On Capacity Under Received-Signal Constraints*

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Abstract

In a world where different systems have to share the same spectrum, the received (interfering) power may be a more relevant constraint than the maximum transmit power. Motivated by such a spectrum-sharing approach, this paper investigates the behavior of capacity under *received-power* constraints, modeling for example the maximum interference that one system may inflict on another. The insight of the paper is that while in the point-to-point case, transmit and received-power constraints are largely equivalent, they can lead to quite different conclusions in network cases, including relay networks, multiple access channels with dependent sources and feedback, and collaborative communication scenarios.

1 Introduction

In this paper, we illustrate that received-signal constraints can lead to substantially different insights and conclusions, as compared to customary transmitted-signal constraints. As examples, we consider Gaussian multiple-access and relay channels, as well as certain scenarios involving collaborative communication. Capacity results and separation theorems are found. Constraints on the received power have been considered in a number of prior investigations, see e.g. [1].

Received-Signal Constraints

Let us denote the signal(s) received by the decoder(s) of a communication system by $\{Y_1[i], \ldots, Y_N[i]\}_{i=1}^n$, where i denotes the (discrete) time index. In this paper, we first consider the scenario where the coding scheme must be designed in such a way as to satisfy constraints of the form

$$E[\rho_k(\{Y_1[i], \ldots, Y_N[i]\}_{i=1}^n)] \quad \leq \quad \Gamma_k, \text{ for } k = 1, 2, \ldots. \tag{1}$$

Moreover, let us denote the signals transmitted by the encoders of a communication system by $\{X_1[i], \ldots, X_M[i]\}_{i=1}^n$. In this paper, we also consider constraints taking the shape

$$E[\rho_{I,k}(\{X_1[i], \ldots, X_M[i]\}_{i=1}^n)] \quad \leq \quad \Gamma_{I,k}, \text{ for } k = 1, 2, \ldots. \tag{2}$$

Here, the function $\rho_{I,k}(\cdot)$ may model the interference that the receiver k of a competing system incurs from the M transmitters of our system. As we illustrate, the latter problem is considerably more intricate than the former.

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Scalar Point-to-point Channels

Consider the standard (memoryless) point-to-point capacity problem, as defined e.g. in [2, p.108], but suppose that the channel input constraint is replaced by a channel output constraint. More precisely, the coding scheme must be designed such that the channel output sequence $\{Y[i]\}_{i=1}^{n}$ satisfies $\frac{1}{n}\sum_{i=1}^{n} E[\rho(Y[i])] \leq \Gamma$. The resulting capacity can be expressed as $C(\Gamma) = \max I(X,Y)$, where the max is taken over all $p(x)$ for which $E[\rho(Y)] \leq \Gamma$, see [2, p.117]. Using this, it is easy to show that the problem of finding capacity under an output constraint is, in most cases, equivalent to the problem of finding capacity under an (appropriately chosen) input constraint.

Example 1. The standard AWGN channel subject to an average input power constraint P (see e.g. [3, p. 239]) is well known to be $C(P) = \log_2(1 + P/\sigma_Z^2)$, where σ_Z^2 is the variance of the additive noise on the channel. For the same channel model, consider now the output (or *received*) power constraint

$$\frac{1}{n}\sum_{i=1}^{n} E\left[|Y[i]|^2\right] \quad \leq \quad Q + \sigma_Z^2. \tag{3}$$

It is easy to establish that $C(Q) = \log_2(1 + Q/\sigma_Z^2)$.

By contrast, we show in the sequel that transmit and receive power constraints lead to *significantly different* insights in certain network capacity problems, including multiple access and relay networks.

2 The Gaussian MIMO Channel

A. Received-Power Constraint

For the standard (linear) Gaussian vector (MIMO) channel with M transmit and N receive antennas and additive noise of variance σ_Z^2 (as defined in [4]), characterized by $Y = HX + Z$, where $X \in \mathbb{C}^M$ and $Y, Z \in \mathbb{C}^N$, where Z is a vector of independent and identically distributed (iid) circularly symmetric complex random variables, and $H \in \mathbb{C}^{N\times M}$ is a fixed matrix, suppose the transmit power constraint is replaced by the received-power constraint[1]

$$\frac{1}{n}\sum_{i=1}^{n} E[\|Y[i]\|^2] \leq Q + N\sigma_Z^2. \tag{4}$$

The capacity can be expressed as follows:

Theorem 1. *The capacity of the standard Gaussian MIMO channel with fixed transfer matrix H under the power constraint (4) is*

$$C \quad = \quad \mathrm{rank}(H)\log_2\left(1 + \frac{Q}{\mathrm{rank}(H)\sigma_Z^2}\right). \tag{5}$$

Note that this capacity formula depends only on the *rank* of the channel matrix, rather than on its singular values (by contrast to the formula in [4]).

[1] The RHS of (4) has been set to $Q + N\sigma_Z^2$ to allow for a slightly more compact result.

Proof sketch. Along the lines of [4], we equivalently determine the capacity of $\tilde{Y} = \Lambda\tilde{X} + \tilde{Z}$, where Λ is a diagonal matrix containing the $\min\{M,N\}$ (non-negative) singular values of H. Denote $k = \mathrm{rank}(H)$. Then, we equivalently determine the capacity of the channel $\tilde{Y}^k = \Lambda\tilde{X}^k + \tilde{Z}^k$. It can be shown that the power constraint on $\tilde{Y}^k$ is simply $Q + k\sigma_Z^2$. Hence,

$$\begin{aligned} I(X^M;Y^N) &= I(\tilde{X}^k;\tilde{Y}^k) = h(\tilde{Y}^k) - h(\tilde{Y}^k|\tilde{X}^k) = h(\tilde{Y}^k) - h(\tilde{Z}^k) \\ &= h(\tilde{Y}^k) - k\log_2(2\pi e\sigma_Z^2) \leq k\log_2\left(2\pi e\frac{Q + k\sigma_Z^2}{k}\right) - k\log_2(2\pi e\sigma_Z^2). \end{aligned}$$

It is easily verified that this is also achievable. $\square$

B. "Spectrum-Sharing": Third-Party Perspective

Suppose that (4) is replaced by the following set of constraints, for $k = 1, 2, \ldots, K$:

$$\frac{1}{n}\sum_{i=1}^{n} E\left[\left|\sum_{m=1}^{M} d_{m,k}^* X_m[i]\right|^2\right] \leq Q_{I,k}. \tag{6}$$

These constraints may model the received interfering power at K competing receivers: the "path loss" coefficient from transmitter m to the competing receiver k is given by $d_{m,k}$. Define the matrix $D \in \mathbb{C}^{M\times K}$ with entries $D_{m,k} = d_{m,k}$. Hence, the constraints can be written as $\{D^\dagger \Sigma_X D\}_{kk} \leq Q_{I,k}$, for $k = 1, \ldots, K$, where Σ_X denotes the covariance matrix of the vector X, and † the Hermitian transpose.

Proposition 2 (spectrum-sharing). *For the MIMO channel under the "spectrum-sharing" constraints characterized by the matrix D,*

(i) if there exists a vector $x = (x_1, \ldots, x_M)$ such that $D^\dagger x = 0$ but $Hx \neq 0$, then the capacity is unbounded.

(ii) if $K = M$ and $\mathrm{rank}(D) = M$, then the capacity $C \leq C_u$, where

$$C_u = \sum_{m=1}^{\mathrm{rank}(HD^{-\dagger})} (\log_2(\mu\lambda_m^{-1}))^+, \tag{7}$$

where μ is chosen such as to satisfy $\sum_{m=1}^{\mathrm{rank}(HD^{-\dagger})}(\mu - \lambda_m^{-1})^+ \leq \sum_{k=1}^{K} Q_{I,k}$, where λ_m, $m = 1, \ldots, \mathrm{rank}(HD^{-\dagger})$, denote the eigenvalues of $HD^{-\dagger}D^{-1}H^\dagger$. Moreover, this upper bound is achievable if the power constraints of Eqn. (6) are weakened to $\mathrm{trace}(D^\dagger\Sigma_X D) \leq \sum_{k=1}^{K} Q_{I,k}$.

This proposition illustrates how spectrum-sharing constraints incorporate both transmit and receive constraints, and hence, permit to interpolate between the two.

Proof. For part *(i)*, simply select the transmitted signal vector to be αx. Clearly, α can be made arbitrarily large without violating the power constraint since the LHS in (6) vanishes for $k = 1, \ldots, K$. Part *(ii)* follows by substituting $\tilde{X} = D^\dagger X$, leading to the new channel $Y = HD^{-\dagger}\tilde{X} + Z$, where the covariance matrix $\Sigma_{\tilde{X}}$ of $\tilde{X}$ must satisfy $\{\Sigma_{\tilde{X}}\}_{kk} \leq Q_{I,k}$, for $k = 1, \ldots, K$. Weakening this constraint by summing on both sides yields a trace constraint on $\Sigma_{\tilde{X}}$. For the latter problem, the solution is well known (see e.g. [4]). $\square$

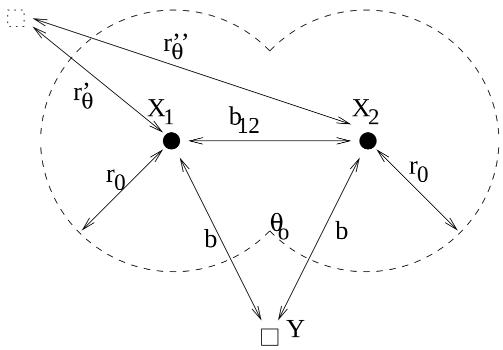

Figure 1: The geometry of the example in Section 2.C.

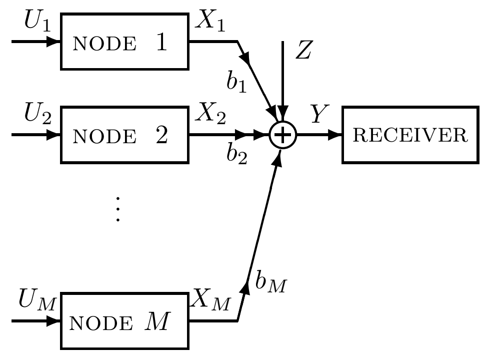

Figure 2: The multiple-access channel considered in Section 3.

C. ***Example:*** Geometric Spectrum-Sharing

While there does not seem to be a simple capacity formula for the general problem considered in Section 2.B, we now illustrate the basic idea via a specific geometric example where a simple solution can be given. Let $M = 2, N = 1$. Consider the two-dimensional geometry depicted in Figure 1. The signal component caused by the two transmitter antennae at any point θ in the plane is determined by

$$V_\theta[i] \quad = \quad f(r'_\theta)X_1[i] + f(r''_\theta)X_2[i], \tag{8}$$

where r'_θ and r''_θ denote the (Euclidean) distances between the point θ and transmitter antennae 1 and 2, respectively. The function $f(r)$ is assumed to be real-valued, non-negative and non-increasing for $r \geq 0$.

Consider the following power constraint: *Outside* of the dashed circles, the received ("interfering") power must not exceed Q_I. That is, we impose a continuum of constraints of the form (6).

Proposition 3. *The capacity of the Gaussian MIMO channel (with $M = 2$ and $N = 1$), under the spectrum-sharing constraint defined in this section, is given by*

$$C \quad = \quad \log\left(1 + \frac{f^2(b)Q_I}{f^2(r_0)\sigma_Z^2}\right). \tag{9}$$

Proof sketch. In this special case, the spectrum-sharing constraint at point θ_0 in Figure 1 can be rewritten as a power constraint on the received signal Y. Hence, Theorem 1 provides an upper bound. It can also be verified that this upper bound is achievable, essentially since no other point on the boundary of the dashed area will received more power than θ_0. $\square$

3 The Gaussian Multiple-Access Channel

A. Independent Messages

Consider the additive white Gaussian multiple-access channel (MAC) as defined in [3, Sec.14.1.2] and illustrated in Figure 2. The receiver observes $Y[i] = Z[i] + \sum_{m=1}^{M} b_m X_m[i]$, where $\{Z[i]\}_{i=1}^n$ is a sequence of iid circularly complex Gaussian random variables of mean zero and variance σ_Z^2. The transmitters have to select their codewords $\{X_m[i]\}_{i=1}^n$, for $m = 1, 2, \ldots, M$, in such a way as to satisfy

$$\frac{1}{n}\sum_{i=1}^{n} E\left[|Y[i]|^2\right] \;\leq\; Q + \sigma_Z^2. \tag{10}$$

In other words, the codewords must satisfy a constraint on the *induced power at the receiver*.[2] This is by contrast to the standard setting as considered in [3, Sec.14.1.2], where the powers *transmitted* by the nodes have to satisfy a set of constraints.

The capacity region (see e.g. [3, Eqns.(14.6)-(14.10)]) is replaced by the following simple characterization:

Theorem 4. *The capacity region of the additive white Gaussian MAC under a received-power constraint contains all rate vectors $(R_1, R_2, \ldots, R_M)$ that satisfy*

$$\sum_{m=1}^{M} R_m \;\leq\; \log\left(1 + \frac{Q}{\sigma_Z^2}\right). \tag{11}$$

In other words, the capacity region is a simplex.

For a proof, see the proof of Thm. 5 below.

Remark 1 (feedback). This result remains unchanged by (causal perfect) feedback from the receiver to each transmitter. In other words, feedback *does not enlarge* the capacity region of the additive white Gaussian MAC under a received-power constraint.[3]

Remark 2 (collaborative communication). The capacity region is not enlarged by arbitrarily collaborating transmitters, either, by contrast to, e.g., [6].

B. Dependent Message Streams, Received-Power Constraint

Reconsider Fig. 2, but suppose now that each node observes a message stream $\{U_m[i]\}_{i=1}^n$. Consider the following simple standard model of dependence: $\{(U_1[i], U_2[i], \ldots, U_M[i])\}_{i=1}^n$ is a sequence of independent and identically distributed (iid) random vectors, according to some fixed joint distribution $p(u_1, u_2, \ldots, u_M)$. The goal of the communication system is to provide estimates $\hat{U}_m[i]$ such that, for $m = 1, 2, \ldots, M$,

$$\lim_{n\to\infty} Prob\left\{\{\hat{U}_m[i]\}_{i=1}^n \neq \{U_m[i]\}_{i=1}^n\right\} \;\rightarrow\; 0. \tag{12}$$

The fundamental performance question is most naturally posed as one of *admissibility*, rather than capacity: Given a distribution $p(u_1, u_2, \ldots, u_M)$ and a Gaussian multi-access channel, is reliable communication possible? Under a transmit power constraint, this question cannot be answered in a conclusive manner [7]. However, under a received-power constraint, the solution turns out to be simple:

Theorem 5 (separation theorem). *Reliable communication is possible if and only if*

$$H\left(\{U_m\}_{m=1}^M\right) \;\leq\; \log\left(1 + \frac{Q}{\sigma_Z^2}\right). \tag{13}$$

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[2]Note that we set the RHS of (10) to $Q + \sigma_Z^2$ merely for notational convenience.

[3]This is by contrast to [5]: Under a transmit power constraint, feedback *does* enlarge the capacity region.

Remark 3 (feedback, collaboration). This result holds even if arbitrary (causal) feedback is available from the receiver to each transmitter, and if (arbitrary) collaboration between the transmitters is allowed.

Remark 4. This theorem says that an optimal strategy is to compress the sources, using Slepian-Wolf coding, and to transmit the compression indices across the multi-access channel using standard multi-access coding for independent sources. In other words, separating source from channel coding is an optimal strategy.

Remark 5. Equality may apply in Eqn. (13) in certain ("degenerate") cases.

Proof sketch. For the converse, note that by the data processing inequality, any code of length n must satisfy

$$\begin{aligned} nH(U_1, U_2, \ldots, U_M) &\leq I(X_1^n, X_2^n, \ldots, X_M^n; Y^n) \leq n(h(Y) - h(Y|X_1, \ldots, X_M)) \\ &\leq n \log\left(1 + \frac{Q}{\sigma_Z^2}\right). \end{aligned} \tag{14}$$

Equality is achievable (approachable) throughout as outlined in Remark 4. □

Unfortunately, this separation theorem does not extend to the fully general case involving distortion. This is discussed in more detail in [8].

C. ***Example:*** Geometric Spectrum-Sharing

Reconsider the example discussed in Section 2.C, and depicted in Figure 1. Now, the nodes X_1 and X_2 are two users in the MAC scenario. Suppose again that the constraint is that the received signal power at any point in the plane outside of the union of the dashed circles may not exceed Q_I.

Proposition 6. *The capacity region of the additive white Gaussian MAC (with $M = 2$), under the spectrum-sharing constraint defined in this section, is given by*

$$R_1 + R_2 \leq \log\left(1 + \frac{f^2(b)Q_I}{f^2(r_0)\sigma_Z^2}\right). \tag{15}$$

Moreover, causal feedback and collaboration cannot enlarge this capacity region, and for dependent sources, a separation theorem of the shape of Theorem 5 applies.

Proof. See Proposition 3. □

4 The General Discrete Memoryless MAC

Consider the general discrete memoryless M-user MAC, defined in straightforward extension of [2, p.271], but subject to the constraint given in Equation (1), with $N = 1$ and $K = 1$. From standard arguments, the capacity region can be described as follows:

Theorem 7. *For the discrete memoryless MAC, characterized by the conditional distribution $p(y|x_1, \ldots, x_M)$, the capacity region is the convex hull of the union over all $p(x_1, \ldots, x_M)$ satisfying $E[\rho(Y)] \leq \Gamma$ of the rate vectors $(R_1, \ldots, R_M)$ satisfying*

$$R_S \leq I(X_S; Y|X_{S^c}), \tag{16}$$

for all $S \subseteq \{1, \ldots, M\}$, where $R_S = \sum_{m \in S} R_m$, and $S^c = \{1, \ldots, M\} \backslash S$.

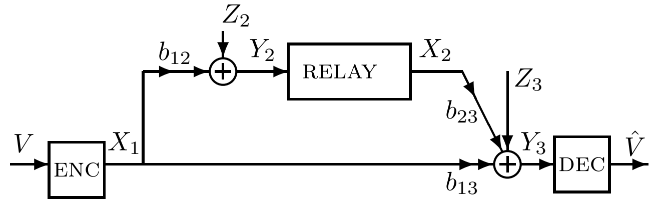

Figure 3: The considered Gaussian single-relay channel.

By contrast to the Gaussian example discussed in the preceding section, this region can generally be enlarged if feedback is available from the receiver to all the transmitters, and no separation theorem holds for the reliable transmission of potentially dependent sources in the general case. At the same time, the Gaussian example is not unique, as the following proposition establishes:

Proposition 8. *If the channel conditional distribution $p(y|x_1, \ldots, x_M)$ satisfies the following symmetry conditions:*

(i) $H(Y|x_1, \ldots, x_M) = H_0$, *for all* $(x_1, \ldots, x_M)$,

(ii) for each m, there exist $x_{m',0}$, for $m' \neq m$, and a distribution $p(x_m)$ such that

$$\sum_{x_m} p(y|x_{1,0}, \ldots, x_{m-1,0}, x_m, x_{m+1,0}, \ldots, x_{M,0})p(x_m) \quad \in \quad \arg \max_{p(y):E[\rho(Y)] \leq \Gamma} H(Y),$$

then the capacity region is $\sum_{m=1}^{M} R_m \leq C \stackrel{def}{=} \max_{p(y):E[\rho(Y)]} H(Y) - H_0$, *and cannot be enlarged by feedback. Moreover, reliable communication is feasible if and only if the joint entropy of the sources is smaller than C, by analogy to Thm. 5.*

5 The Gaussian Relay Network

A. The Considered System

A model for the relay channel was introduced and studied [9].Two fundamental coding strategies were developed in [10]. Due to space constraints, we refer to [11] for an extensive list of the recent literature on relay channels. Here, we adopt the (non-degraded) relay channel model used in [10], and illustrated in Figure 3. At the encoder, a message V is selected uniformly at random out of 2^{nR} possible messages, and a corresponding (complex-valued) codeword of length n, denoted by $\{X_1[i]\}_{i=1}^n$, is transmitted. The relay receives $\{Y_2[i]\}_{i=1}^n = \{b_{12}X_1[i] + Z_2[i]\}_{i=1}^n$, where $\{Z_2[i]\}_{i=1}^n$ is a sequence of iid circularly complex Gaussian random variables of mean zero and variance σ_Z^2. The relay then transmits (complex-valued) symbols $X_2[i] = f_i(Y_2[i-1], \ldots, Y_2[1])$, where $f_i(\cdot)$ is an arbitrary function. Below, we will consider specific constraints on the coding functions at the source and at the relay node.

Based on the received signal $\{Y_3[i]\}_{i=1}^n = \{b_{13}X_1[i] + b_{23}X_2[i] + Z_3[i]\}_{i=1}^n$, where $\{Z_3[i]\}_{i=1}^n$ is a sequence of iid circularly complex Gaussian random variables of mean zero and variance σ_Z^2, the destination determines an estimate $\hat{V}$, and the capacity problem is that of determining the largest rate R for which a sequence of coding schemes exists that satisfies $\lim_{n\to\infty} Prob\{V \neq \hat{V}\} = 0$.

B. Received-Power Constraint At The Destination

Suppose that the codewords $\{X_1[i]\}_{i=1}^n$ and $\{X_2[i]\}_{i=1}^n$, transmitted by the source and the relay, respectively, must be designed such as to satisfy

$$\frac{1}{n}\sum_{i=1}^{n} E\left[|Y_3[i]|^2\right] \quad \leq \quad Q + \sigma_Z^2. \tag{17}$$

Theorem 9. *The capacity of the Gaussian single-relay channel under the received-power constraint (17) is given by*

$$C \quad = \quad \log_2\left(1 + \frac{Q}{\sigma_Z^2}\right). \tag{18}$$

Remark 6 (single-mode relays). This capacity result also applies to the single-mode relay model, where the relay can only either transmit or receive, but not both at the same time.

Proof. Capacity is bounded above by

$$C \quad \leq \quad \max_{p(x_1,x_2)} I(X_1, X_2; Y_3) \leq \max_{p(x_1,x_2)} \left(h(Y_3) - h(Y_3|X_1, X_2)\right) \leq \log_2\left(1 + \frac{Q}{\sigma_Z^2}\right). \tag{19}$$

It is clear that this can be achieved simply by switching off the relay and letting the source node transmit at power $P_1 = Q/|b_{13}|^2$. □

In other words, under this received-power constraint, the relay does not permit to increase capacity. More interestingly, note that the capacity-achieving input distribution (and hence, the coding scheme) is not unique in this case. Among the capacity-achieving schemes, we can now search for the one whose *transmit* power is smallest. This question does not seem to have a simple answer, but the results of [10] lead to bounds:

Proposition 10 (minimum transmit power capacity-achieving scheme). *Among the coding schemes that achieve capacity in Thm. 9, the one that minimizes the total transmit power satisfies*

$$\min_{\tilde{P}_1,\tilde{P}_2:|\rho|\leq 1,(b_{12}^2+b_{13}^2)\tilde{P}_1(1-|\rho|^2)=Q} (\tilde{P}_1 + \tilde{P}_2) \leq \quad P_1 + P_2 \quad \leq \min_{\tilde{P}_1,\tilde{P}_2:|\rho|\leq 1,b_{12}^2\tilde{P}_1(1-|\rho|^2)=Q} (\tilde{P}_1 + \tilde{P}_2),$$

where $|\rho| = (Q - |b_{13}|^2\tilde{P}_1 - |b_{23}|^2\tilde{P}_2)/(2\mathcal{R}(b_{13}b_{23}^*)(\tilde{P}_1\tilde{P}_2)^{1/2})$.

Proof. The proposition follows directly from [11, Sec. V.A]. □

C. *Example:* Geometric Spectrum-Sharing

Reconsider the example discussed in Section 2.C, and depicted in Figure 1. The node X_1 is the source node, and X_2 the relay. Again, suppose that the received interfering power at any point outside of the union of the dashed circles cannot be larger than Q_I.

Proposition 11. *The capacity of the Gaussian single-relay channel, under the spectrum-sharing constraint defined in this section, is given by*

$$C \quad = \quad \log\left(1 + \frac{f^2(b)Q_I}{f^2(r_0)\sigma_Z^2}\right). \tag{20}$$

Remark 7 (relays to reduce interference). The spectrum-sharing perspective gives a novel meaning to relays: Rather than boosting the source node's transmission towards the destination, the relays may be more helpful in *reducing* the signal strength in a required direction.

D. *Example:* Spectrum-Sharing with Transmit Power Constraints

A further variation on the investigations of this paper, and a logical extension of the material in this section, is to simultaneously consider transmit and receive power constraints. A simple example can be characterized as follows:

Proposition 12. *Consider the Gaussian single-relay channel with b_{13}, b_{23} and d_2 real-valued and non-negative, subject to the power constraint $E[|X_1|^2] \leq P$ and the spectrum-sharing constraint $E[|X_1 + d_2X_2|^2] \leq P$. If $b_{23}/b_{13} \geq d_2$, then*

$$C \;=\; \log\left(1 + \frac{b_{13}^2 P}{\sigma_Z^2}\right). \tag{21}$$

Proof sketch. Consider $E[|b_{13}X_1 + b_{23}X_2|^2] = b_{13}b_{23}/d_2 E[|X_1 + d_2X_2|^2] + (b_{13}^2 - b_{13}b_{23}/d_2)$ $E[|X_1|^2] - (b_{13}b_{23}d_2 - b_{23}^2)E[|X_2|^2] \leq (b_{13}b_{23}/d_2 + b_{13}^2 - b_{13}b_{23}/d_2)E[|X_1|^2]$, since $b_{13}b_{23}d_2 - b_{23}^2 \geq 0$ by assumption. This is also achievable, e.g. by turning off the relay. □

E. Gaussian Relay Networks

For Gaussian relay networks, similar capacity results can be obtained. Extend the model of Section 5.A by introducing more relay nodes: Relay node m, $m = 2, \ldots, M-1$, receives $Y_m = Z_m + \sum_{m'=1, m' \neq m}^{M-1} b_{mm'}X_{m'}$. The destination node (node M) receives $Y_M = Z_M + \sum_{m=1}^{M-1} b_{mM}X_m$. Detailed definitions are given in [11].

Suppose that the encoding process must be designed such as to satisfy a received-power constraint

$$\frac{1}{n}\sum_{i=1}^{n} E\left[|Y_M[i]|^2\right] \;\leq\; Q + \sigma_Z^2. \tag{22}$$

Theorem 13. *The capacity of the Gaussian relay network under the received-power constraint (22) is given by*

$$C \;=\; \log\left(1 + \frac{Q}{\sigma_Z^2}\right). \tag{23}$$

Proof. See the proof of Theorem 9. □

6 The General Discrete Memoryless Relay Network

The capacity results obtained in the previous section can be extended beyond the Gaussian case. Upper and lower bounds to capacity can be given in straightforward extension of [10], and the interesting question again becomes: when can a capacity theorem be established? A partial answer can be phrased as follows:

Proposition 14. *For the general relay channel as defined in [10], subject to a constraint on the received signal $E[\rho(Y_M)] \leq \Gamma$, suppose that $p(y_M|x_1, \ldots, x_{M-1})$ is symmetric in the sense that $H(Y_M|x_1, \ldots, x_{M-1}) = H_0$ for all $(x_1, \ldots, x_M)$. If there exist $x_{2,0}, x_{3,0}, \ldots, x_{M-1,0}$ such that*

$$\sum_{x_1} p(y_M|x_1, x_{2,0}, x_{3,0}, \ldots, x_{M-1,0})p(x_1) \;\in\; \arg \max_{p(y_M): E[\rho(Y_M)] \leq \Gamma} H(Y_M), \tag{24}$$

then, the capacity is $C = \max_{p(y_M): E[\rho(Y_M)] \leq \Gamma} H(Y_M) - H_0$.

Note that this is a special class of relay networks for which the relays, in fact, do not permit to increase capacity, extending [10, p.572,case 2)].

7 Conclusions and Extensions

This paper derives capacity results and separation theorems for communication systems where the *received power*, either at the intended receiver, or at some third-party receiver, is *constrained*. This is motivated, in part, by recent *spectrum-sharing* ideas: When multiple systems need to share the same part of the spectrum, it may become meaningful to define a maximum *interference power* that one system may inflict on its neighbors.

It is argued that while this change of perspective does not involve any conceptual differences in the scalar point-to-point case, it provides a new set of conclusive performance results in network cases, some of which are discussed or outlined in this paper. Detailed proofs, along with extensions to other scenarios, will be presented in an upcoming journal version.

It is also found that in certain network problems with received-power constraints, the capacity-achieving distribution is non-unique. For these situations, an interesting follow-up question can be asked by comparing these capacity-achieving distributions and finding, e.g., the one that minimizes transmit power.

Another interesting question concerns networks that are subject to *both* a transmit *and* and received-power constraint. A simple result concerning a relay situation is given.

Extensions may concern more general spectrum-sharing geometries, and an analysis of the capacity of wireless networks under received-power constraints, in the spirit of [12].

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