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Majorization Problems for Certain Classes of Analytic Functions

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Abstract. In this paper, we introduce a new subclass $S^{p,j}_{\lambda,m}(\gamma)$ of certain analytic functions defined by a generalized operator. A majorization problem for functions belonging to class $S^{p,j}_{\lambda,m}(\gamma)$ is considered. Moreover, we point out some new or known consequences of our main result.

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1. Introduction and preliminaries

Let $\mathcal{A}_p$ denote the class of functions $f(z)$ of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_n z^{p+n}, \qquad (p \in \mathbb{N} := \{1, 2, 3, \ldots\}), \tag{1}$$

which are analytic and p-valent in the open unit disk $\mathcal{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Also let $\mathcal{A}_1 =: \mathcal{A}$. For functions $f_j \in \mathcal{A}_p$ given by

$$f_j(z) = z^p + \sum_{n=1}^{\infty} a_{n,j} z^{p+n}, \qquad (j = 1, 2; p \in \mathbb{N}), \tag{2}$$

we define the Hadamard product (or convolution) of f_1 and f_2 by

$$(f_1 * f_2)(z) = z^p + \sum_{n=1}^{\infty} a_{n,1} a_{n,2} z^{p+n} = (f_2 * f_1)(z).$$

Let $f(z)$ and $g(z)$ be analytic in $\mathcal{U}$. Then we say that the function $f(z)$ is subordinate to $g(z)$ in $\mathcal{U}$, if there exists an analytic function $w(z)$ in $\mathcal{U}$ with

$$w(0) = 0, \quad |w(z)| < 1 \quad (z \in \mathcal{U}),$$

such that

$$f(z) = g(w(z)) \qquad (z \in \mathcal{U}).$$

We denote this subordination by $f(z) \prec g(z)$. Furthermore, if the function $g(z)$ is univalent in $\mathcal{U}$, then $f(z) \prec g(z) \quad (z \in \mathcal{U}) \Longleftrightarrow f(0) = g(0)$ and $f(\mathcal{U}) \subset g(\mathcal{U})$.

Suppose that the functions $f(z)$ and $g(z)$ are analytic in the open unit disk $\mathcal{U}$. Then we say that the function $f(z)$ is majorized by $g(z)$ in $\mathcal{U}$ (see [5]) and write

$$f(z) \ll g(z) \qquad (z \in \mathcal{U}), \tag{3}$$

if there exists a function $\varphi(z)$, analytic in $\mathcal{U}$, such that

$$|\varphi(z)| \le 1 \quad \text{and} \quad f(z) = \varphi(z)g(z) \qquad (z \in \mathcal{U}).$$

The majorization (3) is closely related to the concept of quasi-subordination between analytic functions in $\mathcal{U}$.

Let $\alpha_1, \alpha_2, \ldots, \alpha_q$ and $\beta_1, \beta_2, \ldots, \beta_s$ $(q, s \in \mathbb{N} \cup \{0\}, q \le s + 1)$ be complex numbers such that $\beta_l \neq 0, -1, -2, \ldots$ for $l \in \{1, 2, \ldots, s\}$. The generalized hypergeometric function ${}_qF_s$ is given by

$${}_qF_s(\alpha_1, \alpha_2, \ldots, \alpha_q; \beta_1, \beta_2, \ldots, \beta_s; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n(\alpha_2)_n \ldots (\alpha_q)_n}{(\beta_1)_n(\beta_2)_n \ldots (\beta_s)_n} \frac{z^n}{n!}, \qquad (z \in \mathcal{U}),$$

where $(x)_n$ denotes the Pochhammer symbol defined by

$$(x)_n = x(x+1)(x+2)\cdots(x+n-1) \quad \text{for} \quad n \in \mathbb{N} \quad \text{and} \quad (x)_0 = 1.$$

Corresponding to a function $\mathcal{G}^p_{q,s}(\alpha_1; \beta_1; z)$ defined by

$$\mathcal{G}_{q,s}(\alpha_1, \beta_1; z) := z^p \, {}_qF_s(\alpha_1, \alpha_2, \ldots, \alpha_q; \beta_1, \beta_2, \ldots, \beta_s; z), \tag{4}$$

C. Selvaraj and K.R. Karthikeyan [9] recently defined the following generalized differential operator $D^{p,m}_\lambda(\alpha_1, \beta_1)f : \mathcal{A}_p \longrightarrow \mathcal{A}_p$ by

$$\begin{aligned} D^{p,0}_\lambda(\alpha_1, \beta_1)f(z) &= f(z) * \mathcal{G}^p_{q,s}(\alpha_1, \beta_1; z), \\ D^{p,1}_\lambda(\alpha_1, \beta_1)f(z) &= (1-\lambda)(f(z) * \mathcal{G}^p_{q,s}(\alpha_1, \beta_1; z)) \\ &\qquad + \frac{\lambda}{p} z(f(z) * \mathcal{G}^p_{q,s}(\alpha_1, \beta_1; z))', \\ D^{p,m}_\lambda(\alpha_1, \beta_1)f(z) &= D^{p,1}_\lambda(D^{p,m-1}_\lambda(\alpha_1, \beta_1)f(z)), \end{aligned} \tag{5}$$

where $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\lambda \ge 0$.
If $f(z) \in \mathcal{A}_p$, then we have

$$D^{p,m}_\lambda(\alpha_1, \beta_1)f(z) = z^p + \sum_{n=1}^{\infty} \left(\frac{p+\lambda n}{p}\right)^m \frac{(\alpha_1)_n(\alpha_2)_n \ldots (\alpha_q)_n}{(\beta_1)_n(\beta_2)_n \ldots (\beta_s)_n} a_n \frac{z^{p+n}}{n!}. \tag{6}$$

It can be seen that, by specializing the parameters the operator $D^{p,m}_\lambda(\alpha_1, \beta_1)f(z)$ reduces to many known and new integral and differential operators. In particular, when $m = 0$ and $p = 1$ the operator $D^{p,m}_\lambda(\alpha_1, \beta_1)f(z)$ reduces to the well known Dziok- Srivastava operator [3] and for $p = 1$, $q = 2$, $s = 1$, $\alpha_1 = \beta_1$, and $\alpha_2 = 1$, it reduces to the operator introduced by F. AL-Oboudi [1]. Further we remark that, when $p = 1$, $q = 2$, $s = 1$, $\alpha_1 = \beta_1$, $\alpha_2 = 1$, and $\lambda = 1$ the operator $D_\lambda^{p,m}(\alpha_1, \beta_1)f(z)$ reduces to the operator introduced by G. S. Sălăgean [8].

It can be easily verified from (6) that

$$\lambda z(D_\lambda^{p,m}(\alpha_1, \beta_1)f(z))' = pD_\lambda^{p,m+1}(\alpha_1, \beta_1)f(z) - p(1-\lambda)D_\lambda^{p,m}(\alpha_1, \beta_1)f(z). \tag{7}$$

Using the operator $D_\lambda^{p,m}(\alpha_1, \beta_1)f(z)$ we now define the following class of p-valent analytic functions.

Definition 1.1. *A function $f(z) \in \mathcal{A}_p$ is said to be in the class $S_{\lambda,m}^{p,j}(\gamma)$ of p-valent functions of complex order $\gamma \neq 0$ in $\mathcal{U}$ if and only if*

$$Re\left\{1 + \frac{1}{\gamma}\left(\frac{z\big(D_\lambda^{p,m}(\alpha_1, \beta_1)f(z)\big)^{(j+1)}}{\big(D_\lambda^{p,m}(\alpha_1, \beta_1)f(z)\big)^{(j)}} - p + j\right)\right\} > 0 \tag{8}$$

$$(z \in \mathcal{U};\ p \in \mathbb{N};\ m, j \in \mathbb{N}_0;\ \gamma \in \mathbb{C} - \{0\};\ |2\gamma\lambda - p| \leq p).$$

It can be seen that, by specializing the parameters the class $S_{\lambda,m}^{p,j}(\gamma)$ reduces to many known subclasses of analytic functions. In particular, when $m = 0$, $p = 1$, $j = 0$, $q = 2$, $s = 1$, $\alpha_1 = \beta_1$, and $\alpha_2 = 1$, the class $S_{\lambda,m}^{p,j}(\gamma)$ reduces to $\mathcal{S}(\gamma)$, the class of starlike functions of complex order $\gamma \neq 0$ in $\mathcal{U}$ and when $m = 0$, $p = 1$, $j = 1$, $q = 2$, $s = 1$, $\alpha_1 = \beta_1$, and $\alpha_2 = 1$, the class $S_{\lambda,m}^{p,j}(\gamma)$ reduces to $\mathcal{K}(\gamma)$, the class of convex functions of complex order $\gamma \neq 0$ in $\mathcal{U}$. These classes were considered by M. A. Nasr and M. K. Aouf [6] and P. Wiatrowski [10]. Further we note that, when $\gamma = 1 - \alpha$, $m = 0$, $p = 1$, $j = 0$, $q = 2$, $s = 1$, $\alpha_1 = \beta_1$, and $\alpha_2 = 1$, the class $S_{\lambda,m}^{p,j}(\gamma)$ reduces to $\mathcal{S}^*(\alpha)$, the class of starlike functions of order α in $\mathcal{U}$.

2. Majorization problem for the class $S_{\lambda,m}^{p,j}(\gamma)$

Theorem 2.1. *Let the function $f(z)$ be in the class $\mathcal{A}_p$ and suppose that $g(z) \in S_{\lambda,m}^{p,j}(\gamma)$. If $\big(D_\lambda^{p,m}(\alpha_1, \beta_1)f(z)\big)^{(j)}$ is majorized by $\big(D_\lambda^{p,m}(\alpha_1, \beta_1)g(z)\big)^{(j)}$ in $\mathcal{U}$ for $j \in \mathbb{N}_0$, then*

$$\left|\big(D_\lambda^{p,m+1}(\alpha_1, \beta_1)f(z)\big)^{(j)}\right| \leq \left|\big(D_\lambda^{p,m+1}(\alpha_1, \beta_1)g(z)\big)^{(j)}\right| \quad \text{for} \quad |z| \leq r_1, \tag{9}$$

where

$$r_1 = r_1(p, \gamma, \lambda) := \frac{k - \sqrt{k^2 - 4p|2\gamma\lambda - p|}}{2|2\gamma\lambda - p|} \tag{10}$$

$$(k := 2\lambda + p + |2\gamma\lambda - p|; p \in \mathbb{N};\ \gamma \in \mathbb{C} - \{0\};\ \lambda \geq 0).$$

Proof. Let

$$h(z) = 1 + \frac{1}{\gamma}\left(\frac{z\big(D_\lambda^{p,m}(\alpha_1,\beta_1)g(z)\big)^{(j+1)}}{\big(D_\lambda^{p,m}(\alpha_1,\beta_1)g(z)\big)^{(j)}} - p + j\right) \tag{11}$$

$$(p \in \mathbb{N};\ m, j \in \mathbb{N}_0;\ \gamma \in \mathbb{C} - \{0\};\ p > j).$$

Since $g(z) \in S_{\lambda,m}^{p,j}(\gamma)$, we have $Re\big(h(z)\big) > 0$ $(z \in \mathcal{U})$ and

$$h(z) = \frac{1 + w(z)}{1 - w(z)} \qquad (w \in \mathcal{P},) \tag{12}$$

where $\mathcal{P}$ denotes the well known class of bounded analytic functions in $\mathcal{U}$, which satisfy the conditions (cf. [4])

$$w(0) = 0 \quad \text{and} \quad |w(z)| \le |z| \qquad (z \in \mathcal{U}).$$

It follows from (11) and (12) that

$$\frac{z\big(D_\lambda^{p,m}(\alpha_1,\beta_1)g(z)\big)^{(j+1)}}{\big(D_\lambda^{p,m}(\alpha_1,\beta_1)g(z)\big)^{(j)}} = \frac{p - j + (2\gamma - p + j)w(z)}{1 - w(z)} \tag{13}$$

In view of

$$\begin{aligned}\lambda z(D_\lambda^{p,m}(\alpha_1,\beta_1)f(z))^{(j+1)} =& p(D_\lambda^{p,m+1}(\alpha_1,\beta_1)f(z))^{(j)} - \\ & (p - p\lambda + \lambda j)(D_\lambda^{p,m}(\alpha_1,\beta_1)f(z))^{(j)},\end{aligned} \tag{14}$$

(13) immediately yields the following inequality:

$$\left|\big(D_\lambda^{p,m}(\alpha_1,\beta_1)g(z)\big)^{(j)}\right| \le \frac{p(1 + |z|)}{p - |2\gamma\lambda - p||z|}\left|\big(D_\lambda^{p,m+1}(\alpha_1,\beta_1)g(z)\big)^{(j)}\right|. \tag{15}$$

Since $\big(D_\lambda^{p,m}(\alpha_1,\beta_1)f(z)\big)^{(j)}$ is majorized by $\big(D_\lambda^{p,m}(\alpha_1,\beta_1)g(z)\big)^{(j)}$ in $\mathcal{U}$, there exist an analytic function $\varphi(z)$ such that

$$\big(D_\lambda^{p,m}(\alpha_1,\beta_1)f(z)\big)^{(j)} = \varphi(z)\big(D_\lambda^{p,m}(\alpha_1,\beta_1)g(z)\big)^{(j)} \tag{16}$$

and $|\varphi(z)| \le 1$ $(z \in \mathcal{U})$. Thus we have

$$\begin{aligned}z\big(D_\lambda^{p,m}(\alpha_1,\beta_1)f(z)\big)^{(j+1)} =& z\varphi'(z)\big(D_\lambda^{p,m}(\alpha_1,\beta_1)g(z)\big)^{(j)} \\ & + z\varphi(z)\big(D_\lambda^{p,m}(\alpha_1,\beta_1)g(z)\big)^{(j+1)}.\end{aligned} \tag{17}$$

Using (14), in the above equation, we get

$$\begin{aligned}\big(D_\lambda^{p,m+1}(\alpha_1,\beta_1)f(z)\big)^{(j)} =& \frac{\lambda z}{p}\varphi'(z)\big(D_\lambda^{p,m}(\alpha_1,\beta_1)g(z)\big)^{(j)} \\ & + \varphi(z)\big(D_\lambda^{p,m+1}(\alpha_1,\beta_1)g(z)\big)^{(j)}.\end{aligned} \tag{18}$$

Noting that $\varphi(z)$ satisfies (cf. [7])

$$|\varphi'(z)| \le \frac{1 - |\varphi(z)|^2}{1 - |z|^2} \qquad (z \in \mathcal{U}), \tag{19}$$

we see that

$$
\begin{aligned}
&\left|\left(D_\lambda^{p,m+1}(\alpha_1,\beta_1)f(z)\right)^{(j)}\right| \\
&\le \left\{\varphi(z)+\frac{1-|\varphi(z)|^2}{1-|z|}\frac{\lambda|z|}{p-|2\gamma\lambda-p||z|}\right\}\left|\left(D_\lambda^{p,m+1}(\alpha_1,\beta_1)g(z)\right)^{(j)}\right| \\
&= \frac{-\lambda r\rho^2+(1-r)(p-|2\gamma\lambda-p|r)\rho+\lambda r}{(1-r)(p-|2\gamma\lambda-p|r)}\left|\left(D_\lambda^{p,m+1}(\alpha_1,\beta_1)g(z)\right)^{(j)}\right| \\
&\qquad (|z|=r,\quad |\varphi(z)|=\rho) \\
&= \frac{\Theta(\rho)}{(1-r)(p-|2\gamma\lambda-p|r)}\left|\left(D_\lambda^{p,m+1}(\alpha_1,\beta_1)g(z)\right)^{(j)}\right| \qquad (z\in\mathcal{U}),
\end{aligned}
\tag{20}
$$

where the function $\Theta(\rho)$ defined by

$$\Theta(\rho) := -\lambda r\rho^2+(1-r)(p-|2\gamma\lambda-p|r)\rho+\lambda r \qquad (0\le\rho\le 1)$$

takes its maximum value at $\rho=1$ with $r=r_1(p,\gamma,\lambda)$ given by (10). Furthermore, if $0\le\sigma\le r_1(p,\gamma,\lambda)$ where $r_1(p,\gamma,\lambda)$ given by (10), then the function

$$\Phi(\rho) := -\lambda\sigma\rho^2+(1-\sigma)(p-|2\gamma\lambda-p|\sigma)\rho+\lambda\sigma$$

increases in the interval $0\le\rho\le 1$, so that $\Phi(\rho)$ does not exceed

$$\Phi(1)=(1-\sigma)(p-|2\gamma\lambda-p|\sigma) \qquad (0\le\sigma\le r_1(p,\gamma,\lambda)).$$

Therefore, from this fact, (20) gives the inequality (9). □

As a special case of Theorem 2.1, when $p=1$ and $j=0$, we have

Corollary 2.2. *Let the function $f(z)\in\mathcal{A}$ be analytic and univalent in the open unit disk $\mathcal{U}$ and suppose that $g(z)\in S_{\lambda,m}^{1,0}(\gamma)$. If $\left(D_\lambda^{1,m}(\alpha_1,\beta_1)f(z)\right)$ is majorized by $\left(D_\lambda^{1,m}(\alpha_1,\beta_1)g(z)\right)$ in $\mathcal{U}$, then*

$$\left|\left(D_\lambda^{1,m+1}(\alpha_1,\beta_1)f(z)\right)\right| \le \left|\left(D_\lambda^{1,m+1}(\alpha_1,\beta_1)g(z)\right)\right| \quad \textit{for} \quad |z|\le r_2, \tag{21}$$

where

$$r_2 := \frac{k-\sqrt{k^2-4|2\gamma\lambda-1|}}{2|2\gamma\lambda-1|} \tag{22}$$

$$(k := 2\lambda+1+|2\gamma\lambda-1|;\ \gamma\in\mathbb{C}-\{0\};\ \lambda\ge 0).$$

Further putting $\lambda=1$, $m=0$, $q=2$, $s=1$, $\alpha_1=\beta_1$, and $\alpha_2=1$ in Corollary 2.2, we get

Corollary 2.3. [2] *Let the function $f(z)\in\mathcal{A}$ be analytic and univalent in the open unit disk $\mathcal{U}$ and suppose that $g(z)\in S(\gamma)$. If $f(z)$ is majorized by $g(z)$ in $\mathcal{U}$, then*

$$\left|f'(z)\right| \le \left|g'(z)\right| \quad \textit{for} \quad |z|\le r_3, \tag{23}$$

where

$$r_3 := \frac{3 + |2\gamma - 1| - \sqrt{9 + 2|2\gamma - 1| + |2\gamma - 1|^2}}{2|2\gamma - 1|}. \tag{24}$$

For $\gamma = 1$, Corollary 2.3 reduces to the following result:

Corollary 2.4. [5] *Let the function $f(z) \in \mathcal{A}$ be analytic and univalent in the open unit disk $\mathcal{U}$ and suppose that $g(z) \in S^* = S^*(0)$. If $f(z)$ is majorized by $g(z)$ in $\mathcal{U}$, then*

$$\left|f'(z)\right| \leq \left|g'(z)\right| \quad \textit{for} \quad |z| \leq 2 - \sqrt{3}. \tag{25}$$

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