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A new discretized Lyapunov–Krasovskii functional for stability analysis and control design of time-delayed TS fuzzy systems

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SUMMARY

This paper proposes a new Lyapunov–Krasovskii functional to cope with stability analysis and control design for time-delay nonlinear systems modeled in the Takagi–Sugeno (TS) fuzzy form. The delay-dependent conditions are formulated as linear matrix inequalities (LMIs), solvable through several numerical tools. By using the Gu's discretization technique and by employing an appropriated fuzzy functional, less conservative conditions are obtained. Numerical results illustrate the efficiency of the proposed methods. Copyright © 2010 John Wiley & Sons, Ltd.

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1. INTRODUCTION

Nonlinear dynamics are almost ubiquitous in physical and engineering applications. Takagi–Sugeno (TS) fuzzy modeling [1] is an established approach to cope with nonlinearity, consisting in a set of linear models, locally valid, combined by means of membership functions. Exact models can be efficiently obtained by the sector nonlinearity approach [2]. A very powerful strategy to fuzzy controllers is the parallel distributed compensation (PDC) [2, 3], where the state-feedback gains are combined through the same membership functions used in the modeling stage.

Another challenging phenomenon regarding engineering and communications applications is the time-delay. It is the source of instability or performance degradation [4, 5]. However, in a dichotomic fashion, time-delays can guarantee stability or enhance performance, see for instance [6, 7]. Stability criteria for TS time-delay systems are systematically formulated as linear matrix inequalities (LMIs), solvable through linear optimization algorithms. Several results concerning stability analysis, control design and estimation for TS time-delay systems are available [8–20].

For TS systems the prevalent approach is the use of a common Lyapunov functional guaranteeing stability for the entire fuzzy system. This was the earliest approach [21] and draws attention even today [22]. However, due to its inherit conservativeness different types of functionals must be considered. In this scenario the fuzzy Lyapunov functional is a promising candidate [23],

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consisting on a fuzzy combination of Lyapunov functionals that are locally valid and parameterized by the same membership functions adopted in the modeling stage. The time-derivative of the fuzzy Lyapunov function proposed by Tanaka *et al.* [23] contains information on the membership function time variation. Although this information can be used to reduce conservativeness [24], sometimes it is not readily available. In such case, the fuzzy Lyapunov functional proposed in [25] is more suitable. The main drawback found in [25] is that LMI conditions are available only for stability analysis. The work [26] addresses this issue providing LMI-based control design. Another issue concerning [25] is that the states must be chosen as premise variables.

When stability analysis of delayed systems is concerned a very effective strategy is to apply the Gu's Lyapunov–Krasovskii functional discretization technique [4], developed for *linear* systems with *constant* delay. It has been successfully extended for stability analysis of some nonlinear systems classes, specifically the problem of stability of neural networks [27, 28] and synchronization of Lur'e systems [6, 7]. However, control design based on this strategy is not an easy task, since several products between decision variables will be generated, leading to nonconvex formulations [29]. This issue has been addressed in [7, 30] and LMI-based conditions for control design are provided. A relaxation strategy was applied to avoid nonconvexity, by decoupling decision variables related to the Lyapunov–Krasovskii functional from the control gain matrices.

A new fuzzy Lyapunov–Krasovskii functional (FLKF) is proposed in this paper leading to LMI conditions for stability analysis and control design of TS time-delay systems, where its main characteristics are: (i) the time-derivative of this functional is independent of the knowledge of the time variation of membership function, as in [25], and (ii) the Gu's discretization technique is promptly applied, as in [4]. These are the points in common with [4, 25]. Moreover, the proposed functional is suitable for systems subject to uncertain time-delay and extra strategies that allow to obtain LMI-based control design are applied.

Notation: In this paper T represents the transpose of vector and matrices; $M>0$ (<0) means that M is positive (negative) definite; $\mathrm{sym}\{M\}$ stands for $M+M^{\mathrm{T}}$, whereas $\mathrm{diag}\{\cdot\}$ stands for a diagonal matrix; $*$ is used instead of writing terms in a symmetric matrix; denote the following subsets $\{1,2,\ldots,r\}\subset\mathbb{N}^*$, $\{0,1,2,\ldots,N\}\subset\mathbb{N}$ and $\{1,2,\ldots,N\}\subset\mathbb{N}^*$ as $\mathscr{R}$, $\mathscr{N}$ and $\mathscr{N}^*$, respectively.

2. TS FUZZY MODEL AND PROBLEM STATEMENT

Consider a nonlinear time-delayed system that is described by a TS time-delay system with r rules and with the states being premise variables [25]:
System Rule i: IF $x_1(t)$ is $\Theta_1^{\alpha_{i1}}$ and $x_2(t)$ is $\Theta_2^{\alpha_{i2}}$ and ... and $x_n(t)$ is $\Theta_n^{\alpha_{in}}$ THEN

$$\begin{aligned} \dot{x}(t) &= A_i x(t) + A_{di} x(t-\mathrm{d}(t)) + B_i u(t) \\ x(t) &= \phi(t) \quad \forall t\in[-\max(\mathrm{d}(t)),0], \end{aligned} \tag{1}$$

where $x(t)\in\mathbb{R}^{n_x}$, $u(t)\in\mathbb{R}^{n_u}$, $\phi(t)\in\mathbb{R}^{n_x}$, are the state vector, the control input and the initial condition vector, respectively; A_i, A_{di}, and B_i are constant real matrices with proper dimensions; $\Theta_j^{\alpha_{ij}}$ $(i\in\mathscr{R})$ are the fuzzy sets in the ith rule based on the premises $x_j(t)$; the time-delay has the form $\mathrm{d}(t)=\tau+\eta(t)$, where $\eta(t)$ is a time-varying perturbation satisfying $|\eta(t)|\leqslant\upsilon<\tau$ with known upper-bound υ.

Using a standard fuzzy inference method, the global model inferred by (1) is expressed as:

$$\dot{x}(t) = A(t)x(t)+A_d(t)x(t-\mathrm{d}(t))+B(t)u(t), \tag{2}$$

where $Q(t)\triangleq\sum_{i=1}^r \hbar_i(x(t))Q_i$ and $\hbar_i(x(t))$ is the normalized grade of membership, given as

$$\hbar_i(x(t))\triangleq\frac{\omega_i(x(t))}{\sum_{i=1}^r\omega_i(x(t))},\quad \omega_i(x(t))\triangleq\prod_{j=1}^n\mu_{ij}(x_j(t)), \tag{3}$$

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satisfying $\hbar_i(x(t)) \in [0, 1]$, $\sum_{i=1}^{r} \hbar_i(x(t)) = 1$; $\mu_{ij}(x_j(t))$ is the grade of membership of $x_j(t)$ in the fuzzy set $\Theta_j^{\alpha_{ij}}$.

A fuzzy controller for this system can be obtained with the following PDC scheme.
Controller Rule i: IF $x_1(t)$ is $\Theta_1^{\alpha_{i1}}$ and $x_2(t)$ is $\Theta_2^{\alpha_{i2}}$ and ... and $x_n(t)$ is $\Theta_n^{\alpha_{in}}$ THEN

$$u(t) = K_i x(t) + K_{di} x(t-\tau). \tag{4}$$

The state-feedback control law is inferred using the same procedure of the modeling stage

$$u(t) = \sum_{i=1}^{r} \hbar_i(x(t))[K_i x(t) + K_{di} x(t-\tau)]. \tag{5}$$

Notice that the proposed control law takes into account τ, the nominal time-delay value. Therefore, the proposed methodology can be applied to deal with time-delay uncertainty. Moreover, particular cases of the proposed control law are the memoryless control, $K_{di} = 0$, and purely delayed control $K_i = 0$.

The purpose is to determine local state-feedback gains K_i and K_{di} such that the closed-loop system:

$$\begin{aligned}\dot{x}(t) &= [A(t)+B(t)K(t)]x(t)+[A_d(t)+B(t)K_d(t)]x(t-\tau)-A_d(t)\int_{\mathrm{d}(t)}^{\tau}\dot{x}(t-\xi)\,\mathrm{d}\xi \\ &\triangleq \sum_{i=1}^{r}\sum_{j=1}^{r}\hbar_i(t)\hbar_j(t)[(A_i+B_iK_j)x(t)+(A_{di}+B_iK_{dj})x(t-\tau)]-A_{di}\int_{\mathrm{d}(t)}^{\tau}\dot{x}(t-\xi)\,\mathrm{d}\xi\end{aligned} \tag{6}$$

is asymptotically stable. The above closed-loop system was obtained using the identity: $x(t-\mathrm{d}(t)) = x(t-\tau) - \int_{\mathrm{d}(t)}^{\tau} \dot{x}(t-\xi)\,\mathrm{d}\xi$.

3. FUZZY LYAPUNOV–KRASOVSKII FUNCTIONAL

In the sense of obtaining less conservative conditions, the following FLKF is proposed:

$$V(x(t)) = V_L + V_K + V_\eta. \tag{7}$$

The first term of (7) is given as in [25]

$$V_L = 2\int_{\Gamma(0,x)} f(\psi)\psi \cdot \mathrm{d}\psi, \tag{8}$$

where $\Gamma(0, x)$ is a path from the $\mathbb{R}^{n_x}$ origin to the present state; $(\cdot)$ stands for the inner product of vectors; ψ is a vector for the integral and $\mathrm{d}\psi$ is an infinitesimal displacement; the matrix $f(\psi)$ is responsible for the fuzzy characteristic of the functional, since is obtained with the same procedure that gives the TS model:
Matrix Rule i: IF $x_1(t)$ is $\Theta_1^{\alpha_{i1}}$ and $x_2(t)$ is $\Theta_2^{\alpha_{i2}}$ and ... and $x_n(t)$ is $\Theta_n^{\alpha_{in}}$ THEN

$$f(x(t)) = P_i. \tag{9}$$

Therefore, the global fuzzy vector is given by:

$$f(x(t)) = \sum_{i=1}^{r} \hbar_i(x(t)) P_i. \tag{10}$$

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Remark 1

It is worth mentioning that matrices P_i must have a special structure:

$$P_i \triangleq D_0 + D_i, \tag{11}$$

where

$$D_0 = \begin{bmatrix} 0 & d_{12} & \dots & d_{1n} \\ d_{12} & 0 & \dots & d_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ d_{1n} & d_{2n} & \dots & 0 \end{bmatrix}, \quad D_i = \begin{bmatrix} d_{11}^{\alpha_{i1}} & 0 & \dots & 0 \\ 0 & d_{22}^{\alpha_{i2}} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_{nn}^{\alpha_{in}} \end{bmatrix}. \tag{12}$$

This notation indicates that the off-diagonal elements of P_i are the same, $\forall i \in \mathscr{R}$. The diagonal elements $d_{jj}^{\alpha_{ij}}$ of P_i are altered according to the fuzzy sets $\Theta_j^{\alpha_{ij}}$ in the IF-THEN rules. See [25, 26] for further details.

Using Lie derivative results in:

$$\dot{V}_L = \sum_{i=1}^{r} \hbar_i(x(t))\{x^{\mathrm{T}}(t)P_i\dot{x}(t) + \dot{x}^{\mathrm{T}}(t)P_i x(t)\} \tag{13}$$

$$= x^{\mathrm{T}}(t)P(t)\dot{x}(t) + \dot{x}^{\mathrm{T}}(t)P(t)x(t). \tag{14}$$

Notice that time-derivative of the Lyapunov functional V_L proposed in [25] is parameterized by the membership functions without depending on their time-derivative, as occurs in [23].

The second term in (7) is given by

$$V_K = 2x^{\mathrm{T}}(t)\int_{-\tau}^{0} Q(\xi)x(t+\xi)\,\mathrm{d}\xi + \int_{-\tau}^{0}\int_{-\tau}^{0} x^{\mathrm{T}}(t+s)R(s,\xi)\,\mathrm{d}s\,x(t+\xi)\,\mathrm{d}\xi$$
$$+ \int_{-\tau}^{0} x^{\mathrm{T}}(t+\xi)S(\xi)x(t+\xi)\,\mathrm{d}\xi, \tag{15}$$

where $Q(\xi)$, $R(s,\xi)$, and $S(\xi)$ are matrix functions of order $n_x \times n_x$.

These matrix functions hamper to obtain LMI-based stability analysis conditions, leading to complex results. This task is considerably softened if the time-delay interval $[-\tau, 0]$ is splitted into N segments $[\varphi_n, \varphi_{n-1}]$ of equal length $h = \tau/N$, where $\varphi_n = -nh$, as proposed in [4]. The function matrices are chosen piecewise linear, which can be expressed in terms of their values at the diving points using interpolation formula, i.e.

$$Q(\theta_n + \alpha h) = (1-\alpha)Q_n + \alpha Q_{n-1}, \quad S(\theta_n + \alpha h) = (1-\alpha)S_n + \alpha S_{n-1}$$

$$R(\theta_n + \alpha h, \theta_m + \beta h) = \begin{cases} (1-\alpha)R_{n,m} + \beta R_{n-1,m-1} + (\alpha-\beta)R_{n-1,m} & \text{for } \alpha \geqslant \beta \\ (1-\beta)R_{n,m} + \alpha R_{n-1,m-1} + (\beta-\alpha)R_{n,m-1} & \text{for } \alpha < \beta \end{cases}$$

for $0 \leqslant \alpha, \beta \leqslant 1$ and $n, m \in \mathscr{N}$.

Therefore, with the arguments of this section it follows that the terms V_L and V_K of the functional (7) are completely determined by the constant matrices P_i, Q_n, S_m, and $R_{n,m}$. Define matrices $\mathscr{M}_i$ that assemble the functional matrices as

$$\mathscr{M}_i = [P_i, Q_n, S_n, R_{n,m}] \quad \forall i \in \mathscr{R}, \; n, \; m \in \mathscr{N}. \tag{16}$$

The last term V_η is chosen as in [31]: $V_\eta = \int_{-\nu}^{\nu}\int_{t+\xi-\tau}^{t} \dot{x}^{\mathrm{T}}(s)U\dot{x}(s)\,\mathrm{d}s\,\mathrm{d}\xi$, with $U = U^{\mathrm{T}}$.

Finally, the TS time-delay system (1) is asymptotically stable if the following conditions are satisfied: $V(x(t)) \geqslant \varepsilon\|x(t)\|^2$, and $\dot{V}(x(t)) \leqslant -\varepsilon\|x(t)\|^2$, for sufficiently small $\varepsilon > 0$.

4. STABILITY CONDITION

Sufficient condition is proposed in this section guaranteeing stability of the autonomous TS systems subjected to uncertain time-delay, i.e. $u(t)=0$ and $\mathrm{d}(t)=\tau+\eta(t)$ in (2). The following is an LMI-based stability condition derived from the new FLKF in (7).

Theorem 1

Consider the TS fuzzy system in (1) with $u(t)=0$ and $\mathrm{d}(t)\in[\tau-v,\tau+v]$. For given scalars τ and v satisfying $0\leqslant v<\tau$ the autonomous TS system is asymptotically stable if there exist any $n_x\times n_x$ matrices F, G, Q_n, symmetric $n_x\times n_x$ matrices U, S_n, and $R_{n,m}$ ($\forall n,m\in\mathcal{N}$), and special $n_x\times n_x$ matrices P_i, defined as in (11) $\forall i\in\mathcal{R}$, such that the following LMIs are satisfied:

$$\Psi_i(\mathcal{M})=\begin{bmatrix} P_i & Q^s \\ * & R^s+S^s \end{bmatrix}>0, \tag{17}$$

$$\Omega_i(A,A_d,F,G,U,\mathcal{M})=\begin{bmatrix} \begin{pmatrix} \Xi_i & v\Gamma_i^{\mathrm{T}} \\ * & -vU \end{pmatrix} & \begin{pmatrix} D^a \\ 0 \end{pmatrix} & \begin{pmatrix} D^b \\ 0 \end{pmatrix} \\ * & -R_d-S_d & 0 \\ * & * & -3S_d \end{bmatrix}<0, \quad i\in\mathcal{R} \tag{18}$$

where $\Gamma_i=[A_{di}F \;\; A_{di}G \;\; 0]$, $Q^s=[Q_0 \;\; Q_1 \;\; \dots \;\; Q_N]$, $R^s=[R_{n,m}]$, with $R_{n,m}$ at positions (n,m) of R^s ($\forall n,m\in\mathcal{N}$),

$$S^s=\mathrm{diag}\left\{\frac{1}{h}S_0 \;\; \frac{1}{h}S_1 \;\; \dots \;\; \frac{1}{h}S_N\right\}, \tag{19}$$

$$D^a=[D_1^a \;\; D_2^a \;\; \cdots \;\; D_n^a], \quad D^b=[D_1^b \;\; D_2^b \;\; \cdots \;\; D_n^b] \tag{20}$$

$$D_n^a=\begin{bmatrix} \frac{h}{2}[R_{0,n-1}+R_{0,n}]-[Q_{n-1}-Q_n] \\ \frac{h}{2}[Q_{n-1}+Q_n] \\ -\frac{h}{2}[R_{n-1,N}^{\mathrm{T}}+R_{n,N}^{\mathrm{T}}] \end{bmatrix}, \quad D_n^b=\begin{bmatrix} -\frac{h}{2}[R_{0,n-1}-R_{0,n}] \\ -\frac{h}{2}[Q_{n-1}-Q_n] \\ \frac{h}{2}[R_{n-1,N}^{\mathrm{T}}-R_{n,N}^{\mathrm{T}}] \end{bmatrix}, \tag{21}$$

$$S_d=\mathrm{diag}\{S_0-S_1,S_1-S_2,\dots,S_{N-1}-S_N\},$$

$$R_d=[R_{dn,m}], \quad n,m\in\mathcal{N}^*, \tag{22}$$

with $R_{dn,m}=h[R_{n-1,m-1}-R_{n,m}]$, and

$$\Xi_i=\begin{bmatrix} \mathrm{sym}\{F^{\mathrm{T}}A_i+Q_0\}+S_0 & P_i-F^{\mathrm{T}}+A_i^{\mathrm{T}}G & F^{\mathrm{T}}A_{di}-Q_N \\ * & \mathrm{sym}\{-G\}+2vU & G^{\mathrm{T}}A_{di} \\ * & * & -S_N \end{bmatrix}, \tag{23}$$

where $i\in\mathcal{R}$ and $h=\tau/N$.

Proof

First, the positiveness of the proposed functional $V_L+V_K+V_\eta$ in (7) is shown. It is shown that $V_L+V_K+V_\eta>0$, since the LMIs in (17) and (18) assure that $V_L+V_K>0$ and $V_\eta>0$. Consider the term V_L+V_K in (7). According to [25, Appendix B] if $P_i>0$, $i\in\mathcal{R}$, $\exists \underline{P}=\underline{P}^{\mathrm{T}}>0$ such that $V_L\geqslant x^{\mathrm{T}}(t)\underline{P}x(t)\triangleq V_P$, which implies that $V_L+V_K\geqslant V_P+V_K$. Besides note that, V_P+V_K has the same form of the complete Lyapunov–Krasovskii functional in [4].

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As demonstrated in [4, p. 185], $V_P + V_K > 0$ if $S_0 > S_1 > \cdots > S_N > 0$ and

$$\begin{bmatrix} \underline{P} & Q^s \\ * & R^s + S^s \end{bmatrix} > 0. \tag{24}$$

where R^s, Q^s, and S^s are defined as in (17).

Furthermore, if the LMI in (17) holds for $i \in \mathscr{R}$, it guarantees that (24) holds. On the other hand, satisfying (18) results in $S_N > 0$, implying that $S_0 > S_1 > \cdots > S_N > 0$ (see details in [4, Prop. 5.22]). Therefore, if LMIs (17) and (18) hold, then $V_L + V_K > 0$.

Finally, consider the term V_η. Note that, since $v > 0$, if (18) is satisfied then $U > 0$, which implies that $V_\eta > 0$. Therefore if LMIs (17) and (18) hold then $V(x(t)) \geqslant \varepsilon \| x(t) \|^2 \, (\varepsilon > 0)$.

Moreover, taking the time-derivative of the FLKF in (7) results in:

$$\begin{aligned} \dot{V}(x(t)) = {} & 2\dot{x}^{\mathrm{T}}(t)\left[P(t)x(t) + \int_{-\tau}^{0} Q(\xi)x(t+\xi)\,\mathrm{d}\xi\right] + 2x^{\mathrm{T}}(t)\int_{-\tau}^{0} Q(\xi)\dot{x}(t+\xi)\,\mathrm{d}\xi \\ & + 2\int_{-\tau}^{0}\int_{-\tau}^{0} \dot{x}^{\mathrm{T}}(t+s)R(s,\xi)\,\mathrm{d}s\,x(t+\xi)\,\mathrm{d}\xi + 2\int_{-\tau}^{0} x^{\mathrm{T}}(t+\xi)S(\xi)\dot{x}(t+\xi)\,\mathrm{d}\xi \\ & + 2\dot{x}^{\mathrm{T}}(t)vU\dot{x}(t) - \int_{t-\tau-v}^{t-\tau+v} \dot{x}^{\mathrm{T}}(s)U\dot{x}(s)\,\mathrm{d}s. \end{aligned} \tag{25}$$

Now, consider the following identity obtained from the autonomous TS fuzzy system in (2):

$$0 = 2[Fx(t) + G\dot{x}(t)]^{\mathrm{T}}\{-\dot{x}(t) + A(t)x(t) + A_d(t)x(t-\tau)\} + \Delta, \tag{26}$$

where,

$$\Delta = -2[Fx(t) + G\dot{x}(t)]^{\mathrm{T}} A_d(t) \int_{t-\tau-\eta(t)}^{t-\tau} \dot{x}(s)\,\mathrm{d}s.$$

Consider the upper-bound to Δ as in [31]:

$$\begin{aligned} \Delta & \leqslant \left| \int_{t-\tau-\eta(t)}^{t-\tau} [Fx(t) + G\dot{x}(t)]^{\mathrm{T}} A_d^{\mathrm{T}}(t) U^{-1} A_d(t) [Fx(t) + G\dot{x}(t)]\,\mathrm{d}s \right| + \left| \int_{t-\tau-\eta(t)}^{t-\tau} \dot{x}^{\mathrm{T}}(s)U\dot{x}(s)\,\mathrm{d}s \right| \\ & \leqslant v[Fx(t) + G\dot{x}(t)]^{\mathrm{T}} A_d^{\mathrm{T}}(t) U^{-1} A_d(t) [Fx(t) + G\dot{x}(t)] + \int_{t-\tau-v}^{t-\tau+v} \dot{x}^{\mathrm{T}}(s)U\dot{x}(s)\,\mathrm{d}s. \end{aligned} \tag{27}$$

Adding (26) in (25), considering the upper bound (27) and in the sequel following the same steps as in [4, p. 188], it yields:

$$\begin{aligned} \dot{V}(x(t)) \leqslant {} & \zeta^{\mathrm{T}}\{\Xi(t) + v[F\ G\ 0]^{\mathrm{T}} A_d^{\mathrm{T}}(t) U^{-1} A_d(t)[F\ G\ 0]\}\zeta \\ & + \left[2\zeta^{\mathrm{T}} \int_0^1 (D^a + (1-2\alpha)D^b) \int_0^1 \phi^{\mathrm{T}}(\alpha) S_d\right] \phi(\alpha)\,\mathrm{d}\alpha - \int_0^1 \left[\int_0^1 \phi^{\mathrm{T}}(\alpha) R_d \phi(\beta)\,\mathrm{d}\alpha\right] \mathrm{d}\beta \\ = {} & \sum_{i=1}^{r} h_i(x(t)) \Bigg\{ (\zeta^{\mathrm{T}} \Xi_i \zeta + v[F\ G\ 0]^{\mathrm{T}} A_{di}^{\mathrm{T}} U^{-1} A_{di} [F\ G\ 0]) \\ & + \left[2\zeta^{\mathrm{T}} \int_0^1 (D^a + (1-2\alpha)D^b) - \int_0^1 \phi^{\mathrm{T}}(\alpha) S_d\right] \phi(\alpha)\,\mathrm{d}\alpha \\ & - \int_0^1 \left[\int_0^1 \phi^{\mathrm{T}}(\alpha) R_d \phi(\beta)\,\mathrm{d}\alpha\right] \mathrm{d}\beta \Bigg\}, \end{aligned} \tag{28}$$

where $\zeta^{\mathrm{T}}=[x^{\mathrm{T}}(t)\ \dot{x}^{\mathrm{T}}(t)\ x^{\mathrm{T}}(t-\tau)]$, Ξ_i is given in (23), D^a and D^b are given both in (20), S_d, R_d are defined in (21) and (22), respectively, and $\phi^{\mathrm{T}}(\alpha)=[x^{\mathrm{T}}(t-h+\alpha h)\ x^{\mathrm{T}}(t-2h+\alpha h)\ \dots\ x^{\mathrm{T}}(t-Nh+\alpha h)]$.

Now, applying [4, Proposition 5.21] in (28), one concludes if $\Omega(t,A,A_d,F,G,U,\mathscr{M})<0$, then $\dot{V}(x_t)\leqslant-\varepsilon_0\,\|x(t)\|^2-\varepsilon_1\,\|\dot{x}(t)\|^2$, where

$$\Omega(t,A,A_d,F,G,U,\mathscr{M})=\begin{bmatrix}\begin{pmatrix}\Xi(t) & v\Gamma_i^{\mathrm{T}}\\ * & -vU\end{pmatrix} & \begin{pmatrix}D^a\\ 0\end{pmatrix} & \begin{pmatrix}D^b\\ 0\end{pmatrix}\\ * & -R_d-S_d & 0\\ * & * & -3S_d\end{bmatrix}$$

$$=\sum_{i=1}^{r}\hbar_i(t)\Omega_i(A,A_d,F,G,U,\mathscr{M}), \tag{29}$$

and $\Omega_i(A,A_d,F,G,U,\mathscr{M})$ is defined in (18). Therefore, to guarantee that $\dot{V}(x_t)\leqslant-\varepsilon_0\,\|x(t)\|^2-\varepsilon_1\,\|\dot{x}(t)\|^2$, it is sufficient that the LMIs in (18) are satisfied, which completes the proof. □

5. STABILITY ANALYSIS EXAMPLES

In this section stability analysis examples are considered for TS time-delay systems with constant and time-varying delays.

5.1. Example 1

Consider a two rule TS fuzzy system with a constant time-delay

$$\dot{x}(t)=\sum_{i=1}^{2}\hbar_i(x(t))(A_ix(t)+A_{id}x(t-\tau)),$$

where

$$A_1=\begin{bmatrix}-2.1 & 0.1\\ -0.2 & -0.9\end{bmatrix},\quad A_2=\begin{bmatrix}-1.9 & 0\\ -0.2 & -1.1\end{bmatrix},\quad A_{1d}=\begin{bmatrix}-1.1 & 0.1\\ -0.8 & -0.9\end{bmatrix},\quad A_{2d}=\begin{bmatrix}-0.9 & 0\\ -1.1 & -1.2\end{bmatrix}.$$

The objective is to determine the maximum value of constant time-delay τ for which the system is stable. Table I compares works based on common quadratic functionals [8–15] with the fuzzy functional of Theorem 1. It is clear by inspecting Table I that Theorem 1 provides the largest time-delays. Notice that even when $N=1$ the proposed approach outperform the other methods, besides when N increases the result is improved, in the present example for $N>4$ no improvement is obtained.

5.2. Example 2

In this example the proposed approach is compared with the method in [20], based on a different kind of fuzzy functional. The method from [20] requires an upper bound for the time-derivative of the membership functions, whereas the result of Theorem 1 does not.

Table I. Maximum allowable time-delay τ.

									Theorem 1				
Method	[8]	[9]	[10]	[11]	[12]	[13]	[14]	$[15]_{m=2}$	$N=1$	$N=2$	$N=3$	$N=4$	$N=5$
τ	0.65	0.80	1.25	1.85	3.15	3.37	3.85	4.28	4.47	4.58	4.60	4.61	4.61

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Table II. Largest constant time-delay $\mathrm{d}(t)=\tau$ with Theorem 1.

N	1	2	3	4	5
τ	0.3985	0.4059	0.4059	0.4060	0.4060

Table III. Largest time-delay with Theorem 1 and $N=1$.

v	0.01	0.03	0.05	0.10	0.11
τ	0.3734	0.3266	0.2781	0.1719	0.1520

Table IV. Largest time-delay with Theorem 1 and $N=3$.

v	0.01	0.03	0.05	0.10	0.11
τ	0.3781	0.3281	0.2797	0.1719	0.1520

Consider the following time-delayed nonlinear system

$$\ddot{y}(t)=-6\dot{y}(t)\sin^2(y(t))-8y(t)+y(t-\mathrm{d}(t))-2\dot{y}(t-\mathrm{d}(t)),$$

which can be exactly expressed as the TS delayed system in (1), with:

$$A_1=\begin{bmatrix}0 & 1\\ -8 & -6\end{bmatrix},\quad A_2=\begin{bmatrix}0 & 1\\ -8 & 0\end{bmatrix},\quad A_{d1}=A_{d2}=\begin{bmatrix}0 & 0\\ 1 & -2\end{bmatrix},$$

$\hbar_1(x(t))=\sin^2(y(t))$ and $\hbar_2(x(t))=\cos^2(y(t))$. Assume also that $|\dot{h}_i(x(t))|\leqslant 2$ for $i=1,2$.

Considering a constant time-delay, this system has been analyzed in [20] using methods based on fuzzy functionals, namely Theorems 1 and 2, which lead to the largest time-delay $\tau=0.3195$ and $\tau=0.3222$, respectively.

For this example, Theorem 1 is computed for different values of N, resulting in Table II. For every value of N the proposed methodology is less conservative than [20]. Notice also that for $N\geqslant 2$ there is no significant increase in the time-delay.

Now, the time-delay is considered uncertain, i.e. $\mathrm{d}(t)=\tau+\eta(t)$ with $|\eta(t)|\leqslant v$. For some values of the uncertainty bounds v the largest time-delay has been computed with $N=1$ and $N=3$, as show in Tables III and IV. Even in the presence of uncertainty, for $v\leqslant 0.03$, the proposed approach guarantees stability for largest time-delays than the ones obtained by the method from [20], that considers only precisely know time-delays.

6. CONTROL DESIGN CONDITION

The stability result previously presented can be easily extended for control design. The LMI-based procedure for control is presented in the following.

Theorem 2

Consider the TS fuzzy system in (6) with $\mathrm{d}(t)\in[\tau-v,\tau+v]$. For given scalars τ and v satisfying $0\leqslant v<\tau$, and a tuning parameter $\delta\neq 0$, the TS system is stabilizable via the *Controller Rule*, with $K_j=\bar{K}_j\bar{F}^{-1}$ and $K_{dj}=\bar{K}_{dj}\bar{F}^{-1}$, if there exist any $n_x\times n_x$ matrices: $\bar{Q}_n$, $\bar{F}$, any $n_u\times n_x$ matrices: $\bar{K}_j$ and $\bar{K}_{dj}$ $(j\in\mathscr{R})$, symmetric $n_x\times n_x$ matrices $\bar{S}_n$, $\bar{R}_{n,m}$ $(n,m\in\mathscr{N})$, $\bar{U}$, and special $n_x\times n_x$ matrices $\bar{P}_i$ $(i\in\mathscr{R})$, as defined in (11), such that the following LMIs are satisfied:

$$\Psi_i(\bar{\mathscr{M}})=\begin{bmatrix}\bar{P}_i & \hat{Q}\\ * & \hat{R}+\hat{S}\end{bmatrix}>0,\quad i\in\mathscr{R} \tag{30}$$

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$$\Omega_{ij}(A, A_d, \bar{F}, \bar{G}, \bar{U}, \mathscr{M}) = \begin{cases} \bar{\Omega}_{i,i} < 0, & i \in \mathscr{R} \\ \bar{\Omega}_{i,j} + \bar{\Omega}_{j,i} < 0, & i < j, \ i, j \in \mathscr{R} \end{cases} \tag{31}$$

with

$$\bar{\Omega}_{i,j} = \begin{bmatrix} \begin{pmatrix} \Xi_{ij} & v\bar{\Gamma}_i^{\mathrm{T}} \\ * & -v\bar{U} \end{pmatrix} & \begin{pmatrix} \bar{D}^a \\ 0 \end{pmatrix} & \begin{pmatrix} \bar{D}^b \\ 0 \end{pmatrix} \\ * & -\bar{R}_d - \bar{S}_d & 0 \\ * & * & -3\bar{S}_d \end{bmatrix} < 0, \tag{32}$$

where $\bar{\Gamma}_i = [A_{di}\bar{F} \ \ \delta A_{di}\bar{F} \ \ 0]$ and

$$\bar{\Xi}_{i,j} = \begin{bmatrix} \mathrm{sym}\{A_i\bar{F} + B_i\bar{K}_j + \bar{Q}_0\} + \bar{S}_0 & \bar{P}_i - \bar{F} + \delta(\bar{F}^{\mathrm{T}}A_i^{\mathrm{T}} + \bar{K}_j^{\mathrm{T}}B_i^{\mathrm{T}}) & A_{di}\bar{F} + B_i\bar{K}_{dj} - \bar{Q}_N \\ * & -\delta\mathrm{sym}\{\bar{F}\} + 2v\bar{U} & \delta(A_{di}\bar{F} + B_i\bar{K}_{dj}) \\ * & * & -\bar{S}_N \end{bmatrix}. \tag{33}$$

The terms $\hat{Q}$, $\hat{S}$, $\hat{R}$ are given, respectively, as in $\tilde{Q}$, $\tilde{S}$, $\tilde{R}$ in (17). $\bar{D}^a$, $\bar{D}^b$, $\bar{S}_d$, and $\bar{R}_d$ are given, respectively, as D^s, D^a, S_d, R_d in (18), where all elements are written with a superscript bar.

Proof
Initially, note that the block $(2,2)$ in (33) is $-\delta(\bar{F}+\bar{F}^{\mathrm{T}})+2v\bar{U}$. Since $v>0$, $\delta\neq 0$ and $\bar{U}>0$ (due to block $(4,4)$ in (32)), $-\delta(\bar{F}+\bar{F}^{\mathrm{T}})$ must to be negative definite, which implies that $\bar{F}$ is nonsingular. Then, the new variables are defined:

$$\bar{F} \triangleq F^{-1}; \quad [\bar{P}_i \ \ \bar{Q}_n \ \ \bar{S}_n \ \ \bar{R}_{n,m} \ \ \bar{U} \ \ \bar{\Theta}_i] \triangleq \bar{F}^{\mathrm{T}}[P_i \ \ Q_n \ \ S_n \ \ R_{n,m} \ \ U \ \ \Theta_i]\bar{F}$$

for $i \in \mathscr{R}$ and $n, m \in \mathscr{N}$.

Now the remaining of this proof follows directly from Theorem 1. The LMI in (30) is obtained pre- and post-multiplying LMI in (17) by $\mathrm{diag}\{\bar{F}, \ldots, \bar{F}\}^{\mathrm{T}}$ and $\mathrm{diag}\{\bar{F}, \ldots, \bar{F}\}$, respectively.

Finally, setting $G = \delta F$, where δ is a tuning parameter, and defining the new variables $[\bar{K}_j \ \ \bar{K}_{dj}] \triangleq [K_j \ \ K_{dj}]\bar{F}$. Further, pre- and post-multiplying LMIs in (18) by $\mathrm{diag}\{\bar{F}, \ldots, \bar{F}\}^{\mathrm{T}}$ and $\mathrm{diag}\{\bar{F}, \ldots, \bar{F}\}$, respectively, one obtain the LMIs in (32), completing this proof. □

Remark 3
Note that Theorem 2 as stated allows to design state-feedback memory control, as given in (4). However, it can be extended to design two special cases: memoryless control, $K_{di} = 0$ in (4), and purely delayed control, $K_i = 0$ in (4), by setting in Theorem 2: $\bar{K}_{dj} = 0$ or $\bar{K}_j = 0$, respectively.

Remark 4
The matrices P_i must have the structure shown in (11). To obtain P_i from $\bar{P}_i$, the substructure of matrices $\bar{P}_i$ and $\bar{F}$ must change according to each configuration of P_i.

Remark 5
In the control design an additional parameter δ has been introduced to obtain LMI conditions which may introduce some conservativeness. However, this result is more desirable since a line search algorithm can be used to find a suitable choice for δ, instead of solving a set of Bilinear Matrix Inequalities (BMIs), which are in general much more expensive.

7. CONTROL DESIGN EXAMPLES

In this section control design experiments are presented. In the first example the objective is to determine the largest time-delay for which a controller can be designed. In the second one,

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Table V. Maximum constant time-delay τ allowed by several methods m.

m	[11]	[9]	[13]$_{\text{Theorem 3}}$	[13]$_{\text{Co.2}}$	[14]	[15]	[16]	Theorem $2_{N=1}$	Theorem $2_{N=3}$
τ	—	0.1524	0.2574	0.2664	0.6611	0.8420	1.0947	1.2437	1.3169

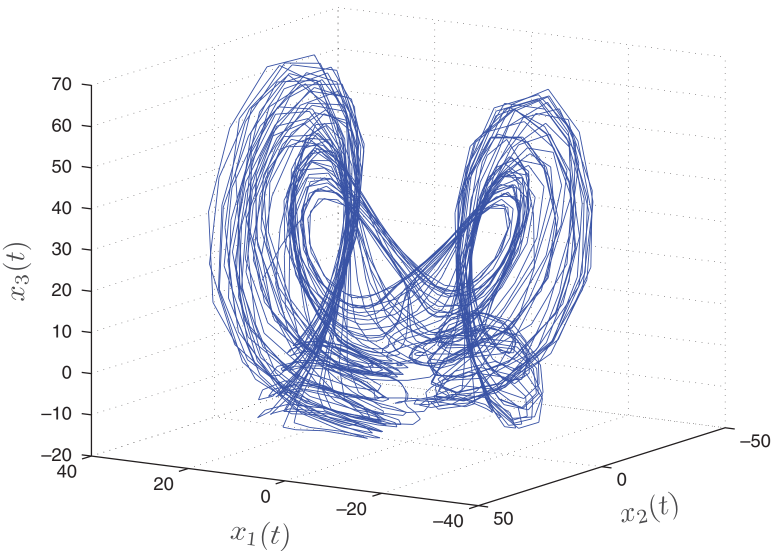

Figure 1. Three-dimensional view on the attractor generated by the delayed Lorenz system for $\mathrm{d}(t) = 0.1 + 0.05\cos(t)$ and initial condition vector $\phi^{\mathrm{T}}(t) = [\cos(t)\ \ \sin(t)\ \ -\cos(t)]\ \forall t \in [-0.15, 0]$.

controllers are designed and simulations show the performance when they are applied to the original nonlinear system.

7.1. Example 3

Consider the TS system in (1) with the following parameters, as in [13]:

$$A_1 = \begin{bmatrix} 0 & 0.6 \\ 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad A_{d1} = \begin{bmatrix} 0.5 & 0.9 \\ 0 & 2 \end{bmatrix}, \quad A_{d2} = \begin{bmatrix} 0.9 & 0 \\ 1 & 1.6 \end{bmatrix}, \quad B_1 = B_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

where the delay is time-invariant, i.e. $v = 0$. The objective is to compare the maximum time-delays allowed by memoryless PDC controllers designed by some methods in the Literature and by Theorem 2. The comparison is presented in Table V, considering $N = 1$ and $N = 3$ and $\delta = 1$.

7.2. Example 4

Consider the following delayed Lorenz chaotic system [32],

$$\begin{aligned} \dot{x}_1(t) &= m(x_2(t - \mathrm{d}(t)) - x_1(t)) + u_1(t) \\ \dot{x}_2(t) &= r x_1(t) - x_2(t) - x_1(t)x_2(t) \\ \dot{x}_3(t) &= x_1(t)x_2(t) - b x_3(t - \mathrm{d}(t)) + u_3(t) \end{aligned}$$

where $\mathrm{d}(t)$ is the uncertain time-delay, m, r, b are the parameters, and $u_1(t)$, $u_3(t)$ are the control signals. The system is in the chaotic state when $m = 10$, $r = 28$, $b = \frac{8}{3}$, $u_1(t) = u_3(t) = 0$. Figure 1 shows the three-dimensional attractor generated for $\mathrm{d}(t)=0.1+0.05\cos(t)$ and initial condition vector $\phi^{\mathrm{T}}(t)=[\cos(t)\ \ \sin(t)\ \ -\cos(t)]\,\forall t\in[-0.15,0]$.

Then by using the sector nonlinearity approach [2], a TS model in the form (1), which exactly represents the nonlinear equation under $x_1(t)\in[-\gamma,\gamma]$, is obtained for the premisse $x_1(t)$:

$$A_1=\begin{bmatrix}-m & 0 & 0\\ r & -1 & -\gamma\\ 0 & \gamma & 0\end{bmatrix},\quad A_2=\begin{bmatrix}-m & 0 & 0\\ r & -1 & \gamma\\ 0 & -\gamma & 0\end{bmatrix},\quad A_{d1}=A_{d2}=\begin{bmatrix}0 & m & 0\\ 0 & 0 & 0\\ 0 & 0 & -b\end{bmatrix},$$

$$B_1=B_2=\begin{bmatrix}1 & 0\\ 0 & 0\\ 0 & 1\end{bmatrix},$$

$$\hbar_1(x_1(t))=\frac{1}{2}\left(1+\frac{x_1(t)}{\gamma}\right),\quad \hbar_2(x_1(t))=\frac{1}{2}\left(1-\frac{x_1(t)}{\gamma}\right).$$

For the set of parameters selected the state $x_1(t)$ is limited within the following range $x_1(t)\in[-32,32]$. Thus, it is selected $\gamma=32$.

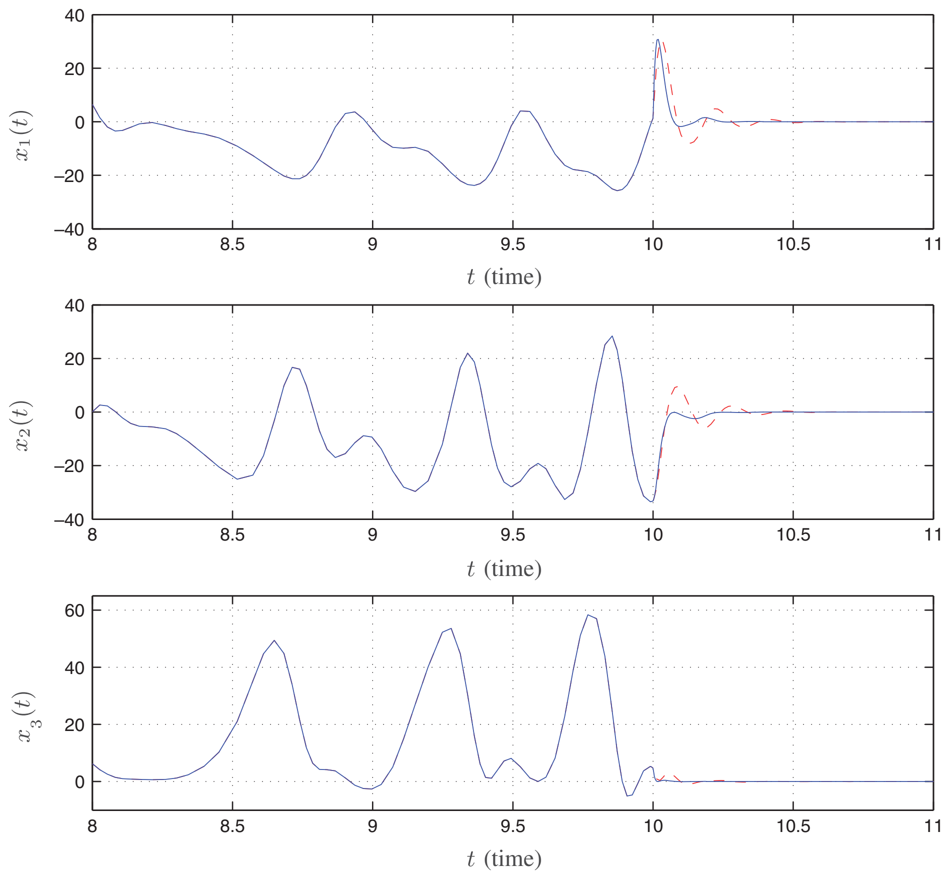

Figure 2. State response. PDC memory control (dashed) and PDC memoryless control (solid).

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Therefore, Theorem 2 is applied to design a PDC controller. Then, assuming $\tau=0.1$, $v=0.05$, $N=1$, and $\delta=0.01$, the following results are obtained by the proposed method:

- PDC memory control:

$$\left[\frac{K_1}{K_2}\right]=\left[\begin{array}{ccc}-28.6986 & -59.4339 & 1.8014\\ 13.4624 & -16.1529 & -78.5853\\ \hline -27.1237 & -53.6536 & 0.4614\\ -0.6831 & 11.3108 & -81.9296\end{array}\right],\quad \left[\frac{K_{d1}}{K_{d2}}\right]=\left[\begin{array}{ccc}0.1053 & -9.8032 & 0.0004\\ 0.0358 & -0.0770 & 2.6813\\ \hline 0.0497 & -9.9963 & 0.0011\\ 0.1304 & 0.5500 & 2.6693\end{array}\right].$$

- PDC memoryless control:

$$\left[\frac{K_1}{K_2}\right]=\left[\begin{array}{ccc}-93.6089 & -133.0704 & 16.4593\\ 8.6036 & -20.9346 & -185.3154\\ \hline -81.7541 & -121.5687 & -41.1131\\ -1.6001 & 13.8152 & -176.9765\end{array}\right].$$

- PDC purely delayed control: infeasible.

The controlled Lorenz system was simulated considering the time-varying delay $\mathrm{d}(t)=0.1+0.05\cos(t)$ and the initial condition vector $\phi^{\mathrm{T}}(t)=[\cos(t)\ \sin(t)\ -\cos(t)]\ \forall t\in[-0.15,0]$. The time response is depicted in Figure 2, for the PDC controllers designed with the control action taking place when $t=10$. In the tests the PDC memoryless and the PDC memory controllers provide similar time response, but the memoryless controller is faster.

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