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TWO-SIDED ESSENTIAL SUBMODULES OF $Q^r(R)$

ULRICH ALBRECHT

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Abstract. The focus of this paper are essential submodules, A, of the maximal right ring of quotients, Q^r_R, of a right non-singular ring R. Since Q^r is a R-R-bimodule, particular attention is given to submodules of Q^r_R which are also submodules of ${}_RQ^r$. In this discussion, properties of R which are inherited by intermediate rings $R \subseteq S \subseteq Q^r$ are investigated. The results obtained are used to discuss homological properties of essential submodules A of Q^r_R. In particular, the paper addresses the question when $\mathcal{S}$-closed submodules of finite direct sums of copies of A are direct summands.

1. Introduction

The classical notion of torsion-freeness for modules over an integral domain can also be formulated for non-commutative rings. However, fundamental difficulties make such an extension meaningful only if one restricts the discussion to modules over semi-prime, right and left Goldie-rings [12]. Because of this, the concept of non-singular modules was introduced as a replacement of torsion-freeness in the non-commutative setting. A right module M over a ring R is *non-singular* if every non-zero element x of M has a non-essential right annihilator $r_R(x) = \{r \in R : xr = 0\}$. The ring R is right non-singular if R_R is a non-singular R-module. Every right non-singular ring R has a right self-injective regular maximal right ring of quotients $Q^r = Q^r(R)$, e.g. see [12] and [16]. The class of right non-singular rings contains the *right p.p.-rings*, i.e. the rings R for which every cyclic right ideal is projective, or equivalently, such that the right annihilator of every element of R is generated by an idempotent. Finally, R is a *Baer-ring* if the right annihilator of every subset of R is generated by an idempotent. In contrast to being non-singular or p.p., the property to be a Baer-ring is right-left-symmetric.

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While avoiding many of the problems associated with the classical notion of torsion-freeness in the non-commutative setting, non-singularity fails to capture some of the homological properties of torsion-free modules over integral domains. For instance, flat modules need not be non-singular. Because of this, Hattori called a right R-module M *torsion-free* if $\mathrm{Tor}_1^R(M, R/Rr) = 0$ for all $r \in R$ [13]. Naturally, the question arises when these two approaches yield the same "torsion-free" modules. Its answer in [3] focused on the class of right Utumi rings: A right non-singular ring R is *right Utumi* if every $\mathcal{S}$-closed right ideal of R is the right annihilator of a subset of R. Here, a submodule U of a right R-module M is *$\mathcal{S}$-closed* if M/U is non-singular. The right and left Utumi-rings are the right and left non-singular rings for which $Q^r = Q^\ell$ [16]. Finally, a right non-singular ring R is *right strongly non-singular* if every finitely generated non-singular right R-module can be embedded into a projective module. Right strongly non-singular rings can be described as the right non-singular rings R for which Q^r is a *perfect left localization* of R, i.e. Q^r is flat as a right R-module and the multiplication map $Q^r \otimes_R Q^r \to Q^r$ is an isomorphism, see [12] and [16].

The rings R for which the concept of non-singularity and Hattori's notion of torsion-freeness coincide are the right Utumi p.p.-ring without an infinite set of orthogonal idempotents [3]. These rings resemble integral domains in many ways, in particular, when they are right strongly non-singular [3]. This becomes apparent when considering subrings S of Q^r which contain R. In case that R is a Prüfer domain, every such S is Prüfer too. Section 3 investigates properties of right non-singular rings R which are inherited by intermediate rings $R \subseteq S \subseteq Q^r$. Proposition 3.1 and Theorem 3.2 show that being right Utumi, right and left Utumi p.p., and right strongly non-singular, right semi-hereditary are such properties provided that R has no infinite set of orthogonal idempotents.

Such intermediate rings arise naturally in the investigation of essential submodules of Q^r_R, as is shown in Section 2. It begins with an example demonstrating the significant differences between the commutative and the non-commutative setting. Because Q^r is a R-R-bimodule, we are particularly interested in *two-sided R-submodules* A of Q^r, i.e. submodules of Q^r_R which also are submodules of ${}_RQ^r$. In case that R a right and left Utumi-ring, we investigate how A_R being essential in Q^r_R is related to ${}_RA$ being essential in ${}_RQ^r$. Furthermore, we describe the endomorphism ring of an essential submodule A_R of Q^r_R. This ring will be one of the subrings investigated in Section 3 exactly if A is a two-sided R-submodule of Q^r.

Section 4 applies these results to the investigation of homological properties of $\mathcal{S}$-closed submodules of finite direct sums of copies of A. We concentrate on the case that R is a right non-singular ring without an infinite set of orthogonal idempotents and that A is a two-sided R-submodule of Q^r which is essential as a submodule of Q^r_R. We call such an A a *right essential submodule*. In this case, $\mathcal{S}$-closed submodules of finite direct sums of copies of A are direct summands exactly if R is a strongly non-singular right semi-hereditary ring. We conclude with further examples, and apply the previous results to modules over integral domains.

2. Essential Submodules of Q^r

Let R be a right non-singular ring, and A be a submodule of Q^r_R. Consider the subring $Fix(A) = \{q \in Q^r | qA \subseteq A\}$ of Q^r and the two-sided ideal $\ell(A) = \{q \in Q^r | qA = 0\}$ of $Fix(A)$. The ring $Fix(A)$ is a subring of Q^r which contains R if and only if A is a two-sided R-submodule of Q^r. For every $q \in Fix(A)$, let $\lambda_q : A \to A$ be left multiplication by q. It is easy to see that $\phi_A(q) = \lambda_q$ defines a ring homomorphism ϕ_A from $Fix(A) \to End_R(A)$ whose kernel is $\ell(A)$. Since Q^r is the injective hull of R_R, every map $\phi : A \to A$ is induced by a R-homomorphism $\hat{\phi} : Q^r \to Q^r$ which can easily be shown to be a Q^r-map. Therefore, there exists $q \in Q^r$ such that $\hat{\phi}(x) = qx$ for all $x \in Q$, and ϕ_A is onto.

To illustrate the difference between the commutative and the non-commutative setting, consider the ring

$$R = \left\{ \begin{pmatrix} n & 0 \\ x & y \end{pmatrix} | n \in \mathbb{Z}, x, y \in \mathbb{Q} \right\}$$

which is right non-singular with maximal right ring of quotients $Q^r = Mat_2(\mathbb{Q})$ [12]. For a subgroup A of $\mathbb{Q}$, let

$$M_A = \left\{ \begin{pmatrix} a & 0 \\ x & y \end{pmatrix} | a \in A, x, y \in \mathbb{Q} \right\} \text{ and } L_A = \left\{ \begin{pmatrix} a & 0 \\ x & 0 \end{pmatrix} | a \in A, x \in \mathbb{Q} \right\}.$$

Observe that L_0 is the nilradical of R. Consider the idempotents

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \ e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } e_{(x)} = \begin{pmatrix} 1 & 0 \\ x & 0 \end{pmatrix}$$

of R where $x \in \mathbb{Q}$.

Example 2.1. a) *The ring R described above is a right strongly non-singular p.p.-ring.*

b) *Q^r, M_A, and L_A, where A is a subgroup of $\mathbb{Q}$, are the non-zero two-sided R-submodules of Q^r. Moreover, M_A and L_A are right essential if and only if $A \neq 0$.*

c) i) *If A is a non-zero subgroup of $\mathbb{Q}$, then*

$$Fix(M_A) = \left\{ \begin{pmatrix} a & 0 \\ x & y \end{pmatrix} | a \in End_{\mathbb{Z}}(A), x, y \in \mathbb{Q} \right\}.$$

ii) *If A is a non-zero, proper subgroup of $\mathbb{Q}$, then*

$$Fix(L_A) = \left\{ \begin{pmatrix} a & 0 \\ x & y \end{pmatrix} | a \in End_{\mathbb{Z}}(A), x, y \in \mathbb{Q} \right\},$$

while $Fix(L_{\mathbb{Q}}) = Q^r$.

iii) $Fix(M_0) = \left\{ \begin{pmatrix} u & 0 \\ x & y \end{pmatrix} | u, x, y \in \mathbb{Q} \right\}.$

iv) $Fix(e_1 R) = \left\{ \begin{pmatrix} n & x \\ 0 & y \end{pmatrix} | n \in \mathbb{Z}, x, y \in \mathbb{Q} \right\}.$

PROOF. a) To see that R is a right p.p.-ring, consider be a non-zero element $t = \begin{pmatrix} n & 0 \\ x & y \end{pmatrix}$ of R. If both, n and y, are non-zero, then t is a regular element of R, and $r(t) = 0$. Thus, we may assume $n = 0$ or $y = 0$.

Every $s = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \in r_R(t)$ yields the equations $na = 0$, $xa + yb = 0$ and $yc = 0$. Suppose $n = 0$. If $y = 0$ too, then the equations reduce to $xa = 0$ which yields $a = 0$ since $t \neq 0$. Hence, $r_R(t) = e_2 R$. On the other hand, if $y \neq 0$, then $yc = 0$ yields $c = 0$ and $b = -xy^{-1}a$. From this, we obtain $r_R(t) = e_{(-xy^{-1})}R$. On the other hand, if $n \neq 0$, then $na = 0$ yields $a = 0$. Since $y = 0$, we have $r_R(t) = e_2 R$. In either case, $r_R(t)$ is generated by an idempotent, i.e. R is a right p.p.-ring.

Observe that $Q^r = e_1 Q^r \oplus e_2 Q^r$. Since $e_2 Q^r = e_2 R$, it is a projective R-module. Moreover, $e_1 Q^r$ is generated by $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ as a R-module. One obtains $e_1 Q^r \cong e_2 R$ as R-modules since $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = e_2$ and $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} e_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Therefore, Q^r is a projective R-module. Since every finitely generated non-singular right R-module can be embedded into a direct sum of copies of Q^r, it is isomorphic to a submodule of a projective module, i.e. R is right strongly non-singular.

b) Direct computation shows that all the listed modules are two-sided R-submodules of Q^r. To see that M_A and L_A are right essential in Q^r_R if $A \neq 0$, let

$u, v, x, y \in \mathbb{Q}$. Observe

$$\begin{pmatrix} u & v \\ x & y \end{pmatrix} e_1 = \begin{pmatrix} u & 0 \\ x & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} u & v \\ x & y \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} v & 0 \\ y & 0 \end{pmatrix}.$$

On the other hand, $e_1 R \cap M_0 = 0$ shows that M_0 and L_0 are not right essential.

Finally, let U be a non-zero two-sided R-submodule of Q^r. If U contains an element of the form $\begin{pmatrix} u & v \\ x & y \end{pmatrix}$ with $v \neq 0$, then $e_1 \begin{pmatrix} u & v \\ x & y \end{pmatrix} e_2 = \begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix} \in U$. But, $\begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix} R = e_1 Q^r$.

On the other hand,

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & v \end{pmatrix} \in U.$$

But, $\begin{pmatrix} 0 & 0 \\ 0 & v \end{pmatrix} R = e_2 Q^r$. Hence, $Q^r = e_1 Q^r \oplus e_2 Q^r \subseteq U$. Therefore, one may assume that U is a submodule of $M_{\mathbb{Q}}$. If U is not contained in $L_{\mathbb{Q}}$, then $e_2 Q^r \subseteq U$, and $U = M_A$ for some subgroup A of $\mathbb{Q}$. On the other hand if $U \subseteq L_{\mathbb{Q}}$, then $L_0 \cap U \neq 0$. Hence, $L_0 \subseteq U$. Therefore, $U = L_A$ for some $A \subseteq \mathbb{Q}$.

c) Suppose $\begin{pmatrix} u & v \\ x & y \end{pmatrix} \in Fix(M_A)$. One immediately obtains $uA + v\mathbb{Q} \subseteq A$ and $v\mathbb{Q} = 0$. Hence, $v = 0$ and $uA = A$, i.e. $u \in End_{\mathbb{Z}}(A)$. On the other hand, if A is a proper subgroup of $\mathbb{Q}$, then $\begin{pmatrix} u & v \\ x & y \end{pmatrix} \in Fix(L_A)$ also yields $uA + v\mathbb{Q} \subseteq A$, which is not possible unless $v = 0$ since A is a proper subgroup of $\mathbb{Q}$. Direct computation shows $Q^r L_{\mathbb{Q}} = L_{\mathbb{Q}}$. This establishes i) - iii). Part iv) is shown similarly. □

Other non-commutative rings with an ample supply of two-sided ideals are the right bounded rings, where a ring R is *right bounded* if every essential right ideal of R contains an essential two-sided ideal (e.g. see [9]).

In the following, let $dim_R M$ denote the Goldie-dimension of a right R-module M.

Theorem 2.2. *Consider the following conditions for a right non-singular ring R:*

a) *R has finite right Goldie dimension.*

b) i) *R has the ACC for right annihilators.*
ii) *A submodule A of Q_R is essential if and only if $\ell(A) = 0$.*

Then, a) always implies b), and the converse is true if R is a right and left non-singular right and left Utumi-ring.

Proof. *a)* $\Rightarrow$ *b)*: Since R has finite right Goldie dimension, Q^r is a semi-simple Artinian ring, e.g. see [16]. By [12], R has the ACC for right annihilators. Suppose that A is an essential submodule of Q^r_R, and $q \in \ell(A)$. Then, $A \cap R$ is an essential right ideal of R, and so $q(A \cap R) = 0$ yields $q = 0$ since Q^r_R is a non-singular module.

Conversely, if $\ell(A) = 0$, select a submodule U of Q^r_R maximal with respect to the property that $A \cap U = 0$. If $x \in AQ^r \cap UQ^r$, then there are $a_1, \ldots, a_m \in A$, $u_1, \ldots, u_n \in U$, and $q_1, \ldots, q_m, s_1, \ldots, s_n \in Q^r$ with $x = \Sigma_{i=1}^m a_i q_i = \Sigma_{j=1}^n u_j s_j$. Choose an essential right ideal J of R such that $u_i J, s_j J \subseteq R$ for all $i = 1, \ldots, m$ and $j = 1, \ldots, n$. Then, $xJ \subseteq A \cap U = 0$. Since Q^r is non-singular, $x = 0$. Since $A \oplus U$ is an essential submodule of Q^r, we have that $AQ^r \oplus UQ^r$ is an essential Q^r-submodule of Q^r [12]. Because Q^r is semi-simple Artinian, $Q^r = AQ^r \oplus UQ^r$. If A is not essential in Q^r, then $U \neq 0$, and the projection of Q^r onto UQ^r with kernel AQ^r induces a non-zero Q^r-endomorphism ϕ of Q^r with $\phi(A) = 0$. However, ϕ is left multiplication by some $q \in Q^r$ as has been shown at the beginning of this section. Then, $qA = 0$ yields $q \in \ell(A) = 0$. Thus, A is essential.

b) $\Rightarrow$ *a)*: Suppose that R is not finite dimensional, and consider a family $\{I_n\}_{n<\omega}$ of non-zero right ideals of R whose sum is direct. Without loss of generality, one may assume that $I = \oplus_n I_n$ is essential in R. Then, $\ell(I) = 0$. Let $J_n = \ell_R(I_1 \oplus \ldots \oplus I_n)$. Since the J_n's form a descending chain of left annihilators of R, it has to becomes stationary at some point, say $J_n = J_{n+k}$ for all $k < \omega$. For this, observe the ACC for right annihilators is equivalent to the DCC for left annihilators. In particular, $J_n I = 0$, and $J_n \subseteq \ell(I) = 0$ yields $J_n = 0$. On the other hand, there is $0 \neq q \in Q^r$ with $q(I_1 \oplus \ldots \oplus I_n) = 0$ since $I_1 \oplus \ldots \oplus I_n$ is not essential in R. Since R is a right and left Utumi-ring, Q^r is also the left ring of quotients of R, and there is an essential left ideal K of R with $Kq \subseteq R$. Since Q^r is a non-singular left R-module, there is a non-zero $x \in K$ with $xq \neq 0$. Then, $xq \in J_n$ contradicts $J_n = 0$. □

Consequently, if R has finite right Goldie dimension, then a submodule A of Q_R is essential if and only if ϕ_A is a monomorphism.

Corollary 2.3. *The following conditions are equivalent for a right and left non-singular ring R which has finite right and left Goldie-dimension.*

a) *Let A be a two-sided R-submodule of Q^r. Then, A is essential as a right R-module if and only it is essential as a left R-module.*
b) *R is a semi-prime right and left Utumi-ring.*

PROOF. *a)* $\Rightarrow$ *b)*: Because R_R is an essential submodule of Q^r_R, it is an essential submodule of ${}_RQ^r$. Observe that Q^r is semi-simple Artinian since R has finite right Goldie-dimension. Thus, it is its own maximal left ring of quotients. On the other hand, R is a left non-singular ring which is essential in ${}_RQ^r$. Thus, Q^r is a left ring of quotients of R. By [12, Theorem 2.30], Q^r is a maximal left ring of quotients of R, and $Q^r = Q^\ell$, i.e. R is a right and left Utumi-ring.

If $N(R) \neq 0$, then there exists a non-zero two-sided ideal I of R with $I^2 = 0$. Select a right ideal J of R such that $I \oplus J$ is essential in R, and consider $A = I + RJ$ which is a two-sided ideal of R. One has $AI \subseteq I^2 + RJI = RJI$. However, $JI \subseteq J \cap I = 0$ yields $0 \neq I \subseteq r(A)$, a contradiction.

b) $\Rightarrow$ *a)*: By symmetry, it suffices to show that A is essential as a submodule of ${}_RQ^r$ if it essential in Q^r_R. Suppose that such an A is not essential in ${}_RQ^r$. Since R is right and left Utumi, Theorem 2.2 applies to A as a submodule of ${}_RQ^r$. It yields $I = r(A)$ is a non-zero right ideal of $Fix({}_RA)$. However, since A is a submodule of Q^r_R, we have $R \subseteq Fix({}_RA)$. Considered as right R-modules, R is essential in $Fix({}_RA)$, and hence $I \cap R$ is a non-zero R-submodule of Q^r_R. However, A_R essential in Q_R yields that $I \cap R \cap A$ is a non-zero right ideal of R. But $(I \cap R \cap A)^2 \subseteq AI = 0$ implies $N(R) \neq 0$, a contradiction. □

Observe that the previous results, in particular, apply to strongly non-singular p.p.-rings without an infinite family of orthogonal idempotents since they have finite right and left Goldie dimension and are right and left Utumi by [3].

Theorem 2.4. *Let R be a right and left non-singular, right and left Utumi ring. The following are equivalent:*

a) *R is a Baer-ring.*
b) *If A is a submodule of Q^r_R, then $\ell(A) = Fix(A)e$ for an idempotent e of R.*

In this case, $End_R(A) \cong eFix(A)e$ where $e \in R$ is an idempotent with $\ell(A) = Fix(A)e$.

PROOF. Since R is right and left Utumi, Q^r is the maximal left ring of quotients of R too. We thus write Q for Q^r. To simplify our notation, S denotes the ring $Fix(A)$.

a) $\Rightarrow$ *b)*: To see that $\ell(A)$ is generated by an idempotent of R, let $q \in Q$ such that $q(A \cap R) = 0$, and consider $a \in A$. There is an essential right ideal I of R such that $aI \subseteq R$. Since $aI \subseteq A \cap R$, one has $qaI = 0$. But this is only possible if $qa = 0$. Hence $\ell(A \cap R) \subseteq \ell(A)$. Therefore, it suffices to show that $\ell(A \cap R)$ is generated by an idempotent e of R. For this, observe that $\ell(A \cap R) \cap R = \ell_R(A \cap R)$. However, since R is a Baer-ring, there is an idempotent $e \in R$ with $\ell_R(A \cap R) = Re$. In particular, $eA = 0$ yields $Se \subseteq \ell(A)$. On the other hand, let $q \in \ell(A)$. Since Q is the maximal left ring of quotients of R, there is an essential left ideal J of R with $Jq \subseteq R$. But $Jq(A \cap R) = 0$ yields $Jq \subseteq Re$. Then, $Jq(1-e) = 0$ which is only possible if $q(1-e) = 0$ since Q is a non-singular left R-module. Therefore, $q = qe + q(1-e) = qe \in Se$.

Since $\ell(A) = Se$ is a two-sided ideal of S, one obtains $eS \subseteq Se$, and hence $eS(1-e) = 0$. Then, $S = \ell(A) \oplus (1-e)S(1-e)$ as abelian groups, and $S/\ell(A) \cong (1-e)S(1-e)$ as rings. Define a map $\lambda : S \to End_R(A)$ by $[\lambda(q)](a) = qa$. By what has been shown at the beginning of this section, λ is an epimorphism of rings with ker $\lambda = \ell(A)$.

b) $\Rightarrow$ *a)*: Assume that $\ell(A) = Se$ for some idempotent $e \in R$ whenever $A \subseteq Q_R$, and let X be a subset of R. The right ideal I of R generated by X satisfies $\ell_R(X) = \ell_R(I) = \ell(I) \cap R$. By b), $\ell(I) = Fix(I)e$ for some idempotent $e \in R$. Hence, $Re \subseteq \ell_R(X)$. On the other hand, if $r \in R$ satisfies $rx = 0$ for all $x \in X$, then $rI = 0$, and $r = qe$ for some $q \in Fix(I)$. Since Q is the maximal left ring of quotient of R, there is an essential left ideal J of R such that $Jq \subseteq R$. Hence, $Jr = Jqe \subseteq Re$. Since R/Re is non-singular, $r \in Re$. Therefore, R is a Baer-ring. □

3. Essential Ring Extensions

This section investigates properties of a right non-singular ring R which are inherited by intermediate rings S between R and Q^r.

Proposition 3.1. *Let R be a right non-singular ring without an infinite set of orthogonal idempotents, and consider an intermediate ring $R \subseteq S \subseteq Q^r$.*

- a) *If R is a right Utumi-ring, then so is S.*
- b) *If R is a semi-prime right Goldie-ring, so is S.*
- c) *If R is a right Utumi p.p.-ring, then so is S.*

Proof. To see that S is a right non-singular ring, consider $x \in S$, and assume $xI = 0$ for some essential right ideal I of S. Then, $x(I \cap R) = 0$ yields $x = 0$ since Q^r_R is a non-singular module and $I \cap R$ obviously is essential in R.

a) Let I be a $\mathcal{S}$-closed right ideal of S. To see that $I \cap R$ is $\mathcal{S}$-closed in R, choose $r \in R$ such that $rJ \subseteq I \cap R$ for some essential right ideal J of R. For every non-zero $s \in S$, there exists an essential right ideal K_s of R such that $sK_s \subseteq J$ since J is essential in R_R and R_R is essential in Q_R. Then, $sK_sS \subseteq JS$. Moreover, since Q_R is non-singular, there exists $r_s \in K_s$ such that $sr_s \neq 0$. Because $0 \neq sr_s \in JS \cap sS$, the right ideal JS of S is essential. Hence, $rJS \subseteq I$ yields $r \in I$ since S/I is a non-singular S-module. Consequently, $r \in I \cap R$, and $I \cap R$ is $\mathcal{S}$-closed in R. Since R is a right Utumi-ring, there is a subset X of R such that $I \cap R = r_R(X)$.

If $s \in I$, then there is an essential right ideal J of R such that $sJ \subseteq R$. Hence, $sJ \subseteq I \cap R$, and $XsJ = 0$. By the non-singularity of S as an R-module, $Xs = 0$, and $I \subseteq r_S(X)$. On the other hand, for $s_1 \in r_S(X)$, choose an essential right ideal K of R with $s_1K \subseteq R$. For each $y \in K$, one has $X(s_1y) = 0$. This yields $s_1K \subseteq I \cap R$, and hence $s_1K \subseteq I$. Then, $s_1KS \subseteq I$ from which $s_1 \in I$ follows since KS is an essential right ideal of S and S/I is non-singular as a S-module. Thus, S is right Utumi.

b) If R is a semi-prime Goldie-ring, then R is right non-singular and finite-dimensional. Clearly, every such S is a finite dimensional R-module, and consequently has finite right Goldie dimension as a S-module too. Since we have already seen that S is a right non-singular ring, it remains to show that it is semi-prime by [12, Corollary 3.32]. For a right ideal I of S with $I^2 = 0$, we have that $I \cap R$ is a right ideal of R with $(I \cap R)^2 = 0$. This yields $I \cap R = 0$ because R is semi-prime. Since R is essential in S, one has $I = 0$.

c) Because of a), it remains to show that S is a right p.p.-ring. Let $x \in S$. By [3], every non-singular cyclic right R-module is projective. Hence, there is an idempotent $e \in R$ such that $r_R(x) = eR$. Consequently, $eS \subseteq r_S(x)$. Conversely, suppose that $xt = 0$ for some $t \in S$. There exists an essential right ideal I of R with $tI \subseteq R$. Then, $tI \subseteq eR \subseteq eS$. But $S/eS \cong (1-e)S$ is a non-singular R-module, and hence $t \in eS$ as desired. □

Theorem 3.2. *Let R be a right and left non-singular ring without an infinite set of orthogonal idempotents. Consider an intermediate ring $R \subseteq S \subseteq Q^r$.*

a) *If R is a right and left Utumi p.p.-ring, then S is a right and left Utumi p.p-ring.*
b) *If R is a right and left Utumi-ring, then S has the ACC (DCC) for right (left) annihilators provided that R has it.*
c) *If R is a right strongly non-singular right semi-hereditary ring, then so is S.*

Proof. a) By [3], R has finite right Goldie-dimension; and Q^r is a semi-simple Artinian ring. Observe that R is also a left p.p.-ring. Therefore, R is a right and left Utumi p.p-ring, and the same holds for S by Proposition 3.1b. Because Q^r is the maximal right ring of quotients of S, it is also its maximal left ring of quotients.

b) Since the ACC (DCC) for right annihilators is equivalent to the DCC (ACC) for left annihilators, it suffices to consider subsets X_1 and X_2 of S such that $\ell_S(X_1) \subseteq \ell_S(X_2)$. For each $x \in X_i$, choose an essential right ideal J_x of R with $xJ_x \subseteq R$, and let $S_i = \Sigma_{x \in X_i} xJ_x \subseteq R$. Clearly, $\ell_S(X_i) \subseteq \ell_S(S_i)$. On the other hand, if $sS_i = 0$ for some $s \in S$, then $sxJ_x = 0$ for all $x \in X_i$. Since S is non-singular, $sx = 0$, and $\ell_S(X_i) = \ell_S(S_i)$. Hence, $\ell_R(S_1) = \ell_S(S_1) \cap R \subseteq \ell_S(S_2) \cap R = \ell_R(S_2)$. If $\ell_S(X_1) \neq \ell_S(X_2)$, then there is $s \in S$ such that $sS_1 = 0$ but $sS_2 \neq 0$. Since R is a left and right Utumi-ring, Q^r is the maximal left ring of quotient of R, and there exists an essential left ideal I of R with $Is \subseteq R$. Then, $IsS_1 = 0$, but $IsS_2 \neq 0$ since S is a non-singular left R-module. Hence, we can find $r \in I$ such that $rsS_2 \neq 0$ and $rs \in R$, i.e. $\ell_R(S_1) \neq \ell_R(S_2)$. The rest of b) follows immediately.

c) To show that S is a right semi-hereditary ring for which Q^r is a perfect left localization of S, it suffices to establish that every finitely generated non-singular right S-module M is projective [16]. Since Q^r is semi-simple Artinian, and S_S is essential in Q^r_S, one obtains that Q^r is the maximal right ring of quotients of S [12]. We first consider the case that M is a S-submodule of Q^r. If $M = x_1S + \ldots + x_nS$, then $U = x_1R + \ldots + x_nR$ is an essential R-submodule of Q^r. Because R is a right strongly non-singular, right semi-hereditary ring, every finitely generated non-singular right R-module is projective. Hence, there exists a right R-module W such that $U \oplus W \cong \oplus_n R$. Then, $(U \otimes_R S) \oplus (W \otimes_R S) \cong \oplus_n S$ yields that $U \otimes_R S$ is a projective right S-module.

The map $\phi : U \otimes_R S \to M$ defined by $\phi(x \otimes s) = xs$ is onto because $US = M$. It remains to show that ϕ is one-to-one. Since $U \otimes_R S$, as a projective S-module, is R-non-singular, it suffices to show $dim_R U \otimes_R S = dim_R M$ observing that the latter is finite by [3]. Since U is an essential submodule of M, they have the same Goldie-dimension over R. The inclusion $R \subseteq S$ of R-R-bimodules induces an exact sequence $0 \to U \otimes_R R \to U \otimes_R S \to U \otimes_R (S/R) \to 0$ of right R-modules since U is flat. In view of the fact that S/R is singular, we have $(U \otimes_R S)/(U \otimes_R R)$ is singular, and $dim_R U = dim_R(U \otimes_R S)$. Therefore, S is right semi-hereditary and right strongly non-singular.

If M is a finitely generated non-singular right S-module, then $M \subseteq \oplus_n Q^r$ for some $n < \omega$. We induct on n, and consider $U = M \cap \oplus_{n-1} Q^r$. Since $M/U \cong [M + \oplus_{n-1} Q^r] / \oplus_{n-1} Q^r$ is isomorphic to a finitely generated submodule of Q^r, it is projective by what has been shown so far. Thus, $M = U \oplus P$ for some projective module P. $\square$

By [3], a right Utumi p.p.-ring without an infinite set of orthogonal idempotents is Baer. Thus, Part c) of Lemma 3.1 shows that S is a Baer right Utumi-ring provided R is.

Lemma 3.3. *Let R be a right strongly non-singular, right semi-hereditary ring without an infinite set of orthogonal idempotents, and $R \subseteq S \subseteq Q^r$ an intermediate ring. Then, S is a perfect right localization of R.*

Proof. By [3], R also is left strongly non-singular and left semi-hereditary. In particular, its maximal right and left rings of quotients coincide. Denote this ring by Q. Then, Q is a perfect left localization of R. Furthermore, every finitely generated non-singular left R-module is projective, and S is flat as a left R-module. By [16, XI.2.4], S is a perfect right localization of R. $\square$

A ring R has the *restricted right minimum condition* if R/I is Artinian for every essential right ideal I of R. Right and left Noetherian hereditary rings have the restricted right minimum condition [9].

Proposition 3.4. *Let R be a right strongly non-singular p.p.-ring without an infinite set of orthogonal idempotents. Consider an intermediate ring $R \subseteq S \subseteq Q^r$ such that S is a perfect right localization of R.*

a) *The multiplication map $M \otimes_R S \to M$ is an isomorphism for all right S-modules M.*
b) *If R has the restricted right minimum condition, so does S.*

Proof. a) Consider an exact sequence $P \to F \to M \to 0$ where P and F are free S-modules. Since S is a perfect right localization of R, the multiplication map $S \otimes_R S \to S$ is an isomorphism. Then, the multiplication maps $P \otimes_R S \to P$ and $F \otimes_R S \to F$ are isomorphisms too, and fit into the commutative diagram

$$\begin{array}{ccccccc} P \otimes_R S & \longrightarrow & F \otimes_R S & \longrightarrow & M \otimes_R S & \longrightarrow & 0 \\ \wr\downarrow & & \wr\downarrow & & \downarrow & & \\ P & \longrightarrow & F & \longrightarrow & M & \longrightarrow & 0. \end{array}$$

By the 5-Lemma, the multiplication map $M \otimes_R S \to M$ is an isomorphism.

b) We first show that $(I \cap R)S = I$ for every right ideal I of S. Associated with every submodule A of S_R is a natural map $\sigma_A : A \otimes_R S \to AS$ defined by $\sigma_A(a \otimes s) = as$ for all $a \in A$ and $s \in S$. Since S is a perfect right localization of

R, the map σ_S is an isomorphism. The flatness of S as a left R-module gives the exactness of the top-row of the commutative diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & A \otimes_R S & \longrightarrow & S \otimes_R S \\ & & \big\downarrow \sigma_A & & \wr \big\downarrow \sigma_S \\ 0 & \longrightarrow & AS & \longrightarrow & S \end{array}$$

whose rows are induced by the inclusion map. Thus, σ_A is an isomorphism.

By [16, XI.1.2], we have $(S/R) \otimes_R S = 0$. As an R-module, $I/(I \cap R) \cong (I+R)/R \subseteq S/R$ yields the exact sequence $0 \to [I/(I \cap R)] \otimes_R S \to (S/R) \otimes_R S = 0$ from which we get the exact sequence $0 \to (I \cap R) \otimes_R S \to I \otimes_R S \to [I/(I \cap R)] \otimes_R S = 0$. The isomorphisms $\sigma_{I \cap R}$ and σ_I fit into the commutative diagram

$$\begin{array}{ccc} (I \cap R) \otimes_R S & \xrightarrow[\sim]{} & I \otimes_R S \\ \wr \big\downarrow \sigma_{I \cap R} & & \wr \big\downarrow \sigma_I \\ (I \cap R)S & \xrightarrow[\iota]{} & IS = I \end{array}$$

where ι denotes the inclusion map. Thus, ι is an isomorphism, and $I = (I \cap R)S$ as desired.

Let J be an essential right ideal of S. Arguing as before, $J \cap R$ is an essential right ideal of R. If $I_0 \supseteq \ldots \supseteq I_n \supseteq \ldots \supseteq J$ is a descending chain of right ideals of S, then the descending chain $\ldots \supseteq R \cap I_n \supseteq R \cap I_{n+1} \supseteq \ldots$ becomes stationary, say $R \cap I_m = R \cap I_{m+k}$ for all $k < \omega$. But then, $I_m = (R \cap I_m)S = (R \cap I_{m+k})S = I_{m+k}$. □

Corollary 3.5. *Let R be a right strongly non-singular, right hereditary ring without an infinite set of orthogonal idempotents. Every intermediate ring $R \subseteq S \subseteq Q^r$ is right hereditary.*

Proof. Since R is right hereditary and has finite right Goldie-dimension [3], it is right Noetherian by Sandomirski's Theorem. By Theorem 3.2 and [16], S is a right Noetherian, right semi-hereditary ring. □

4. $\mathcal{S}$-Closed Submodules of A-Projective Modules

Let A and M be right R-modules. The *A-radical of M* is $\rho_A(M) = \cap\{\ker \alpha | \alpha \in \mathrm{Hom}(_R(M, A)\}$. A right R-module P is *A-projective* if it is a direct summand of $\oplus_I A$ for some index-set I (see [6] and [7]). Finally, a R-module M is *A-generated* if it is an epimorphic image of $\oplus_I A$ for some index-set I. If I can be chosen to be finite, then M is *finitely A-generated*.

Theorem 4.1. *The following are equivalent for a right non-singular ring R without an infinite set of orthogonal idempotents:*

a) *R is right strongly non-singular and right semi-hereditary.*
b) *Let A be a right essential two-sided R-submodule of Q^r. For all $n < \omega$, an $\mathcal{S}$-closed submodule of A^n is a direct summand.*
c) *Let A be a right essential two-sided R-submodule of Q^r. Every finitely A-generated non-singular right R-module is A-projective.*
d) i) *R is right strongly non-singular.*
   ii) *Let A be a right essential two-sided R-submodule of Q^r. A finitely A-generated right R-module M of finite Goldie dimension such that $\rho_A(M) = 0$ is A-projective.*

Proof. *a)* $\Rightarrow$ *b)*: By [3], R also is a left strongly non-singular left semi-hereditary ring whose maximal right and left ring of quotients coincide. We denote the latter by Q. Section 2 shows that $S = End_R(S) = Fix(A)$ is a subring of Q which contains R. Because of Proposition 3.1 and Theorem 3.2, S is a right and left strongly non-singular, right and left semi-hereditary ring without an infinite family of orthogonal idempotents. Associated with the S-R-bimodule A is a pair of adjoint functors $H_A(-) = Hom_R(A, -)$ and $T_A = - \otimes_S A$ between the categories of right R-modules and right S-modules respectively.

Denote the embedding $A \subseteq Q$ by α. Since it is both a right R-module and a left S-module map, the induced map $\alpha^* : \mathrm{Hom}_R(Q, Q) \to Hom_R(A, Q)$ is a map of right S-modules. Moreover, it is an isomorphism since it fits into the exact sequence $0 = \mathrm{Hom}_R(Q/A, Q) \to \mathrm{Hom}_R(Q, Q) \xrightarrow{\alpha^*} \mathrm{Hom}_R(A, Q) \to \mathrm{Ext}^1_R(Q/A, Q) = 0$ where the first term vanishes by the singularity of Q/R, while the last term does the same since Q_R is an injective R-module. However, $\mathrm{Hom}_R(Q, Q) \cong Q$ as an S-module since R-maps $Q \to Q$ are Q-homogeneous. On the other hand, $Q \otimes_S A$ is the injective hull of A as an S-module by [16]. The map α induces a monomorphism $Q \otimes_S A \to Q \otimes_S Q$ of right R-modules because Q is flat over S. Since Q is a perfect right and left localization of S, the multiplication map $Q \otimes_S Q \to Q$ is an isomorphism. Hence, $dim_R Q \otimes_S A \leq dim_R Q$. On the other hand, the natural map $\theta_Q : \mathrm{Hom}_R(A, Q) \otimes_S A \to Q$ is an epimorphism because Q is A-generated. We have seen that $H_A(Q) \cong Q$ as right S-modules. Thus, $dim_R Q \otimes_S A = dim_R Q + dim_R \ker\ \theta_Q < \infty$ since Goldie-dimension is additive over $\mathcal{S}$-closed sequences. Thus, $\ker\ \theta_Q = 0$, and θ_M is an isomorphism whenever $M \cong Q^m$ for some $m < \omega$.

Since S is a right and left strongly non-singular right and left semi-hereditary ring, every finitely generated non-singular S-module is projective, and all non-singular S-modules are flat. Because A is an S-submodule of Q, we obtain that the functor T_A is exact. Consider an $\mathcal{S}$-closed submodule U of A^n for some $n < \omega$. Since A^n/U is a non-singular module of finite Goldie dimension, there is a monomorphism $\alpha : A^n/U \to Q^\ell$ for some $\ell < \omega$. It induces the commutative diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & T_AH_A(A^n/U) & \xrightarrow[H_A(\alpha)]{} & T_AH_A(Q^\ell) \\ & & \downarrow{\scriptstyle \theta_{A^n/U}} & & \wr\downarrow{\scriptstyle \theta_{Q^\ell}} \\ 0 & \longrightarrow & A^n/U & \xrightarrow[\alpha]{} & Q^\ell \end{array}$$

whose rows are exact. Since $\theta_{A^n/U}$ is onto, it is actually an isomorphism.

Let $\pi : A^n \to A^n/U$ be a projection map with kernel U. It induces the exact sequence $0 \to H_A(U) \to H_A(A^n) \overset{H_A(\pi)}{\to} X \to 0$ of right S-modules where $X = im\ H_A(\pi)$ is a finitely generated S-submodule of $H_A(A^n/U)$. Since A^n/U is isomorphic to a submodule of Q^ℓ for some $\ell < \omega$, we obtain that $H_A(A^n/U)$ is a non-singular right S-module by what has been shown so far. Because S is right strongly non-singular and right semi-hereditary, X is a projective right S-module; and the last sequence splits. But then, the top-row of the following commutative diagram will also split:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T_AH_A(U) & \longrightarrow & T_AH_A(A^n) & \xrightarrow[T_AH_A(\pi)]{} & T_A(X) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \theta_U} & & \wr\downarrow{\scriptstyle \theta_{A^n}} & & \downarrow{\scriptstyle \theta} & & \\ 0 & \longrightarrow & U & \longrightarrow & A^n & \xrightarrow[\pi]{} & A^n/U & \longrightarrow & 0 \end{array}$$

in which the induced map θ is defined by $\theta(\phi \otimes a) = \phi(a)$ for all $\phi \in X \subseteq H_A(A^n/U)$ and $a \in A$. Since the top-row of the diagram splits, the same will hold for the bottom, once we have shown that θ is a monomorphism, which follows immediately from the commutative diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & T_A(X) & \longrightarrow & T_AH_A(A^n/U) \\ & & \downarrow{\scriptstyle \theta} & & \wr\downarrow{\scriptstyle \theta_{A^n/U}} \\ & & A^n/U & \xrightarrow[1_{A^n/U}]{} & A^n/U. \end{array}$$

b) $\Rightarrow$ *c)* follows directly from the fact that M fits into an exact sequence $0 \to U \to A^n \to M \to 0$ in which U is an $\mathcal{S}$-closed submodule of A^n.

For $c) \Rightarrow d)$, it remains to show that R is right strongly non-singular. Since R is one of the modules A to which c) can be applied, the latter yields that finitely generated non-singular right R-modules are projective, i.e. R is right strongly non-singular and right semi-hereditary. Finally, to see $d) \Rightarrow a)$, again consider the case $A = R$. □

As in [7], call an R-module A self-small if, for every index-set I and every map $\alpha : A \to \oplus_I A$, there is a finite subset J of I with $\alpha(A) \subseteq \oplus_J A$. It is easy to see that non-singular modules which have finite Goldie-dimension are self-small.

Corollary 4.2. *Let R be a right strongly non-singular, right semi-hereditary ring without an infinite family of orthogonal idempotents, and consider a right essential two-sided R-submodule A of Q^r. Every A-projective right R-module is a direct sum of submodules of Q^r.*

Proof. Since A is self-small, $H_A(P)$ is a projective right $S = End_R(A)$-module whenever P is A-projective [7]. Therefore, there exists right ideals $\{I_j | j \in J\}$ of S with $H_A(P) \cong \oplus_J I_j$ since S is right semi-hereditary [16]. Hence, $P \cong T_A H_A(P) \cong \oplus_J T_A(I_j)$. Since A is flat as left S-module, one has $T_A(I_j) \cong I_j A \subseteq Q$. □

Corollary 4.3. *The following are equivalent for a right non-singular ring R without an infinite set of orthogonal idempotents:*

a) *R is a right Utumi p.p.-ring.*
b) *Let A be a right essential two-sided R-submodule of Q^r. Every A-generated $\mathcal{S}$-closed submodule of A is a direct summand.*

Proof. $a) \Rightarrow b)$: Denote the endomorphism ring of A by S, and let Q^r be the maximal right ring of quotient of R. Let U be an $\mathcal{S}$-closed A-generated submodule of A, and consider the induced diagram

$$\begin{array}{ccccccc} T_A H_A(U) & \longrightarrow & T_A H_A(A) & \longrightarrow & X & \longrightarrow & 0 \\ \downarrow \theta_U & & \wr\downarrow \theta_A & & \downarrow \theta & & \\ U & \longrightarrow & A & \longrightarrow & A/U & \longrightarrow & 0 \end{array}$$

where X is a cyclic submodule of $H_A(A/U)$. As in the proof of Theorem 4.1, $H_A(Q^r)$ is a non-singular right S-module. Hence, $H_A(A/U)$ is non-singular. By [3], cyclic non-singular modules over right Utumi p.p.-rings without an infinite set of orthogonal idempotents are projective. Since θ is an isomorphism by the Snake-Lemma, U is a direct summand of A.

b) $\Rightarrow$ *a)*: Condition b) applies in particular to the case $A = R$, and yields that every $\mathcal{S}$-closed right ideal of R is a direct summand of R. But a ring with this property clearly is a right Utumi p.p.-ring. □

Corollary 4.4. *Let R be a right strongly non-singular p.p.-ring without an infinite set of orthogonal idempotents. The following are equivalent:*

a) *R is right hereditary.*
b) *Let A be a right essential two-sided R-submodule of Q^r. Every A-generated right R-module M of finite right Goldie dimension such that $\rho_A(M) = 0$ is A-projective.*

Proof. *a)* $\Rightarrow$ *b)*: Let M be an A-generated right R-module with $\rho_A(M) = 0$ which has finite Goldie-dimension. There exist an index-set I and a monomorphism $M \to A^I$. Suppose that I cannot be chosen to be finite, and set $U_0 = A$. Assume that we have constructed a strictly descending chain $U_0 \supseteq \ldots \supseteq U_n$ of $\mathcal{S}$-closed submodules $U_0, \ldots, U_n$ of M such that M/U_n is isomorphic to a submodule of A^n. Since $U_n \neq 0$, select $0 \neq u \in U_n$, for which we can find a map $\alpha_n : M \to A$ with $\alpha_n(u) \neq 0$. Setting $U_{n+1} = U_n \cap \ker\ \alpha_n$, one obtains a monomorphism $M/U_{n+1} \to A^{n+1}$. Furthermore, $0 \neq U_n/U_{n+1} \cong [U_n + \ker\ \alpha_n]/\ker\ \alpha_n \subseteq A$ is non-singular. Since Goldie-dimension is additive over $\mathcal{S}$-closed submodules, $dim_R M \geq n$ for all $n < \omega$, a contradiction. Therefore, $M \subseteq A^n$ for some n.

Consider an epimorphism $\pi : \oplus_I A \to M$ for some $m < \omega$. As in the proof of Theorem 4.1, A is flat as a right $S = End_R(A) = Fix(A)$-module, and the map θ_M is an isomorphism. Arguing similar to the proof of *a)* $\Rightarrow$ *b)* in Theorem 4.1, one obtains that M is A-projective since $H_A(M) \subseteq H_A(A^n)$ yields that $H_A(M)$ is a projective S-module because S is right hereditary by Corollary 3.5. □

The modules A under consideration behave very much like submodules of $Q(D)$ for an integral domain D:

Theorem 4.5. *Let R be a right strongly non-singular, right semi-hereditary ring without an infinite family of orthogonal idempotents. If A is a right essential two-sided R-submodule of Q^r, then the following hold:*

a) *θ_M is an isomorphism for each non-singular A-generated R-module.*
b) *$\mathcal{S}$-closed submodules of A-generated modules are A-generated.*
c) *If M is a finitely presented non-zero right $S = Fix(A)$-module, then $M \otimes_S A \neq 0$.*
d) *If M is a non-singular non-zero right $S = Fix(A)$-module, then $M \otimes_S A \neq 0$.*

Proof. a) As in the proof of Theorem 4.1, one obtains that A is flat as an S-module, and that θ_Q is an isomorphism. In the same way, submodules of a module M with θ_M an isomorphism have this property too. Since Q is a semi-simple Artinian ring, the injective hull of a non-singular module M is a direct summand of a module of the form $\oplus_I Q$ for some index-set I. Since A has finite Goldie-dimension, it follows that $\theta_{\oplus_I Q}$ is an isomorphism.

b) Consider an exact sequence $0 \to B \to C \xrightarrow{\pi} M \to 0$ in which C is A-generated and M is non-singular. By a), θ_M is an isomorphism. With $X = im\ H_A(\pi) \subseteq H_A(M)$, we obtain the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T_AH_A(B) & \longrightarrow & T_AH_A(C) & \xrightarrow{T_AH_A(\pi)} & T_A(X) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \theta_B} & & \downarrow{\scriptstyle \theta_C} & & \downarrow{\scriptstyle \theta} & & \\ 0 & \longrightarrow & B & \longrightarrow & C & \xrightarrow{\pi} & M & \longrightarrow & 0 \end{array}$$

in which θ_C is onto. By the Snake Lemma, the map θ_B will be onto provided that θ is an isomorphism. To see this, observe that θ satisfies $\theta_M T_A(\iota) = \theta$ where $\iota : X \to H_A(M)$ is the inclusion map. Since A is flat, $T_A(\iota)$ is one-to-one, and the same holds for θ.

c) Suppose that M is a non-zero finitely presented right S-module such that $M \otimes_S A = 0$, and consider a projective resolution $0 \to U \to F \to M \to 0$ in which F is a finitely generated free module. Then, U is finitely generated, and hence projective since R is right semi-hereditary. Since A is flat as a left S-module, we obtain the exact sequence $0 \to T_A(U) \to T_A(P) \to T_A(M) = 0$ which yields the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_AT_A(U) & \longrightarrow & H_AT_A(F) & \longrightarrow & 0 & & \\ & & \wr\downarrow{\scriptstyle \theta_U} & & \wr\downarrow{\scriptstyle \theta_F} & & & & \\ 0 & \longrightarrow & U & \longrightarrow & F & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

from which $M = 0$ follows immediately.

d) Let M be a non-singular right S-module with $T_A(M) = 0$. Since A is S-flat, $T_A(U) = 0$ for all finitely generated submodules U of M. However, every finitely generated non-singular module is projective and hence finitely presented. By c), $U = 0$, and the same holds for M. □

Corollary 4.6. *Let R be a right strongly non-singular semi-hereditary ring without an infinite family of orthogonal idempotents such that every maximal right ideal of R is principal. If A is a right essential two-sided R-submodule of Q^r, then $M \otimes_S A \neq 0$ for all non-zero right $S = End_R(A_R)$-modules.*

PROOF. Since A is flat as a right S-module, it suffices to show that $IA = A$ yields $I = S$ for all right ideal I of S. Suppose that I is a proper right ideal of S with $IA = A$. Arguing as in the proof of Proposition 3.4, one obtains $I = JS$ where $J = I \cap R$ is a proper right ideal of R. One has $A = IA = JSA = JA$. Choose a proper maximal right ideal J_0 of R containing J. Then, $J_0A = A$, and there exists $c \in J_0$ with $J_0 = cR$. Therefore, $A = c(A)$ where we identify c with the endomorphism of A induced by left multiplication with c. Now, $dim_R(\ker\ c) + dim_R A = dim_R A < \infty$ since A is non-singular as a right and left R-module. Consequently, $\ker\ c = 0$, and c is a unit of R, which is not possible since J_0 is proper. □

Corollary 4.7. *Let R be a right strongly non-singular, right hereditary ring without an infinite family of orthogonal idempotents. If A is an essential two-sided R-submodule of Q^r, then $M \otimes_S A \neq 0$ for all non-zero right $S = End_R(A_R)$-modules.*

PROOF. Let M be a right S-module with $M \otimes_S A = 0$. The ring S is right hereditary by Corollary 3.5. Consider a projective resolution $0 \to U \to F \to M \to 0$ in which F is free, and U is projective. Now argue as in the proof of Theorem 4.5c. □

We now turn to examples of rings which satisfy Theorem 4.1. A ring R without zero-divisors is *a right chain domain* if, for all right ideals I and J of R, we have $I \subseteq J$ or $J \subseteq I$.

Example 4.8. *A right and left chain domain R is right strongly non-singular and right semi-hereditary since it has right and left Goldie-dimension* 1 *and every finitely generated right ideal is isomorphic to R_R. Every two-sided ideal of R, e.g. $J(R)$, is a right essential two-sided submodule of Q^r. Such rings have been constructed by Neumann in* [15] *and as localizations of groups algebras over right ordered groups in* [4].

Another class of rings, to which Theorem 4.1 can be applied, arises from the discussion of right and left Noetherian hereditary rings R. By [9, Theorem 5.4], such a ring R is the product of prime rings and right Artinian rings. Furthermore, every right Artinian ring in this product is also left Artinian.

If R is a right and left Artinian, hereditary ring, then R is right strongly non-singular if and only if it is left strongly non-singular [3]. But then, R is a right and left Utumi-ring, and all non-singular right R-modules are projective by [12, Theorem 5.23]. Because of [12, Theorem 5.28], R is Morita equivalent to a finite product of lower triangular matrix rings over division rings. On the other hand, a prime right and left Noetherian ring has a semi-simple Artinian right and left classical ring of quotients, and hence is strongly non-singular. We thus obtain:

Theorem 4.9. *The following condition are equivalent for a right and left Noetherian ring R:*

a) *Let A be a right essential two-sided R-submodule of Q^r. For all $n < \omega$, an $\mathcal{S}$-closed submodule of A^n is a direct summand.*
b) *R is a product of prime hereditary rings and rings Morita-equivalent to lower triangular matrix rings over division algebras.*

□

Examples of right and left Noetherian hereditary primes rings include maximal S-orders in a finite-dimensional $\mathbb{Q}$-algebra K where S is a subring of $Center(K)$ with $\mathbb{Q} = Center(K)$ (e.g., see [8, Chapter 11]).

Turning to the commutative setting, observe that every commutative strongly non-singular semi-hereditary ring without an infinite family of orthogonal idempotents is the finite product of Prüfer domains by [3]. In addition, every ring R which is Morita-equivalent to a Prüfer domain is a right and left strongly non-singular, semi-hereditary ring of finite Goldie-dimension. Hence, we shall restrict our discussion to domains in the following.

A submodule U of an R-module M is *an rd-submodule of M* if $rM \cap U = rU$ for all $r \in R$. If M is torsion-free, rd-submodules of M are $\mathcal{S}$-closed and vice-versa.

Corollary 4.10. *The following are equivalent for an integral domain R:*

a) *R is Prüfer.*
b) *Let A be a submodule of Q. Every rd-submodule of A^n for some $n < \omega$ is a direct summand.*
c) *Let A be a submodule of Q. Every finitely A-generated torsion-free R-module is A-projective.*
d) *Let A be a submodule of Q. Every finitely A-generated R-module M of finite rank such that $\rho_A(M) = 0$ is A-projective.*

□

In particular, an integral domain R is Dedekind if and only if, for every submodule A of Q, every A-generated right R-module M of finite rank such that $\rho_A(M) = 0$ is A-projective.

Corollary 4.11. *Let R be a Prüfer domain, and A a rank 1 R-module.*

a) *θ_M is an isomorphism for each torsion-free A-generated R-module.*

b) *Rd-submodules of A-generated modules are A-generated.*
c) *If M is a right $S = Fix(A)$-module with $M \otimes_S A = 0$, then M is torsion.*

□

Corollary 4.12. *Let R be a Prüfer domain such that R/rR is Artinian for each non-zero $r \in R$. Then, every rank 1 torsion-free R-module is faithful.*

Proof. Since R/rR is Artinian for each non-zero r, the ring R satisfies the restricted minimum condition, and the same holds for $S = End_R(A)$ by Corollary 3.4. Let I be an ideal of S with $IA = A$. Select a non-zero $s \in I$, and consider the descending chain $\ldots (I/sS)^n \supseteq (I/sS)^{n+1} \ldots$ of ideals of the Artinian ring R/sS. There is $m < \omega$ with $(I/sS)^m = (I/sS)^{m+1}$. Since each Artinian ring is Noetherian, $(I/sS)^m$ is finitely generated. By [8, Lemma 5.8], there is $y \in I$ such that $(1 + y + sS)(I/sS)^m = 0$. Therefore, $(1 + y)I^m \subseteq sS$ and $(1 + y)(A) = (1 + y)I^m A \subseteq sA$. For each $a \in A$, choose $a' \in A$ with $(1 + y)(a) = sa'$. Define $\phi \in S$ by $\phi(a) = a'$. Then, $1 + y = s\phi \in sS \subseteq I$ yields $1 \in I$. □

Corollary 4.13. *Let R be a Prüfer domain such that every maximal ideal is principal. If A is a non-zero submodule of Q, then $M \otimes_S A \neq 0$ for all non-zero right $S = End_R(A_R)$-modules.*

□

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Department of Mathematics and Statistics, Auburn University, Auburn, AL 36849, U.S.A.