{m=1}^{M} a_m \frac{d^2C}{d\bar{F}^{(m)2}} \text{Var}\{\hat{\Omega}^{(m)}\} \quad (24)$$

$$\text{Var}\{\hat{C}\} = \sum_{m=1}^{M} a_m^2 \left[\frac{dC}{d\bar{F}^{(m)}}\right]^2 \text{Var}\{\hat{\Omega}^{(m)}\} \quad (25)$$

The measurements of the first two natural frequencies obtained in the last section will now be used to calculate an estimate of the beam's restraint. The estimates of the mth restraint are obtained by substituting $\hat{\Omega}^{(m)}$ from Table 1 into (20), yielding $\hat{C}^{(1)} = 8.694$, and $\hat{C}^{(2)} = 5.902$.

Note that the estimate of restraint computed from the second natural frequency is lower than the estimate calculated from the fundamental one. This is related to the effects of shear deformation, rotatory inertia, and accelerometer mass which cause the observations of the higher natural frequencies to be increasingly lower than those predicted by the physical model [2]. The estimates of the weighting factors are obtained from (21) and (23) by substituting $s^2\{Z^{(m)}\}$ for Var $\{\hat{\Omega}^{(m)}\}$ and evaluating the derivatives at $\hat{\Omega}^{(m)}$. These estimates are $\hat{a}_1 = 0.4135$, and $\hat{a}_2 = 0.5865$. Observe that more weight is given to the second natural frequency than to the fundamental. This is the case since $dC/d\bar{F}$ is smaller at this beam's second natural frequency than it is at its fundamental and since this in turn causes the restraint estimated from the second natural frequency to have a smaller estimated variance. The estimate, $\hat{C}$, of the partial restraint, C, is then found from (20) and (22). The result is $\hat{C} = 7.056$. An unbiased restraint estimate can be computed from (24), yielding the same value. This indicates that the bias is negligible. Similarly, the variance of $\hat{C}$ is found from (25) to be $s^2\{\hat{C}\} = 1.6850 \times 10^{-4}$. This variance is smaller than the variance of the restraint estimated from the fundamental frequency alone, 4.075×10^{-4}. In this sense, the use of the observations of the beam's first two natural frequencies gave a better identification of the boundary condition than the identification obtained from the observations of the fundamental frequency alone.

Using the first three natural frequencies, this trend continues [3]. However, a problem is encountered when the beam's fourth natural frequency is employed in the estimation model. It leads to physically unrealizable (negative) values for $\hat{C}^{(4)}$. Clearly it is expected that at some natural frequency the technique would break down, for the higher the frequency the less dependence on the boundary conditions.

In an attempt to improve upon the model presented above, we investigated the case where k is finite, thus studying the model whose characteristic equation is (3). This work is contained in [3]. With this model, using the first two natural frequencies led to predictions of the higher natural frequencies which were not as accurate as those obtained using the model where k was assumed infinite.

Beam Constrained at Both Ends

Our purpose here is to develop a model for a system constrained at more than one point using the procedures developed in the preceding sections. Again, we will validate the model by comparing results obtained from it to experimentally obtained results.

Recall, we mentioned at the outset that the ultimate use for the models developed herein is to make predictions of a structure's buckling characteristics. In the experiments we performed we validated the mathematical model by actually buckling the specimens. Thus, to maintain accuracy, the structure was mounted such that the vibration tests and buckling tests could be made without disturbing the means of support. This necessitated that one end of the beam be connected to a load cell as illustrated in Fig. 4.

In Fig. 4 the rotational spring c_2 represents the support provided by the head of the universal testing machine. The rota-

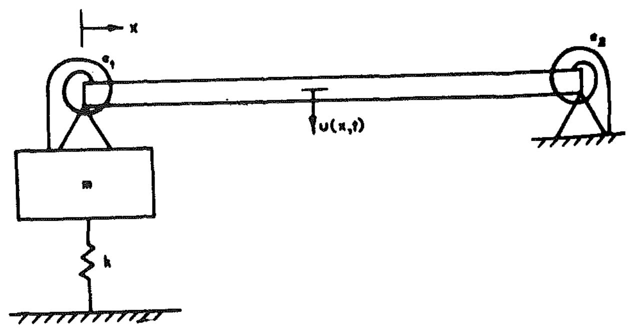

Fig. 4 Beam restrained at both ends

tional spring c_1 describes the connection between the end of the beam and the load cell. The translational spring k and the concentrated mass m are a mathematical model of this load cell. The free vibration of this beam must satisfy (1) subject to the boundary conditions

$$
\begin{aligned}
&EIu_{xxx}(o, t) + ku(o, t) + mu_{tt}(o, t) = 0 \\
&EIu_{xx}(o, t) - c_1u_x(o, t) = 0 \qquad (26) \\
&u(L, t) = 0,\ EIu_{xx}(L, t) + c_2u_x(L, t) = 0
\end{aligned}
$$

The characteristic equation for this boundary value problem is

$$
\begin{aligned}
&(K - MF^4)[C_1C_2q_5(F) + (C_1 + C_2)Fq_2(F) + 2F^2q_4(F)] \\
&+ F^3\{C_1C_2q_3(F) + C_1F[q_1(F) - q_5(F)] + C_2Fq_1(F) - F^2q_2(F)\} = 0
\end{aligned}
\qquad (27)
$$

and the eigenfunction is

$$
\begin{aligned}
X(x) = B(\{&(K - MF^4)C_1[C_2(\cos F - \cosh F) \\
&- F(\sin F + \sinh F)] + F^3[2C_1C_2 \sinh F + 2C_1F \cosh F \\
&- C_2F(\cos F + \cosh F) + F^2(\sin F + \sinh F)]\} \cos \lambda x \\
&+ \{(K - MF^4)[C_1C_2(\sin F + \sinh F) + C_1F(\cos F + \cosh F) \\
&+ 2C_2F \cosh F + 2F^2 \sinh F] + F^4[C_2(\sin F - \sinh F) \\
&+ F(\cos F - \cosh F)]\} \sin \lambda x \\
+ \{&(K - MF^4)C_1[C_2(\cosh F - \cos F) + F(\sinh F + \sin F)] \\
&+ F^3[2C_1C_2 \sin F + 2FC_1 \cos F + FC_2(\cos F + \cosh F) \\
&+ F^2(\sinh F - \sin F)]\} \cosh \lambda x + \{(K - MF^4)[- C_1C_2(\sin F \\
&+ \sinh F) - C_1F(\cos F + \cosh F) - 2C_2F \cos F + 2F^2 \sin F] \\
&+ F^4[C_2(\sin F - \sinh F) + F(\cos F - \cosh F)]\} \sinh \lambda x) \quad (28)
\end{aligned}
$$

in which

$$
C_1 = \frac{c_1L}{EI}, \qquad C_2 = \frac{c_2L}{EI}, \qquad M = \frac{m}{\rho AL} \qquad (29)
$$

The load cell used had the dimensionless mass parameter $M = 3.290$.

An experiment was conducted in which the first five natural frequencies of a beam restrained at both ends were observed. This beam was of 4150 steel, density 7834 kg/m^3, and Young's modulus 2.113×10^{11} N/m^2. The beam was 89.97 cm long and had a rectangular cross section which averaged 2.544 cm by 1.279 cm. One end of the beam was supported as shown in Fig. 2. The other end of the beam was bolted to a similar set of steel blocks which were attached to the load cell as shown in Fig. 5. The first five natural frequencies of this beam were each measured five times in the same manner as outlined in the previous sections. The results are summarized in Table 3.

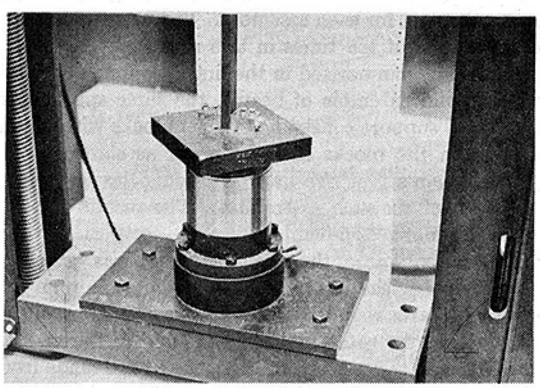

Fig. 5 Attachment of beam to load cell

Table 3 Summary of data from 5 experiments on a beam restrained at both ends

Mode k	Sample mean of frequency $\hat{\Omega}^{(k)}$	Sample variance of frequency $\frac{s^2\{Z^{(k)}\}}{10^{-5}}$	Fractional standard deviation $\frac{s\{Z^{(k)}\}}{\Omega^{(k)}}$
1	4.296	1.4782	0.09%
2	7.059	1.3828	0.05%
3	10.234	2.670	0.05%
4	13.335	2.480	0.04%
5	16.047	1.9074	0.03%

Let the translational restraint, k, of the load cell support be an unknown parameter and assume that the rotational restraints of both supports are equal. It then follows that estimators, $\hat{C}$ and $\hat{K}$, of the beam's two unknown support parameters are found from the simultaneous solution of the following special case of the characteristic equation.

$$
\begin{aligned}
&[\hat{K} - M\hat{\Omega}^{(m)}]\{\hat{C}^2q_5(\hat{\Omega}^{(m)}) + 2\hat{\Omega}^{(m)}\hat{C}q_2(\hat{\Omega}^{(m)}) + 2\hat{\Omega}^{(m)2}q_4(\hat{\Omega}^{(m)})\} \\
&\quad + \hat{\Omega}^{(m)3}(\hat{C}^2q_3(\hat{\Omega}^{(m)}) + \hat{\Omega}^{(m)}\hat{C}\{2q_1(\hat{\Omega}^{(m)}) - q_5(\hat{\Omega}^{(m)})\} \\
&\quad - \hat{\Omega}^{(m)2}q_2(\hat{\Omega}^{(m)})) = 0 \qquad m = 1, 2 \qquad (30)
\end{aligned}
$$

Using the beam's first and second sample mean frequencies from Table 3 yields $\hat{C} = 8.888$ and $\hat{K} = 833.0$.

Forming Taylor series expansions of $\hat{C}$ and $\hat{K}$ about $\hat{\Omega}^{(1)} = \bar{F}^{(1)}$ and $\hat{\Omega}^{(2)} = \bar{F}^{(2)}$ the expectations and variances are

$$
\begin{aligned}
E\{\hat{C}\} &= C + \frac{1}{2J}\left[\frac{\partial^2 C}{\partial \bar{F}^{(1)2}} \mathrm{Var}\,\{Z^{(1)}\} + \frac{\partial^2 C}{\partial \bar{F}^{(2)2}} \mathrm{Var}\,\{Z^{(2)}\}\right] \\
E\{\hat{K}\} &= K + \frac{1}{2J}\left[\frac{\partial^2 K}{\partial \bar{F}^{(1)2}} \mathrm{Var}\,\{Z^{(1)}\} + \frac{\partial^2 K}{\partial \bar{F}^{(2)2}} \mathrm{Var}\,\{Z^{(2)}\}\right] \qquad (31)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Var}\,\{\hat{C}\} &= \left[\frac{\partial C}{\partial \bar{F}^{(1)}}\right]^2 \frac{\mathrm{Var}\,\{Z^{(1)}\}}{J} + \left[\frac{\partial C}{\partial \bar{F}^{(2)}}\right]^2 \frac{\mathrm{Var}\,\{Z^{(2)}\}}{J} \\
\mathrm{Var}\,\{\hat{K}\} &= \left[\frac{\partial K}{\partial \bar{F}^{(1)}}\right]^2 \frac{\mathrm{Var}\,\{Z^{(1)}\}}{J} + \left[\frac{\partial K}{\partial \bar{F}^{(2)}}\right]^2 \frac{\mathrm{Var}\,\{Z^{(2)}\}}{J} \qquad (32)
\end{aligned}
$$

Unbiased estimates of these restraints can now be calculated. They are $\hat{C} = 8.888$ and $\hat{K} = 833.1$. Hence, the bias of these restraint estimates is negligible. The estimates for the variances of $\hat{C}$ and $\hat{K}$ are $s^2\{\hat{C}\} = 3.606 \times 10^{-3}$ and $s^2\{\hat{K}\} = 82.90$.

Table 4 contains the measurements of the beam's third through fifth natural frequencies and estimates calculated from the two parameter model.

Table 4 Comparison of measured and estimated natural frequencies for the beam constrained at both ends

k	$\hat{\Omega}^{(k)}$	$\hat{F}^{(k)}$	$\frac{\hat{F}^{(k)} - \hat{\Omega}^{(k)}}{\hat{\Omega}^{(k)}}$ Percent
3	10.234	10.044	−1.9
4	13.335	13.080	−1.9
5	16.047	16.146	−0.6

Observe from Table 4 that the two-parameter model predicts the third through fifth natural frequencies within 1.9 percent.

A model assuming C_1 is not equal to C_2 which incorporates the first three natural frequencies leads to estimates of the parameters which are not physically realizable (negative or imaginary values). Using the experimental data of other investigators for beams supported at more than one point [4], [5] also yielded physically unrealizable results. Clearly this area warrants further investigation.

An Ensemble of Beams

Consider a collection of similarly manufactured beam-columns designed to have similar support boundary conditions. One suspects that when these structures are installed that differences will be found in the boundary support parameters. Our purpose here is to model this phenomenon.

The measured fundamental frequencies of an ensemble of cantilever beams can be used to obtain estimates for the mean and the variance of the boundary restraint parameter within the ensemble. Returning to the model represented by Fig. 1, let the translational restraints of an ensemble of beams be infinite and allow the rotational restraint of each beam to be different due, for example, to unavoidable small differences in assembly. It is convenient to assume that the restraint is a random variable, C_i, represented as

$$C_i = \bar{C} + X_i \qquad i = 1, \ldots I \tag{33}$$

in which $\bar{C}$ is the ensemble's mean value of restraint and the X_i, $i = 1, \ldots I$, are a sequence of independent, identically distributed random variables such that

$$E\{X_i\} = 0 \qquad i = 1, \ldots I \tag{34}$$

$$E\{X_{i_1}X_{i_2}\} = \mathrm{Var}\,\{X\}\delta_{i_1i_2} \qquad i_1, i_2 = 1, \ldots I \tag{35}$$

Denoting the kth natural frequency of the ith member of the ensemble by the random variable $F_i^{(k)}$, and truncating a Taylor series expansion of $F_i^{(k)}$ about $C_i = \bar{C}$ (using (33) through (35)) the mean and the variance of $F_i^{(k)}$ are given by

$$E\{F^{(k)}\} = \bar{F}^{(k)} = \tilde{F}^{(k)} + \frac{1}{2}\frac{d^2\tilde{F}^{(k)}}{d\bar{C}^2}\,\mathrm{Var}\,\{X\} \qquad k = 1, \ldots \tag{36}$$

$$\mathrm{Var}\,\{F^{(k)}\} = \left[\frac{d\tilde{F}^{(k)}}{d\bar{C}}\right]^2 \mathrm{Var}\,\{X\} \qquad k = 1, \ldots \tag{37}$$

in which $\tilde{F}^{(m)}$ is the mth natural frequency corresponding to $\bar{C}$ in the characteristic equation (7). The subscript i has been omitted since the result is the same for all members. In these expressions the derivatives are evaluated at $C = \bar{C}$ and can be obtained from the implicit differentiation of the characteristic equation. Now assume that the jth measurement of the fundamental frequency of the ith member of the ensemble is a random variable denoted by $\hat{\Omega}_{ij}^{(1)}$ which can be represented as

$$\hat{\Omega}_{ij}^{(1)} = F_i^{(1)} + Z_{ij}^{(1)} \qquad i = 1, \ldots I; \; j = 1, \ldots J \tag{38}$$

where $Z_{ij}^{(1)}$, $j = 1, \ldots J$, are $i = 1, \ldots I$ sequences of independent, identically distributed random variables such that

$$E\{Z_{ij}^{(1)}\} = 0 \qquad i = 1, \ldots I; \; j = 1, \ldots J \tag{39}$$

$$E\{Z_{i_1j_1}^{(1)}Z_{i_2j_2}^{(1)}\} = \mathrm{Var}\,\{Z^{(1)}\}\delta_{i_1i_2}\delta_{j_1j_2} \qquad i_1, i_2 = 1, \ldots I; \; j_1, j_2 = 1, \ldots J \tag{40}$$

Thus, each frequency measurement is the sum of two random variables. One of these is the fundamental frequency of the ith member of the ensemble, $F_i^{(1)}$. The randomness of this variable is due to the restraint's difference from beam to beam. The other random variable is $Z_{ij}^{(1)}$ and represents the scatter which appears in a sequence of identical experiments for the ith member. The latter random variable is assumed to be independent from member to member and from measurement to measurement. For simplicity, it is also assumed that this random variable's distribution is the same from member to member. Further assume that the random variables representing restraint randomness are independent of those which account for measuring errors. Thus

$$E\{F_{i_1}^{(1)}Z_{i_2j}^{(1)}\} = 0 \qquad i_1, i_2 = 1, \ldots I; \; j = 1, \ldots J \tag{41}$$

The estimator, $\hat{C}$, of the ensemble's mean restraint, $\bar{C}$, is found by substituting the ensemble sample mean, $\hat{\Omega}^{(1)}$, of the sample mean, $\hat{\Omega}_i^{(1)}$, of the fundamental frequency measurements for each beam into the characteristic equation. That is

$$\hat{C}q_1(\hat{\Omega}^{(1)}) - \hat{\Omega}^{(1)}q_2(\hat{\Omega}^{(1)}) = 0 \tag{42}$$

in which

$$\hat{\Omega}^{(1)} = \frac{1}{I}\sum_{i=1}^{I}\hat{\Omega}_i^{(1)} \tag{43}$$

and

$$\hat{\Omega}_i^{(1)} = \frac{1}{J}\sum_{j=1}^{J}\hat{\Omega}_{ij}^{(1)} \qquad i = 1, \ldots I \tag{44}$$

The mean and the variance of $\hat{C}$ can be shown to be

$$E\{\hat{C}\} = \bar{C} + \frac{1}{2}\frac{d^2\bar{C}}{d\bar{F}^{(1)2}}\left\{\frac{1-I}{I}\left[\frac{d\bar{F}^{(1)}}{d\bar{C}}\right]^2 \mathrm{Var}\,\{X\} + \frac{1}{IJ}\mathrm{Var}\,\{Z^{(1)}\}\right\} \tag{45}$$

$$\mathrm{Var}\,\{\hat{C}\} = \frac{1}{I}\mathrm{Var}\,\{X\} + \frac{1}{IJ}\left[\frac{d\bar{C}}{d\bar{F}^{(1)}}\right]^2 \mathrm{Var}\,\{Z^{(1)}\} \tag{46}$$

An estimator of the ensemble's restraint variance, Var $\{X\}$, is derived by substituting the estimate of the ensemble variance of the fundamental frequency, $s^2\{F^{(1)}\}$, into (37). That is

$$s^2\{C\} = \left[\frac{d\hat{C}}{d\hat{\Omega}^{(1)}}\right]^2 s^2\{F^{(1)}\} \tag{47}$$

in which the derivative is evaluated at $F^{(1)} = \hat{\Omega}^{(1)}$ and

$$s^2\{F^{(1)}\} = \frac{1}{I-1}\sum_{i=1}^{I}[\hat{\Omega}_i^{(1)} - \hat{\Omega}^{(1)}]^2 - \frac{1}{J(J-1),}\sum_{j=1}^{J}[\hat{\Omega}_{ij}^{(1)} - \hat{\Omega}_i^{(1)}]^2 \qquad i = 1, \ldots I \tag{48}$$

An experiment was performed in which the steady state vibration data for three different ensembles of cantilever beams was obtained. To obtain the first ensemble, the cantilever specimen previously discussed was unbolted from its supports and reassembled in an "identical" manner using a torque wrench. The first five natural frequencies of the reassembled beam were measured. It was soon apparent from the observations that the restraint's randomness was contributing significantly more to the variance of each frequency measurement than was the measurement's scatter. Thus each natural frequency of the beam was measured two times for each assembly. The beam was assembled and tested a total of ten times in this manner. The results for this ensemble are summarized in the first 3 columns of Table 5.

For the second ensemble of beams, the same specimen beam was bolted to a support consisting of steel blocks with aluminum inserts between the blocks and the beam as shown in Fig. 2. Each of the beam's first five natural frequencies was measured twice for each of six such assemblies. The results for this ensemble are summarized in Table 6.

For the final ensemble, the specimen beam was bolted to the support with plexiglass inserts replacing the aluminum inserts. A new set of inserts was used in each assembly to avoid a systematic effect on the rotational restraint due to the viscoelastic behavior of the plexiglass. All of the inserts were made from the same sheet of material to minimize the variation of the support's boundary condition. The results of two measurements of each

Table 5 Comparison of estimates and measurements for the ensemble using steel inserts

Mode k	Sample mean of frequency $\hat{\Omega}^{(k)}$	Sample variance of frequency $s^2\{\hat{\Omega}^{(k)}\}$ 10^{-4}	Estimate of mean frequency $\hat{F}^{(k)}$	$\frac{\hat{F}^{(k)} - \hat{\Omega}^{(k)}}{\hat{\Omega}^{(k)}}$ Percent	Estimate of variance of frequency $s^2\{F^{(k)}\}$ 10^{-4}	Measurement of variance of frequency $s^2\{\hat{\Omega}^{(k)}\}$ 10^{-4}
1	1.6939	4.752	--	--	--	--
2	4.280	10.405	4.360	1.9	8.42	10.39
3	7.225	21.85	7.409	2.5	9.03	21.83
4	10.096	24.82	10.483	3.8	7.73	24.78
5	12.963	26.90	13.579	4.8	6.34	26.85

Table 6 Comparison of estimates and measurements for the ensemble using aluminum inserts

Mode k	Sample mean of frequency $\hat{\Omega}^{(k)}$	Sample variance of frequency $s^2\{\hat{\Omega}^{(k)}\}$ 10^{-4}	Estimate of mean frequency $\hat{F}^{(k)}$	$\frac{\hat{F}^{(k)} - \hat{\Omega}^{(k)}}{\hat{\Omega}^{(k)}}$ Percent	Estimate of variance of frequency $s^2\{F^{(k)}\}$ 10^{-4}	Measurement of variance of frequency $s^2\{\Omega^{(k)}\}$ 10^{-4}
1	1.6627	3.237	--	--	--	--
2	4.215	10.959	4.320	2.5	4.72	10.94
3	7.139	16.658	7.369	3.2	4.53	16.63
4	9.999	27.47	10.445	4.5	3.60	27.43
5	12.861	26.22	13.545	5.3	2.81	26.17

natural frequency for each of six assemblies are summarized in Table 7.

Substitution of the sample mean of the average of the fundamental frequency measurements for each member from Table 6 into the characteristic equation gives the estimate $\hat{C} = 8.056$ for the mean restraint in the ensemble having steel supports. Placing $s^2\{Z^{(1)}\}$ from Table 1 and $s^2\{\hat{\Omega}^{(1)}\}$ from Table 6 in (47) and (48) yields the estimate $s^2\{C\} = 1.5189$ for the variance of the restraint. Recall from (45) that $\hat{C}$ is a biased estimator of the mean restraint $\bar{C}$. However, an unbiased estimate of the mean restraint can be calculated from this equation by using the value of $\hat{C}$ for $E\{\hat{C}\}$, using $s^2\{C\}$ for Var $\{X\}$, using $s^2\{Z^{(1)}\}$ for Var $\{Z^{(1)}\}$, and evaluating the derivatives of the restraint at $\hat{\Omega}^{(1)}$. The result is 8.191. The accuracy of the identification of mean restraint can be assessed by using this unbiased estimate of restraint in the characteristic equation to predict the means of higher natural frequencies. These estimates and the sample means of the corresponding measurements for the ensemble's second through fifth modes are presented in Table 5. The difference between the predictions and the measurements ranges from 1.9 percent in the second mode to 4.8 percent in the fifth mode. The accuracy of the estimate of the restraint variance can similarly be assessed by using $s^2\{C\}$ in (37) to estimate the variances of the higher natural frequencies. These estimates are also listed in Table 5. The quantity

$$s^2\{\Omega^{(k)}\} = \frac{1}{I-1}\sum_{i=1}^{I}[\hat{\Omega}_i^{(k)} - \hat{\Omega}^{(k)}]^2 - \frac{1}{J(J-1)}\sum_{j=1}^{J}[\hat{\Omega}_{ij}^{(k)} - \hat{\Omega}_i^{(k)}]^2 \quad i = 1, \ldots I; \quad k = 1, \ldots \tag{49}$$

is derived from the measurements and can be shown to be an unbiased estimate of the variance of the ensemble's higher natural frequencies. This measurement and its corresponding standard deviation is also presented in Table 5.

A similar set of calculations for the ensemble with aluminum inserts shows the unbiased estimate of the mean restraint to be 6.606, and the estimate of the variance of the restraint to be $s^2\{C\} = 0.5432$. The estimates of the higher natural frequencies for this ensemble are compared to the experimental measurements in Table 6.

In the case of the plexiglass inserts, the unbiased estimate of the mean restraint was 5.297 and the estimate of the variance of

Table 7 Comparison of estimates and measurements for the ensemble using plexiglass inserts

Mode k	Sample mean of frequency $\hat{\Omega}^{(k)}$	Sample variance of frequency $s^2\{\hat{\Omega}^{(k)}\}$ 10^{-4}	Estimate of mean frequency $\hat{F}^{(k)}$	$\frac{\hat{F}^{(k)} - \hat{\Omega}^{(k)}}{\hat{\Omega}^{(k)}}$ Percent	Estimate of variance of frequency $s^2\{F^{(k)}\}$ 10^{-4}	Measurement of variance of frequency $s^2\{\Omega^{(k)}\}$ 10^{-4}
1	1.6263	1.3430	--	--	--	--
2	4.174	2.054	4.278	2.5	1.56	2.04
3	7.098	2.935	7.328	3.2	1.33	2.91
4	9.979	5.427	10.410	4.3	0.99	5.38
5	12.808	1.7864	13.514	5.5	0.73	1.74

the restraint was $s^2\{C\} = 0.1180$. Estimates of the properties of the higher frequencies of this ensemble are compared to the experimental measurements of these properties in Table 7.

Concluding Remarks

Our purpose in this paper was to demonstrate the feasibility of identifying the boundary conditions of constrained beams from vibration test data. An important application of this technique is the prediction of buckling loads; the subject of a further investigation by the authors.

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