HOMOGENEOUS VECTOR BUNDLES

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1. Root Space Decompositions

Let us first examine the root space decomposition of a semisimple complex Lie group and the corresponding decompositions for certain important subgroups. These decompositions are based on the action of G on itself via conjugation, $c_g : G \to G$, $c_g(h) = ghg^{-1}$, for $g, h \in G$. Let $\mathfrak{g}$ be the tangent space T_G at the identity. The differential of c_g at the identity gives a linear map, $\mathrm{Ad}(g) : \mathfrak{g} \to \mathfrak{g}$. The map $g \to \mathrm{Ad}(g)$ is multiplicative in $g \in G$ and defines the adjoint representation, $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$. The differential of this homomorphism is a linear map $\mathrm{ad} : \mathfrak{g} \to \mathfrak{g}$ that defines a Lie algebra structure on $\mathfrak{g}$: for $X, Y \in \mathfrak{g}$, $[X, Y] = \mathrm{ad}(X)(Y)$. If H is a complex subgroup of G then its tangent space at the identity defines a complex subalgebra $\mathfrak{h} \subset \mathfrak{g}$. Conversely, any complex subalgebra $\mathfrak{h} \subset \mathfrak{g}$ gives rise to a (not necessarily closed) connected complex subgroup $H \subset G$ generated by $\exp(X)$, $X \in \mathfrak{h}$. Here, $\exp : \mathfrak{g} \to G$ is the usual exponential map (for matrix groups $\exp(M) = \sum_{k=0}^{\infty} \frac{1}{k!} M^k$). We shall follow the convention that the Lie algebra corresponding to a subgroup will be denoted by the corresponding German letter.

Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{g}$, that is, a maximal abelian subalgebra such that $\mathrm{ad}(\mathfrak{t})$ is diagonalizable. There is a finite set of 'roots' $\Phi \subset \mathfrak{t}^*$ such that

$$\mathfrak{g} = \mathfrak{t} + \sum_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

Here, for each $\alpha \in \Phi$, $\mathfrak{g}_\alpha$ is a 1-dimensional 'root space' satisfying $[H, X_\alpha] = \alpha(H) X_\alpha$ for all $H \in \mathfrak{t}$ and $X_\alpha \in \mathfrak{g}_\alpha$.

The roots of G have the property that if $\alpha \in \Phi$ then $-\alpha \in \Phi$, and if $X_\alpha \in \mathfrak{g}_\alpha$ and $X_{-\alpha} \in \mathfrak{g}_{-\alpha}$, then $Z_\alpha = [X_\alpha, X_{-\alpha}]$ is a non-zero element of $\mathfrak{t}$. It is possible to choose the X_α for all $\alpha \in \Phi$ so that $\alpha(Z_\alpha) = 2$. The other root spaces are linked by $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$ if $\alpha + \beta \in \Phi$, and $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = 0$ otherwise. We denote by T the subgroup of G generated by $\mathfrak{t}$. Such a subgroup is called a maximal torus of G; it is isomorphic to $(\mathbb{C}^*)^\ell$ where $\ell = \dim \mathfrak{t}$ is the rank of G. We denote by U_α the 1-dimensional subgroup of G generated by $\mathfrak{g}_\alpha$. In this case, the exponential map $\exp : \mathfrak{g}_\alpha \to U_\alpha$ is an isomorphism, so that $U_\alpha \cong \mathbb{C}$ for all $\alpha \in \Phi$.

A subset $\Psi \subset \Phi$ is said to be closed under addition of roots if α, $\beta \in \Psi$ and $\alpha + \beta \in \Phi$ imply $\alpha + \beta \in \Psi$. The above rules for Lie brackets show that for any subset of roots $\Psi \subset \Phi$ that is closed under addition of roots, the subspace

$$\mathfrak{q}_\Psi = \mathfrak{t} + \sum_{\alpha \in \Psi} \mathfrak{g}_\alpha$$

is a complex subalgebra of $\mathfrak{g}$. The subgroup Q_Ψ of G generated by $\mathfrak{q}_\Psi$ is connected and closed and has the same dimension as $\mathfrak{q}_\Psi$, namely $\ell + |\Psi|$ where $|\Psi|$ is the number of elements in Ψ. In fact, Q_Ψ is generated by T and U_α for $\alpha \in \Psi$. Conversely, if Q is a closed connected complex subgroup of G that is normalized by T, then the Lie algebra $\mathfrak{q}$ of Q necessarily decomposes into root spaces of G, $\mathfrak{q} = \mathfrak{q} \cap \mathfrak{t} + \sum_{\alpha \in \Psi} \mathfrak{g}_\alpha$ for some closed set of roots Ψ.

We divide the roots into two disjoint subsets, $\Phi = \Phi^+ \cup \Phi^-$, by choosing a fixed linear functional $f : E \to \mathbb{R}$ on the $\mathbb{R}$-linear span $E = \Phi \otimes_{\mathbb{Z}} \mathbb{R}$ that does not vanish on any of the roots. Those roots for which $f(\alpha) > 0$ we call the positive roots, Φ^+, and those for which $f(\alpha) < 0$ we call the negative roots, Φ^-. We write simply $\alpha > 0$ for $\alpha \in \Phi^+$, and $\alpha < 0$ for $\alpha \in \Phi^-$. The positive roots are clearly closed under addition of roots, as are the negative roots. A minimal set of generators for

Φ^+ contains exactly $\ell = \operatorname{rank} G$ roots, $\alpha_1, \dots, \alpha_\ell$, called the simple roots. Thus every $\alpha > 0$ (resp. $\alpha < 0$) has a unique decomposition $\alpha = \sum_{i=1}^{\ell} n_i(\alpha)\alpha_i$ where $n_i(\alpha) \geq 0$ (resp. $n_i(\alpha) \leq 0$) for $1 \leq i \leq \ell$. The height of α is $h(\alpha) = \sum_{i=1}^{\ell} n_i(\alpha)$.

Let $\mathfrak{k}$ be the real Lie subalgebra of $\mathfrak{g}$ generated by iZ_α, $X_\alpha - X_{-\alpha}$ and $i(X_\alpha + X_{-\alpha})$ for $\alpha > 0$. The subgroup K of G corresponding to $\mathfrak{k}$ is compact. In fact, K is a maximal compact subgroup and any two such subgroups are conjugate.

We denote the complex subalgebra generated by $\mathfrak{t}$ and the negative root spaces by $\mathfrak{b} = \mathfrak{t} \oplus \sum_{\alpha<0} \mathfrak{g}_\alpha$. Let $\mathfrak{u}_k$ be the subalgebra of $\mathfrak{b}$ generated by the root spaces $\mathfrak{g}_\alpha$ such that $h(\alpha) \leq -k$. Then $\mathfrak{b}^{(1)} = [\mathfrak{b}, \mathfrak{b}] \subset \mathfrak{u}_1$ and $\mathfrak{b}^{(k+1)} = [\mathfrak{b}^{(k)}, \mathfrak{b}^{(k)}] \subset \mathfrak{u}_{k+1}$. Since $\mathfrak{u}_{k+1} = 0$ for k sufficiently large, the subalgebra $\mathfrak{b}$ is solvable. Let B be the closed connected solvable subgroup of G associated to $\mathfrak{b}$. We call B and any of its conjugates in G a Borel subgroup. Consider the complex homogeneous manifold $Y = G/B$. Let $\mathfrak{l} = \mathfrak{k} \cap \mathfrak{b} = \mathfrak{k} \cap \mathfrak{t}$ and let L be the corresponding subgroup of K. By inspecting the dimensions of the Lie algebras it follows that $\dim K/L = \dim G/B$. Hence the K-orbit of the identity coset in G/B is both open and compact from which we deduce that $Y = G/B = K/L$ is compact.

A parabolic subgroup P of G is a connected subgroup that contains a conjugate of B. For simplicity, we shall usually assume that P contains the Borel group B just constructed. Notice that the coset space $X = G/P$ is also compact since the natural coset map $\pi : Y = G/B \to X = G/P$ is surjective. By the above remarks, the Lie algebra $\mathfrak{p}$ of P decomposes into root spaces of G:

$$\mathfrak{p} = \mathfrak{t} + \sum_{\alpha>0} \mathfrak{g}_{-\alpha} + \sum_{\alpha \in \Phi_P^+} \mathfrak{g}_\alpha$$

where Φ_P^+ is a closed set of *positive* roots. Since $\Phi_P = \Phi^- \cup \Phi_P^+$ must also be a closed set of roots and since Φ_P contains all the negative simple roots $-\alpha_i$ for $1 \leq i \leq \ell$, it follows that Φ_P^+ is generated by a set of positive simple roots, say α_i for some subset of indexes I. We thus have a one-to-one correspondence between the 2^ℓ subsets $I \subset \{1, \dots, \ell\}$ and conjugacy classes of parabolic subgroups P. To emphasize this connection we sometimes write P_I for P and Φ_I^+ for Φ_P^+. We also let $\Phi_P = \Phi_P^+ \cup -\Phi_P^+$ denote the roots of P. The complementary set of roots, $\Phi^+ \setminus \Phi_P^+$ is denoted Φ_X and called the roots of X.

It will be convenient to have standard names for certain subgroups of P. The first is the semisimple subgroup S_P whose lie algebra is

$$\mathfrak{s}_P = \sum_{\alpha \in \Phi_P^+} \mathbb{C}Z_\alpha + \mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha}$$

Let T_P be the subgroup of T whose lie algebra is generated by Z_α for $\alpha \in \Phi^+ \setminus \Phi_P^+$, i.e.,

$$\mathfrak{t}_P = \sum_{i \notin I} \mathbb{C}Z_{\alpha_i}$$

The unipotent radical U_P of P is the subgroup of U generated by the root groups $U_{-\alpha}$ for $\alpha \in \Phi^+ \setminus \Phi_P^+$ and has Lie algebra

$$\mathfrak{u}_P = \sum_{\alpha \in \Phi^+ \setminus \Phi_P^+} \mathfrak{g}_{-\alpha}$$

The radical of P is $R_P = T_P U_P$ with lie algebra $\mathfrak{r}_P = \mathfrak{t}_P + \mathfrak{u}_P$; it is the maximal connected normal solvable subgroup of P. The Levi factor $L_P = T_P S_P$ of P

is the reductive subgroup of P whose lie algebra is $\mathfrak{l}_P = \mathfrak{t}_P + \mathfrak{s}_P$. Note that $P = R_P S_P = T_P U_P S_P = U_P L_P$ with corresponding decompositions of $\mathfrak{p}$.

2. Weights

Let $\hat{\phi} : G \to \mathrm{GL}(V)$ be an algebraic representation of G on a finite dimensional complex vector space V, and let $\phi : \mathfrak{g} \to \mathrm{GL}(V)$ be the corresponding representation of the Lie algebra of G,

$$\phi(x) = \frac{d}{dt}\hat{\phi}(\exp(tx))\bigg|_{t=0}$$

We shall also refer to V as a G-module (resp. $\mathfrak{g}$-module). The restricted homomorphism, $\hat{\phi}|_T$, is diagonalizable since T is abelian and consists of semisimple elements, and therefore determines characters $\hat{\lambda} : T \to \mathbb{C}^*$. The corresponding linear functionals, $\lambda : \mathfrak{t} \to \mathbb{C}$, are called the weights of the representation. These weights are constrained in the following way: For any root $\alpha \in \Phi$, let $\mathfrak{s}_\alpha = \mathbb{C}Z_\alpha + \mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha}$ be the subalgebra isomorphic to $\mathfrak{sl}_2(\mathbb{C})$. Let λ be a weight of V with corresponding weight vector v. Then, for $H \in \mathfrak{h}$,

$$HX_\alpha v = [H, X_\alpha]v + X_\alpha H v = (\lambda(H) + \alpha(H))X_\alpha v$$

so, either $X_\alpha v$ (resp. $X_{-\alpha}v$) is zero or a weight vector of weight $\lambda+\alpha$ (resp. $\lambda-\alpha$). Therefore the weights of the $\mathfrak{s}_\alpha$-submodule generated by v are

$$\lambda - q\alpha, \ldots, \lambda - \alpha, \lambda, \lambda + \alpha, \ldots, \lambda + p\alpha$$

where p (resp. q) is the largest integer k for which $X_\alpha^k v \neq 0$, (resp. $X_{-\alpha}^k v \neq 0$). Since $\mathfrak{s}_\alpha$ is semisimple, the trace of ϕ restricted to this submodule is zero and hence the sum of these weights evaluated on Z_α must be 0,

$$(p+q+1)\lambda(Z_\alpha) + \left(\frac{p(p+1)}{2} - \frac{q(q+1)}{2}\right)\alpha(Z_\alpha) = 0$$

Since $p(p+1) - q(q+1) = (p-q)(p+q+1)$ and $\alpha(Z_\alpha) = 2$, we conclude that

$$\lambda(Z_\alpha) = q - p \in \mathbb{Z} \quad \text{for all } \alpha \in \Phi$$

The weight lattice of G is defined to be $\Lambda = \{\lambda \in \mathfrak{t} \mid \lambda(Z_\alpha) \in \mathbb{Z} \text{ for all } \alpha \in \Phi\}$. The rank of this lattice is $\ell = \dim T = \operatorname{rank} G$. The roots of G are just the weights of the adjoint representation and they generate a sublattice $\Lambda_0 \subset \Lambda$ of finite index. If G is simply-connected, then every weight in Λ is actually attained in some representation of G.

The Lie algebra $\mathfrak{g}$ has a canonical inner product, called the Killing form, which is defined by $(X, Y) = \mathrm{Tr}(\mathrm{ad}(X)\,\mathrm{ad}(Y))$. It follows from the definition that for $W, X, Y \in \mathfrak{g}$, $(W, [X, Y]) = ([W, X], Y)$. Also, since

$$\mathrm{ad}(X_\alpha)\,\mathrm{ad}(X_\beta)(X_\gamma) = [X_\alpha, [X_\beta, X_\gamma]] = cX_{\alpha+\beta+\gamma}$$

we see that $(X_\alpha, X_\beta) = \mathrm{Tr}(\mathrm{ad}(X_\alpha)\,\mathrm{ad}(X_\beta)) = 0$ unless $\beta = -\alpha$. To compute $(X_\alpha, X_{-\alpha})$ notice that

$$(Z_\alpha, Z_\beta) = (Z_\alpha, [X_\beta, X_{-\beta}]) = ([Z_\alpha, X_\beta], X_{-\beta}) = \beta(Z_\alpha)(X_\beta, X_{-\beta})$$

In particular, $(X_\alpha, X_{-\alpha}) = (Z_\alpha, Z_\alpha)/2$. Moreover, for X, $Y \in \mathfrak{t}$,

$$\mathrm{ad}(X)\,\mathrm{ad}(Y)(X_\alpha) = [X, [Y, X_\alpha]] = [X, \alpha(Y)X_\alpha] = \alpha(X)\alpha(Y)X_\alpha$$

so that

$$(X, Y) = \sum_{\alpha \in \Phi} \alpha(X)\alpha(Y)$$

and hence the Killing form is positive definite on the $\mathbb{R}$-span of the Z_α. It is not hard to check that if $T_\alpha = 2Z_\alpha/(Z_\alpha, Z_\alpha)$ for $\alpha \in \Phi$, then $(T_\alpha, Z) = \alpha(Z)$ for all

$Z \in \mathfrak{t}$. The Killing form can then be extended to a non-degenerate positive definite form on $E = \Phi \otimes_{\mathbb{Z}} \mathbb{R}$ by defining for $\alpha, \beta \in \Phi$, $(\alpha, \beta) = (T_\alpha, T_\beta)$. In this way E becomes a standard normed euclidean space.

For $\lambda \in E$, $\alpha \in \Phi$, we get $\lambda(Z_\alpha) = 2(\lambda, \alpha)/(\alpha, \alpha)$. This expression occurs frequently so we denote it by $\langle \lambda, \alpha \rangle$. In particular, the weight lattice is defined by $\Lambda = \{\lambda \in E \,|\, \langle \lambda, \alpha \rangle \in \mathbb{Z} \text{ for all } \alpha \in \Phi\}$. Also, it follows from the above description of the weights of a representation of $\mathfrak{s}_\alpha$ that if $\lambda \in \Lambda$ then so is $\lambda - \langle \lambda, \alpha \rangle \alpha$, for all $\alpha \in \Phi^+$.

A convenient basis, $\lambda_1, \ldots, \lambda_\ell$, to use for E when working with weights is the one that is dual to the simple roots in the following way: $\langle \lambda_i, \alpha_j \rangle = \delta_{ij}$. These weights are called the fundamental dominant weights and they obviously generate the lattice Λ. An arbitrary weight $\lambda \in \Lambda$ can be written as $\lambda = n_1\lambda_1 + \cdots + n_\ell \lambda_\ell$ where $n_i = \langle \lambda, \alpha_i \rangle \in \mathbb{Z}$. The weight λ is called dominant if $\lambda \in \mathcal{C}$, i.e., if $n_i \geq 0$ for all i. We denote the subset of dominant weights by $\Lambda^+ \subset \Lambda$.

Some additional notation that involves weights: For $\lambda \in \Lambda$ we denote by P_λ the parabolic subgroup of G generated by the simple roots perpendicular to λ (if there are no such simple roots then $P_\lambda = B$, a Borel subgroup). If $\lambda = n_1\lambda_1 + \cdots n_\ell \lambda_\ell$ is dominant, then $P_\lambda = P_I$ where I is the set of indexes i such that $n_i = 0$.

Let $X = G/P$ where P is a parabolic subgroup of G. The tangent bundle T_X of X is homogeneous with respect to G. The tangent space at $x_0 = 1P$ is naturally isomorphic to the vector space quotient $\mathfrak{g}/\mathfrak{p}$. Since the action of $p \in P$ on a point x in a neighborhood of x_0 is induced by conjugation, $p \cdot x = pgP = pgp^{-1}P$, we see that the action of P on T_{X,x_0} is given by the adjoint representation of G restricted to P and projected to the quotient $\mathfrak{g}/\mathfrak{p}$. The weights of the representation of P on T_{X,x_0} are thus the roots of G that are not in P, $\Phi_X = \Phi^+ \setminus \Phi_P^+$. Note that T_{X,x_0} is dual to $\mathfrak{u}_P$.

The Dynkin diagram of a Lie group G is a graph whose nodes are the simple roots. The node corresponding to α_i is connected to the node for α_j by $\langle \alpha_i, \alpha_j \rangle \langle \alpha_i, \alpha_j \rangle$ lines. An inequality sign, $>$ or $<$ appropriately oriented, is superimposed on the lines connecting roots of different lengths. The connected components of a Dynkin diagram correspond to the simple factors of G. For future reference we have included the Dynkin diagrams for the simple Lie groups below. The number adjacent to a node is the index i of the corresponding simple root α_i.

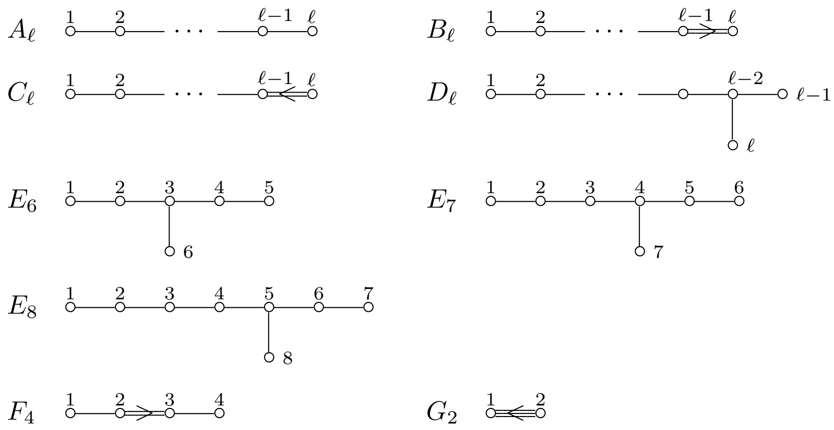
The Cartan matrix $[\langle \alpha_i, \alpha_j \rangle]$ can be easily recovered from the diagram by using the fact that for simple roots $\alpha_i \neq \alpha_j$, $\langle \alpha_i, \alpha_j \rangle \leq 0$. The positive roots of G can be constructed from the Cartan matrix by careful application of the above formula,

$$\langle \alpha, \alpha_j \rangle = q - p$$

where p (resp. q) is the number of times α_j can be added (resp. subtracted) from α and still obtain a root.

For example, if $G = G_2$, then $\langle \alpha_1, \alpha_2 \rangle = -1$, and $\langle \alpha_2, \alpha_1 \rangle = -3$. We cannot subtract α_1 from α_2 (and vice versa) to get another root, so these negative numbers imply that $\alpha_1 + \alpha_2$, $2\alpha_1 + \alpha_2$, and $3\alpha_1 + \alpha_2$ are positive roots. Since we know how many times we can subtract any simple root from one of these newly computed roots α, we can determine how many times we can add a simple root to α. Indeed, we find that for $\alpha = 3\alpha_1 + \alpha_2$, we cannot subtract α_2 but $\langle \alpha, \alpha_2 \rangle = -3 + 2 = -1$, so $\beta = 3\alpha_1 + 2\alpha_2$ is another positive root. Similar considerations for other choices yield no new roots. Finally, we check that no new root can be generated from β. Since we know α_1 cannot be subtracted from β and $\langle \beta, \alpha_1 \rangle = 6 - 6 = 0$, we

Table 1. Dynkin Diagrams



conclude that α_1 cannot be added to β get a positive root either. Similarly, α_2 can be subtracted once from β and $\langle \beta, \alpha_2 \rangle = -3 + 4 = 1$, so again we find that α_2 cannot be added to β. We conclude that the positive roots of G_2 are α_1, α_2, $\alpha_1 + \alpha_2$, $2\alpha_1 + \alpha_2$, $3\alpha_1 + \alpha_2$, and $3\alpha_1 + 2\alpha_2$.

3. Weyl Group

Geometrically, the linear map $\sigma_\alpha : E \to E$ defined by

$$\sigma_\alpha(\lambda) = \lambda - \langle \lambda, \alpha \rangle \alpha$$

is a reflection of E through the hyperplane $E_\alpha = \{\lambda \in E \,|\, (\lambda, \alpha) = 0\}$ which preserves the weight lattice of G and the Killing form. The group of reflections generated by σ_α for $\alpha \in \Phi$ is a finite group W called the Weyl group of G.

This group is in fact generated by the simple reflections $\sigma_i = \sigma_{\alpha_i}$, $i = 1, \ldots, \ell$. The length of $\omega \in W$, denoted $\ell(\omega)$, is defined to be the minimal number t of simple reflections needed to express ω as a product of simple reflections, $\omega = \sigma_{i(1)} \cdots \sigma_{i(t)}$.

The Weyl chambers of E are the closures of the connected components of $E \setminus \cup_{\alpha \in \Phi^+} E_\alpha$. The fundamental chamber is defined to be $\mathcal{C} = \{\lambda \in E \,|\, (\lambda, \alpha) \geq 0 \text{ for all } \alpha \in \Phi^+\}$. Let $\lambda \in E$ and consider a straight line from λ to a generic point of $\mathcal{C}$. If the hyperplanes crossed by this line are, in order, $E_{\beta(1)}, \ldots, E_{\beta(t)}$, then clearly $\omega = \sigma_{\beta(t)} \cdots \sigma_{\beta(1)}$ takes λ to $\mathcal{C}$. This shows, in particular, that the Weyl group W acts transitively on the Weyl chambers (it can be shown that this action is also simple). We call $\omega(\lambda)$ the dominant conjugate of λ and sometimes denote it by $[\lambda]$. The index of λ is defined to be

$$\operatorname{ind}(\lambda) = \min\{\ell(\omega) \,|\, \omega\lambda \in \Lambda^+\}$$

Note that ω can also be written $\omega = \sigma_{i(1)} \cdots \sigma_{i(t)}$ where $\sigma_{i(j)}$ is the simple reflection $\sigma_{\beta(t)} \cdots \sigma_{\beta(j+1)} \sigma_{\beta(j)} \sigma_{\beta(j+1)} \cdots \sigma_{\beta(t)}$, $j = 1, \ldots, t$. Therefore, if we define $\Phi_\omega^+ = \Phi^+ \cap \omega^{-1}\Phi^-$, then

$$\Phi_\omega^+ = \{\alpha \in \Phi^+ \,|\, \omega(\alpha) < 0\} = \{\sigma_{i(1)} \cdots \sigma_{i(j)} \alpha_{i(j)} \,|\, j = 1, \ldots, t\}$$

and $\ell(\omega) = |\Phi_\omega^+|$, the number of elements in Φ_ω^+. The index of λ can be similarly identified with the number of positive roots $\alpha > 0$ such that $(\lambda, \alpha) < 0$. We define the index of a set of weights $A \subset \Lambda$ to be the minimum of the indexes of weights in A, $\operatorname{ind} A = \min\{\operatorname{ind} \mu \,|\, \mu \in A\}$.

To find the index of a weight $\lambda \in \Lambda$, we write λ in the basis of fundamental dominant weights: $\lambda = n_1\lambda_1 + \cdots + n_\ell\lambda_\ell$. If the first negative coordinate is $n_j = \langle \lambda, \alpha_j \rangle < 0$, then the weight $\sigma_j(\lambda)$ will have index one less than λ. This is because σ_j permutes the positive roots other than α_j and hence the number of negative inner products, $(\lambda, \alpha) = (\sigma_j\lambda, \sigma_j\alpha)$, is reduced by exactly one. In coordinates, $\sigma_j(\lambda) = \lambda - n_j\alpha_j = \sum_i (n_i - c_{ij}n_j)\lambda_i$ where the constants $c_{ij} = \langle \alpha_i, \alpha_j \rangle$ come from the Cartan matrix $[c_{ij}]$. This reflection is easy to calculate since $c_{ii} = 2$, so $n_j \to -n_j$ and only the coordinates n_i 'adjacent' to n_j in the Dynkin diagram for G must be adjusted, $n_j \to n_j - c_{ij}n_j$. Repeating the above process on $\sigma_j\lambda$, i.e., reflecting it by σ_k where k is the index of the first negative coordinate, gives a weight of index 2 less than λ, and so on. Eventually we obtain the dominant conjugate of λ and the number of simple reflections used to get there is the index of λ.

Since the Weyl group acts simply transitively on the Weyl chambers, the Weyl group itself can be 'coded' into the set of vectors $\omega\delta$, $\omega \in W$, where $\delta = \lambda_1 + \cdots + \lambda_\ell$. The length of ω as well as its decomposition into a product of simple roots can be determined by applying the above algorithm to $\omega\delta$. In particular, $\ell(\omega) = \operatorname{ind}(\omega\delta)$.

There is a unique longest element $\omega_0 \in W$ distinguished by the fact that $\omega_0\Lambda^+ = -\Lambda^+$. The length of ω_0 equals the number of positive roots in the group:

$$\ell(\omega_0) = |\Phi^+| = \dim G/B$$

Since $\omega_0^2 = 1$, ω_0 induces a two-fold symmetry in W. For example, any $\omega \in W$ can be written $\omega = \omega'\omega_0$ with $\ell(\omega) = \ell(\omega_0) - \ell(\omega')$.

For any set of indexes $I \subset \{1, \ldots, \ell\}$ we let W_I be the subgroup of W generated by the simple reflections σ_i, $i \in I$. We also denote this subgroup by W_P where P is the parabolic subgroup of G generated by I, since W_P is isomorphic to the Weyl group of the semisimple factor $S(P)$. A formula similar to that above holds for the longest element $\omega_1 \in W_P$:

$$\ell(\omega_1) = |\Phi_P^+| = \dim P/B$$

More generally, for any dominant weight $\lambda \in \Lambda$ we have the relations

$$\operatorname{ind}(\omega\lambda) \leq \ell(\omega) \leq \operatorname{ind}(\omega\lambda) + \dim P_\lambda/B$$

and both extremes are taken on by some $\omega \in W$. The upper bound is due to the fact that $\omega'\lambda = \lambda$ for all $\omega' \in W_{P_\lambda}$, and the maximum length of $\omega' \in P_\lambda$ is $\dim P_\lambda/B$. Note that $\dim P_\lambda/B$ is just the number of positive roots perpendicular to λ.

Let $N_G(T)$ be the normalizer of T in G. For any representative n of a coset in $N_G(T)/T$, it is clear that the linear map $\mathrm{Ad}(n)|_{\mathfrak{t}} : \mathfrak{t} \to \mathfrak{t}$ is independent of the choice of representative. This correspondence between $N_G(T)/T$ and automorphisms of $\mathfrak{t}$ actually defines an isomorphism from $N_G(T)/T$ to the Weyl group W defined above, although we shall not try to justify this here. However, we shall make use of the fact that for every $\omega \in W$, there is a $n_\omega \in N_G(T)$ such that $n_\omega U_\alpha n_\omega^{-1} = U_{\omega\alpha}$. If $U = \prod_{\alpha\in\Phi^-} U_\alpha$ is the maximal unipotent subgroup of G generated by the negative root groups, then it follows that

$$n_\omega^{-1} U n_\omega = \prod_{\alpha\in\Phi^-} U_{\omega^{-1}\alpha} = \prod_{\beta\in\Phi_1} U_\beta \prod_{\gamma\in\Phi_2} U_\gamma$$

where $\Phi_1 = \Phi^- \cap \omega^{-1}\Phi^-$ and $\Phi_2 = \Phi^+ \cap \omega^{-1}\Phi^- = \Phi_\omega^+$. Note that $|\Phi_2| = |\Phi_\omega^+| = \ell(\omega)$. This is the starting point for the following decomposition of G whose proof we omit:

Theorem 3.1 (Bruhat Decomposition)**.** *Let G be a connected semisimple complex Lie group, let B be a Borel subgroup of G, and let $U \subset B$ be the maximal unipotent subgroup. Then G decomposes into a disjoint union:*

$$G = \bigcup_{\omega\in W} U n_\omega B$$

where $Un_\omega B = Un_\sigma B$ if and only if $\omega = \sigma$ in W. Moreover, $\dim Un_\omega B = \ell(\omega) + \dim B$ *for $\omega \in W$.*

This decomposition reflects a corresponding cellular decomposition of the homogeneous space $X = G/B$. Indeed, the orbit $X_\omega = U \cdot (n_\omega B) \subset X$ for $\omega \in W$ is isomorphic to $\prod_{\gamma\in\Phi_\omega^+} U_\gamma \cong \mathbb{C}^{\ell(\omega)}$ and the disjoint union of these orbits is X. A similar decomposition can be obtained for $X = G/P$ where P is any parabolic subgroup of G by using coset representatives $\omega(\tau) \in W$ of minimal length for $\tau \in W/W_P$:

$$X = \bigcup_{\tau\in W/W_P} U n_{\omega(\tau)} P$$

Since $\omega(\tau)$ has minimal length, the dimension of the cell $X_\tau = Un_{\omega(\tau)}P$ is $\ell(\omega(\tau))$. Thus we find that the betti numbers, $b_i(X) = \dim H^i(X, \mathbb{R}) = \dim H_i(X, \mathbb{R})$, satisfy $b_{2i+1}(X) = 0$ and $b_{2i}(X)$ is the number of elements in $\{\tau \in W/W_P \mid \ell(\omega(\tau)) = i\}$

A good way to determine the coset representatives $\omega(\tau)$ for $\tau \in W/W_P$ is to consider the W-orbit of $\lambda_P = \sum_{i \notin I} \lambda_i$ where I is the set of indexes defining P. The orbit is isomorphic to W/W_P and for any $\omega\lambda_P \in W/W_P$ a minimal coset representative is given by finding the sequence of simple reflections that take $\omega\lambda_P$ to its dominant conjugate as above. One can construct the orbit of a dominant weight $\lambda \in \Lambda^+$ directly by reversing the above algorithm: First, let $L_0 = \{\lambda\}$ and define L_i for $i \geq 1$ recursively as the set of weights $\sigma_j\mu$ where $\mu \in L_{i-1}$ and the j-th coordinate of μ is positive, $\langle \mu, \alpha_j \rangle > 0$. The weights in L_i have index i. In general, a single weight in L_i can come from many different weights in L_{i-1} and this becomes a significant computational problem when the orbits are large. There are efficient ways to deal with this however, see [47].

Some simple examples to illustrate how these ideas can be applied:

- $b_2(X)$ is the number of positive coordinates of λ_P, so $b_2(X) = \operatorname{rank} G - |I|$. In particular, P is a maximal parabolic subgroup ($\lambda_P = \lambda_i$ for some i) if and only if $b_2(X) = 1$.
- If $X = \mathbb{P}^\ell$ then G is type A_ℓ and $\lambda_P = \lambda_1$. In this case, $L_i = \{\sigma_i\sigma_{i-1}\cdots\sigma_1\lambda_1 = -\lambda_i + \lambda_{i+1}\}$, and $b_{2i} = 1$ for $0 \leq i \leq \ell$ as expected.
- If X is an n-dimensional non-singular quadric hypersurface in projective space then G is of type B_ℓ ($n = 2\ell - 1$) or D_ℓ ($n = 2\ell - 2$) and again $\lambda_P = \lambda_1$. For type B_ℓ we find $L_i = \{\sigma_i\sigma_{i-1}\cdots\sigma_1\lambda_1 = -\lambda_i + \lambda_{i+1}\}$ ($L_{\ell-1} = \{-\lambda_{\ell-1} + 2\lambda_\ell\}$) and $L_{n-i} = -L_i$ so that $b_{2i}(X) = 1$ for $0 \leq i \leq \ell - 1$. Type D_ℓ is the same except $L_{n/2} = L_{\ell-1} = \{-\lambda_{\ell-1} + \lambda_\ell, \lambda_{\ell-1} - \lambda_\ell\}$ and $b_{n/2}(X) = 2$.

4. Irreducible Representations

Weyl's theorem states that every G-module V can be decomposed into a direct sum of irreducible G-modules. More precisely,

$$V \cong \bigoplus_{i=1}^{t} V_i \otimes E_i$$

where the V_i are pairwise non-isomorphic irreducible G-modules and the E_i are trivial G-modules. The dimension of E_i is called the multiplicity of V_i in V. Any G-invariant subspace W of V—for example, the image or kernel of a G-homomorphism—must be of the form

$$W \cong \bigoplus_{i=1}^{t} V_i \otimes F_i$$

where the F_i are (possibly trivial) subspaces of E_i. A special case of this is known as Shur's Lemma: If $\phi : V \to W$ is a G-homomorphism of irreducible G-modules, then ϕ is either trivial or an isomorphism.

We denote the weights of the G-module V by $\Lambda(V) \subset \Lambda$. Recall from 2 that if v is a weight vector for $\lambda \in \Lambda(V)$, then for all roots $\alpha \in \Phi$, $X_\alpha v$ is either 0 or has weight $\lambda + \alpha$. This leads to a partial ordering of the weights: we say $\mu < \lambda$ if $\lambda - \mu$ is a positive linear combination (possibly zero) of positive roots. This ordering can be applied to all of Λ, of course, and is compatible with the notation for positive roots, $\alpha > 0$.

Let $\lambda \in \Lambda(V)$ be a weight that is maximal with respect to the above partial order and let v be the corresponding weight vector. Since $X_\alpha v = 0$ for all positive roots $\alpha > 0$ by maximality, we see that the weights in the G-submodule generated by v_λ are all $< \lambda$. Notice also that such a v would be fixed by the maximal unipotent subgroup U^+ of G generated by the positive root groups U_α, $\alpha > 0$. Let V^{U^+} denote the set of fixed points of U^+. The set of maximal weights $\Lambda_{\max}(V)$ in V is thus $\Lambda(V^{U^+})$. We also refer to the weights $\Lambda_{\max}(V)$ as the highest weights of V. Equivalently, if U denotes the usual maximal unipotent subgroup generated by the negative root groups U_α, $\alpha < 0$, then $\Lambda_{\max}(V) = -\Lambda(V^{*U})$. This last equality allows us to extend the definition of maximal weights in a meaningful way to any P-module where P is a parabolic subgroup of G, see §8.

If V is irreducible, any two maximal weights, λ, $\mu \in \Lambda_{\max}(V)$, would both have weight vectors that generate all of V, and so $\mu < \lambda$ and $\lambda < \mu$. Therefore, for any irreducible G-module there is a unique maximal weight λ. The G-module V is in fact uniquely determined by λ and we use the notation V^λ to make this association explicit. The notation for the weights $\Lambda(V^\lambda)$ will be shortened to $\Lambda(\lambda)$. Since the weights $\Lambda(\lambda)$ and their multiplicities are invariant under the action of the Weyl group, see §2, they are determined by their dominant conjugates: If $\Lambda^+(\lambda) = \Lambda(\lambda) \cap \Lambda^+$, then $\Lambda(\lambda) = W \cdot \Lambda^+(\lambda)$. In particular, since $\sigma\lambda < \lambda$ for $\sigma \in W$, λ itself must be dominant. We refer to the weights $\Lambda^+(\lambda)$ as the subdominant weights of the representation.

The subdominant weights $\Lambda^+(\lambda)$ are easily determined. Starting with λ, we collect all weights of the form $\lambda - \alpha$, $\alpha > 0$, that are dominant into a list L_1. We know from §2 that L_1 must consist of subdominant weights, $L_1 \subset \Lambda^+(\lambda)$. Continuing in this way, we create a list L_{k+1} consisting of all weights of the form

$\mu - \alpha$, $\mu \in L_k$, $\alpha > 0$, that are dominant. Again, $L_{k+1} \subset \Lambda^+(\lambda)$. We eventually reach the state where $L_{t+1} = \emptyset$. It is clear that all subdominant weights must have the form $\mu = \lambda - \beta_1 - \cdots - \beta_k$. However, the 'path' from λ to μ may, in general, leave the fundamental chamber, depending on the roots $\beta_1, \ldots, \beta_k$ and their order. Nevertheless, it can be shown there is always a path from λ to any subdominant weight μ that passes through only dominant weights so that $\Lambda^+(\lambda) = \{\lambda\} \cup L_1 \cup \cdots \cup L_t$, see, e.g., [41].

While the above discussion shows how to reconstruct the weights of any representation with minimal effort, its says nothing about how to determine the multiplicities of the weights. There are several formulas which give these multiplicities, but they involve a fair amount of computation compared to the above procedure. Weyl's character formula is the simplest to write:

$$\chi(\lambda) = \frac{\sum_{\omega \in W} (-1)^{\ell(\omega)} e^{\omega(\lambda+\delta)}}{\sum_{\omega \in W} (-1)^{\ell(\omega)} e^{\omega(\delta)}}$$

where $\chi(\lambda) = \operatorname{Tr} \phi = \sum_{\mu \in \Lambda(\lambda)} m_\mu e^\mu$ is the character of the representation of $\phi : \mathfrak{g} \to \mathrm{GL}(V^\lambda)$ and $\delta = \lambda_1 + \cdots + \lambda_\ell$. The symbols e^μ are to be manipulated using the usual rules of exponents. The multiplicity of μ in V^λ is the coefficient m_μ of e^μ in the resulting quotient which is computed like a quotient of polynomials. This formula is not very practical for computing multiplicities for groups of high rank since the size of W is exponential in ℓ making the above sums and quotient difficult to handle. Freudenthal's multiplicity formula is recursive and serves this purpose much better, see [27, 41]. Using a limiting process, Weyl derived from his character formula the following expression for the dimension of a representation which is relatively easy to compute:

$$\dim V^\lambda = \frac{\prod_{\alpha>0} \langle \lambda + \delta, \alpha \rangle}{\prod_{\alpha>0} \langle \delta, \alpha \rangle}$$

Writing $\lambda = \sum_{i=1}^{\ell} n(\lambda_i)\lambda_i$ and $\alpha = \sum_{i=1}^{\ell} m_i(\alpha)\alpha_i$, the dimension formula becomes

$$\dim V^\lambda = \prod_{\alpha>0} \left[1 + \frac{\sum_{i=1}^{\ell} n_i(\lambda) m_i(\alpha)}{\sum_{i=1}^{\ell} m_i(\alpha)} \right]$$

Recall from §1 that the denominators in this expression are the heights of the roots, $h(\alpha) = \sum_{i=1}^{\ell} m_i(\alpha)$.

Let V be a G-module and let V^* denote the dual module. The action of G on $f \in V^*$ is defined by $(g \cdot f)(v) = f(g^{-1} \cdot v)$ for $v \in V$ so that the weights of V^* are the negatives of the weights of V: $\Lambda(V^*) = -\Lambda(V)$. If $V = V^\lambda$ is irreducible with highest weight λ, then it is clear that $-\lambda$ is the 'lowest' weight of V^* in the sense that $-\lambda - \alpha$ is not a weight of V^* for any positive root $\alpha > 0$. Let $\omega_0 \in W$ be the unique Weyl group element that takes Λ^+ to $-\Lambda^+$, see §2. Since $\omega_0 \Lambda(V^*) = \Lambda(V^*)$ and $\omega_0 \Phi^+ = \Phi^-$, it follows that $-\omega_0 \lambda$ must be the highest weight of V^* and $V^* = V^{-\omega_0 \lambda}$.

The involution $-\omega_0$ defines an involution of the Dynkin diagram of G. For groups with diagram components lacking two-fold symmetry, i.e., whose components have types B_ℓ, C_ℓ, E_7, E_8, F_4, and G_2, this involution must be trivial and $-\omega_0 = 1$. For components of type A_ℓ the involution $-\omega_0$ is given by $\lambda_i \leftrightarrow \lambda_{\ell-i+1}$, $1 \le i \le \ell$, for type D_ℓ it is given by $\lambda_{\ell-1} \leftrightarrow \lambda_\ell$, and for E_6 it is given by $\lambda_1 \leftrightarrow \lambda_6$, $\lambda_2 \leftrightarrow \lambda_5$.

5. Basic Concepts of Vector Bundles

Let X be a connected compact complex manifold that is homogeneous under a complex Lie group G so that $X \cong G/P$ where P is the isotropy subgroup of a point $x_0 \in X$. A holomorphic vector bundle $\pi : E \to X$ is *homogeneous* if the group of bundle automorphisms of E acts transitively on the set of fibers of E. We say that E is *homogeneous with respect to* G if the action of G on X lifts to a compatible action of G on E via holomorphic bundle automorphisms. If G is a connected, simply-connected semisimple complex Lie group, then a homogeneous vector bundle $\pi : E \to X$ is always homogeneous with respect to G. This is not necessarily the case for other types of groups. We shall always assume that E is homogeneous with respect to G.

The isotropy subgroup P acts linearly on the fiber $E_0 = \pi^{-1}(x_0)$. Consider the action map

$$G \times E_0 \xrightarrow{\mu} E$$

defined by $\mu(g, z) = g \cdot z$. Since G acts transitively on X, the map μ is clearly surjective. The fibers of μ are the orbits of P on $G \times E_0$ under the diagonal action $(g, z) \to (gp^{-1}, p \cdot z)$, $p \in P$. In fact, we may represent any point in E as an equivalence class $[g, z]$ for $(g, z) \in G \times_P E_0$ where $[gp, z] = [g, p \cdot z]$ for all $p \in P$. In this way E is isomorphic to the quotient

$$E = G \times_P E_0 = (G \times E_0)/P$$

Conversely, given a holomorphic representation $P \to \mathrm{GL}(E_0)$, $E = G \times_P E_0$ is a holomorphic vector bundle over X that is homogeneous with respect to G. The bundle map $\pi : E \to X$ is given by projection, $\pi([g, z]) = gP \in G/P$. Thus, the holomorphic vector bundles on X that are homogeneous with respect to G are in one-to-one correspondence with holomorphic representations $P \to \mathrm{GL}(E_0)$ and two such bundles are isomorphic if and only if the representations of P are conjugate in $GL(E_0)$.

Let $q : G \to G/P$ be the quotient map. Notice that the pull-back q^*E is isomorphic to $G \times E_0$ and the bundle map $q^*E \to E$ is simply the action map μ. Given a holomorphic section of E, its pull back to $q^*E \cong G \times E_0$ defines a holomorphic map $s : G \to E_0$ such that $s(gp^{-1}) = p \cdot s(g)$. Conversely, any such function defines a section of E. Therefore,

$$H^0(X, E) \cong \{s : G \to E_0 \mid s(gp^{-1}) = p \cdot s(g) \text{ for all } g \in G, p \in P\}$$

The latter set of functions on G has a natural G-module structure and is known as the induced G-module of the P-module E_0, denoted $E_0|^G$. The evaluation map defines a P-module homomorphism $\mathrm{ev}_0 : E_0|^G \longrightarrow E_0$, $\mathrm{ev}_0(s) = s(1)$, that satisfies the following universal property (see for example [11]):

Proposition 5.1 (Universality). *If W is a finite dimensional G-module and $\phi : W \to E_0$ is a P-module homomorphism, then there exists a unique G-module homomorphism $\psi : W \to E_0|^G$ such that $\phi \circ \psi = \mathrm{ev}_0$.*

The map ψ is straightforward to construct. For $w \in W$, define $\psi(w) : G \to E_0$ by $\psi(w)(g) = \phi(g^{-1} \cdot w)$. Since $\psi(w)(gp^{-1}) = p \cdot \psi(w)(g)$ for all $p \in P$, we see that $\psi(w)$ does indeed lie in $E_0|^G$. In fact, it is often convenient to start with a G-module V and a P-module homomorphism $\phi : V \to E_0$ and then construct sections $s : G \to E_0$ for $G \times_P E_0$ by defining $s(g) = \phi(g^{-1} \cdot v)$ for $v \in V$.

Another useful property of induced modules is the following.

Proposition 5.2 (Transitivity). *If Q is a closed complex subgroup of G containing P, then $E_0|^G \cong E_0|^Q|^G$.*

Let $Y = G/Q$ and let $\tau : X \to Y$ be the natural coset map with fiber $Z = Q/P$. Since the direct image sheaf, $\tau_* E$, is isomorphic to $H^0(Z, E|_Z) \otimes \mathcal{O}_Y \cong E_0|^Q \otimes \mathcal{O}_Y$, the above statement simply expresses the fact that $H^0(X, E) \cong H^0(Y, \tau_* E)$. The degree to which the bundle E is 'trivial' is reflected in the degree to which the P-module E_0 can be extended to larger subgroups of G. For example, if E_0 can be extended to a Q-module, then E is the pull back of the bundle $G \times_Q E_0$ over Y under the coset map $X \to Y$ and $E|_Z$ is isomorphic to the trivial bundle, $Z \times E_0$. In particular, the P-module E_0 can be extended to a G-module if and only if the bundle E is globally trivial, $E \cong X \times E_0$.

The above evaluation map, ev_0, of course, corresponds to the usual evaluation map of sections, $\mathrm{ev} : X \times V \to E$ defined by $\mathrm{ev}(x, s) = s(x)$, $x \in X$, $s \in V = H^0(X, E)$. The bundle E is said to be *spanned by global sections* (or simply *spanned*) if ev is surjective,

$$X \times V \xrightarrow{\mathrm{ev}} E \longrightarrow 0$$

If E is spanned, then the dual map, ev^*, imbeds E^* into $X \times V^*$. Composing ev^* with projection onto V^* yields a map

$$\nu : E^* \longrightarrow V^*$$

which imbeds the fibers E_0^* of E^* as linear subspaces of V^* and sends the zero section Z_X of E^* to the origin of V^*. By projectivizing we also obtain a map

$$\mathbb{P}\nu : \mathbb{P}(E^*) \longrightarrow \mathbb{P}(V^*)$$

which imbeds the fibers $\mathbb{P}(E_0^*)$ of $p : \mathbb{P}(E^*) \to X$ linearly into $\mathbb{P}(V^*)$. Notice that ν thus defines an equivariant map of X to a Grassmann manifold by sending $x \in X$ to the point represented by the subspace $\nu(E_x^*) \subset V^*$. In the case where E is a line bundle, the projectivization $\mathbb{P}(E^*)$ is isomorphic to X, and this map to a Grassmann manifold coincides with $\mathbb{P}\nu$ giving the canonical map of X to projective space defined by the sections of E.

A canonical hermitian metric for E can be induced from a hermitian metric in V as follows. First observe that a hermitian metric, similar to the case of sections, is given by a map $h_E : G \to \mathrm{GL}(E_0)$ satisfying $\overline{h_E}^t = h_E$ and $h_E(gp) = \overline{\varphi(p)}^t h_E(g)\varphi(p)$ for $g \in G$, $p \in P$ where $\varphi : P \to \mathrm{GL}(E_0)$ defines the P-module structure of E_0. The latter condition ensures that for $[g, (z, w)] \in G \times_P (E_0 \times E_0)$ the hermitian product

$$(z, w)_E = \bar{w}^t h_E(g) z$$

is well-defined. Choosing a basis of weight vectors $s_1, \ldots, s_m$ for V and a dual basis $\eta_1, \ldots, \eta_m$ for V^* allows us to write ν as $\nu[g, z] = \sum_k z(s_k(g))\eta_k$ for $[g, z] \in G \times_P E_0$. A metric h_{E^*} for E^* is then defined by

$$\bar{w}^t h_{E^*}(g) z = \overline{\nu[g, w]}^t \nu[g, z] = \sum_{k=1}^{m} \overline{w(s_k(g))} z(s_k(g))$$

where the transpose is from a column vector to a row vector. Equivalently,

$$h_{E^*}(g) = \sum_{k=1}^{m} \overline{s_k(g)} s_k(g)^t$$

The metric for E is

$$h_E = (h^t_{E^*})^{-1} = \left(\sum_{k=1}^{m} s_k(g) \overline{s_k(g)^t} \right)^{-1}$$

This construction does not depend on the bundle being homogeneous—similar definitions for h_{E^*} can be made using local trivializations of E^* instead—and can be extended to any vector bundle E on X as long as X is a projective manifold: Let L be an ample line bundle on X such that $E \otimes L$ is spanned. The metrics $h_{E\otimes L}$ and h_L are defined as above, and then a metric for E is given by $h_E = (h_{E\otimes L})h_L^{-1}$.

The inverse image under the map ν of a ball centered at the origin in V^* gives a tube neighborhood of the zero section Z_X in E^*. The Levi form, $\mathcal{L}$, of the boundary of this tube retains the positive eigenvalues of the ball it came from. Therefore, there can be at most k non-positive eigenvalues of $\mathcal{L}$ where k is the maximum fiber dimension of ν. This number k can also be determined from the eigenvalues of the curvature form $\Theta_E = \bar{\partial}(h_E^{-1}\partial h_E)$, see §8. Such information has implications for the cohomology of E and its symmetric powers $E^{(m)}$. For example, the theorem of Andreotti-Grauert [1] in this case states that $H^q(X, E^{(m)}) = 0$ for $q > k$ if m is sufficiently large.

Maps similar to ν and $\mathbb{P}\nu$ can be constructed in a slightly more general setting. Let $\mathbb{P}(E^*)$ be the projectivization of E^*, that is, the quotient of the natural $\mathbb{C}^*$ action on $E^* \setminus Z_X$, and let $\xi_E \to \mathbb{P}(E^*)$ be the associated tautological line bundle that is isomorphic to the hyperplane section bundle, $\mathcal{O}_{\mathbb{P}(E_0^*)}(1)$, on the projective space fibers $\mathbb{P}(E_0^*)$ of $p : \mathbb{P}(E^*) \to X$. As manifolds, $E^* \setminus Z_X \cong \xi_E \setminus Z_{\mathbb{P}(E^*)}$ where $Z_{\mathbb{P}(E^*)}$ is the zero section of ξ_E. Moreover, for any positive integer m there is a natural isomorphism of sheaves, $p_*\xi_E^m \cong E^{(m)}$. Instead of requiring that E^* be spanned, we may assume the weaker condition that some power, say ξ_E^m, is spanned over $\mathbb{P}(E^*)$. Letting $V_m = H^0(\mathbb{P}(E^*), \xi_E^m) \cong H^0(X, E^{(m)})$ we obtain the maps

$$\mathbb{P}\nu_m : \mathbb{P}(E^*) \longrightarrow \mathbb{P}(V_m^*) \quad \text{and} \quad \nu_m : E^* \longrightarrow V_m^*$$

The latter is obtained from the former by lifting. The fibers of ν_m are finite on the fibers E_0^* so the analysis of the eigenvalues of the Levi form of a tube neighborhood goes through as before, see §7.3. These ideas lead to the the notion of k-ampleness, see [53].

Definition 5.3. A line bundle $L \to X$ is k-ample if some power L^m is spanned and the fibers of $\mathbb{P}\nu_m : X \to \mathbb{P}(V_m^*)$ have dimension at most k. A vector bundle E is k-ample if $\xi_E \to \mathbb{P}(E^*)$ is k-ample.

Classically, a line bundle $L \to X$ is said to be very ample if $\mathbb{P}\nu : X \to \mathbb{P}(V^*)$ is an imbedding and L is said to be ample if some power L^m is very ample. Thus, 0-ample in the above sense is equivalent to ample in the classical sense (if the fibers of $\mathbb{P}\nu_m$ are zero dimensional, some higher value of m yields an imbedding).

Most of the familiar vanishing theorems for ample bundles have versions for k-ample bundles. For example:

Theorem 5.4 ([36, 53]). *If $E \to X$ is a k-ample vector bundle, then $H^{p,q}(X, E) = H^q(X, \Omega^p_X \otimes E) = 0$ for $p + q \geq k + \operatorname{rank} E + \dim X$.*

A homogeneous vector bundle $E = G \times_P E_0$ is said to irreducible if the representation of P on E_0 is irreducible. For example, any homogeneous line bundle on G/P is necessarily irreducible since the representation of P is one-dimensional. Since a Borel subgroup $B \subset G$ is solvable, an irreducible representation of B is necessarily one-dimensional and so homogeneous line bundles are the only irreducible homogeneous vector bundles on G/B.

We shall see that irreducible homogeneous bundles often have the sharpest theorems and the most detailed formulas associated to them. Unfortunately, many interesting bundles are not irreducible, for example, most tangent bundles on G/P. Nevertheless, it is it is often possible to draw conclusions about an arbitrary homogeneous vector bundle $E = G \times_P E_0$ from the irreducible case by considering a filtration of E_0 by P-submodules, as follows.

If L_P is a Levi-factor of P, then the module E can be decomposed into a direct sum irreducible L_P-modules, $E_0 = F_1 \oplus \cdots \oplus F_t$, see §1. Furthermore, since the unipotent radical, U_P, is normal in P and acts on E_0 in a 'triangular' fashion, it is clear that these L_P-irreducible factors can be arranged (non-uniquely) such that $U_P F_i \subset F_j$ with $j \geq i$. Therefore, E_0 has a filtration by P-submodules

$$E_0 \supset E_1 \supset \cdots \supset E_t \supset E_{t+1} = 0$$

such that $E_i/E_{i+1} \cong F_i$ is an irreducible P-module with maximal weight μ_i, $i \leq i \leq t$.

Definition 5.5. The maximal weights of a P-module E_0, denoted $\Lambda_{\max}(E_0)$, are the maximal weights of the irreducible factors associated to a filtration of E_0 by P-submodules, as above.

Equivalently, we could define $\Lambda_{\max}(E_0) = -\Lambda(E_0^{*U})$, where $U \subset P$ is the maximal unipotent subgroup of G generated by all the negative root groups, see §4.

6. Line Bundles

We now concentrate on the case of homogeneous line bundles. The following well-known lemma is the first step in classifying projective homogeneous manifolds.

Lemma 6.1. *Let X be a compact complex space and let R be a connected solvable complex Lie group acting holomorphically on X. Assume that with respect to this action X is equivariantly imbedded into some complex projective space $\mathbb{P}^N$. Then R has a fixed point on X, that is, there is a point $x \in X$ such that $r \cdot x = x$ for all $r \in R$.*

Proof. By Lie's Theorem, R stabilizes a flag of linear subspaces

$$P_0 \subset P_1 \subset \cdots \subset P_N = \mathbb{P}^N$$

where $P_k \cong \mathbb{P}^k$, $0 \leq k \leq N$. Let $X_k = X \cap P_k$ and define $X_{-1} = \emptyset$. Let k be the least integer for which $X_k \neq \emptyset$ and $X_{k-1} = \emptyset$. Then X_k is a compact complex space contained in $P_k \setminus P_{k-1} \cong \mathbb{C}^k$. Therefore, X_k must be a finite set of points stabilized by R. Since R is connected, it must fix each of the points in X_k. □

Recall that if L is an ample line bundle on X, then the sections $V_m = H^0(X, L^m)$ of some power of L imbed X into $\mathbb{P}(V_m^*)$. If L is homogeneous with respect to G then G acts linearly on V_m and the imbedding is naturally G-equivariant. The following proposition shows that there is a significant restriction on the type of spaces for which this can occur.

Proposition 6.2. *Suppose there exists an ample homogeneous line bundle on a connected homogeneous compact complex manifold X. Then X is homogeneous under a connected, simply-connected, semisimple complex Lie group G, $X = G/P$, and the isotropy subgroup P is parabolic.*

Proof. Let X be homogeneous under a complex Lie group G such that there exists a G-equivariant imbedding of X into some projective space. Since X is connected we may assume G is connected. Let $G = R \cdot S$ be a Levi-Malčev decomposition of G where R is the radical of G (a maximal connected normal solvable subgroup) and S is a connected semisimple complex subgroup of G. By Lemma 6.1, R fixes a point in X. But then R fixes every point of X since R is normal in G. Hence, the semisimple group S acts transitively on X. It is an easy matter to lift the action to the universal cover $\tilde{S}$ of S which is still a semisimple complex Lie group. Now let B be a Borel subgroup of $\tilde{S}$. Again by 6.1, B fixes some point in X and thus is contained in the isotropy subgroup of that point. □

From now on we shall assume that $X = G/P$ where G is a connected simply-connected semisimple complex Lie group and P is a parabolic subgroup. Our next goal is to understand the additive group of holomorphic line bundles, $\mathrm{Pic}(X)$, on X.

Let $L \to X = G/P$ be a homogeneous line bundle on X. Then $L = G \times_P \mathbb{C}$ determines a homomorphism $\hat{\lambda} : P \to \mathrm{GL}(1, \mathbb{C}) \cong \mathbb{C}^*$. Since $\hat{\lambda}|_{S_P}$ and $\hat{\lambda}|_{U_P}$ are necessarily trivial, we see that $\hat{\lambda}$ is determined by its restriction to T_P, which in turn defines a weight $\lambda \in \Lambda$ that is perpendicular to the weights of P: $(\lambda, \alpha) = 0$ for all $\alpha \in P$. Conversely, starting with weight λ perpendicular to the weights of P, we can construct a character $\hat{\lambda} : P \to \mathbb{C}^*$ that defines a homogeneous line bundle on X. Thus we see that the homogeneous line bundles on X are in one-to-one correspondence with the set of weights

$$\Lambda_X = \{\lambda \in \Lambda \mid (\lambda, \alpha) = 0 \text{ for all } \alpha \in \Phi_P\}$$

which we call the weights of X. Note that $(\lambda, \alpha) = 0$ for all $\alpha \in \Phi_P$ is equivalent to $\langle \lambda, \alpha_i \rangle = 0$ for all $i \in I$, where I is the subset of indexes that defines P. Thus, with respect to the basis of fundamental dominant weights, the weights of X are the weights with i-th coordinate zero for $i \in I$.

A remarkable fact about line bundles on $X = G/P$ is that they must always be homogeneous with respect to G. This is certainly not the case for vector bundles of higher rank. We first prove a lemma.

Lemma 6.3. *Let $X = G/P$ with G semisimple and P a parabolic subgroup. Then $\pi_1(X) = 0$ and $H^1(X, \mathcal{O}_X) = 0$.*

Proof. Since X is compact, we need only show that $\pi_1(X) = 0$. For if $H^1(X, \mathcal{O}_X) \cong H^0(X, \Omega_X)$ were then not also trivial, we could construct non-constant holomorphic functions on X by integrating closed holomorphic forms. By Theorem 3.1, there is an open dense subset of X isomorphic to $\mathbb{C}^n$. Since the inclusion map of this cell into X lifts to an inclusion map of the cell to a dense open subset of the universal cover $\pi : \tilde{X} \to X$, we see that π must be one-to-one and X is simply-connected. □

Theorem 6.4. *Let $X = G/P$ with G semisimple and P a parabolic subgroup. If L is any holomorphic line bundle on X, then L is homogeneous with respect to G. In particular,* $\operatorname{Pic}(X) \cong \Lambda_X$.

Proof. Holomorphic line bundles on X correspond to elements in $H^1(X, \mathcal{O}_X^*)$. The short exact sequence $0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0$ leads to natural maps $H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X^*) \to H^2(X, \mathbb{Z})$, the latter sending a holomorphic line bundle to its topological class. Since $H^2(X, \mathbb{Z})$ is discrete and G is connected, the topological class of g^*L must be the same as L. By Lemma 6.3, $H^1(X, \mathcal{O}_X) = 0$, so the two bundles L and g^*L are in fact isomorphic as holomorphic line bundles. Since there are no non-trivial homomorphisms $G \to \mathbb{C}^*$, these isomorphisms for $g \in G$ define an action of G on L by bundle automorphisms and L is homogeneous with respect to G. □

Now that we know that line bundles on $X = G/P$ correspond to weights Λ_X, we can investigate how various properties of the line bundle can be translated into properties of weights. The next theorem addresses the question of when a line bundle is spanned or is ample.

Theorem 6.5 (Borel-Weil [42])**.** *Let L be a holomorphic line bundle on $X = G/P$ where G is semisimple and P is a parabolic subgroup defined by a set of indexes I. Let $\lambda = \sum_{i \notin I} n_i \lambda_i \in \Lambda_X$ be the weight associated to L. Then*

1. *L is spanned at one point of X iff L is spanned at every point of X iff $n_i \geq 0$ for $i \notin I$ (i.e. iff λ is dominant).*
2. *L is ample iff L is very ample iff $n_i > 0$ for $i \notin I$.*
3. *If λ is dominant, then $H^0(X, L)$ is isomorphic to the irreducible G-module V^λ.*
4. *If λ is not dominant, then $H^0(X, L) = 0$.*

Proof. 1. ($\Rightarrow$) Let $s \in V = H^0(X, L)$ be a section that is non-zero at $x \in X$ and let $y = g \cdot x$, $g \in G$, be any other point of X. Here s corresponds to a function $s : G \to \mathbb{C}$ such that $s(gp^{-1}) = \hat{\lambda}(p)s(g)$ for all $g \in G$ and $p \in P$ (see §5). Since L is homogeneous with respect to G, $g \cdot s$ is also a section of L that is non-zero at y: $g \cdot s(y) = s(g^{-1}y) = s(x)$. It follows that the P-equivariant evaluation map $\mathrm{ev}_0 : V \to \mathbb{C}$ is non-zero. Let s be a weight vector in V of weight $\hat{\mu}$ such that $\mathrm{ev}_0(s) = s(1) \neq 0$ and let W be the irreducible G-submodule generated by s. Now for all $t \in T$, $(t \cdot s)(1) = s(t^{-1} \cdot 1) = \hat{\mu}(t)s(1)$. On the other hand, $s(1 \cdot t^{-1}) = \hat{\lambda}(t)s(1)$ for all $t \in T \subset P$, so $\mu = \lambda$. Moreover, since λ is perpendicular to Φ_P, $\mathrm{ev}_0(X_\alpha s) = \lambda(X_\alpha)\,\mathrm{ev}_0(s) = 0$ for all roots $\alpha \in \Phi_P$. Since the kernel of ev_0 is P-invariant and all the negative roots are in Φ_P, we conclude that s is a maximal weight vector and $W \cong V^\lambda$. Moreover, by Proposition 5.1, $\mathrm{ev}_0 : V \to \mathbb{C}$ factors through the projection of W onto its maximal weight space $\mathbb{C} \cdot s$. In particular, λ is dominant.

($\Leftarrow$) Let $\lambda \in \Lambda_X^+$ be dominant, let $v \in V^\lambda$ be a maximal weight vector with weight $\hat{\lambda}$, and let $f \in (V^\lambda)^*$ be dual to v: $f(v) = 1$ and $f(u) = 0$ for u in any other weight space. Since λ is perpendicular to the weights of P, it can be extended to a character $\hat{\lambda} : P \to \mathbb{C}^*$. For $X \in \mathfrak{p}$, we have $X.v = \lambda(X)v + u(X)$ where $u(X)$ is a linear combination of vectors from weight spaces other than λ. Hence, $f(p.v) = \hat{\lambda}(p)f(v)$ for all $p \in P$. Define a map $s : G \to \mathbb{C}$ by $s(g) = f(g^{-1}.v)$. Then $s(gp^{-1}) = \hat{\lambda}(p)s(g)$ for all $g \in G$ and $p \in P$, see Proposition 5.1. Hence, there is a non-zero section of the line bundle on G/P defined by $\hat{\lambda}$.

2. Since L^m corresponds to the weight $m\lambda$, it is clear that we need only show that L being very ample is equivalent to $n_i > 0$ for all $i \notin I$. We may also assume λ is dominant, so that L is spanned.

Suppose $n_j = 0$ for some $j \notin I$. Let $J = I \cup \{j\}$ and let Q be the parabolic subgroup defined by J. Since the roots of Q are perpendicular to the weight λ, the character $\hat{\lambda}$ can be extended to Q. This implies that L is the pull-back of a line bundle on G/Q, and hence L is not very ample, see §5.

Conversely, if L is not very ample then the map given by the sections of L, $\pi : G/P \to G/Q$, is not an imbedding. The character $\hat{\lambda}$ therefore extends to a parabolic subgroup Q that properly contains P. Hence there is a simple root α_j not in P such that $n_j = \langle \lambda, \alpha_j \rangle = 0$.

3. Let $W \cong V^\mu$ be any non-trivial irreducible G-submodule of V. There must be at least one weight vector s in W such that $\mathrm{ev}_0(s) \neq 0$. Otherwise, for any weight vector s in W, $\mathrm{ev}_0(g \cdot s) = 0$ for all $g \in G$ and hence $s = 0$, a contradiction. By part 1, we know that $W \cong V^\lambda$, so $\mu = \lambda$, and $\mathrm{ev}_0 : V \to \mathbb{C}$ factors through the projection of W onto $\mathbb{C} \cdot s$. Since W is arbitrary, we conclude that $V \cong V^\lambda$.

4. This is a simple consequence of 1.

□

The proof of 2 above shows how to describe the map of $X = G/P$ to projective space defined by the sections of a spanned line bundle.

Corollary 6.6. *Let L be a line bundle on $X = G/P$ defined by a weight $\lambda \in \Lambda_X$. Assume that L is spanned by global sections so that λ is dominant. Then the map $\pi : X \to Y \subset \mathbb{P}(V^*)$ defined by the sections of L, $V = H^0(X, L)$, is a homogeneous fibration $G/P \to G/Q$ where Q is the parabolic subgroup of G defined by the simple roots perpendicular to λ. In particular, λ defines a very ample line bundle L' on Y and $L = \pi^* L'$.*

7. Curvature

We now discuss some aspects of the curvature of a homogeneous vector bundle $E = G \times_P E_0$ on $X = G/P$ where P is a parabolic subgroup of G. We first compute the curvature form $\Theta_E = \bar{\partial}(h_E^{-1} \partial h_E)$ associated to a natural left-invariant hermitian metric h_E on E. Since Θ_E is also left-invariant, it is determined by its form at the identity coset. To carry out this computation, we rely on the notation of §2 and §5, and use local coordinates x_α, $\alpha \in \Phi_X$, defined for $g \in G$ near 1 by

$$g = \prod_{\alpha \in \Phi_X} \exp(x_\alpha X_\alpha)$$

In the next theorem we assume that E is spanned by global sections. The curvature form of an arbitrary homogeneous vector bundle E can then be derived, as usual, by $\Theta_E = \Theta_{E \otimes L} - I_E \Theta_L$ where L is an ample line bundle on X such that $E \otimes L$ is spanned by global sections.

Theorem 7.1. *Let $E = G \times_P E_0$ be a homogeneous vector bundle on $X = G/P$ where P is a parabolic subgroup of G. Assume E is spanned by global sections so that the P-module homomorphism $\phi : V = H^0(X, E) \to E_0$ is surjective. Let $v_1, \ldots, v_m$ be a basis of weight vectors for V and $e_1, \ldots, e_r$ for E_0 such that $\phi(v_k) = e_k$, $1 \leq k \leq r$. Then there is a natural left-invariant hermitian metric for E whose associated curvature form is given at the identity coset by*

$$\Theta_E = \sum_{\alpha, \beta \in \Phi_X} \sum_{k > r} \phi(X_\alpha \cdot v_k) \overline{\phi(X_\beta \cdot v_k)^t} dx_\alpha \wedge d\bar{x}_\beta$$

Proof. As in §5, we define sections of E, $s_k : G \to E_0$, by $s_k(g) = \phi(g^{-1} \cdot v_k)$ for $1 \leq k \leq m$ so that a hermitian metric $h_E : G \to \mathbb{R}$ for E is given by

$$h_E = \Big(\sum_{k=1}^m s_k \overline{s_k^t} \Big)^{-1}$$

Let

$$A = \sum_{k=1}^m s_k \overline{s_k^t} = h_{E^*}^t$$

Since $\partial h_E = -h_E (\partial A) h_E$, we obtain

$$\Theta_E = \bar{\partial}(h_E^{-1} \partial h_E) = (\partial \bar{\partial} A) h_E - (\partial A) h_E \wedge (\bar{\partial} A) h_E$$

Note that $h_E(1) = I$, $s_k(1) = e_k$, and

$$\frac{\partial s_k}{\partial x_\alpha}(1) = \phi(-X_\alpha \cdot v_k)$$

We abbreviate $\phi(X_\alpha \cdot v_k)$ by $\phi_{\alpha,k}$ so that at the identity coset the above expression for Θ_E becomes

$$\begin{aligned} \Theta_E &= \partial \bar{\partial} A - \partial A \wedge \bar{\partial} A \\ &= \sum_{\alpha, \beta \in \Phi_X} \left[\sum_{k=1}^m \phi_{\alpha,k} \overline{\phi_{\beta,k}^t} - \sum_{j,k=1}^m \phi_{\alpha,k} e_k^t e_j \overline{\phi_{\beta,j}^t} \right] dx_\alpha \wedge d\bar{x}_\beta \\ &= \sum_{\alpha, \beta \in \Phi_X} \sum_{k > r} \phi_{\alpha,k} \overline{\phi_{\beta,k}^t} dx_\alpha \wedge d\bar{x}_\beta \end{aligned}$$

as claimed. □

The curvature Θ_E can be written in a particularly simple way. Let F_0 denote the kernel of $\phi : V \to E_0$, i.e., the subspace of V spanned by v_k, $k > r$. For $\alpha \in \Phi_X$ define linear maps $\phi_\alpha : F_0 \to E_0$ by

$$\phi_\alpha(v) = \phi(X_\alpha \cdot v)$$

and maps $\phi_\alpha^t : E_0 \to F_0$ dual to ϕ_α with respect to the above bases. Then

$$\Theta_E = \sum_{\alpha,\beta\in\Phi_X} \phi_\alpha \overline{\phi_\beta^t} dx_\alpha \wedge d\bar{x}_\beta = \Psi \wedge \overline{\Psi^t}$$

where

$$\Psi = \sum_{\alpha\in\Phi_X} \phi_\alpha dx_\alpha$$

This description makes it easier to see the properties of the curvature form

$$\Theta_E(z) = z^t \Theta_E \bar{z}$$

for $z \in E_0$.

Definition 7.2. We say that $\Theta_E(z)$ is positive semidefinite if $\Theta_E(z)(\eta, \bar{\eta}) \geq 0$ for all $\eta \in (T_X)_0$. The kernel of $\Theta_E(z)$ is the subspace of $\eta \in (T_X)_0$ such that $\Theta_E(z)(\eta, \bar{\eta}) = 0$. The flatness of Θ_E is defined to be

$$\mathrm{fl}\, \Theta_E = \max_{z\in E_0\setminus 0} \dim\ker \Theta_E(z)$$

If (,) denotes the hermitian metric on V with respect to the basis v_k, $1 \leq k \leq m$, then

$$\Theta_E(z) = \sum_{\alpha,\beta\in\Phi_X} (\phi_\alpha^t z, \phi_\beta^t z) dx_\alpha \wedge d\bar{x}_\beta$$

and for $\eta = \sum_{\alpha\in\Phi_X} \eta_\alpha \partial/\partial x_\alpha \in (T_X)_0$,

$$\Theta_E(z)(\eta, \bar{\eta}) = \sum_{\alpha,\beta\in\Phi_X} (\eta_\alpha \phi_\alpha^t z, \eta_\beta \phi_\beta^t z) = |\Psi^t(z)\eta|^2$$

Thus we may conclude the following.

Corollary 7.3. *If E is spanned by global sections, the curvature form $\Theta_E(z)$ is positive semidefinite for all $z \in E_0$, its kernel equals the kernel of $\Psi^t(z) : (T_X)_0 \to F_0$, $\Psi^t(z) = \sum_{\alpha\in\Phi_X} \phi_\alpha^t(z) dx_\alpha$, and the flatness of Θ_E is given by*

$$\mathrm{fl}\, \Theta_E(z) = \max_{z\in E_0\setminus 0} \dim\ker \Psi^t(z)$$

The ‘flatness’ of Θ_E is related to the ‘ampleness’ of the bundle E, see §8.

Remark 7.4. It is possible to construct a holomorphic connection θ_E and its associated curvature form $\Theta_E = \bar{\partial}\theta_E$ for a homogeneous vector bundle $E = G \times_P E_0$ using a basis of left-invariant forms, their duals, and the representation of $\mathfrak{p}$ on E_0, see, e.g., [24, 22]. In general, such a connection is a metric connection only when the representation of P on E_0 is *irreducible*. In this case, the resulting curvature form Θ_E is also non-degenerate, and its signature is given by the index of the maximal weight of E_0, see [24, Theorem 4.17]. While some homogeneous vector bundles (e.g., line bundles) are known to be irreducible, many other important bundles (e.g., most tangent bundles) are not.

7.1. **Curvature of tangent bundles.** As an example, let us compute the curvature form for the holomorphic tangent bundle $E = T_X$ of a homogeneous manifold of the form $X = \mathrm{SL}(\ell+1,\mathbb{C})/P$ where P is parabolic subgroup. In this case $E_0 = (T_X)_0$ can be viewed as a vector space of certain strictly upper triangular matrices by identifying the root vectors X_α, $\alpha \in \Phi_X$, with the elementary matrices $e_{\mu\nu}$ indexed as follows. Define 'row' and 'column' indexes by

$$R_\nu = \{i \mid e_{i\nu} = X_\alpha \text{ for some } \alpha \in \Phi_X\}$$
$$C_\mu = \{j \mid e_{\mu j} = X_\alpha \text{ for some } \alpha \in \Phi_X\}$$

Both R_ν and C_μ may be empty, but if they are not then they have the form $R_\nu = \{1,\dots,r_\nu\}$ and $C_\mu = \{c_\mu,\dots,\ell+1\}$. In fact, if I is the set of indexes that defines P, then

$$r_\nu = \max\{i \in I \mid i < \nu\}, \quad c_\mu = \min\{i \in I \mid i \geq \mu\} + 1$$

For $\alpha \in \Phi_X$ we have $X_\alpha = e_{\mu\nu}$ if and only if $1 \leq \mu \leq \ell+1$ and $\nu \in C_\mu$ (or $1 \leq \nu \leq \ell+1$ and $\mu \in R_\nu$). Note that $\mu \in R_\nu$ if and only if $\nu \in C_\mu$.

The space of sections $V = H^0(X, T_X)$ is isomorphic to the Lie algebra of $G = \mathrm{SL}(\ell+1,\mathbb{C})$. The map $\phi : V \to (T_X)_0$ is just the natural projection of matrices and the module structure is given by Lie brackets of matrices. Thus, if $X_\alpha = e_{\mu\nu}$, and $v_k = e_{ij}$, then '$k > r$' means $i \notin R_j$ (or $j \notin C_i$), and

$$\phi_{\alpha,k} = \phi(X_\alpha \cdot v_k) = \phi([e_{\mu\nu}, e_{ij}]) = \begin{cases} e_{\mu j} & \text{if } i = \nu \text{ and } j \in C_\mu \setminus C_\nu \\ -e_{i\nu} & \text{if } j = \mu \text{ and } i \in R_\nu \setminus R_\mu \\ 0 & \text{otherwise} \end{cases}$$

Let us write $x_{\mu\nu}$ for the coordinate x_α, and let $f^{\rho\sigma}_{\mu\nu}$ denote the $r \times r$ elementary matrix $e_{\mu\nu} e^t_{\rho\sigma}$ (the transpose here is of a column vector to a row vector not the usual transpose of a matrix, and $r = \dim X$). Let $\beta \in \Phi_X$ be another root with $X_\beta = e_{\rho\sigma}$. If $\nu = \sigma$ then

$$\phi_\alpha \overline{\phi^t_\beta} = \sum_{k>r} \phi_{\alpha,k} \overline{\phi^t_{\beta,k}} = \sum_{j \in C_\mu \cap C_\rho \setminus C_\nu} f^{\rho j}_{\mu j}$$

while if $\mu = \rho$, then

$$\phi_\alpha \overline{\phi^t_\beta} = \sum_{i \in R_\nu \cap R_\sigma \setminus R_\mu} f^{i\sigma}_{i\nu}$$

Finally, if $\nu \in R_\sigma \setminus R_\rho$ and $\rho \in C_\mu \setminus C_\nu$, then $\phi_\alpha \overline{\phi^t_\beta} = -f^{\nu\sigma}_{\mu\rho}$, while if $\sigma \in R_\nu \setminus R_\mu$ and $\mu \in C_\rho \setminus C_\sigma$, then $\phi_\alpha \overline{\phi^t_\beta} = -f^{\sigma\nu}_{\rho\mu}$. For any other combination of μ, ν, ρ and σ, $\phi_\alpha \overline{\phi^t_\beta} = 0$. Let δ_{pq} have the value 1 if $p = q$ and 0 otherwise. Define ε^{st}_{pq} to be 1 if $s \in R_t \setminus R_q$ and $q \in C_p \setminus C_s$, and to be 0 otherwise. According to Theorem 7.1 and the above description of $\phi_\alpha \overline{\phi^t_\beta}$ we obtain

$$\Theta_{T_X} = \sum_{\mu \in R_\nu, \rho \in R_\sigma} \Bigg[\delta_{\nu\sigma} \sum_{j \in C_\mu \cap C_\rho \setminus C_\nu} f^{\rho j}_{\mu j} + \delta_{\mu\rho} \sum_{i \in R_\nu \cap R_\sigma \setminus R_\mu} f^{i\sigma}_{i\nu} - \varepsilon^{\nu\sigma}_{\mu\rho} f^{\nu\sigma}_{\mu\rho} - \varepsilon^{\sigma\nu}_{\rho\mu} f^{\sigma\nu}_{\rho\mu} \Bigg] \, dx_{\mu\nu} \wedge d\bar{x}_{\rho\sigma}$$

Another way to express this is to view Θ_{T_X} as a $r \times r$ matrix of $(1,1)$-forms and let Θ^{st}_{pq} denote the entry in the pq-row and st-column of Θ_{T_X}. Then

$$\Theta^{st}_{pq} = \delta_{qt} \sum_{\nu \notin R_q} dx_{p\nu} \wedge d\bar{x}_{s\nu} \quad + \quad \delta_{ps} \sum_{\mu \notin C_p} dx_{\mu q} \wedge d\bar{x}_{\mu t} \quad - \quad \varepsilon^{st}_{pq} dx_{ps} \wedge d\bar{x}_{qt} - \quad \varepsilon^{st}_{pq} dx_{qt} \wedge d\bar{x}_{ps}$$

The map $\phi_\alpha : F_0 \to E_0$ is given by

$$\phi_\alpha(v) = \sum_{j \in C_\mu \setminus C_\nu} v_{\nu j} e_{\mu j} - \sum_{i \in R_\nu \setminus R_\mu} v_{i\mu} e_{i\nu}$$

where $v = \sum_{i \notin R_j} v_{ij} e_{ij} \in F_0$. The dual $\phi^t_\alpha : E_0 \to F_0$ is

$$\phi^t_\alpha(z) = \sum_{j \in C_\mu \setminus C_\nu} z_{\mu j} e_{\nu j} - \sum_{i \in R_\nu \setminus R_\mu} z_{i\nu} e_{i\mu}$$

where $z = \sum_{i \in R_j} z_{ij} e_{ij} \in E_0$. Thus, for $p \in R_q$,

$$\Psi^t(e_{pq}) = \sum_{\nu \notin R_q} e_{\nu q} dx_{p\nu} - \sum_{\mu \notin C_p} e_{p\mu} dx_{\mu q}$$

It follows that the kernel of $\Psi^t(e_{pq})$ is the set of tangent vectors η whose coordinates $\eta_{\mu q} = 0$ for $\mu \notin C_p$ and $\eta_{p\nu} = 0$ for $\nu \notin R_q$. Therefore

$$\dim \ker \Psi^t(e_{pq}) = \dim X - (\ell + c_p - r_q - 1)$$

The maximum of this kernel dimension for $z \in (T_X)_0$ is achieved at $z = e_{i,i+1}$, $i \in I$, where $c_i = i + 1$ and $r_{i+1} = i$, and thus

$$\operatorname{fl} \Theta_E = \max_{z \in (T_X)_0 \setminus 0} \dim \ker \Psi^t(z) = \dim X - \ell$$

a fact that will be recalled in §8. Notice that among these examples the only case where $\Theta_E(z)$ is positive definite is for complex projective space, $X = \mathbb{P}^\ell$.

Two extreme examples of such homogeneous spaces are the flag manifold and a Grassmann manifold. The flag manifold is the space of all full 'flags' of linear subspaces in $(\ell+1)$-space: $V_0 \subset V_1 \subset \cdots \subset V_{\ell+1}$, $\dim V_i = i$. For this manifold, P is a Borel subgroup and $I = \{1, \ldots, \ell\}$. The row and column indexes are

$$R_\nu = \{1, \ldots, \nu - 1\}, \quad C_\mu = \{\mu + 1, \ldots, \ell + 1\}$$

which are easily applied to the above formulas for Θ_{T_X}.

For the Grassmann manifold $X = \mathrm{Gr}(\omega, \ell + 1)$ of ω-planes in $(\ell+1)$-space, P is a maximal parabolic subgroup and $I = \{\ell + 1 - \omega\}$. The row and column indexes in this case are

$$\begin{aligned} R_\nu &= \{1, \ldots, \ell + 1 - \omega\} && \text{for } \nu \geq \ell + 2 - \omega \\ C_\mu &= \{\ell + 2 - \omega, \ldots, \ell + 1\} && \text{for } \mu \leq \ell + 1 - \omega \end{aligned}$$

The formula for Θ_{T_X} simplifies somewhat:

$$\Theta_{T_X} = \sum_{\mu,\rho=1}^{\ell+1-\omega} \sum_{\nu,\sigma=\ell+2-\omega}^{\ell+1} \left[\delta_{\nu\sigma} \sum_{j=\ell+2-\omega}^{\ell+1} f^{\rho j}_{\mu j} \quad + \quad \delta_{\mu\rho} \sum_{i=1}^{\ell+1-\omega} f^{i\sigma}_{i\nu} \right] dx_{\mu\nu} \wedge d\bar{x}_{\rho\sigma}$$

and

$$\Theta^{st}_{pq} = \delta_{qt} \sum_{\nu=\ell+2-\omega}^{\ell+1} dx_{p\nu} \wedge d\bar{x}_{s\nu} \quad + \quad \delta_{ps} \sum_{\mu=1}^{\ell+1-\omega} dx_{\mu q} \wedge d\bar{x}_{\mu t}$$

The tangent bundle of T_X is naturally isomorphic to $E \otimes Q^*$ where E is the tautological bundle of rank ω and Q is the quotient bundle of rank $\ell + 1 - \omega$. As a further example, let us now work out the curvature form for E—the analysis for Q^* is similar—and compare the result with the above expression for Θ_{T_X}.

A basis for $V = H^0(X, E)$ is given by the elementary matrices, $v_k = e_{1,k}$, $2 \leq k \leq \ell + 1$, a basis for E_0 is given by $e_k = v_k$, $\ell + 2 - \omega \leq k \leq \ell + 1$, and the map $\phi : V \to E_0$ is the natural projection of matrices. The action of a root vector $X_\alpha = e_{\mu\nu}$ on v_k is given by matrix multiplication:

$$X_\alpha \cdot v_k = -e_{1,k} \cdot e_{\mu\nu} = -\delta_{k,\mu} e_{1,\nu}$$

If f^q_p represents the $\omega \times \omega$ matrix $e_p e^t_q$, then Theorem 7.1 gives

$$\Theta_E = \sum_{\mu=1}^{\ell+1-\omega} \sum_{\nu,\sigma=\ell+2-\omega}^{\ell+1} f^\sigma_\nu dx_{\mu\nu} \wedge d\bar{x}_{\rho\sigma}$$

and

$$(\Theta_E)^t_q = \sum_{\mu=1}^{\ell+1-\omega} dx_{\mu q} \wedge d\bar{x}_{\mu t}$$

(compare with [23, p.195]). A similar calculation for Q^* yields

$$(\Theta_{Q^*})^s_p = \sum_{\nu=\ell+2-\omega}^{\ell+1} dx_{p\nu} \wedge d\bar{x}_{s\nu}$$

and we see that these two expressions give $\Theta_{T_X} = \Theta_E \otimes I_{Q^*} + I_E \otimes \Theta_{Q^*}$ as expected.

7.2. **Curvature of line bundles.** We now refine Theorem 7.1 for the case of line bundles. Of course, any holomorphic line bundle L on $X = G/P$, P parabolic, is homogeneous with respect to G, see Theorem 6.4. The curvature form Θ_L defines the first Chern class $c_1(L)$ up the constant factor $\sqrt{-1}/2\pi$, see §9. In order to have a convenient expression for Θ_L, we shall renormalize the metric used. Similar descriptions can be found in [7, Proposition 14.5] or [21, Proposition 7.1].

Theorem 7.5. *Let P be a parabolic subgroup of G and let L be a line bundle on $X = G/P$ defined by a weight $\lambda \in \Lambda_X$. There is a hermitian metric for L such that at the identity coset the curvature form is given by*

$$\Theta_L = \sum_{\alpha \in \Phi_X} \langle \lambda, \alpha \rangle dx_\alpha \wedge d\bar{x}_\alpha$$

Proof. Let I be the set of indexes that defines P and let L_i be the line bundle defined by λ_i for $i \notin I$. Let $\lambda = \sum_{i \notin I} n_i \lambda_i \in \Lambda_X$ be the weight associated to L. If we prove that there are hermitian metrics h_i for L_i, $i \notin I$, satisfying the statement of the theorem, then the theorem also holds for the hermitian metric $h_L = \prod_{i \notin I} h_i^{n_i}$

on $L = \bigotimes_{i \notin I} L_i^{n_i}$ since

$$\begin{aligned}\Theta_L = -\partial\bar{\partial}\log h &= \sum_{i\notin I} n_i(-\partial\bar{\partial}\log h_i) = \sum_{i\notin I} n_i \Theta_{L_i} \\ &= \sum_{i\notin I} n_i \sum_{\alpha\in\Phi_X} \langle \lambda_i, \alpha\rangle dx_\alpha \wedge d\bar{x}_\alpha = \sum_{\alpha\in\Phi_X} \langle \lambda, \alpha\rangle dx_\alpha \wedge d\bar{x}_\alpha\end{aligned}$$

Fix $i \notin I$ and let $V = V^{\lambda_i} = H^0(X, L_i)$. Choose a basis, $v_1, \dots, v_m$, of weight vectors for V so that v_1 has weight λ_i and such that all non-zero vectors of the form $X_{-\alpha} \cdot v_1$ for $\alpha > 0$ are included in the list. We further normalize any such vector v_k by defining $v_k = k_\alpha X_{-\alpha} \cdot v_1$ where

$$k_\alpha = \begin{cases} \langle \lambda_i, \alpha\rangle^{-1/2} & \text{if } \langle \lambda_i, \alpha\rangle \neq 0 \\ 1 & \text{otherwise} \end{cases}$$

The line bundle L_i is spanned by global sections so we may apply Theorem 7.1:

$$\Theta_{L_i} = \sum_{\alpha\in\Phi_X} \sum_{k>1} |\phi(X_\alpha \cdot v_k)|^2 dx_\alpha \wedge d\bar{x}_\alpha$$

where $\phi : V \to \mathbb{C}$ is the P-module homomorphism defined by the dual of v_1. Now, $\phi(X_\alpha \cdot v_k)$ vanishes unless $v_k = k_\alpha X_{-\alpha} \cdot v_1$ in which case

$$\begin{aligned}\phi(X_\alpha \cdot v_k) &= k_\alpha \phi(X_\alpha X_{-\alpha} \cdot v_1) = k_\alpha \phi([X_\alpha, X_{-\alpha}] \cdot v_1) \\ &= k_\alpha \lambda_i(Z_\alpha) = k_\alpha \langle \lambda_i, \alpha\rangle = \langle \lambda_i, \alpha\rangle^{1/2}\end{aligned}$$

We thus obtain $\Theta_{L_i} = \sum_{\alpha\in\Phi_X} \langle \lambda_i, \alpha\rangle dx_\alpha \wedge d\bar{x}_\alpha$ as claimed. □

7.3. **Levi curvature.** Let $E = G \times_P E_0$ be a homogeneous vector bundle with left-invariant hermitian metric h_E. There is a natural exhaustion function $\varphi_E : E \to \mathbb{R}$ defined by

$$\varphi_E[g, z] = |z|_E^2 = \bar{z}^t h_E(g) z$$

The tubular neighborhood N_ε of the zero section in E is given by $\varphi_E[g, z] < \varepsilon$. The Levi form $\mathcal{L}(\varphi_E) = \partial\bar{\partial}\varphi_E$ evaluated on the tangent space to the boundary ∂N_ε is left invariant. To calculate $\mathcal{L}(\varphi_E)$ at points $[1, z] \in E_0$ it is convenient to use local coordinates $\xi = (\xi_1, \dots, \xi_n)$ near the identity coset $x_0 \in X = G/P$ for which $h_E(x_0) = I$ and $dh_E(x_0) = 0$, see, e.g., [23, p.195]. In these coordinates, the tangent space to the boundary ∂N_ε at $[1, z]$ consists of tangent vectors of the form

$$\eta = \sum_{i=1}^r a_i \frac{\partial}{\partial z_i} + \sum_{j=i}^n b_j \frac{\partial}{\partial \xi_j}$$

such that

$$\sum_{i=1}^r \bar{z}_i a_i = 0$$

Also, at $[1, z]$

$$\mathcal{L}(\varphi_E) = \partial\bar{\partial}\varphi_E = \sum_{i=1}^r dz_i \wedge d\bar{z}_i - \bar{z}^t \Theta_E z$$

since in these coordinates $\Theta_E = -\partial\bar{\partial} h_E$ at $[1, z]$.

For line bundles, this curvature information is easy to derive and gives the following simple vanishing theorem.

Theorem 7.6. *Let P be a parabolic subgroup of G and let L be a line bundle on $X = G/P$ defined by a weight $\lambda \in \Lambda_X$ The number of positive eigenvalues of $\mathcal{L}(\varphi_L)$ equals* $\operatorname{ind}(\lambda)$ *and the number of negative eigenvalues of $\mathcal{L}(\varphi_L)$ equals* $\operatorname{ind}(-\lambda)$. *In particular, if m is sufficiently large, then $H^q(X, L^m) = 0$ for $q < \operatorname{ind}(\lambda)$ or $q > \dim X - \operatorname{ind}(-\lambda)$, and if λ is non-singular, then $H^q(X, L^m) = 0$ for $q \neq \operatorname{ind}(\lambda)$.*

Proof. By Theorem 7.5 and the above calculation, the Levi form at $[1, z]$ is

$$\mathcal{L}(\varphi_L) = -|z|^2\Theta_L = -|z|^2 \sum_{\alpha\in\Phi_X} \langle\alpha, \lambda\rangle \, dx_\alpha \wedge d\bar{x}_\alpha$$

Thus, the number of positive (resp. negative) eigenvalues of $\mathcal{L}(\varphi_L)$ equals the number of positive roots $\alpha \in \Phi_X$ such that $\langle\alpha, \lambda\rangle < 0$ (resp. $\langle\alpha, \lambda\rangle > 0$), which is the index of λ (resp. the index of $-\lambda$). If λ is non-singular, then $\mathcal{L}(\varphi_L)$ is also everywhere non-degenerate and $\operatorname{ind}(\lambda) + \operatorname{ind}(-\lambda) = \dim X$. The statement then follows from Andreotti-Grauert [1]. □

The above theorem also holds for an irreducible homogeneous vector bundle on $X = G/P$ with maximal weight λ, see [24, p.275]. Bott's theorem [9] is a more precise vanishing theorem that is related to the above statement in an elegant way, see §11.

Let $\xi_E \to \mathbb{P}(E^*)$ be the tautological line bundle over the projectivization of E^*, see §5. Since, as manifolds, $E^* \setminus Z_X \cong \xi_E \setminus \mathbb{P}(E^*)$ with the zero sections Z_X and $Z_{\mathbb{P}(E^*)}$ at "opposite ends," the Levi form of the boundary of a tube neighborhood of Z_X in E^* is the negative of the Levi form of the boundary of a tube neighborhood of $Z_{\mathbb{P}(E^*)}$ in ξ_E. In fact, a positive function can be defined on $\xi_E \setminus Z_{\mathbb{P}(E^*)}$ by

$$h_{\xi_E}(p) = (\bar{z}^t h_{E^*}(x) z)^{-1}$$

where $p \in \xi_E \setminus Z_{\mathbb{P}(E^*)}$ corresponds to $[g, z] \in G \times_P E_0^* \setminus Z_X$ and $x \in X$ is the coset $gP \in G/P$. Clearly, h_{ξ_E} defines a metric for ξ_E and the Levi form of the boundary of the *unit* tube neighborhood of $Z_{\mathbb{P}(E)}$ is $\partial\bar{\partial}\log h_{\xi_E} = -\Theta_{\xi_E}$. Thus,

$$\begin{aligned}\Theta_{\xi_E} &= \partial\bar{\partial}\log(\bar{z}^t h_{E^*} z) \\ &= -\frac{1}{|z|^2}\bar{z}^t\Theta_{E^*}z + \frac{1}{|z|^4}\big(|z|^2\sum_i dz_i \wedge d\bar{z}_i - \sum_{ij}\bar{z}_i z_j dz_i \wedge d\bar{z}_j\big)\end{aligned}$$

This observation allows us to conclude the following.

Proposition 7.7. *Let $E = G \times_P E_0$ be a homogeneous vector bundle on $X = G/P$ where P is a parabolic subgroup of G. If the tautological line bundle ξ_E^m on $\mathbb{P}(E^*)$ is spanned by global sections for some $m > 0$, then $\Theta_E(z)$ is positive semidefinite for $z \in E_0$.*

Proof. Since ξ_E^m is spanned by global sections, $\Theta_{\xi_E^m} = m\Theta_{\xi_E}$ is positive semidefinite by Corollary 7.3. Therefore, by the above calculation, $\bar{z}^t\Theta_E z = -\bar{z}^t\Theta_{E^*}z$ is also positive semidefinite. □

This statement also holds for an arbitrary holomorphic vector bundle $E \to X$ over a compact complex manifold X, since Θ_E is positive semidefinite whenever E is spanned by global sections, see, e.g., [20, p.80], and the above calculation of Θ_{ξ_E} remains the same, see e.g., [23, p.202].

8. Ampleness Formulas

Let $E = G \times_P E_0$ be a homogeneous vector bundle on $X = G/P$ where P is a parabolic subgroup. We shall now work out a formula for the ampleness of E in terms of the indexes of certain extremal weights of E_0.

Definition 8.1. The ampleness of E, denoted $a(E)$, is defined to be the minimum k such that E is k-ample (see §5).

Note that E_0 is also a B-module where $B \subset P$ is a Borel subgroup. The corresponding vector bundle $G \times_B E_0|_B$ is isomorphic to the pull back p^*E under the projection $p : G/B \to G/P$, see §5. It follows immediately from the definition of ampleness that

$$a(E) = a(E_B) - \dim G/P$$

so that the ampleness of E depends primarily on the B-module structure of E_0.

Let U be the maximal unipotent subgroup of B generated by the negative root groups U_α, $\alpha < 0$. For any B-module E_0, let E_0^U be the set of points in E_0 fixed by U, $E_0^U = \{v \in E_0 \,|\, u \cdot v = v \text{ for all } u \in U\}$. The action of B on E_0^U reduces to the action of the maximal torus $T \subset B$: $E_0^U = E_1 + \cdots + E_s$ and $b \cdot v = \hat{\lambda}_i(b)v$ for $v \in E_i$, $b \in B$, $1 \le i \le s$. Conversely, if $v \in E_0^U$ and $b.v = \hat{\lambda}_i(b)v$ for all $b \in B$ then $v \in E_i$. Thus, E_0^U consists of the B-stable lines in E_0. Recall that the maximal weights of E_0 are

$$\Lambda_{\max}(E_0) = -\Lambda(E_0^{*U}),$$

and that the index of a set of weights is the minimum of the indexes of the weights in the set, see §5.

Definition 8.2. Let E_0 be a B-module and let W be the Weyl group of G. The extremal weights of E_0 are

$$\Lambda_{\text{ext}}(E_0) = W(\Lambda_{\max}(E_0)) \cap \Lambda(E_0)$$

Theorem 8.3 ([44])**.** *Let $E = G \times_P E_0$ be a homogeneous vector bundle on $X = G/P$ where P is a parabolic subgroup of G. Assume the tautological line bundle ξ_E^m on $\mathbb{P}(E^*)$ is spanned by global sections for some $m > 0$. Then, the maximal weights of E_0 are dominant, $\Lambda_{\max}(E_0) \subset \Lambda^+$, and the ampleness of E is given by*

$$a(E) = \dim X - \operatorname{ind}(-\Lambda_{\text{ext}}(E_0))$$

Proof. Let $\lambda \in \Lambda_{\max}(E_0)$ and let $C_{-\lambda}$ be a corresponding B-stable line in E_0^*. By assumption, the map $\nu_m : E^* \to V_m^*$ defined by the sections $V_m = H^0(\mathbb{P}(E^*), \xi_E^m) \cong H^0(X, E^{(m)})$ is finite on the fibers, $\nu_m|E_0^* \to V_m^*$, taking the B-stable line $C_{-\lambda} \subset E_0^*$ to a B-stable line $C_{-m\lambda} \subset V_m^*$, see §5. Therefore, the B-module homomorphism $V_m \to C_{m\lambda}$ is surjective. Since V_m is a G-module, the line bundle defined by $m\lambda$ is spanned by Proposition 5.1. Therefore $m\lambda \in \Lambda^+$ by Theorem 6.5, and hence $\lambda \in \Lambda^+$.

By definition, $a(E)$ is the maximum fiber dimension of the G-equivariant map $\mathbb{P}\nu_m : \mathbb{P}(E^*) \to \mathbb{P}(V^*)$. Since the fiber dimension is upper semi-continuous and constant along G-orbits, its maximum can be found by specializing to points in the closures of G-orbits. In particular, the maximum occurs at a point $[v] = \mathbb{P}\nu_m[g,[z]]$ such that $G \cdot [v]$ is a closed G-orbit in $\mathbb{P}(V_m^*)$. We may then additionally choose $[v]$ to be a fixed point of B, see Lemma 6.1. Therefore, we may assume $z \in E_0^{*U}$, z has weight $\lambda \in \Lambda(E_0^{*U}) = -\Lambda_{\max}(E_0)$, and $v \in V_m^{*U}$ has weight $m\lambda$.

The fiber dimension over $[v]$ is:

$$\begin{aligned}\dim \mathbb{P}\nu_m^{-1}[v] &= \dim \nu_m^{-1}(v)\\ &= \dim\{[g,z]\in G\times_P E_0^* \mid \nu_m[g,z]=v\}\\ &= \dim\{(g,z)\in G\times E_0^* \mid \nu_m(1,z)=g^{-1}\cdot v\} - \dim P\\ &= \dim\{g\in G \mid g\cdot v\in \nu_m E_0^*\} - \dim P\end{aligned}$$

By the Bruhat decomposition, see Theorem 3.1, each $g \in G$ lies in some $Un_\omega B$, $\omega \in W$, with $\dim Un_\omega B = \ell(\omega) + \dim B$. Then $g \cdot v \in \nu_m E_0^*$ iff $n_\omega \cdot v \in \nu_m E_0^*$ iff $\omega\lambda \in \Lambda(E_0^*)$. Thus, $\dim \mathbb{P}\nu_m^{-1}[v]$ is the maximum of $\ell(\omega) + \dim B - \dim P$ over all $\omega \in W$ such that $\omega\lambda \in \Lambda(E_0^*)$, and we obtain

$$a(E) = \max\{\ell(\omega) \mid \omega \in W,\ \omega(\Lambda(E_0^{*U})) \cap \Lambda(E_0^*) \neq \emptyset\} - \dim P/B$$

Of course, $\omega(\Lambda(E_0^{*U}))\cap\Lambda(E_0^*) \neq \emptyset$ is equivalent to $\omega(\Lambda_{\max}(E_0))\cap\Lambda(E_0) \neq \emptyset$. Now, the maximum of $\ell(\omega)$ such that $\mu = \omega\lambda \in \Lambda(E_0)$ for some $\lambda \in \Lambda_{\max}(E_0)$ is the same as the maximum of $\ell(\omega)$ such that $\omega\mu \in \Lambda^+$ for some $\mu \in \Lambda_{\text{ext}}(E_0)$.

Let ω_0 denote the longest element of W so that $\ell(\omega_0) = |\Phi^+| = \dim G/B$. For any $\omega \in W$ we can write $\omega = \omega'\omega_0$ so that $\ell(\omega) = \ell(\omega_0) - \ell(\omega') = \dim G/B - \ell(\omega')$ and $\operatorname{ind}(-\mu) = \operatorname{ind}(\omega_0\mu)$, see §3. Then,

$$\begin{aligned}&\max\{\ell(\omega) \mid \mu \in \Lambda_{\text{ext}}(E_0),\ \omega\mu \in \Lambda^+\} - \dim P/B\\ &= \dim G/B - \min\{\ell(\omega') \mid \mu \in \Lambda_{\text{ext}}(E_0),\ \omega'(\omega_0\mu) \in \Lambda^+\} - \dim P/B\\ &= \dim G/P - \min\{\operatorname{ind}(\omega_0\mu) \mid \mu \in \Lambda_{\text{ext}}(E_0)\}\\ &= \dim G/P - \operatorname{ind}(-\Lambda_{\text{ext}}(E_0))\end{aligned}$$

□

We have already remarked in Proposition 7.7 that if ξ_E^m is spanned by global sections for some $m > 0$, then $\Theta_E(z)$ is positive semidefinite for $z \in E_0$. Moreover, if E is k-ample, then E^* is $(k+1)$-convex in the sense of Andreotti-Grauert, see [54] for details. From the calculations in §7.3 it is then clear that the maximum number of zero eigenvalues of $\Theta_E(z)$ for $z \in E_0 \setminus 0$ is $a(E)$. Therefore, rather than computing the maximum dimension of the kernel of $\Psi^t(z) : F_0 \to E_0$ for $z \in E_0 \setminus 0$, we may calculate the flatness of Θ_E the same way as $a(E)$:

Corollary 8.4. *If ξ_E^m is spanned by global sections for some $m > 0$, then*

$$\operatorname{fl}\Theta_E = a(E) = \dim X - \operatorname{ind}(-\Lambda_{\text{ext}}(E_0))$$

8.1. **Ampleness of irreducible bundles.** It is not clear if there is any simple data about the P-module E_0 that is equivalent to ξ_E^m being spanned for some $m > 0$. Nevertheless, when the bundle is irreducible the previous theorem can be strengthened to the following.

Corollary 8.5. *Let E_0 be an irreducible P-module with highest weight λ and lowest weight μ ($-\mu$ is the highest weight of E_0^*). Then ξ_E^m is spanned for some $m > 0$ iff E is spanned iff λ is dominant. Moreover, $a(E) = \dim X - \operatorname{ind}(-\mu)$ and if λ is non-singular, then E is ample, $a(E) = 0$.*

Proof. To prove the first assertion, we need only show that if $\lambda \in \Lambda^+$, then E is spanned. Since E_0 is irreducible, it is induced from the one dimensional P-module C_λ associated to λ: $E_0 = C_\lambda|^P$. By transitivity, Theorem 5.2,

$$E_0|^G = C_\lambda|^P|^G = C_\lambda|^G$$

Since the B-module homomorphism $E_0|^G \to C_\lambda$ is surjective and factors through the B-module epimorphism $E_0 = C_\lambda|^P \to C_\lambda$, it follows that the P-module homomorphism $E_0|^G \to E$ cannot be the zero map and is therefore surjective.

Since $\Lambda_{\max}(E_0) = \{\lambda\}$ and $\Lambda_{\text{ext}}(E_0) = W_P\lambda$, we find that the minimum of $\text{ind}(-\omega\lambda)$ occurs when $\omega = \omega_1$, the longest element in W_P and $\omega_1\lambda = \mu$, the lowest weight of E_0. If λ is in addition non-singular, then $\dim G/B = \text{ind}(-\lambda) = \text{ind}(-\omega_1\lambda) + \dim P/B$ since $\ell(\omega_1) = \dim P/B$. This implies $a(E) = \dim X - \text{ind}(-\mu) = 0$.

□

For the case of a line bundle L, which is in particular an irreducible bundle, it is quite easy to determine $a(L)$. If λ is the weight associated to L, then λ is perpendicular to the roots of P, $P \subset P_\lambda$. The corollary asserts that if λ is dominant, then $a(E) = \dim X - \text{ind}(-\lambda)$. In this situation $\text{ind}(-\lambda) = \dim G/P_\lambda$, so

$$a(E) = \dim P_\lambda/P$$

This, of course, is the expected value for $a(E)$ since it is the dimension of the fiber of the map $X \to \mathbb{P}(V^*)$ defined by the sections $V = H^0(X, L)$, see §6.

A final remark about the irreducible case: The value of $a(E)$ or even whether ξ_E^m is spanned cannot be determined by considering the same questions for the irreducible bundles $G \times_P (E_i/E_{i-1})$ associated to a filtration $E_0 \supset E_1 \supset \cdots \supset E_t \supset E_{t+1} = 0$ by P-submodules, see §5. For example, if E_0 is a B-module, then the irreducible factors are 1-dimensional weight spaces for all the weights of E_0, yet the ampleness of E_0 is determined by the index of certain extremal weights of E_0. Moreover, E can be spanned without all its weights being dominant. Nevertheless, it is easy to modify the proof of Corollary 8.5 to show that if the maximal weights of all the factors E_i/E_{i-1} are dominant then the bundle E is spanned, see §11.

8.2. **Ampleness of tangent bundles.** The derivation of the ampleness of the tangent bundle T_X of $X = G/P$ was originally carried out by Goldstein [26] and it was this work that provided the inspiration for Theorem 8.3. Let $G = G_1 \times \cdots \times G_t$ be the decomposition of G into its simple factors so that $X = X_1 \times \cdots \times X_t$, where $X_i = G_i/P \cap G_i$, $i = 1, \ldots, t$. The weights of $T_{X,0}$ are just the roots of X, $\Lambda(T_{X,0}) = \Phi_X$, and $\Lambda_{\max}(T_{X,0}) = \{\beta_1, \ldots, \beta_k\}$ where β_i is the longest root of G_i. Then $\Lambda_{\text{ext}}(T_{X,0})$ is the set of long roots in Φ_X. By Theorem 8.3, to compute the ampleness of the tangent bundle we must compute the minimum of $\text{ind}(-\mu)$ for the long roots μ in Φ_X. Since the dominant conjugate of μ is always one of the β_i's, this is the same as computing the minimum of $\ell(\omega)$ over $\omega \in W$ such that $\omega\beta_i < 0$ for some i and $\omega\beta_i \in -\Phi_X$.

In finding this minimum, the first condition, $\omega\beta_i < 0$, can be met by having $\omega\beta_i$ equal to any long simple root. Table 8.2 gives the minimum, which we denote by $m(G_i)$, of $\ell(\omega)$ such that $\omega\beta_i < 0$ for each simple type G_i. The second condition, $\omega\beta_i \in -\Phi_X$, is automatically met when G_i has type A_ℓ, D_ℓ, or E_ℓ since all roots are long. For the other types, if Φ_X does not contain a long simple root in G_i, then

Table 2. $m(G)$ for a simple group G

A_ℓ	B_ℓ	C_ℓ	D_ℓ	E_6	E_7	E_8	F_4	G_2
ℓ	$2\ell-2$	ℓ	$2\ell-3$	11	17	29	8	3

a certain number of further reflections are necessary to bring $\omega\beta_i$ into $-\Phi_X$. A case by case study shows that only one additional reflection is necessary for types B_ℓ and G_2 when I, the set of indexes that defines P, contains the indexes of all the simple roots of G_i except the long simple root. For types C_ℓ and F_4, $d(G_i, I)$ additional reflections are necessary where $d(G_i, I)$ is the number of nodes in the diagram for G_i from the complement of I to the nearest long simple root. Using the same definition for $d(G_i, I)$ for all the simple types we then we have the formula,

$$a(T_X) = \dim X - \min_i\{m(G_i) + d(G_i, I)\}$$

It is also convenient to use the easily derived fact

$$\dim X - a(T_X) = \min_i\{\dim X_i - a(T_{X_i})\}$$

so that the calculation of $a(T_X)$ can be reduced to the irreducible factors X_i of X.

For example, if X is a product of $X = X_1 \times \cdots \times X_t$, where $X_i = G_i/P_i$, P_i a parabolic subgroup of $G_i = \mathrm{SL}(\ell_i + 1, \mathbb{C})$, (which includes products of Grassmann manifolds) then

$$a(T_X) = \dim X - \min_i \ell_i$$

Note how this agrees with the calculation of flatness of Θ_{T_X} in §7.1. If X is an n-dimensional non-singular quadric hypersurface, then $X = G/P$ where G is of type B_ℓ ($n = 2\ell - 1$) or D_ℓ ($n = 2\ell - 2$). In either case, the complement of I is a long root so by Table 8.2 $a(T_X) = 1$. It is straightforward to derive from the above formula that among homogeneous manifolds, $a(T_X) = 0$ only when $X = \mathbb{P}^\ell$. Of course, it is well-known that for any projective manifold X, T_X is ample if and only if $X = \mathbb{P}^\ell$, see [40].

If $Y \subset X$ is a complex submanifold of $X = G/P$, then the ampleness of the normal bundle N_Y of Y in X is related to the convexity of the complement of Y. In fact, if N_Y is k-ample, then $X \setminus Y$ is $k + \mathrm{codim}_X Y$ convex in the sense of Andreotti-Grauert. Moreover, if $Z \subset X$ is another complex submanifold then

$$2\dim Y \geq k + \dim X, \dim Z \leq \dim Y + 1 \Longrightarrow \pi_i(Z, Z \cap Y) = 0, i \leq \dim Y - k$$

see [56]. Since the tangent bundle of X surjects onto N_Y,

$$0 \to T_Y \to T_X|_Y \to N_Y \to 0$$

the ampleness of T_X provides a convenient upper bound for the ampleness of N_Y. Thus, for example, if $X = \mathrm{SL}(\ell+1, \mathbb{C})/P$ (e.g., a Grassmann manifold) and $Y \subset X$, then $X \setminus Y$ is $\ell + \mathrm{codim}_X Y$ convex.

"Connectedness" theorems are also related to the ampleness of T_X: Suppose W is an irreducible subvariety of $X \times X$ where $X = G/P$, and let $\Delta \subset X \times X$ be the diagonal. Then

$$\dim W \geq \dim X + a(T_X) \Longrightarrow W \cap \Delta \neq \emptyset$$

Moreover, strict inequality implies $W \cap \Delta$ is connected, see [26, 18]. For example, let X be a $(2\ell - 2)$-dimensional quadric so that X is a homogenoeus space of an orthogonal group of type D_ℓ and $a(T_X) = 1$. If $Y, Z \subset X$ are two subvarieties, then

$$\dim Y + \dim Z \geq \dim X + 1 \Longrightarrow Y \cap Z \neq \emptyset$$

(take $W = \{(y, z) \mid y \in Y, z \in Z\}$).

9. Chern Classes

Let $E \to X$ be a holomorphic vector bundle on X. The Chern classes of E, $c_q(E) \in H^{2q}(X, \mathbb{R})$, $1 \le q \le r = \operatorname{rank} E$, can be defined in many ways, see [7, 19, 23]. For example, $c_q(E)$ can be defined as the class of a differential form of type (q, q) determined by

$$c(E) = \sum_{q=0}^{r} c_q(E) t^{r-q} = \det\Big(tI + \frac{i}{2\pi}\Theta_E\Big)$$

where $\Theta_E = \bar{\partial}(h_E^{-1}\partial h_E)$ is the curvature form associated to a hermitian metric h_E on E, see §7. The expression $c(E)$ is called the total Chern class of E and behaves well with respect to exact sequences and tensor products: If

$$0 \longrightarrow F_1 \longrightarrow E \longrightarrow F_2 \longrightarrow 0$$

is an exact sequence of holomorphic vector bundles on X then

$$c(E) = c(F_1)c(F_2)$$

Moreover, for any two holomorphic vector bundles E and F on X of ranks r and s respectively,

$$c(E \otimes F) = \prod_{i=1}^{r}\prod_{j=1}^{s}\Big(t + \gamma_i(E) + \gamma_j(F)\Big)$$

Here the $\gamma_i(E)$ are defined by factoring $c(E) = \prod_{i=1}^{r}(t + \gamma_i(E))$, and similarly for $c(F)$. In particular, the Chern classes of E are equivalent to the Chern classes a direct sum of line bundles $L_1 \oplus \cdots \oplus L_r$ where L_i has Chern class $\gamma_i(E)$, $1 \le i \le r$.

If X is a projective manifold, there is an alternate geometric definition for the Chern classes of E. Let L be an ample line bundle such that $E \otimes L$ is spanned and let $\xi_1, \ldots, \xi_{r-q+1}$ be generic sections of $E \otimes L$. Then $\xi_1 \wedge \cdots \wedge \xi_{r-q+1}$ is a section of $\bigwedge^{r-q+1} E \otimes L$ whose zero locus defines a class $S_q \in H_{2n-2q}(X, \mathbb{R})$. The Chern class $c_q(E \otimes L) \in H^{2q}(X, \mathbb{R})$ is defined to be the Poincaré dual of S_q, $0 \le q \le r$. Finally, the total Chern class of E is defined by

$$c(E) = \prod_{i=1}^{r}(t + \gamma_i(E \otimes L) - c_1(L))$$

The Chern classes of a homogeneous vector bundle $E = G \times_P E_0$ on $X = G/P$, where P is a parabolic subgroup of G, can be expressed in terms of the weights $\Lambda(E_0)$ of the representation of P on E_0; an explicit calculation of the curvature form is not needed. This is due to the fact that the Chern classes are determined by line bundles as just described, and Theorem 7.5 allows us to naturally identify the Chern class of a line bundle L,

$$c_1(L) = (i/2\pi)\Theta_L = (i/2\pi)\sum_{\alpha \in \Phi_X^+} \langle \lambda, \alpha \rangle dx_\alpha \wedge \bar{x}_\alpha$$

with its associated weight $\lambda \in \Lambda_X \cong \operatorname{Pic}(X)$, both representing the same element in $H^2(X, \mathbb{R})$. In fact, we shall just write

$$c_1(L) = \lambda$$

The next theorem gives a simple construction of $c(E)$ using the language and notation of §2 and §5. For a more detailed account see [7]. Let $\operatorname{pr}_X : \Lambda \to \Lambda_X$ denote the projection of weights of G onto the the weights perpendicular to the roots of P with respect to the basis of fundamental dominant weights, i.e., if $\lambda = \sum_i n_i\lambda_i \in \Lambda$, and P is defined by the set of indexes I, then

$$\mathrm{pr}_X(\lambda) = \sum_{i \notin I} n_i\lambda_i$$

Theorem 9.1. *Let $E = G \times_P E_0$ be a homogeneous vector bundle on $X = G/P$ where P is a parabolic subgroup of G. Then*

$$c(E) = \prod_{\mu \in \Lambda(E_0)} (t + \mathrm{pr}_X\, \mu)^{m_\mu}$$

where m_μ is the multiplicity of the weight $\mu \in \Lambda(E_0)$.

Proof. Let $B \subset P$ be a Borel subgroup of G and let $p : G/B \to G/P$ be the natural projection. Let $p^*E = G \times_B E_0|_B$ be the pull back vector bundle on $Y = G/B$. Let $E_0|_B \supset E_1 \supset \cdots \supset E_t$, be a filtration of $E_0|_B$ by with irreducible and hence one-dimensional quotients $F_i = E_i/E_{i-1}$, see §5. The maximal weights of $E_0|_B$ are thus all the weights of E_0 as a P-module, $\Lambda(E_0)$. Since the total Chern class respects exact sequences we obtain

$$c(p^*E) = \prod_{i=0}^{t} c(F_i) = \prod_{\mu \in \Lambda(E_0)} (t + \mu)^{m_\mu}$$

The map $\mathrm{pr}_X : \Lambda \to \Lambda_X$ splits the natural inclusion $\Lambda_X \to \Lambda$ induced by $p^* : H^2(X, \mathbb{R}) \to H^2(Y, \mathbb{R})$ and so $c(E) = \prod_{\mu \in \Lambda(E_0)} (t + \mathrm{pr}_X\, \mu)^{m_\mu}$ as claimed. □

The product of weights given in the previous theorem, of course, must be carried out in $H^*(X, \mathbb{R})$.

9.1. **Chern classes of tangent bundles.** The Chern classes of the tangent bundle T_X of a complex manifold X are called simply the Chern classes of X, $c(X) = c(T_X)$. Since $\Lambda(T_{X,0}) = \Phi_X^+$, the total Chern class $c(X)$ of $X = G/P$, P parabolic, can be given in terms of the roots of X by Theorem 9.1:

$$c(X) = \prod_{\alpha \in \Phi_X^+} (t + \mathrm{pr}_X\, \alpha)$$

The first Chern class of X is easily derived from this to be $c_1(X) = \sum_{\alpha \in \Phi_X^+} \mathrm{pr}_X\, \alpha$. The projection pr_X in this case is superfluous, since the sum $\sum_{\alpha \in \Phi_X^+} \alpha$ already lies in Λ_X. In fact, it is clearly the weight of the line bundle $\bigwedge^n T_X$, $n = \dim X$, which must be perpendicular to the roots of P. Thus,

$$c_1(X) = \sum_{\alpha \in \Phi_X^+} \alpha$$

Definition 9.2. Let $S \subset \Lambda$ be a set of weights. We denote the sum of the weights of S by

$$\langle S \rangle = \sum_{\lambda \in S} \lambda$$

In particular, $c_1(X) = \langle \Phi_X^+ \rangle$. An important special case is when $P = B$ is a Borel subgroup of G so that the roots of X are all the positive roots of G.

Lemma 9.3. *Let $X = G/B$ where B is a Borel subgroup of G. Then*

$$c_1(X) = \sum_{i=1}^{\ell} 2\lambda_i$$

Proof. Let $\mu = \langle \Phi^+ \rangle = \sum_{i=1}^{\ell} \langle \mu, \alpha_i \rangle \lambda_i$. The simple reflection σ_i permutes the positive roots $\Phi^+ \setminus \{\alpha_i\}$ and sends the simple root α_i to $-\alpha_i$. Therefore,

$$\sigma_i \mu = \langle \sigma_i \Phi^+ \rangle = \langle \Phi^+ \rangle - 2\alpha_i = \mu - 2\alpha_i$$

Since $\sigma_i \mu = \mu - \langle \mu, \alpha_i \rangle \alpha_i$ we get $\langle \mu, \alpha_i \rangle = 2$ for $i = 1, \ldots, \ell$. □

The weight

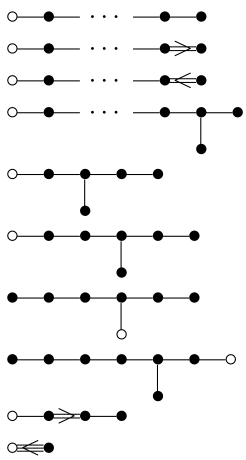
$$\delta = \frac{1}{2}\langle \Phi^+ \rangle = \frac{1}{2}\sum_{\alpha > 0} \alpha = \sum_{i=1}^{\ell} \lambda_i$$

is, of course, the weight of the "minimal" ample line bundle on $X = G/B$ and plays an important role in the cohomology of homogeneous spaces of G and in the representation theory of G.

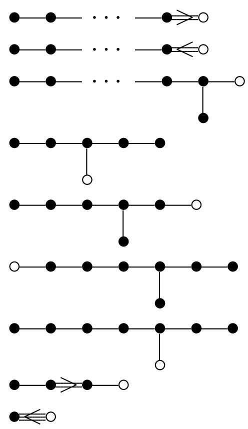
A similar expression for $c_1(X)$ in the general case $X = G/P$ depends on the set of indexes I that defines the parabolic subgroup P.

Definition 9.4. Let $1, \ldots, \ell$ be the indexes of the simple roots $\alpha_1, \ldots, \alpha_\ell$ of a Lie group G. A subset of indexes $J \subset \{1, 2, \ldots, \ell\}$ is called connected if the subdiagram of the Dynkin diagram of G corresponding to the simple roots α_j, $j \in J$, is connected. An index i is said to be adjacent to J if $i \notin J$ and $J_0 \cup \{i\}$ is connected for some connected component J_0 of J. The set of indexes adjacent to J is denoted by ∂J.

Let J be a connected set of indexes. For $i \notin \partial J$ define $\nu_i(J) = 0$. For $i \in \partial J$ define $\nu_i(J)$ to be the number next to the appropriate diagram below. The black nodes correspond to J and the white node corresponds to i. Symmetry of Dynkin diagrams is tacitly assumed.



$|J|$,
$2|J| - 1$,
$2|J|$,
$2|J| - 2$,
10,
16,
12,
21,
6,
3,



$2|J|$,
$|J|$,
$2|J| - 2$,
9,
15,
27,
15,
9,
1,

For an arbitrary set of indexes I define $\nu_i(I) = \sum_k \nu_i(I_k)$ where $I_1, \ldots, I_t$ are the connected components of I.

Theorem 9.5. *Let $X = G/P$ where P is a parabolic subgroup of G defined by a set of indexes I. Then the first Chern class of X is*

$$c_1(X) = \sum_{i \notin I} (2 + \nu_i(I))\lambda_i$$

In particular, $c_1(X)$ is positive semidefinite.

Proof. Let $\mu = \langle \Phi_X^+ \rangle = \sum_{i=1}^{\ell} m_i \lambda_i$. Note that $m_i = \langle \mu, \alpha_i \rangle = 0$ for $i \in I$ because $\mu \in \Lambda_X$. Since $\Phi^+ = \Phi_X^+ \cup \Phi_P^+$, Lemma 9.3 implies $\langle \Phi_X^+ \rangle + \langle \Phi_P^+ \rangle = 2\delta$. Hence, for $i \notin I$,

$$m_i = \langle \langle \Phi_X^+ \rangle, \alpha_i \rangle = 2 - \langle \langle \Phi_P^+ \rangle, \alpha_i \rangle$$

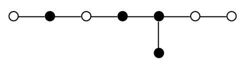
Now, a simple root α_j for $j \in I$ has non-zero coordinates $\langle \alpha_j, \alpha_i \rangle \neq 0$ only for indexes i adjacent to j in the Dynkin diagram of G. Consequently, $\langle \langle \Phi_P^+ \rangle, \alpha_i \rangle = 0$ if i is not adjacent to I, since the roots in Φ_P^+ are sums of such simple roots.

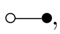
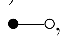
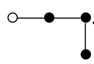
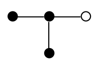
If i is adjacent to I, then each connected component I_k of I to which i is adjacent will contribute to the sum:

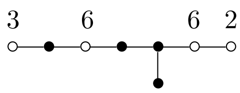
$$\langle \langle \Phi_P^+ \rangle, \alpha_i \rangle = \sum_k \langle \langle \Phi_{I_k}^+ \rangle, \alpha_i \rangle$$

Each connected diagram, $I_k \cup \{i\}$, corresponds to a simple group, and for each simple group it is easy to add the positive roots generated by α_j, $j \in I_k$, and determine the inner product of the sum with α_i. The results for all possible configurations of $I_k \cup \{i\}$ are given in Definition 9.4: $\langle \langle \Phi_{I_k}^+ \rangle, \alpha_i \rangle = -\nu_i(I_k)$. Therefore, $m_i = 2 + \sum_k \nu_i(I_k) = 2 + \nu_i(I)$ as claimed. □

9.2. **Example.** Let us demonstrate the formula of Theorem 9.5 with an explicit example. Suppose the group is $G = E_8$ and that P is defined by the set of indexes $I = \{2, 4, 5, 8\}$ corresponding to the black nodes in the following diagram.



Let $X = G/P$, $\dim X = 113$. The set I has two connected components. The coefficient of $c_1(X) = \langle \Phi_X^+ \rangle$ for the first white node is determined by a diagram of type A_2, , and is therefore $2 + 1 = 3$. The second white node is adjacent to both components of I. The coefficient of $c_1(X)$ for this node is $2 + 1 + 3 = 6$ as determined by two diagrams, one of type A_2, , and one of type A_4, . The coefficient of $c_1(X)$ corresponding to the third white node is determined by a diagram of type D_4,  and is $2 + (2 \cdot 3 - 2) = 6$. The last white node is not adjacent to I and the coefficient of $c_1(X)$ for this node is 2. The coefficients of $c_1(X)$ are thus



that is, $c_1(X) = 3\lambda_1 + 6\lambda_3 + 6\lambda_6 + 2\lambda_7$.

9.3. **Maximal Parabolics.** As another example, let us apply the previous theorem when $X = G/P$ and P is a maximal parabolic subgroup of G. Thus P is defined by a set of indexes I that excludes exactly one index, say i. The maximality of P is equivalent to $b_1(X) = 1$ and allows us to assume that G is simple. In this case, the first Chern class is determined by a single integer: $c_1(X) = m\lambda_i$. Consulting Definition 9.4 and Theorem 9.5 we arrive at the following table for the value of m, see also [7, 49]. We let $n = \dim X$.

(1) $G = A_\ell$: $n = i(\ell + 1 - i)$, $m = \ell + 1$.
(2) $G = B_\ell$: $n = i(4\ell + 1 - 3i)/2$, $m = 2\ell - i$, unless $i = \ell$ in which case $m = 2\ell$.
(3) $G = C_\ell$: $n = i(4\ell + 1 - 3i)/2$, $m = 2\ell - i + 1$.
(4) $G = D_\ell$: $n = i(4\ell - 1 - 3i)/2$, $m = 2\ell - i - 1$, unless $i = \ell - 1$ or ℓ in which case $m = 2\ell - 2$.
(5) $G = E_6$:

i	1,5	2,4	3	6
n	16	25	29	21
m	12	9	7	11

(6) $G = E_7$:

i	1	2	3	4	5	6	7
n	27	42	50	53	47	33	42
m	18	13	10	8	11	17	14

(7) $G = E_8$:

i	1	2	3	4	5	6	7	8
n	57	83	97	104	106	98	78	92
m	29	19	14	11	9	13	23	17

(8) $G = F_4$:

i	1	2	3	4
n	15	20	20	15
m	8	5	7	11

(9) $G = G_2$:

i	1	2
n	5	5
m	5	3

Note that for any Grassmann manifold X (case 1 above) the value of m is independent of i: $c_1(X) = (\ell + 1)\lambda_i$.

10. Nef Value

Let X be a smooth projective variety. A line bundle on X is called numerically effective, or nef for short, if its restriction to any effective curve in X yields a line bundle of non-negative degree. If X is homogeneous, it is easy to decide when a line bundle is nef.

Proposition 10.1. *Let P be a parabolic subgroup of G and let L be a line bundle on $X = G/P$ defined by a weight $\lambda \in \Lambda_X$. Then L is nef if and only if λ is dominant.*

Proof. Let I be the set of indexes that defines P and write $\lambda = \sum_{i \notin I} n_i \lambda_i$. For each $i \notin I$ let S_i be the subgroup of G generated by the roots $\pm\alpha_i$. Thus, S_i has a Lie algebra isomorphic to $\mathfrak{sl}_2(\mathbb{C})$ and $S_i \cap P$ is a Borel subgroup of S_i. It follows that the orbit of the identity coset under S_i is isomorphic to a rational curve, $C_i = S_i/S_i \cap P \cong \mathbb{P}^1$. Let L_i be the line bundle associated to λ_i for $i \notin I$. Then $L_i|_{C_i}$ has degree 1, and since $H^2(X, \mathbb{Z}) \cong \Lambda_X = \oplus_{i \notin I} \mathbb{Z}\lambda_i$ it is clear the rational curves C_i, $i \notin I$, topologically generate all the curves in X. In particular, if $C = \sum_{i \notin I} m_i C_i$ is an effective curve, $m_i \geq 0$ for $i \notin I$, then $\deg L|C = \sum_{i \notin I} m_i n_i$. Consequently, L is nef if and only if $n_i \geq 0$ for $i \notin I$ which is the same as λ being dominant. □

Note that, by Theorem 6.5, L being nef is equivalent to the a priori stronger condition that L is spanned by global sections.

The canonical bundle of X is defined to be $K_X = \bigwedge^n T_X^*$, $n = \dim X$, and is an important intrinsic line bundle on any projective manifold. For $X = G/P$, where P is a parabolic subgroup of G, K_X is generated at the identity coset by the exterior product of the duals of X_α, $\alpha \in \Phi_X^+$, so that the weight associated to K_X is

$$c_1(K_X) = -\sum_{\alpha \in \Phi_X^+} \alpha = -\sum_{i \notin I} (2 + \nu_i(I))\lambda_i$$

The last equality follows from Theorem 9.5; see Definition 9.4 for the meaning of $\nu_i(I)$. In particular, we see that K_X is never nef for a homogeneous manifold $X = G/P$.

Definition 10.2. Let X be a projective manifold whose canonical bundle K_X is not nef and let L be an ample line bundle on X. The nef value of L, denoted $\tau(X, L)$ is defined to be

$$\tau = \tau(X, L) = \inf\{p/q \in \mathbb{Q} \mid K_X^q \otimes L^p \text{ is nef}\}$$

The map $X \to Y \subset \mathbb{P}^N$ with connected fibers and normal image defined by the sections of some power of $K_X^q \otimes L^p$ with $\tau = p/q$ is called the nef value map of L.

By the Kawamata rationality theorem, $\tau(X, L) \in \mathbb{Q}$. Moreover, $\tau(X, L)$ can be characterized as the smallest rational number p/q such that $K_X^q \otimes L^p$ is nef but not ample, see [2]. For homogeneous manifolds, these facts are easy to deduce. In fact, we have the following formula to calculate $\tau(X, L)$, see [49, 50].

Theorem 10.3. *Let $X = G/P$ where P is a parabolic subgroup of G defined by a set of indexes I. Let L be an ample line bundle on X with associated weight $\lambda = \sum_{i \notin I} n_i \lambda_i \in \Lambda_X^+$. Then*

$$\tau(X, L) = \max_{i \notin I} \frac{2 + \nu_i(I)}{n_i}$$

Moreover, the nef value map $X \to Y$ is a homogeneous fibration $G/P \to G/Q$ where Q is defined by the indexes $I \cup J$, $J = \{j \notin I \mid \tau(X, L) = (2 + \nu_j(I))/n_j\}$.

Proof. Since L is ample, $n_i > 0$ for all $i \notin I$. The weight of $K_X^q \otimes L^p$ is

$$\sum_{i \notin I} (-q(2 + \nu_i(I)) + pn_i)\lambda_i$$

and this gives a nef but not ample line bundle precisely when the above coefficients are all non-negative and at least one of them is zero, see Theorem 6.5. This translates into

$$\frac{p}{q} \geq \frac{2 + \nu_i(I)}{n_i} \quad \text{for all} \quad i \notin I$$

with equality for some $i \notin I$. Therefore, $\tau(X, L)$ is simply the maximum of $(2 + \nu_i(I))/n_i$, $i \notin I$. By construction, the coefficients of the weight $\mu = \sum m_i \lambda_i$ associated to $K_X^q \otimes L^p$ with $\tau(X, L) = p/q$ satisfy $m_i = 0$ for $i \in I$, $m_i \geq 0$ for $i \notin I$, and $m_i = 0$ for $i \notin I$ if and only if $(2 + \nu_i(I))/n_i = \tau(X, L)$. The map to projective space defined by the sections of the spanned bundle $K_X^q \otimes L^p$ is thus the homogeneous fibration $G/P \to G/Q$ where $Q = P_\mu = P_J$ as claimed, see Corollary 6.6. □

Corollary 10.4. *Let L_X be the minimal ample line bundle on $X = G/P$, i.e., the weight of L_X is $\sum_{i \notin I} \lambda_i$. Then $\tau(X, L_X) = 2 + \max_{i \notin I} \nu_i(I) \geq 2$. Moreover, for any ample line bundle L on X, $\tau(X, L) \leq \tau(X, L_X)$.*

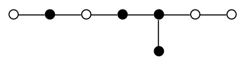
Recall that any $X = G/P$ can be decomposed into a product according to the simple factors of G: $X = X_1 \times \cdots \times X_t$ where $X_i = G_i/P_i$ and P_i is a parabolic subgroup of a simple complex Lie group G_i, $i = 1, \ldots, t$. If L is an ample line bundle on X, then $L_i = L|_{X_i}$ is ample on X_i for all i, and $\tau(X, L) = \max_i \tau(X_i, L_i)$. This observation shows that the following proposition is false if G is not assumed to be simple.

Proposition 10.5. *Let P be a parabolic subgroup of a simple Lie group G and let L_X be the minimal ample line bundle on $X = G/P$. If $\pi : X \to Y$ is a non-trivial homogeneous fibration, then $\tau(X, L_X) \leq \tau(Y, L_Y)$ where L_Y is the minimal ample line bundle on Y.*

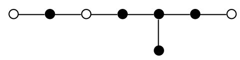
Proof. The fibration π is given by a coset map $G/P \to G/Q$, where Q is a parabolic subgroup of G defined by a set of indexes $J \supset I$. From Definition 9.4, we see that $\nu_i(I) \leq \nu_i(J)$ for $i \notin J$. Moreover, Definition 9.4 also shows that for any index $i \in J \setminus I$, there is always an index $j \in \partial J$ such that $\nu_j(J) \geq \nu_i(I)$ as long as $J \neq \{1, \ldots, \ell\}$. The latter possibility cannot occur, since π is non-trivial. By Corollary 10.4, $\tau(X, L_X) = 2 + \max_{i \notin I}\{\nu_i(I)\} \leq 2 + \max_{j \notin J}\{\nu_j(J)\} = \tau(Y, L_Y)$. □

This proposition and the remarks preceding it imply that for a given group G the maximum value of $\tau(X, L)$ occurs when P is a maximal parabolic subgroup and $L = L_X$ is the corresponding minimal ample line bundle on $X = G/P$. In fact, the nef value in this case is given by the first Chern number, $\tau(X, L_X) = c_1(X)$, see §9.3.

10.1. **Example.** As in Example 9.2, let $X = G/P$ where $G = E_8$ and P is defined by the indexes $I = \{2, 4, 5, 8\}$ corresponding to the black nodes

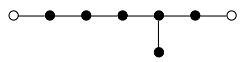


Suppose L is the ample line bundle on X defined by the weight $\lambda = 7\lambda_1+6\lambda_3+5\lambda_6+4\lambda_7$. We have already seen that $c_1(X) = 3\lambda_1 + 6\lambda_3 + 6\lambda_6 + 2\lambda_7$. The nef value of L is the maximum of the quotients of the coefficients of $c_1(X)$ by the corresponding coefficients of L: $\tau(X, L) = \max\{\frac{3}{7}, 1, \frac{6}{5}, \frac{1}{2}\} = \frac{6}{5}$. This maximum occurs for the third white node corresponding to λ_6 and so the indexes which define the parabolic subgroup Q of the nef value map $G/P \to G/Q$ are $\{2, 4, 5, 6\}$ corresponding to the black nodes



The fiber $Z = Q/P$ of the nef value map is isomorphic to a quotient of D_4 by a maximal parabolic subgroup so that $b_1(Z) = 1$. In fact, Z is a 6-dimensional quadric.

For the minimal ample line bundle L_X of weight $\lambda_1 + \lambda_3 + \lambda_6 + \lambda_7$, the nef value is $\tau(X, L_X) = 6$, and this is the maximum nef value for any ample line bundle on X. In this case, the parabolic subgroup Q_0 that defines the nef value map for L_X, $G/P \to G/Q_0$ picks up two nodes,



so that the fiber $Z = Q_0/P$ is isomorphic to the quotient of D_6 by a non-maximal parabolic subgroup and $b_2(Z) = 2$.

10.2. **Nef value and dual varieties.** Let X be a smooth projective variety and let L be a very ample line bundle on X giving an imbedding $X \subset \mathbb{P}(V^*)$, $V = H^0(X, L)$. As demonstrated in [2, 3], there is a connection between the nef value, $\tau(X, L)$, and the codimension of the variety $X' \subset \mathbb{P}^N$ of hyperplanes tangent to X, known as the dual or discriminant variety of X. The defect of (X, L) is defined to be

$$\operatorname{def}(X, L) = \operatorname{codim} X' - 1$$

Most smooth varieties have defect 0. If $\operatorname{def}(X, L) > 0$, then the defect is determined by the nef value [2]:

$$\operatorname{def}(X, L) = 2(\tau(X, L) - 1) - \dim X$$

Moreover, if Z is a general fiber of the nef value map $X \to Y$, then $\operatorname{def}(X, L) = \operatorname{def}(Z, L_Z) - \dim Y$ and $\operatorname{Pic}(Z) \cong \mathbb{Z}$. If the defect of X is greater than 2, then a smooth hyperplane section of X also has positive defect, see [16]. Up to such hyperplane sections and fibrations, the only known examples of smooth varieties with positive defect are linear projective spaces, $\mathbb{P}^n$, the Plücker imbedding of the Grassmann variety, $Gr(2, 2m+1)$, and the 10-dimensional spinor variety S_4 in $\mathbb{P}^{15}$. These last examples are all homogeneous spaces. In fact, they and product spaces built from them are the only homogeneous projective varieties with $\operatorname{def}(X, L) > 0$.

Theorem 10.6 ([30, 49])**.** *Let $X = G/P$ where P is a parabolic subgroup of G and let L be an ample line bundle on X defining an imbedding $X \subset \mathbb{P}^N$. Then the defect $k = \operatorname{def}(X, L)$ is positive if and only if X is one of the following:*

(1) *A linear projective space* $\mathbb{P}^n$ *in* $\mathbb{P}^N$, $k = n$.
(2) *The Plücker imbedding of the Grassmann variety* $Gr(2, 2m+1)$ *in* $\mathbb{P}^{m(2m+1)-1}$, $k = 2$.
(3) *The 10-dimensional spinor variety* S_4 *in* $\mathbb{P}^{15}$, $k = 4$
(4) $X_1 \times X_2$ *where* X_1 *is one of the varieties in 1–3 and* $\operatorname{def} X = \operatorname{def} X_1 - \dim X_2 > 0$.

The classification given in [30] is based on invariant theory and proceeds through many cases based on the type of the group. The above relationship between the defect and the nef value, however, can be exploited to give a rather straightforward classification which we now sketch.

If X is contained in a hyperplane H then X' is a cone over the dual variety of $X \subset H$ with vertex equal to the point dual to H. Conversely, if X' is a cone, then X is contained in the hyperplane dual to the vertex of the cone, see [16]. The defect remains the same whether we consider X as a subvariety of $\mathbb{P}^N$ or of $H \cong \mathbb{P}^{N-1}$. Projective space $\mathbb{P}^n$ with $L = \mathcal{O}_{\mathbb{P}^n}(1)$ is a special case. Since there are no singular hyperplane sections, the dual variety is empty. In order to be compatible with later formulas, we adopt the convention that $\operatorname{def} \mathbb{P}^n = n$ which by the above definition is the same as assigning the dimension -1 to the empty set.

An important part of the classification of homogeneous spaces with positive defect involves products: Let X_1 and X_2 be smooth projective varieties with very ample line bundles L_1 and L_2, respectively, and $\dim X_1 \geq \dim X_2$. Let $X = X_1 \times X_2$ and $L = \operatorname{pr}_1^* L_1 \otimes \operatorname{pr}_2^* L_2$. Then $\operatorname{def}(X, L) > 0$ if and only if $\operatorname{def}(X_1, L_1) > \dim X_2$. When this is the case, $\operatorname{def}(X, L) = \operatorname{def}(X_1, L_1) - \dim X_2 > 0$, see [49, 2].

Now let $X = G/P$ where P is a parabolic subgroup of G and let L be an ample line bundle on X. Let $X = X_1 \times \cdots \times X_s$ be the decomposition of X into irreducible factors $X_i = G_i/P_i$ where G_i is simple and $P_i \subset G_i$ is parabolic, $i = 1, \ldots, s$. The above statements imply that $\operatorname{def}(X, L) > 0$ if and only if $\operatorname{def}(X_i, L_i) > \operatorname{codim}_X X_i$ for some $1 \leq i \leq s$. In this case

$$\operatorname{def}(X, L) = \operatorname{def}(X_i, L_i) - \operatorname{codim}_X X_i > 0$$

Moreover, the nef value map $X \to Y$ associated to L is a homogeneous fibration $G/P \to G/Q$ with fiber $Z = Q/P$ where Q is a parabolic subgroup of G, P is a maximal parabolic subgroup of Q, and

$$\operatorname{def}(X, L) = \operatorname{def}(Z, L_Z) - \dim Y$$

This shows that, up to products and fibrations, it is sufficient to classify homogeneous spaces $X = G/P$ with positive defect for the case where G is simple and P is a maximal parabolic subgroup. Other standard facts about dual varieties allow us to assume in addition that L is an ample generator of the line bundles on X, see [17, Theorem 1.3(b)]. Note that when L is such a generator with weight λ, its defect is the integer τ given by $c_1(X) = \tau\lambda$; see §9.3 for the particular values τ can assume. Using only the numerical criterion that $k = 2(\tau - 1) > 0$ for spaces of positive defect k, we arrive at the following list of candidates.

- $A_\ell/P_1 \cong A_\ell/P_\ell$ (projective space), $\tau = \ell + 1$, $n = k = \ell$
- $A_\ell/P_2 \cong A_\ell/P_{\ell-1}$ (Grassmann), $n = 2(\ell - 1)$, $\tau = \ell + 1$, $k = 2$
- A_5/P_3 (Grassmann), $n = 9$, $\tau = 6$, $k = 1$
- B_ℓ/P_1 (quadric), $n = \tau = 2\ell - 1$, $k = 2\ell - 3$
- $B_2/P_2 = D_4/P_4$ (projective space), $n = 3$, $\tau = 4$, $k = 3$

- $B_3/P_3 = D_4/P_4$ (quadric), $n = \tau = 6$, $k = 4$
- $B_4/P_4 = D_5/P_5$ (spinor), $n = 10$, $\tau = 8$, $k = 4$
- $B_5/P_5 = D_6/P_6$ (spinor), $n = 15$, $\tau = 10$, $k = 3$
- $B_6/P_6 = D_7/P_6$ (spinor), $n = 21$, $\tau = 12$, $k = 1$
- C_ℓ/P_1, (projective space) $n = 2\ell - 1$, $\tau = 2\ell$, $k = 2\ell - 1$
- C_ℓ/P_2, $n = 4\ell - 5$, $\tau = 2\ell - 1$, $k = 1$
- D_ℓ/P_1 (quadric), $n = \tau = 2\ell - 2$, $k = 2\ell - 4$
- $E_6/P_1 \cong E_6/P_5$, $n = 16$, $\tau = 12$, $k = 6$
- E_7/P_1, $n = 27$, $\tau = 18$, $k = 7$
- F_4/P_4, $n = 15$, $\tau = 11$, $k = 5$
- G_2/P_1 (quadric), $n = \tau = 5$, $k = 3$

The listed value of k is the defect only if the defect is positive, which is not the case for most of the entries. For example, quadrics are hypersurfaces and obviously have defect 0 (they are self-dual). The fact that $A_\ell/P_2 = Gr(2, \ell+1)$ has positive defect only when $\ell + 1$ is odd has been known for some time [19, 35]. The 10-dimensional spinor variety $B_4/P_4 = D_5/P_5$ is also known to have defect 4, see e.g., [16, 34].

The space C_ℓ/P_2 is easily seen to have defect 0. Recall that $C_\ell = Sp(2\ell)$ is the stabilizer of a generic 2-form in $\bigwedge^2 \mathbb{C}^{2\ell}$ and stabilizes a (complementary) hyperplane through a point of the form $[v_1 \wedge v_2] \in \mathbb{P}(\bigwedge^2 \mathbb{C}^{2\ell})$. The isotropy subgroup of $[v_1 \wedge v_2]$ is conjugate to P_2 and therefore C_ℓ/P_2, which has dimension $4\ell - 5$, is a hyperplane section in $Gr(2, 2\ell)$. This Grassmann variety has defect 0 as we just pointed out, and so the defect of the hyperplane section C_ℓ/P_2 must also be 0 by [17, (1.3)].

The exceptional variety E_6/P_1 cannot have positive defect because its Betti numbers do not follow the expected pattern for spaces of positive defect k, see [29]:

$$b_n = b_{n-2}, \quad b_{n-1} = b_{n-3}, \cdots, \quad b_{n-k+1} = b_{n-k-1}$$

The odd Betti numbers of E_6/P_1 are zero, and the pertinent even Betti numbers are $b_{16} = 3$, $b_{14} = 2$, and $b_{12} = 2$, see, e.g., [46]. Therefore, the defect of E_6/P_1 is 0.

The remaining cases are the spaces D_6/P_6, D_7/P_7, E_7/P_1, and F_4/P_4. These spaces do not violate any of the simple criteria for positive defect. They can be shown to have defect 0, however, by computing the fiber dimension (= defect) of the duality map $\mathbb{P}(N_X^*(1)) \to \mathbb{P}(V)$ where $N_X^*(1)$ is the conormal bundle of $X \subset \mathbb{P}^N$ twisted by $\mathcal{O}_X(1)$. This calculation can be reduced to determining the rank of matrix involving certain roots and structure constants, see [49].

At this point we can conclude that the only homogeneous spaces G/P, P maximal, having an ample line bundle with positive defect are the familiar examples: $\mathbb{P}^k$, $\mathrm{Gr}(2, 2m+1)$ or S_4. The general case is handled as follows.

Suppose $X = G/P$ and L is a minimal ample line bundle on X with $\mathrm{def}(X, L) > 0$. Since the case of products has already been discussed above, we may assume that G is simple. The nef value map $X = G/P \to Y = G/Q$ has fiber $Z = Q/P$ where P is a maximal parabolic subgroup of Q. Since $\mathrm{def}(X, L) = \mathrm{def}(Z, L_Z) - \dim Y > 0$ implies $\mathrm{def}(Z, L_Z) > 0$, the only possibilities for Z are the three spaces just mentioned. If $Z = \mathrm{Gr}(2, 2m+1)$ then $\mathrm{def}(Z, L_Z) = 2$ forcing $\dim Y < 2$. If $\dim Y = 1$ then $Y \cong \mathbb{P}^1$, and $G = \mathrm{SL}(2, \mathbb{C})$ which is too small to provide a non-trivial fibration. Thus, $\dim Y = 0$ and $X = Z$. Similarly, if $Z = S_4$, then $\dim Y < 4$. The limited possibilities for Y imply that G is of type A_r, $r = 1, 2, 3$ or B_2. None of these can contain either of the two simple Lie groups that act transitively on the

10-dimensional spinor variety S_4 (B_4 or D_5). So again we deduce that the fibration is trivial and $X = Z$. The final possibility of a non-trivial fibration with $Z = \mathbb{P}^k$ can be ruled out since for any such fibration one must have $\dim Y > k$ and this contradicts $\operatorname{def}(Z, L_Z) - \dim Y > 0$, see [49, 4.1]. This concludes the sketch of the proof of Theorem 10.6.

The following statement holds for non-linear smooth projective varieties $X \subset \mathbb{P}^N$ such that $\dim X = \dim X' \leq \frac{2}{3}N$, see [16]. The version we present here for homogeneous spaces is simply a corollary of the previous classification. This list also classifies those non-linear homogeneous spaces with non-singular dual varieties, since a non-singular dual implies that $\dim X = \dim X'$, see [16]. If X is a linear projective space, $X = \mathbb{P}^n \subset \mathbb{P}^N$, then the tangent hyperplanes are clearly parameterized by a complementary projective space $X' = \mathbb{P}^{N-n-1}$.

Corollary 10.7. *Let $X = G/P$ be a non-linear homogeneous space imbedded in $\mathbb{P}^N$ by the sections of an ample line bundle L on X. If $\dim X = \dim X'$ then X is one of the following:*

1. *A quadric hypersurface in $\mathbb{P}^{n+1}$*
2. *The Segre imbedding of $\mathbb{P}^{n-1} \times \mathbb{P}^1$ in $\mathbb{P}^{2n-1}$*
3. *The Plücker imbedding of $Gr(2,5)$ in $\mathbb{P}^9$*
4. *The 10-dimensional spinor variety S_4 in $\mathbb{P}^{15}$*

Proof. The listed varieties are well-known to be self-dual, see e.g., [16, 19, 34]. If $X = Gr(2, 2m+1)$ then $\dim X = 2(2m-1)$ and $\dim X' = m(2m+1) - 4$ and these are equal only when $m = 2$. By Theorem 10.6 it remains to check the case $X = X_1 \times X_2$. Since $L = \operatorname{pr}_1^* L_1 \otimes \operatorname{pr}_2^* L_2$ and hence $H^0(X, L) \cong H^0(X_1, L_1) \otimes H^0(X_2, L_2)$, the imbedding dimension of X satisfies $N+1 = (N_1+1)(N_2+1)$ where N_i is the imbedding dimension of X_i under L_i, $i = 1, 2$. We know that $\operatorname{def} X = N - \dim X - 1 = \operatorname{def} X_1 - \dim X_2$. Hence, $N_1 N_2 + N_1 + N_2 = \dim X_1 + \operatorname{def} X_1 + 1$. Since $N_i \geq \dim X_i$ this equation becomes $\dim X_2(\dim X_1 + 1) \leq \operatorname{def} X_1 + 1$. Now, X_1 must be a projective space, for otherwise $\operatorname{def} X_1 + 1 \leq 5$ and this would imply that $\dim X_1 = 1$. Therefore, $(X_1, L_1) \cong (\mathbb{P}^{n_1}, \mathcal{O}_{\mathbb{P}^{n_1}}(1))$ and the previous equation yields $N_2 = 1$. This implies that $(X_2, L_2) = (\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$, and $X = \mathbb{P}^{n-1} \times \mathbb{P}^1$ as claimed. □

Certain real hypersurfaces in complex projective space are tubes over complex submanifolds, see [12]. This fact along with the above classification of self-dual homogeneous spaces can be used to classify homogeneous real hypersurfaces in complex projective space.

Corollary 10.8 ([57])**.** *Let M be a homogeneous complete real hypersurface imbedded equivariantly in $\mathbb{P}^N$. Then M is a tube over a linear projective space or one of the 4 self-dual homogeneous spaces $X \subset \mathbb{P}^N$ listed in Corollary 10.7.*

Proof. Let $M = K/L$ where K is a compact Lie group and let ξ denote the normal vector field to M. If J denotes the complex structure operator, then $W = -J\xi$ is a tangent vector field. Because the imbedding is equivariant, W is left invariant under K and therefore its integral curves are given by 1-parameter subgroups of K and are geodesics. By [12], M is a tube over a complex submanifold $X \subset \mathbb{P}^N$ (a focal submanifold). In particular, X is homogeneous, $X = G/P$, and G acts transitively on the normal directions to X. It follows that the conormal variety $\mathbb{P}(N_X^*(1))$ itself is a homogeneous space, G/P_0, and therefore the image of $\mathbb{P}(N_X^*(1)) = G/P_0 \longrightarrow X'$

must also be homogeneous, $X' = G/P'$. If X is not a linear projective space then $\dim X' = \dim X$, and so X is one of 4 self-dual homogenoeus spaces by Corollary 10.7. □

Conversely, the classification of homogeneous real hypersurfaces [57] can also be used to obtain the list of self-dual homogeneous spaces. For, if a homogeneous space $X = G/P$ is self-dual, then by a symmetry argument, the conormal variety must be a homogeneous space under G. This implies that G acts transitively on the normal directions to X and hence a maximal compact subgroup of G must have a hypersurface orbit in the normal bundle of X. The resulting orbit is a homogeneous real hypersurface in $\mathbb{P}^N$ realized as a tube over X and therefore must be on the list given in [57].

11. Cohomology

Let $E = G \times_P E_0$ be a homogeneous vector bundle on $X = G/P$ where P is a parabolic subgroup of G. We denote by $H^q(X, E)$ the q-th Čech cohomology with coefficients in the sheaf of local holomorphic sections of E. In [9], Bott gives a precise way to compute these cohomology groups in the case where the representation of P on E_0 is irreducible. Subsequently other proofs have been found, see, e.g., [31, 15]. We shall now sketch the proof of Bott's theorem and give some applications.

Recall from §9.1 that the weight $\delta \in \Lambda^+$ is defined by

$$\delta = \sum_{i=1}^{\ell} \lambda_i = \frac{1}{2} \sum_{\alpha > 0} \alpha$$

so that $\langle \delta, \alpha \rangle = 1$ and $\sigma_\alpha(\delta) = \delta - \alpha$ for any simple root α. Recall also that the dominant conjugate of a weight $\lambda \in \Lambda$ is denoted by $[\lambda] \in \Lambda^+$, see §3.

The proof of Bott's theorem relies on Serre Duality and an application of Leray's spectral sequence. Serre duality states that for a holomorphic vector bundle E on a compact complex manifold X of dimension n,

$$H^q(X, E) \cong H^{n-q}(X, K_X \otimes E)^*$$

where $K_X = \bigwedge^n T_X^*$ is the canonical bundle of X, see [43]. The version of Leray's spectral sequence we need is the following, see [25, Thèoréme 4.17.1].

Lemma 11.1. *Let $\pi : Y \to X$ be a locally trivial holomorphic fibration with a connected compact fiber Z and let F be a holomorphic vector bundle on Y. There is a spectral sequence E_r with*

$$E_2^{q,p} = H^q(X, H^p(Z, F_Z))$$

whose final term is associated to $H^*(Y, F)$. In particular, if $H^p(Z, F_Z) = 0$ except for $p = k$, then the spectral sequence is trivial and therefore

$$H^q(Y, F) \cong H^{q-k}(X, H^k(Z, F_Z))$$

for all $q \geq 0$

Using this spectral sequence, Bott's theorem can be easily reduced to the case of line bundles on G/B, a case we must handle first.

Proposition 11.2. *Let B be a Borel subgroup of G and let L be a holomorphic line bundle on $Y = G/B$. Let $\lambda \in \Lambda$ be the weight associated to L.*

(1) *If $\lambda + \delta$ is singular, then $H^q(Y, L) = 0$ for all q.*
(2) *If $\lambda + \delta$ is not singular, then $H^q(Y, L) = 0$ for all $q \neq \operatorname{ind}(\lambda + \delta)$, and for $q = \operatorname{ind}(\lambda + \delta)$, $H^q(Y, L)$ is isomorphic to the irreducible G-module with highest weight $[\lambda + \delta] - \delta$.*

Proof. We first make some general observations. Let α be a simple root and let P_α be the minimal parabolic subgroup defined by α. Consider the fibration $Y = G/B \to Y_\alpha = G/P_\alpha$ with fiber $Z_\alpha = P_\alpha/B \cong \mathbb{P}^1$. Let K_α be the canonical bundle of Z_α, so the weight associated to K_α is $-\alpha$.

If $\langle \lambda + \delta, \alpha \rangle \leq 0$ then $\langle \lambda, \alpha \rangle \leq -1$. This implies that $H^0(Z_\alpha, L) = 0$ by Theorem 6.5. Applying Lemma 11.1 we see that for $q \geq 0$,

$$H^q(Y, L) \cong H^{q-1}(Y_\alpha, H^1(Z_\alpha, L))$$

By Serre duality,

$$H^1(Z_\alpha, L) \cong H^0(Z_\alpha, K_\alpha \otimes L^{-1})^*$$

The weight of the line bundle $K_\alpha \otimes L^{-1}$ is $-\lambda - \alpha$. We consider the two cases:

(i) If $\langle \lambda + \delta, \alpha\rangle = 0$ then $\langle -\lambda - \alpha, \alpha\rangle = -1$ and hence $H^0(Z_\alpha, K_\alpha \otimes L^{-1}) = 0$ by Theorem 6.5. Therefore, $H^q(Y, L) = 0$ for all $q \geq 0$.

(ii) If $\langle \lambda+\delta, \alpha\rangle < 0$ then $\langle -\lambda-\alpha, \alpha\rangle \geq 0$ so that $H^0(Z_\alpha, K_\alpha \otimes L^{-1})$ is isomorphic to the irreducible P_α-module with maximal weight $-\lambda - \alpha$ by Theorem 6.5. The dual of this module has maximal weight

$$-\sigma_\alpha(-\lambda - \alpha) = \sigma_\alpha(\lambda) - \alpha = \sigma_\alpha(\lambda + \delta) - \delta$$

see §4. Let $\sigma_\alpha L$ denote the line bundle defined by the weight $\sigma_\alpha(\lambda + \delta) - \delta$. Then, for all q,

$$\begin{aligned} H^q(Y, L) &\cong H^{q-1}(Y_\alpha, H^0(Z_\alpha, K_\alpha \otimes L^{-1})^*) \\ &\cong H^{q-1}(Y_\alpha, H^0(Z_\alpha, \sigma_\alpha L)) \\ &\cong H^{q-1}(Y, \sigma_\alpha L) \end{aligned}$$

The last isomorphism follows from Lemma 11.1 again because $H^1(Z_\alpha, \sigma_\alpha L)$ vanishes: by Serre Duality, $H^1(Z_\alpha, \sigma_\alpha L) \cong H^0(Z_\alpha, K_\alpha \otimes \sigma_\alpha L^{-1})^*$ and the latter group is zero by Theorem 6.5 since $\langle -\sigma_\alpha(\lambda + \delta) + \delta - \alpha, \alpha\rangle = \langle \lambda, \alpha\rangle < 0$.

To prove the theorem, let $\alpha_{i(1)}, \ldots, \alpha_{i(t)}$ denote a sequence of simple roots such that $\langle \lambda+\delta, \alpha_{i(1)}\rangle < 0$, $\langle \sigma_{i(1)}(\lambda+\delta), \alpha_{i(2)}\rangle < 0, \ldots, \langle \sigma_{i(t-1)} \cdots \sigma_{i(1)}(\lambda+\delta), \alpha_{i(t)}\rangle < 0$, and $\langle \sigma_{i(t)} \cdots \sigma_{i(1)}(\lambda+\delta), \alpha\rangle \geq 0$ for all simple roots α. Then $t = \mathrm{ind}(\lambda+\delta)$ and the dominant conjugate of $\lambda+\delta$ is $[\lambda+\delta] = \sigma(\lambda+\delta)$ where $\sigma = \sigma_{i(t)} \cdots \sigma_{i(1)}$, see §3. Therefore, repeatedly applying the above isomorphism gives

$$H^q(Y, L) \cong H^{q-1}(Y, \sigma_{i(1)} L) \cong H^{q-2}(Y, \sigma_{i(2)}\sigma_{i(1)} L) \cong \cdots \cong H^{q-t}(Y, \sigma L)$$

The weight associated to σL is $[\lambda+\delta] - \delta$. This shows that $H^q(Y, L) = 0$ for $q < t$.

If $\langle [\lambda + \delta], \alpha\rangle = 0$ for some simple root α—this is equivalent to $\lambda + \delta$ being singular—then the spectral sequence argument of (i) above shows that $H^q(Y, \sigma L) = 0$ for all $q \geq 0$ so that $H^q(Y, L) = 0$ for all $q \geq 0$. If $\langle [\lambda + \delta], \alpha\rangle \neq 0$ for all simple roots α, then $[\lambda+\delta] - \delta$ is dominant and hence $H^t(Y, L) = H^0(Y, \sigma L)$ is isomorphic to the irreducible G-module with highest weight $[\lambda+\delta] - \delta$ by Theorem 6.5. To see that $H^q(Y, L) = 0$ for $q > t$, apply Serre duality:

$$H^{q-t}(Y, \sigma L) \cong H^{n-q+t}(Y, K_Y \otimes \sigma L^{-1})^*$$

where $n = \dim Y$. The weight of $K_Y \otimes \sigma L^{-1}$ is $\mu = -[\lambda + \delta] - \delta$ since the weight of K_Y is -2δ, see §9. Now, because $[\lambda + \delta]$ is non-singular and dominant, the dominant conjugate of $\mu + \delta = -[\lambda + \delta]$ is $[\mu + \delta] = -\omega_0[\lambda + \delta]$ where ω_0 is the longest element in the Weyl group, $\ell(\omega_0) = \mathrm{ind}(\mu + \delta) = n$, see §3. Thus, by what we have just proved,

$$H^{n-q+t}(Y, K_Y \otimes \sigma L^{-1}) \cong H^{t-q}(Y, \tau L)$$

where τL is the line bundle on Y associated to the weight $[\mu + \delta] - \delta$. Therefore, $H^q(Y, L) = 0$ for $q > t$ as well. Note that the dual of the G-module $H^0(Y, \tau L)$ is $H^0(Y, \sigma L)$ since $-\omega_0([\mu + \delta] - \delta) = [\lambda + \delta] - \delta$, see §4. □

Theorem 11.3 (Bott). *Let $E = G \times_P E_0$ be a homogeneous holomorphic vector bundle on $X = G/P$ where P is a parabolic subgroup of G. Assume E_0 is an irreducible representation of P with maximal weight $\lambda \in \Lambda_X$. Let B be a Borel* *subgroup of G and let $L = G \times_B \mathbb{C}$ be the line bundle on $Y = G/B$ associated to the weight λ. Then, for all q*

$$H^q(X,E) \cong H^q(Y,L)$$

In particular,

(1) *If $\lambda+\delta$ is singular, then $H^q(X,E) = 0$ for all q.*
(2) *If $\lambda+\delta$ is not singular, then $H^q(X,E) = 0$ for all $q \neq \mathrm{ind}(\lambda+\delta)$, and for $q = \mathrm{ind}(\lambda+\delta)$, $H^q(X,E)$ is isomorphic to the irreducible G-module with highest weight $[\lambda+\delta]-\delta$.*

Proof. Consider the equivariant fibration $Y = G/B \to X = G/P$ with fiber $Z = P/B$. Since λ is P-dominant, Proposition 11.2 implies that $H^0(Z, L_Z)$ is isomorphic to the irreducible representation of P with maximal weight λ, and $H^q(Z, L_Z) = 0$ for $q > 0$. Therefore, $\pi_* L \cong E$ and by Lemma 11.1,

$$H^q(Y,L) \cong H^q(X, H^0(Z, L_Z)) = H^q(X,E)$$

The last assertions then follow from Proposition 11.2 applied to L. □

Bott's original proof used the Kodaira vanishing theorem, but this can be avoided by applying Serre duality and exploiting the two-fold symmetry in the Weyl group as in the proof of Proposition 11.2. This approach also allows a purely algebraic proof of Bott's theorem, see [15]. The above proof retains the spirit of of Bott's argument which shows how weights of the form $\sigma(\lambda+\delta)-\delta$ appear due to Serre Duality.

Bott's theorem allows us to calculate the cohomology a homogeneous vector bundle that is the direct sum of irreducible homogeneous bundles, $E = m_1F_1 \oplus \cdots \oplus m_tF_t$, with multiplicities m_i and maximal weights μ_i, $1 \le i \le t$. If $M(q)$ denotes the subset of these weights μ_i such that $\mathrm{ind}(\mu_i+\delta) = q$ and $\mu_i+\delta$ is non-singular, $1 \le i \le t$, then

$$H^q(X,E) \cong \bigoplus_{\mu_i \in M(q)} m_i V^{[\mu_i+\delta]-\delta}$$

and the G-module structure of $H^q(X,E)$ is completely determined. In particular, the dimension of $H^q(X,E)$ can calculated using Weyl's dimension formula, see §4. We shall see an example of this later: on a compact hermitian symmetric space the bundle of holomorphic p-forms, Ω^p_X, is a direct sum of irreducible vector bundles of known maximal weights. The cohomology of $\Omega^p_X \otimes L^k$ can then be computed as just described for any line bundle L on X.

Of course, many homogeneous vector bundles of interest are not direct sums of irreducible homogeneous vector bundles. Nevertheless, using filtrations, Bott's theorem can often be applied indirectly to predict when certain cohomology groups vanish, cf. [21].

Proposition 11.4. *Let $E = G \times_P E_0$ be a homogeneous vector bundle on $X = G/P$ where P is a parabolic subgroup of G. Let $M(E_0)$ denote the set of indexes, $\mathrm{ind}(\mu+\delta)$, for all maximal weights $\mu \in \Lambda_{\max}(E_0)$ such that $\mu+\delta$ is non-singular. Then $H^q(X,E) = 0$, if $q \notin M(E_0)$.*

Proof. Let $E_0 \supset E_1 \supset \cdots \supset E_t \supset E_{t+1} = 0$ be a filtration of E_0 by P-submodules with irreducible subquotients $F_i = E_i/E_{i+1}$ of maximal weights μ_i, $i = 0, \ldots, t$.

The short exact sequences $0 \to E_{i+1} \to E_i \to F_i \to 0$, $i = 0, \ldots, t$, lead to the following long exact sequences

$$\begin{array}{ccccccc} \ldots \to & H^q(X, F_t) & \to H^q(X, E_{t-1}) & \to H^q(X, F_{t-1}) & \to \ldots \\ \ldots \to & H^q(X, E_{t-1}) & \to H^q(X, E_{t-2}) & \to H^q(X, F_{t-2}) & \to \ldots \\ & & \cdots & & \\ \ldots \to & H^q(X, E_1) & \to \quad H^q(X, E) & \to \quad H^q(X, F_0) & \to \ldots \end{array}$$

Bott's Theorem 11.3 implies that $H^q(X, F_i) = 0$ if $q \neq \mathrm{ind}(\mu_i + \delta)$ or if $\mu_i + \delta$ is singular, $0 \leq i \leq t$. Therefore, if $q \notin M(E_0)$, then $H^q(X, E_{t-1}) = 0$, $H^q(X, E_{t-2}) = 0$, …, $H^q(X, E) = 0$. □

In the special case when $M(E_0) = \{0\}$ a stronger statement holds which can be viewed as a generalization of the Borel-Weil Theorem 6.5 to vector bundles.

Corollary 11.5. *Let $E = G \times_P E_0$ be a homogeneous vector bundle on $X = G/P$ where P is a parabolic subgroup of G. If the maximal weights of E_0 are dominant, $\Lambda_{\max}(E_0) \subset \Lambda^+$, then E is generated by global sections and $H^q(X, E) = 0$ for $q > 0$.*

Proof. As in Propostition 11.4 we have $H^q(X, E) = 0$ for $q > 0$ and $H^0(X, E) \cong \bigoplus_i H^0(X, F_i)$. Moreover, since the μ_i are dominant, the evaluation maps $X \times H^0(X, F_i) \to F_i$ are surjective. Therefore, by universality of induced modules, see Proposition 5.1, the evaluation map $X \times H^0(X, E) \to E$ is surjective and E is generated by global sections. □

A version of the this theorem for Dolbeault cohomology will be presented in § **??**.

11.1. **Tangent Bundles and Rigidity.** One of the applications of Bott's theorem given in his original paper [9] is to show that the complex structure on $X = G/P$ cannot be locally deformed. This result relies on Kodaira-Spencer theory, see [32], which says that if X_t is a smooth deformation of the complex structure of $X = X_0$ and if $H^1(X, T_X) = 0$, then X_t is holomorphically isomorphic to X for all t near 0.

Recall that the tangent bundle on $X = G/P$ is naturally a homogeneous vector bundle. The representation of P on the the tangent space at the identity coset is isomorphic to the P-module $\mathfrak{g}/\mathfrak{p}$ with the induced adjoint action, see §2. The exact sequence of P-modules

$$0 \longrightarrow \mathfrak{p} \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{p} \longrightarrow 0$$

thus induces a corresponding exact sequence of vector bundles on X

$$0 \longrightarrow F \longrightarrow E \longrightarrow T_X \longrightarrow 0$$

Since the adjoint representation of P on $\mathfrak{g}$ extends to the adjoint representation of G on $\mathfrak{g}$, the bundle E is trivial and hence $H^q(X, E) = 0$ for $q \geq 1$, see §5. Therefore, from the associated long exact sequence we obtain

$$H^q(X, T_X) \cong H^{q+1}(X, F) \qquad q \geq 1$$

The weights of $F = G \times_P \mathfrak{p}$ are all the negative roots of G, 0, and all the positive roots of P. To apply Proposition 11.4 we need to know the index of non-singular weights of the form $\alpha + \delta$ for all such roots α.

This calculation is not hard to carry out explicitly. The general result is that for any root $\alpha \in \Phi$, if $\alpha + \delta$ is non-singular then $\mathrm{ind}(\alpha + \delta) \leq 1$. For example, if all roots have the same length (i.e., G is composed of types A_ℓ, D_ℓ or E_ℓ) then the coordinates of any root in the basis of fundamental dominant weights are always 0,

$+1$, or -1, except for the simple roots whose coordinates are given by the rows of the Cartan matrix. Since δ has all coordinates equal to 1, it is clear that $\alpha+\delta$ is either singular (a coordinate is 0) or has index 0 (all coordinates positive), unless $-\alpha$ is a simple root. When $-\alpha$ is simple, $\delta+\alpha=\sigma_{-\alpha}(\delta)$ and so $\delta+\alpha$ has index 1. Similar arguments can be worked out for the other types; a detailed proof can be found in [9, p.220].

We conclude from Proposition 11.4 that $H^q(X,F)=0$ for $q\geq 2$ which establishes the following proposition and corollary.

Proposition 11.6. *Let $X=G/P$ where P is a parabolic subgroup of G. Then $H^q(X,T_X)=0$ for $q\geq 1$.*

Corollary 11.7. *The complex structure of $X=G/P$ is locally rigid.*

11.2. **Line Bundles.** Since every holomorphic line bundle L on $X=G/P$ is homogeneous and is thus defined by an irreducible 1-dimensional representation of P, i.e., by a weight $\lambda\in\Lambda_X$, Bott's theorem applies particularly well to this case. If $\lambda+\delta$ is singular, then all the cohomology groups $H^q(X,L)$ vanish. Otherwise, precisely one cohomology group is non-zero in degree $\operatorname{ind}(\lambda+\delta)$, and this group is isomorphic to the irreducible G-module $V^{[\lambda+\delta]-\delta}$.

We would now like to relate Bott's theorem for line bundles to the vanishing theorem proved in §7.3 based on the curvature of L. Recall Theorem 7.6 states that the number of positive eigenvalues of the Levi form, $\mathcal{L}(\phi_L)$, of a tube neighborhood of the zero section in L is given by $\operatorname{ind}(\lambda)$ and the number of negative eigenvalues by $\operatorname{ind}(-\lambda)$. The theorem of Andreotti-Grauert then implies that for sufficiently large m, $H^q(X,L^m)=0$ if $q<\operatorname{ind}(\lambda)$ or $q>\dim X-\operatorname{ind}(-\lambda)$. Moreover, if λ is non-singular, then $\operatorname{ind}(\lambda)+\operatorname{ind}(-\lambda)=\dim X$ and $H^q(X,L^m)=0$ for $q\neq\operatorname{ind}(\lambda)$.

The weight of L^m is $m\lambda$ and it is interesting to observe that for sufficiently large m, both $m\lambda$ and $m\lambda+\delta$ must lie in the same Weyl chamber. We need only choose m large enough so that if $\langle\alpha,m\lambda\rangle<0$ then $\langle\alpha,m\lambda\rangle<-1$ for all positive roots $\alpha>0$. Notice that this also guarantees that $m\lambda+\delta$ is non-singular. Thus,

$$\operatorname{ind}(m\lambda+\delta)=\operatorname{ind}(m\lambda)=\operatorname{ind}(\lambda)$$

By Bott's theorem, $H^q(X,L^m)=0$ for $q\neq\operatorname{ind}(\lambda)$ and $H^q(X,L^m)\cong V^{[m\lambda+\delta]-\delta}$ when $q=\operatorname{ind}(\lambda)$.

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