Default, liquidity and crises: an econometric framework*

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Preliminary version

Abstract: In this paper, extending the work by Gourieroux, Monfort and Polimenis (2006) [78], we present a general discrete-time affine framework aimed at jointly modeling yield curves that are associated with different issuers. The underlying fixed-income securities may differ in terms of credit quality and/or in terms of liquidity. The risk factors follow discrete-time Gaussian processes, with drifts and variance-covariance matrices that are subject to regime shifts described by a Markov chain with (historical) non-homogenous transition probabilities. While flexible, the model remains tractable and amenable to empirical estimation. This is illustrated by an application on euro-area government yields. Specifically, the common dynamics of ten euro-area government yield curves are estimated using weekly data spanning the period from 1999 to early 2010. The dynamics of the yield curves are satisfyingly explained by both observable factors and unobservable ones. Among the latter, we exhibit a euro-area-wide liquidity-related latent factor. The estimation results suggest that an important share of the changes in euro-area yield differentials is liquidity-driven.

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1. Introduction

Though already strong before the recent financial crisis, the case for including regime shifts within term-structure models for defaultable bonds is obviously stronger now (see, amongst many others, Christensen, Lopez and Rudebusch, 2009 [34]). This paper proposes a general affine term-structure framework aimed at jointly modeling several yield curves associated with different obligors or different types of securities, in the presence of regime switching.

In this reduced-form framework, the default probabilities are modeled directly instead of defining a stochastic process for the obligor's asset value that triggers default when the process reaches some threshold (as in Merton, 1974 [111]). The focus is on default modeling, but the specifications can also account for the pricing of some liquidity premia using the same machinery.

Our framework provides us with a great flexibility in how we specify the behaviour of the state variables which simultaneously determines the risk-free term structure and the default intensities –or hazard rates. The state variables, also termed with "risk factors", follow discrete-time Gaussian processes.[1] Extending the work of Gourieroux, Monfort and Polimenis (2006) [78], the Gaussian processes present drifts and variance-covariance matrices that are subject to regime shifts. The latter are described by a Markov chain with (historical) non-homogenous transition probabilities.

The modeling of defaults is based on the so-called "doubly-stochastic" assumption: correlations between default events arise solely through dependence on some common underlying stochastic factors which influence the default probabilities of every single loans.[2] Some of the factors may be unobserved. In this sense, our model accomodates frailty. This feature is advocated by recent papers suggesting that including only observable covariates in default-intensity specifications results in poorly-estimated conditional probabilities of default (see e.g. Lando and Nielsen, 2008 [99] or Duffie et al., 2009 [58]). In our framework, frailty may stem from two types of shocks: some are Gaussian and others correspond to regime shifts. Since hazard rates can be affected by the latter, our model is appropriate to capture default clustering: indeed, if one regime implies very high default intensities for a large number of obligors, then clusters of defaults will be observed in this regime.[3]

Particular attention is paid to the tractability of the model and its estimation. Tractability is notably obtained through an extensive use of Car's –Compound autoregressive processes– properties (see, e.g. Darolles, Gourieroux and Jasiak, 2006 [45]), which leads to quasi-explicit fomulas for bond prices. Both historical and risk-neutral dynamics are explicitely modeled, which is helpful for chosing appropriate specifications under the historical measure, for dealing simultaneously with pricing and forecasting or also for Value-at-Risk calculations. We propose an estimation strategy based on several steps. This procedure is intended to facilitate the estimation of unobservable factors (including latent risk factors and regimes).

The framework is exploited to investigate the common dynamics of euro-area government yield curves. We consider the yield curves of ten euro-area countries: Austria, Belgium,

[1] While most of the earliest affine defaultable-bond term-struture models are in continuous-time form (see e.g. Duffie and Singleton (1999) [60]), Gourieroux, Monfort and Polimenis (2006) [78] have shown that discrete-time affine models are well-suited to credit-risk modeling and that they present higher flexibility than their continuous-time counterparts. In particular, the discrete-time framework makes it easier to properly specifiy the dynamics of the observable risk factors under the historical probability measure.

[2] These shocks include both Gaussian shocks and regime-shift shocks.

[3] Beyond the modeling of correlated defaults, including unobserved variables in term-structure models of defaultable-bond yields is also important to satisfyingly capture the credit-spread dynamics (see Collin-Dufresne, Goldstein and Martin, 2001 [39]).

Finland, France, Germany, Greece, Italy, the Netherlands, Portugal and Spain. The model includes three observed macroeconomic variables (a European market volatility index, a business-cycle indicator and the short-term risk-free rate) and three latent factors. Three Markovian regimes are considered in the model, two of them corresponding to market-stress periods. The hazard rate of each country depends on the factors as well as on the regimes. Only one source of country specificity is taken into account (through business-cycle indicators). Therefore, the prices of bonds issued by the different countries are mostly exposed to the same risk factors. However, substantial differences among yields arise because of different exposures of the countries' debts to the risk factors. Among the three latent factor, one is identified as a liquidity-related factor. To that end, we use information that is incorporated in yield differentials between the bonds issued by the Federal Republic of Germany (the *Bunds*) and those issued by KfW (Kreditanstalt für Wiederaufbau), a German agency. Indeed, to the extent that the latter are guaranteed by the German government, the KfW-*Bund* spread should be mainly affected by liquidity pricing. Over the last decade, most of the euro-area yield differentials are captured by the model. In addition, the results suggest that important shares of yield differentials are liquidity driven.

The current section is followed by three brief reviews of the literature that is related to our framework. They respectively deal with (1) the decomposition of the spreads into default and liquidity components using affine-term structure models, (2) the introduction of regime shifts in the dynamics of the term-structure of interest rates and (3) credit-migration modeling. The remainder of the paper is organized as follows. Sections 2 and 3 respectively present the historical and risk-neutral dynamics of the variables. Section 4 gives the bond-pricing formulas. Section 5 deals with internal-consistency restrictions that arise when asset prices are included amongst the risk factors. In Section 6, we propose an estimation strategy. Section 7 shows how the model accomodates the pricing of liquidity. Section 8 investigates possible extensions of the framework: Subsection 8.1 deals with multi-lag dynamics of the risk factors; Subsection 8.2 deals with the specific case where one of the Markov chains coincides with the default state of a given entity and Subsection 8.3 shows how to introduce rating-migration modeling in the framework. Finally, Section 9 presents an application to the modeling of euro-area yield differentials.

1.1. Decomposing spreads in affine term-structure framework

Motivated by derivative-pricing or credit-risk-management objectives, a large strand of the recent literature related to fixed-income securities has focused on the joint modeling of several yield curves. In this context, Jarrow, Lando, Turnbull (1997) [90], Lando (1998) [98] or Duffie and Singleton (1999) [60] have highlighted the potential of affine term-structure frameworks to model jointly yield curves associated with various obligors subject to default risk. Their intensity-based –or reduced-form– approaches used to model defaults differ from the more structural approaches originating in Black and Scholes (1973) [22] and Merton (1974) [111].[4] As shown by Duffie and Singleton (1999) [60], in an intensity-based framework, the modeling of defaultable claims is based on the standard affine term-structure machinery readily available for default risk modeling and estimation. Since then, numerous further developments have illustrated the flexibility and tractability of affine-term structure

[4] In the latter, the default of a firm is modeled in terms of the relationsip between its assets and liabilities. The asset value process is modeled as a geometric Brownian motion and default occurs when the asset value at maturity is lower than the liabilities. Important industry models like KMV's Portfolio Manager or the JP Morgan's CreditMetrics model are based on this approach (see Crouhy, Glai and Mark, 2000 [42]for a comparative analysis of industry credit-risk models). Cathcart and El-Jahel, 2006 [30]) have shown that the two approaches (reduced-form and structural) are somewhat reconcilable.

models to jointly model different yield curves (see e.g. Duffee, 1999 [57], Collin-Dufresne and Solnik, 2001 [40], Dai and Singleton, 2003 [43], Collin-Dufresne, Goldstein and Hugonnier, 2004 [38] and Gourieroux, Monfort and Polimenis, 2006 [78]).

In recent studies, some authors rely on the affine-term structure framework to model yield curves associated not only with different obligors but also with different fixed-income instruments (e.g. bonds, repos, swaps). Further, the authors exploit this modeling to breakdown credit spreads or swap spreads into different components. Specifically, Liu, Longstaff and Mandell (2006) [101] use a five-factor affine framework to jointly model Treasury, repo and swap term structures. One of their factors is related to the pricing of the Treasury-securities liquidity and another factor reflects default risk.[5] Feldhütter and Lando (2009) [69] develop a six-factor model for Treasury bonds, corporate bonds and swap rates that makes it possible to decompose swap spreads into three components: a convenience yield from holding Treasuries, a credit-element associated with the underlying LIBOR rate, and a factor specific to the swap market. They find that the convenience yield is by far the largest component of spreads. Longstaff, Mithal and Neis (2005) [103] use information in credit default swaps –in addition to bond prices– to obtain measures of the nondefault components in corporate spreads. They find that the nondefault component is time-varying and strongly related to measures of bond-specific illiquidity as well as to macroeconomic measures of bond-market liquidity.

The approaches implemented in the previous papers consist in estimating the default and liquidity risk factors in a first step and to find relationships between these estimates and observable proxies for liquidity or default measures or determinants in a second step. Alternatively, one could directly include observable liquidity-related variables among the risk factors.[6]

1.2. Yield-curve dynamics and regime switching

1.2.1. Regime shifts in default-free yield-curve dynamics

Strong evidence points to the existence of regime switching in the dynamics of the term structure of interest rates. Thus, Hamilton (1988) [83] finds that changes in the Federal reserve operating procedures leads to regime-switching in the dynamics of the term structure of interest rates. In addition to such a shift, Cai (1994) [28] finds that the 1974 oil shock resulted in a regime shift in the asymptotic volatility of the three-month Treasury bill. Gray (1996) [79] shows that the assumption of a single regime is a source of misspecification in models of the short rate. Adding term spread in their estimation, Ang and Bekaert (2002) [5] identify regimes that are closely linked to business cycles, suggesting that large periodic

[5] As noted by Feldhütter and Lando (2009) [69], the identification of the liquidity and credit risk factors in Liu et al. relies critically on the use of the 3-month general-collateral repo rate (GC repo) as a short-term risk-free rate and of the 3-month LIBOR as a credit-risky rate. Liu et al. define the liquity factor as the spread between the 3-month GC repo and the 3-month Treasury-bill yield (and is therefore observable). In each yield, their liquidity component is the share of the yield that is explained by this factor.

[6] Including observable factors in affine-term structure models was pionnered by Ang and Piazzesi (2003) [9] (see also Jardet, Monfort and Pegoraro, 2009 [89]). Such an approach was implemented to investigate credit-spread dynamics by Amato and Luisi (2006) [4] and Mueller (2009) [115]. Amato and Luisi (2006) estimate a six-factor term-structure model of US Treasury yields and spreads on BBB and B-rated corporate bonds. Three out of their six factors are observable factors: indicators of real activity, inflation and financial conditions. They show in particular that macro factors are largely responsible for variation in the prices of systematic risk. Mueller (2009) estimates a five-factor model on US data, two factors are observable: GDP growth and inflation. He finds that the macro factors contribute to the predictive power of credit spreads.

shifts in interest rates across distinct regimes present a systematic risk to investors (see also Wu and Zeng, 2005 [131] or Bansal and Zhou, 2002 [12]). The same authors (2002) [6] show that regime switching is efficient in capturing nonlinear dynamics exhibited by Aït-Sahalia (1996) [2]. Christiansen (2004) [36] estimates a two-state markov-switching model for the short-rate and the slope of the yield curve: his estimated regimes turn out to depict low and high variances regimes for short-rate changes. The economy appears to have been in the high-variance state during unusual economic periods such as oil or stock-market crises, or more generally during the official recession periods. Monfort and Pegoraro (2007) [114] show that the introduction of regime switching in term-structure models leads to term-structure models that are well-specified under the historical probability and that are able to explain the expectation-hypothesis puzzle (why the long and short term interest rate differential does not predict the future interest rate changes), over short and long horizons. Following Veronesi and Yared (1999) [128] and Evans (2003) [67], Ang Bekaert and Wei (2008) [8] develop term structure models with regime shifts to investigate the joint dynamics of real and nominal yields. They identify inflation and real factor sources behind regime shifts and analyze how they contribute to nominal interest-rate variations. Dai, Singleton and Yang (2007) [44] develop a model with regime-shift risks that are priced by investors. Allowing for state-dependent transition probabilities, their model makes it possible to conveniently capture asymmetry in the cyclical behavior of interest rates.

1.2.2. Regime shifts in spreads' dynamics

While the previous subsection puts forward the importance of modeling regime switching in yield-curve models, a few has been done to integrate such a feature in term-structure models of defaultable bonds. However, empirical studies point to the existence of different regimes in the default risk valuation. Davies (2004 [49] and 2008 [50]) uses Markov-Switching Vector Auto-Regression (MS-VAR) estimation techniques and finds that credit spreads exhibit distinct high- and low-volatility regimes. Alexander and Kaeck (2008) [3] detect a pronounced regime-specific behaviour of Credit default swap (CDS) spreads. Hackbarth, Miao and Morellec (2006) [82] provide a theoretical model to explain the dependence of credit spread on business-cycle regimes. In the same vein, Bhamra, Kuehn and Strebulaev (2007) [19], Chen (2008) [31] and David (2008) [48] also adopt a Merton structural model including regime switching to assess the influence of different states of the economic cycles on the credit-risk premia. Without deriving a complete model of the credit-spread term structure, Maalaoui, Dionne and François (2009) [108] estimate Markov-switching specifications to investigate the links between credit spreads and its determinants. Their results suggest that the failure of single-regime models to find significant links between potential determinants (see e.g. Collin-Dufresne, Goldstein and Martin, 2001 [39]) may stem from the fact that these determinants have opposite average effects in the two regimes they identify.

The recent financial crisis has eventually highlighted the need for taking into account crisis regimes in yield-curve models. For instance, without using a model based on hidden Markov chains, Christensen, Lopez and Rudebusch (2009) [34] provide evidence of a shift in their six-factor affine term-structure model in 2007.[7] The next paragraph deals more specifically with the potential role of regime-switching features for modeling yield-curve dynamics during crises.

[7] Christensen, Lopez and Rudebusch (2009) [34] model weekly U.S. Treasury yields, financial corporate bond yields, and term interbank rates. Their estimation period starts in 1995 and is based on weekly data.

1.2.3. The potential of regime switching to capture systemic risk, or contagion effects

Including regime shifts in a discrete-time term-structure model may affect pricing through several channels: (i) regimes affect the historical and risk-neutral dynamics of the risk factors, (ii) regimes appear in the stochastic discount factor (s.d.f.) –which implies that regime-transition risk is priced– and (iii) regimes appear in the default-intensity functions. In the following, we connect these characteristics with the literature that jointly addresses credit-risk models and crisis.[8] This literature focuses on systemic risk and contagion effects. Systemic risk differs from systematic risk in terms of the severity and frequency of the associated shocks. More precisely, systematic shocks are frequent and not extreme while systemic shocks are infrequent and extreme (see e.g. Das and Uppal, 2004 [47] or Baur and Schulze, 2009 [15]).[9] In a model accomodating regime shifts, it is natural to associate systematic and systemic risk with the Gaussian shocks and the regime shifts, respectively. Obviously, distinguishing between the two kinds of risks may not be a trivial task. In particular, difficulties arise from the fact that systematic shocks can turn into systemic ones. For instance, in some contexts –notably when the level of uncertainty is high–, temporary systematic shocks can lead to defaults and generate significant negative aftershocks, including liquidity spirals.[10] To the extent that we allow the probability of switching to a crisis regime to be influenced by some systematic risk factors, such sequences could be captured in a framework like ours.

The contagion literature focuses on the interdependencies between the defaults of different debtors, which is sometimes referred to as *counterparty risk*. In the so-called contagion models, if one of the debtor defaults, it affects the hazard rates of the other debtors (their default intensity jumps upwards). Contagion effects, whose consequences are cascades of subsequent spread changes, is explained by the existence of close ties between firms.[11] Jarrow and Yu (2001) [92] develop a *primary-secondary* approach: in case a primary entity defaults, the spreads of other debtors jump upwards; meanwhile, default of secondary firms do not have any impact on other debtors in the portfolio. In the *infectious-default* model developed by Davies and Lo (2001) [51], the default of a debtor triggers a regime shift: in the high-risk regime, the default intensitites of all debtors are increased.[12] Given that our baseline model relies on the doubly-stochastic or conditional-dependence assumption –which states that, conditional to the underlying factors and regimes, the default events of the firms in a portfolio are independent– it does not capture such contagion effects. Nevertheless, as developed in Subsection 8.2, our framework can still accomodate the specific contagion case where one entity (or, for the sake of tractability, only a small number of them) affects

[8] Note that regime-switching features are not useful solely to deal with crisis modeling: it is also required to model regimes that correspond to the position within the business cycle (see Subsection 1.2.1). Bangia et al. (2002) [11] illustrate the importance of distinguishing between expansion and contraction phases for the assessment of loss distribution of credit portfolios.

[9] For de Bandt and Hartmann (2000) [52], a systemic event is an event where the release of bad news about a financial institution, or even its failure, or the crash of a financial market leads in a sequential fashion to considerable adverse effects on one or several other financial institutions or markets, e.g. their failure or crash. They further define the systemic risk as the risk of experiencing systemic events. For an introduction to the contagion literature, see e.g. Lütkebohmert (2009) [107].

[10] See Brunnermeier and Pedersen, 2009 [26] for a structural analysis of this and (e.g.) Hesse and Gonzalo-Hermosillo, 2009 [87] for empirical evidence.

[11] These ties may be of legal (e.g. parent-subsidiary), financial (e.g. trade credit), or business nature (e.g. buyer-supplier). Through these channels, economic distress of one firm can have an immediate adverse effect on the financial health of that firm's business partners (Giesecke, 2004 [75], Egloff, Leippold and Vanini, 2005 [61]).

[12] Other contagion mechanisms based on the same kinds of approaches are proposed by Frey and Backhaus (2003) [73] or Yu (2007) [134].

the default probability of the others: it suffices to make one of the regimes corresponds to the default state of this entity.

Das et al. (2007) [46] test whether default events can reasonably be modeled as dependent solely on exogenous observable factors.[13] As Duffie et al. (2009) [58] and Giesecke and Kim (2010) [76], they find that doubly-stochastic settings perform badly if no latent covariates –also called frailty components– enter the intensity specifications. Duffie et al. (2009) further argue that including frailty covariates in the hazard-rate specifications is necessary to accomodate default clustering.[14][15] Koopman, Lucas and Schwaab (2009) [97] show that modeling frailty contributes to obtain a proper modeling of default rates during crisis. Consistentlty with these findings, our framework accomodates latent regimes and/or factors (cf. Section 6). In particular, the fact that some regimes could correspond to simultaneous and dramatic increases in the default probabilities of all or part of the debtors (cf. point (iii) above) implies that our framework is appropriate to generate default clustering.

1.3. Credit-migration modeling

In Subsection 8.3, we show how our framework can be adapted to accomodate credit-rating migration. Our baseline model considers only one credit event: the default of the debtor. However, credit events include more generally the changes in credit ratings like these attributed by agencies like Moody's, Standard & Poor's or Fitch. There are several reasons why it may be desirable to model not only default events but also rating transitions (see Cantor, 2004 [29] or Gagliardini and Gourieroux, 2001 [74]).[16] First, because of their importance in terms of risk management, modeling credit migration is key for practitioners.[17] Second, such models are obviously required to price credit-event options. Third, when complete default historical data sets are not available (or do not go back far in time), exploiting credit-migration matrices may allow to extrapolate long-term default predictions from short-term credit risk dynamics. Similarly, to the extent that rating classes are seen as approximately homogenous, having a rating-based term structure model at one's disposal makes it quick to get a rough estimate of the fair value of a bond (given the rating of the issuer).

In their seminal study of credit spread, Jarrow, Lando and Turnbull (1997) [90] model rating transitions as a time-homogenous Markov chain. That is, in their model, whether a firm's rating will change in the next period depends on its current rating only and the

[13] Nevertheless, using a different specification of the default intensity, Lando and Nielsen (2008) [99] cannot reject the assumption of conditional independence for default histories recorder by Moody's between 1982 and 2006. Lando and Nielsen conclude that the test proposed by Das et al. (2007) is mainly a misspecification test.

[14] Frailty models come from the biostatistics literature. In these models, the intensity of a point process is proportional to an unobservable variable, the frailty parameter. For a survey of frailty models, see Hougaard (2000) [88].

[15] Collin-Dufresne, Goldstein and Helwege (2008) [37] and Jorion and Zhang (2007) [93] also find that default events are associated with significant increases in the credit spreads of other firms, consistent with default clustering in excess of that suggested by the standard doubly stochastic models. Azizpour and Giesecke (2008) [10] find that contagion effects represent a significant additional source of default clustering (over and beyond the effect due to firms' exposure to observable and frailty risk factors).

[16] Several of the main credit models currently being used in the industry, such as J.P. Morgan's CreditMetrics (1997) [94], draw on the credit-migration approach. For presentation, comparison and evaluation of these models, see e.g. Crouhy, Glai and Mark (2000) [42], Gordy (2000) [77] or Lopez and Saidenberg (2000) [105].

[17] For instance, the VaR or capital adequacy numbers may be based on a portfolio rating's distribution (see Saienberg and Schuermann (2003) [122]). In addition, some portfolio managers are constrained by limits based on the ratings of the bond they held.

probability of changing from one rating to the other remains the same over time. In addition, in their setting, the market risk and the credit risk are assumed to be independent. Different studies suggest however that –per-period– transition probabilities are time-varying (see e.g. Lucas and Lonski, 1992 [106], Belkin, Suchower and Wagner, 1998 [17], Farnsworth and Li, 2007 [68] or Feng, Gourieroux and Jasiak, 2008 [70]). In addition to time-variability, Nickell, Perraudin and Varotto 2000 [119] show that conditioning a transition matrix on the industry (to which the company belongs) is desirable.

Lando (1998) [98] extends the framework developed by Jarrow, Lando and Turnbull (1997) [90] by allowing for dependence between the market risk and the credit risk,[18] and by making the rating-transition probabilities depend on the state variables.[19] As in Lando (1998), we look for specific forms of the rating-transition matrix that lead to quasi-explicit bond-pricing formulas.

2. Information and historical dynamics

2.1. Information

The new information of the investors at date t is $w_t' = (z_t', y_t', x_t', d_t')$ where z_t is a regime variable that can take a finite number J of values, y_t is a multivariate macroeconomic factor, $x_t' = (x_{1,t}', \dots, x_{N,t}')$ is a set of specific multivariate factors $x_{n,t}$ associated with debtor n, and $d_t' = (d_{1,t}, \dots, d_{N,t})$ is a set of binary variables indicating the default ($d_{n,t} = 1$) or the non-default ($d_{n,t} = 0$) state of entity n. The whole information set of the investors at date t is $\underline{w}_t' = (w_1', \dots, w_t')$. At this stage, we do not make any assumption about the observability of these variables by the econometrician (this is done below in Section 6). As outlined at the beginning of Subsection 1.2.3, these regimes influence bond pricing through different channels (they can appear in the stochastic discount factor, in the default-intensity functions and in the dynamics of the risk factors y_t and $x_{n,t}$'s). In the baseline framework, the regimes are viewed as transitory: none of these regimes is absorbing (this restriction is relaxed in a specific case presented in Subsection 8.2).

2.2. Historical dynamics

The regime variable z_t is valued in $\{e_1, \dots, e_J\}$, the set of column vectors of the identity matrix I_J. The conditional distribution of z_t given $\underline{w}_{t-1}$ is characterized by the probabilities:

$$p\left(z_t \mid \underline{w}_{t-1}\right) = \pi\left(z_t \mid z_{t-1}, y_{t-1}\right). \tag{1}$$

The probability $\pi(e_j \mid e_i, y_{t-1})$ that z_t shifts from regime i to regime j between period $t-1$ and t, conditional on y_{t-1}, is also denoted by $\pi_{ij,t-1}$. These specifications allow for state-dependent transition probabilities, as in Gray (1996) [79], Ang and Bekaert (2002) [6] or Dai, Singleton and Yang (2007) [44].

The conditional distribution of y_t given z_t and $\underline{w}_{t-1}$ is Gaussian and given by:

$$y_t = \mu\left(z_t, z_{t-1}\right) + \Phi y_{t-1} + \Omega\left(z_t, z_{t-1}\right)\varepsilon_t \tag{2}$$

where the ε_t are independently and identically $N(0, I)$ distributed. Specifications (1) and (2) imply that, in the universe (z_t, y_t), z_t Granger-causes y_t, y_t causes z_t and there is

[18] Amongst the earliest studies suggesting that such a feature is required, see Longstaff and Schwartz (1995) [104] or Duffee (1998) [56].

[19] Other examples of term-structure models allowing for time-varying transition probabilities include Bielecki and Rutkowski (2000) [20] and Wei (2003) [130].

instantaneous causality between z_t and y_t. Moreover, in the universe $w_t = (z_t, y_t, x_t, d_t)$, (x_t, d_t) does not cause (z_t, y_t). As noted by Ang, Bekaert and Wei (2008) [7], instantaneous causality between z_t and y_t implies that the variances of the factors y_t, conditional on $\underline{w}_{t-1}$, embed a jump term reflecting the difference in drifts μ accross regimes. Such a feature, that allows for conditional heteroskedasticity, is absent from the Dai, Singleton and Yang (2007) [44] setting. However, it should be noted that our framework nests the case where there is no instantaneous causality between z_t and y_t in the historical dynamics.[20] Contrary to Bansal and Zhou (2002) [12], matrix Φ is not regime-dependent: this is for the sake of tractability when it comes to bond pricing.[21]

The $x_{n,t}$'s, $n = 1, \dots, N$ are assumed to be independent conditionally to $(z_t, y_t, \underline{w}_{t-1})$. The conditional distribution of $x_{n,t}$ is Gaussian and defined by:

$$x_{n,t} = q_{1n}\left(z_t, z_{t-1}\right) + Q_{2n} y_t + Q_{3n} y_{t-1} + Q_{4n} x_{n,t-1} + Q_{5n}\left(z_t, z_{t-1}\right) \eta_{n,t} \tag{3}$$

where the shocks $\eta_{n,t}$ are $IIN(0, I)$. Specifications(1), (2) and (3) imply that, in the universe $(z_t, y_t, x_{n,t})$, (z_t, y_t) causes $x_{n,t}$, $x_{n,t}$ does not cause (z_t, y_t) and there is instantaneous causality between (z_t, y_t) and $x_{n,t}$. Moreover, denoting with $\overline{x}_{n,t}$ the vector x_t excluding $x_{n,t}$, $(\overline{x}_{n,t}, d_t)$ does not cause $(z_t, y_t, x_{n,t})$ in the whole universe w_t.

Finally, the $d_{n,t}$'s, $n = 1, \dots, N$, are independent conditionally to$(z_t, y_t, x_t, \underline{w}_{t-1})$ and the conditional distribution of $d_{n,t}$ is such that:

$$p\left(d_{n,t} = 1 \mid z_t, y_t, x_t, \underline{w}_{t-1}\right) = \begin{cases} 1 & \text{if } d_{n,t-1} = 1, \\ 1 - \exp\left(-\lambda_{n,t}\right) & \text{otherwise,} \end{cases} \tag{4}$$

with $\lambda_{n,t} = \alpha_n^{'} z_t + \beta_n^{'} y_t + \gamma_n^{'} x_{n,t}$.

In other words, state 1 of $d_{n,t}$ is an absorbing state and $\exp\left(-\lambda_{n,t}\right)$ is the survival probability. Since the default probability $1 - \exp\left(-\lambda_{n,t}\right)$ is close to $\lambda_{n,t}$ if $\lambda_{n,t}$ is small, $\lambda_{n,t}$ is called the default intensity. The default intensity is expected to be postive, which is not necessarily the case since the ε_t's are Gaussian. However, the parameterization of the model may make this extremely unfrequent.

So, in the universe $(z_t, y_t, x_{n,t}, d_{n,t})$, $(z_t, y_t, x_{n,t})$ causes $d_{n,t}$ whereas $d_{n,t}$ does not causes $(z_t, y_t, x_{n,t})$ and there is instantaneous causality. In the whole universe w_t, $(\overline{x}_{n,t}, \overline{d}_{n,t})$ does not cause $(z_t, y_t, x_{n,t}, d_{n,t})$.

The causality scheme is summarized in Figure 1.

Finally, let us consider the conditional Laplace transform of the distribution of (z_t, y_t) given $\underline{w}_{t-1}$:

$$\varphi_{t-1}(u, v) = E_{t-1}\left[\exp\left(u' z_t + v' y_t\right)\right].$$

Proposition 1. *The conditional Laplace transform of (z_t, y_t) given $\underline{w}_{t-1}$ is:*

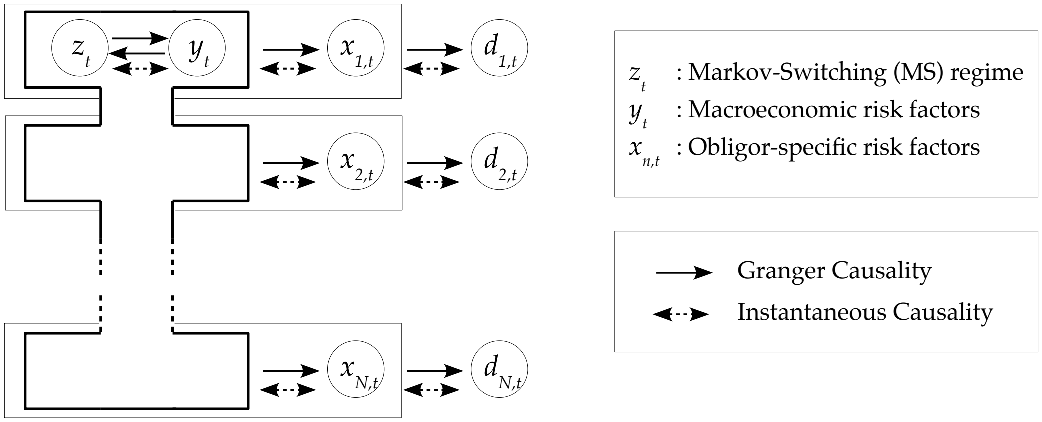
$$\varphi_{t-1}\left(u, v\right) = \exp\left(v'\Phi y_{t-1} + \left[l_1, \dots, l_J\right] z_{t-1}\right), \tag{5}$$

where $l_i = \log \sum_{j=1}^{J} \pi_{ij,t-1} \exp\left\{u_i + v'\mu(e_j, e_i) + \frac{1}{2} v'\Omega\left(e_j, e_i\right)\Omega'\left(e_j, e_i\right) v\right\}$.

[20] Formally, this corresponds to $\mu\left(z_t, z_{t-1}\right) = \mu\left(z_{t-1}\right)$ and $\Omega\left(z_t, z_{t-1}\right) = \Omega\left(z_{t-1}\right)$.

[21] Indeed, the model of Bansal and Zhou (2002) [12] does not admit a closed-form exponential affine solution (they proceed by linearizing the discrete-time Euler equations and by solving the resulting linear relations for prices.

Figure 1: Causality scheme



Proof. We have

$$\begin{aligned}
\varphi_{t-1}(u,v) &= E_{t-1}\left(\exp\left[u'z_t + v'y_t\right]\right) \\
&= E_{t-1}\left(\exp\left[u'z_t + v'\mu(z_t,z_{t-1}) + v'\Phi y_{t-1} + v'\Omega(z_t,z_{t-1})\varepsilon_t\right]\right) \\
&= E\left(E\{\exp\left[u'z_t + v'\mu(z_t,z_{t-1}) + v'\Phi y_{t-1} +\right.\right. \\
&\quad \left.\left. v'\Omega(z_t,z_{t-1})\varepsilon_t\right] \mid \underline{w}_{t-1}, z_t\} \mid \underline{w}_{t-1}\right) \\
&= \exp(v'\Phi y_{t-1}) E\left(\exp\left\{u'z_t + v'\mu(z_t,z_{t-1})\right\} \times\right. \\
&\quad \left. E\left(\exp\left\{v'\Omega(z_t,z_{t-1})\varepsilon_t \mid \underline{w}_{t-1}, z_t\right\}\right) \mid \underline{w}_{t-1}\right) \\
&= \exp(v'\Phi y_{t-1}) E\left(\exp\left\{u'z_t + v'\mu(z_t,z_{t-1})\right\} \times\right. \\
&\quad \left. \frac{1}{2} v'\Omega(z_t,z_{t-1})\Omega'(z_t,z_{t-1}) v \mid \underline{w}_{t-1}\right) \\
&= \exp\left(v'\Phi y_{t-1} + [l_1,\ldots,l_J]\, z_{t-1}\right).
\end{aligned}$$

Using the expression given for the l_i's lead to the result. □

This Laplace transform is not, in general, exponential affine in (z_{t-1}, y_{t-1}), since y_{t-1} appears in the $\pi_{ij,t}$'s. However, this is the case if the $\pi_{ij,t}$'s do not depend on y_{t-1} and then, the dynamics of (z_t, y_t) is Car(1) (see Darolles, Gourieroux and Jasiak, 2006[45] or Bertholon, Monfort and Pegoraro (2008) [18] for in-depth presentations of Car processes).

3. Stochastic discount factor and risk-neutral dynamics

3.1. Stochastic discount factor

We complete the model by specifying the stochastic discount factor $M_{t-1,t}$ between $t-1$ and t:

$$\begin{aligned}
M_{t-1,t} &= \exp\left[-a_1' z_{t-1} - b_1' y_{t-1} - \frac{1}{2}\nu'(z_t,z_{t-1},y_{t-1})\,\nu(z_t,z_{t-1},y_{t-1}) +\right. \\
&\quad \left. +\nu'(z_t,z_{t-1},y_{t-1})\,\varepsilon_t + \delta'(z_{t-1},y_{t-1})\, z_t\right], \qquad (6)
\end{aligned}$$

with the constraints:

$$\sum_{j=1}^{J} \pi_{ij,t-1} \exp\left[\delta_j\left(e_i, y_{t-1}\right)\right] = 1, \, \forall i, y_{t-1}, \tag{7}$$

where δ_j is the j^{th} component of δ. Using Equation (7), it is easily seen that $E_{t-1}(M_{t-1,t}) = \exp(-a_1^{'} z_{t-1} - b_1^{'} y_{t-1})$. Therefore, the riskless short rate between $t-1$ and t is:

$$r_t = a_1^{'} z_{t-1} + b_1^{'} y_{t-1}. \tag{8}$$

In our framework, the variables $(x_{n,t}, d_{n,t})$, specific to entity n, do not appear in the stochastic discount factor. This means that these entities have no impact at the macroeconomic level.[22] This can be formalised in the following way. Let us assume that the N entities appearing in the modeling belong to a large homogenous population of size $\tilde{N}$. This large population could be included in $M_{t-1,t}$, for instance by adding a term of the form

$$G_t(\tilde{N}) = \frac{1}{\tilde{N}} \sum_{n=1}^{\tilde{N}} \left(\nu_n^{'} x_{n,t} + \nu_{0n}^{'} d_{n,t}\right).$$

Since the $(x_{n,t}, d_{n,t})$, $i = 1, \ldots, \tilde{N}$ are independent conditonally to $\underline{z}_t, \underline{y}_t$, we have, denoting respectively by E_t and V_t the conditional expectation and variance (or variance-covariance matrix) given $\underline{z}_t, \underline{y}_t$:

$$V_t\left(G_t\left(\tilde{N}\right)\right) = \frac{1}{\tilde{N}^2} \sum_{n=1}^{\tilde{N}} \left[\nu_n^{'}, \nu_{0,n}\right] V_t\left(x_{n,t}^{'}, d_{n,t}\right) \left[\nu_n^{'}, \nu_{0,n}\right]^{'}.$$

Assuming that the terms in the sum are bounded when $\tilde{N}$ goes to infinity, which means that all the entities have a bounded weight in the infinite population, $V_t(G_t(\tilde{N}))$ goes to zero, when $\tilde{N}$ goes to infinity and $G_t(\tilde{N})$ converges in mean square to $\lim_{\tilde{N}\to\infty} E_t(G_t(\tilde{N}))$ (which is assumed to exist). Therefore, $G_t(\tilde{N})$ asymptotically depends only on $(\underline{z}_t, \underline{y}_t)$. which already appears in $M_{t-1,t}$. In some sense, the impact of these entities has been diversified away.

So the framework of this paper can be used in the context described above, the entities appearing in the modeling are those of specific interest, and the sequential inference method proposed in section 6 shows that these entities can be incorporated progressively in the model.

3.2. Risk-neutral dynamics

3.2.1. The conditional risk-neutral distribution of (z_t, y_t) given $\underline{w}_{t-1}$

Let us now consider the conditional risk-neutral Laplace transform of (z_t, y_t) given $\underline{w}_{t-1}$, $\varphi_{t-1}^{\mathbb{Q}}(u, v) := E_{t-1}^{\mathbb{Q}}\left(\exp\left[u' z_t + v' y_t\right]\right)$, and let us introduce the notations:

$$\begin{aligned} \mu_t &= \mu\left(z_t, z_{t-1}\right) \\ \Omega_t &= \Omega\left(z_t, z_{t-1}\right), \, \Sigma(z_t, z_{t-1}) = \Omega_t \Omega_t' = \Sigma_t \\ \nu_t &= \nu\left(z_t, z_{t-1}, y_{t-1}\right) \\ \delta_{t-1} &= \delta\left(z_{t-1}, y_{t-1}\right). \end{aligned}$$

[22] Diversifiability assumptions and the implied restrictions on default risk premia are studied in details by Jarrow, Lando and Yu (2005) [91] (in a continuous-time setting).

Proposition 2. *The conditional risk-neutral Laplace transform of* (z_t, y_t) *given* $\underline{w}_{t-1}$ *is:*

$$\varphi_{t-1}^{\mathbb{Q}}(u,v) = \exp\left[v'\Phi y_{t-1} + \begin{pmatrix} A_{1,t-1}(u,v) & \ldots & A_{J,t-1}(u,v) \end{pmatrix} z_{t-1}\right], \tag{9}$$

where

$$\begin{aligned} A_{i,t-1}(u,v) \quad &= \quad \log(\sum_{j=1}^{J} \pi_{ij,t-1} \exp\left\{v'\Omega\left(e_j,e_i\right)\nu\left(e_j,e_i,y_{t-1}\right) + \frac{1}{2}v'\Sigma\left(e_j,e_i\right)v + \right. \\ &\left. v'\mu\left(e_j,e_i\right) + u_j + \delta_j\left(e_i,y_{t-1}\right)\right\}) . \end{aligned}$$

Proof.

$$\begin{aligned} \varphi_{t-1}^{\mathbb{Q}}(u,v) &= E_{t-1}^{\mathbb{Q}}\left(\exp\left[u'z_t + v'y_t\right]\right) \\ &= E_{t-1}\left(\exp\left[-\frac{1}{2}\nu_t^{'}\nu_t + \nu_t^{'}\varepsilon_t + \delta_{t-1}^{'}z_t + u'z_t + v'y_t\right]\right) \\ &= \exp\left(v'\Phi y_{t-1}\right) \times \\ &\quad E_{t-1}\left(\exp\left[-\frac{1}{2}\nu_t^{'}\nu_t + \nu_t^{'}\varepsilon_t + \delta_{t-1}^{'}z_t + u'z_t + v'\mu_t + v^{'}\Omega_t\varepsilon_t\right]\right) \\ &= \exp\left(v'\Phi y_{t-1}\right) \times \\ &\quad E_{t-1}\left(\exp\left[-\frac{1}{2}\nu_t^{'}\nu_t + \frac{1}{2}\left(\nu_t^{'} + v^{'}\Omega_t\right)\left(\nu_t^{'} + v^{'}\Omega_t\right)^{'} + v'\mu_t + u'z_t + \delta_{t-1}^{'}z_t\right]\right) \\ &= \exp\left(v'\Phi y_{t-1}\right) E_{t-1}\left(\exp\left[v'\Omega_t\nu_t + \frac{1}{2}v'\Sigma_t v + v'\mu_t + u'z_t + \delta_{t-1}^{'}z_t\right]\right). \end{aligned}$$

Using the expression given for $A_{i,t-1}(u,v)$ leads to the result. □

We immediately deduce the following Corollary.

Corollary 1. *The risk-neutral dynamics of* (z_t, y_t) *is Car(1) if the s.d.f. satisfies the constraints (for any* i, j *and* t*):*

$$\begin{cases} \pi\left(e_j \mid e_i, y_{t-1}\right)\exp\left[\delta_j\left(e_i,y_{t-1}\right)\right] & = \pi_{ij}^{*} \\ \Omega\left(e_j,e_i\right)\nu\left(e_j,e_i,y_{t-1}\right) & = \Phi^{*}y_{t-1} + \mu^{*}\left(e_j,e_i\right), \end{cases} \tag{10}$$

where $\pi_{ij}^{*} = \pi^{*}(e_j \mid e_i)$ *does not depend on* y_{t-1}, Φ^{*} *is any matrix and* μ^{*} *is any function.*

If such constraints are satisfied, the risk-neutral conditional Laplace transform becomes:

$$\varphi_{t-1}^{\mathbb{Q}}(u,v) = \exp\left[v'\left(\Phi + \Phi^{*}\right)y_{t-1} + \begin{pmatrix} A_1^{*}(u,v) & \ldots & A_J^{*}(u,v) \end{pmatrix} z_{t-1}\right], \tag{11}$$

with $A_i^{*}(u,v) = \log\left(\sum_{j=1}^{J}\pi_{ij}^{*}\exp\left\{u_j + v'\left[\mu\left(e_j,e_i\right) + \mu^{*}\left(e_j,e_i\right)\right] + \frac{1}{2}v'\Sigma\left(e_j,e_i\right)v\right\}\right)$.

Comparing with equation (5), we deduce that the risk-neutral dynamics of (z_t, y_t) is then defined by:

$$y_t = \mu(z_t, z_{t-1}) + \mu^*(z_t, z_{t-1}) + (\Phi + \Phi^*) y_{t-1} + \Omega(z_t, z_{t-1}) \varepsilon_t^*, \tag{12}$$

where, under $\mathbb{Q}$, z_t is an homogenous Markov chain defined by the transition matrix $\{\pi_{ij}^*\}$, and ε_t^* –defined by $\varepsilon_t^* = \varepsilon_t - \Omega^{-1}(z_t, z_{t-1})[\mu^*(z_t, z_{t-1}) + \Phi^* y_{t-1}]$– is $IIN(0, I)$.

The previous results show that an appropriate choice of the s.d.f., that is an appropriate choice of the risk sensitivity vectors ν and δ pricing respectively the (standardized) innovations ε_t of y_t and z_t, allows to obtain a joint risk-neutral dynamics of (z_t, y_t) defined by any transition matrix $\{\pi_{ij}^*\}$ and any equation:

$$y_t = \tilde{\mu}(z_t, z_{t-1}) + \tilde{\Phi} y_{t-1} + \Omega(z_t, z_{t-1}) \varepsilon_t^*,$$

where ε_t^* is $IIN(0, I)$. Note that the Ω function is the same in the historical and risk-neutral worlds.

3.2.2. The risk-neutral distribution of (x_t, d_t) given $(z_t, y_t, \underline{w}_{t-1})$

Lemma 1. *Let us consider a partition of* $w_t = \left(w_{1,t}', w_{2,t}'\right)'$. *If* $M_{t-1,t}$ *is a function of* $\left(w_{1,t}, \underline{w}_{t-1}\right)$, *the risk-neutral probability density function, or p.d.f. , of* $w_{1,t}$ *given* $\underline{w}_{t-1}$ *is:*

$$f_{\mathbb{Q}}\left(w_{1,t} \mid \underline{w}_{t-1}\right) = f\left(w_{1,t} \mid \underline{w}_{t-1}\right) M_{t-1,t} \exp(-r_t)$$

(where f is the historical conditional p.d.f. of $w_{1,t}$ given $\underline{w}_{t-1}$) and the conditional risk-neutral distribution of $w_{2,t}$ given $\left(w_{1,t}, \underline{w}_{t-1}\right)$ is the same as the corresponding historical distribution.

Proof. See Appendix A. □

Since $M_{t-1,t}$ is a function of (z_t, y_t) but not of (x_t, d_t), the previous lemma shows that the risk-neutral distribution of (x_t, d_t) given $\left(z_t, y_t, \underline{w}_{t-1}\right)$ is the same as the historical one and it is given by equations (3) and (4). In particular, the functional forms of the default intensities $\lambda_{n,t}$ are the same as in the historical world. Of course, since the dynamics of (z_t, y_t) are different in the two worlds, the same is true for the $x_{n,t}$'s and the $\lambda_{n,t}$'s.

In addition, it can be shown that $(z_t, y_t, x_{n,t})$ is Car(1) under the risk-neutral measure (see Appendix C). However, it is not the case for $(z_t, y_t, x_{n,t}, d_{n,t})$.

It is also clear that the causality structure of the risk-neutral dynamics is similar to the historical one, the only difference being the non-causality from y_t to z_t implied by the homogeneity of the matrix $\{\pi_{ij}^*\}$.

3.3. Discussion of the constraints on the SDF

Constraints (10) can be written:

$$\begin{cases} \delta_j(z_{t-1}, y_{t-1}) & = \log\left(\frac{\pi^*(e_j|z_{t-1})}{\pi(e_j|z_{t-1}, y_{t-1})}\right) \\ \nu(z_t, z_{t-1}, y_{t-1}) & = \Omega^{-1}(z_t, z_{t-1})\left[\Phi^* y_{t-1} + \mu^*(z_t, z_{t-1})\right], \end{cases} \tag{13}$$

where the transition matrix $\{\pi^*_{ij}\}$, the matrix Φ^* and the vectors $\mu^*(e_j, e_i)$ are arbitrary. Note that constraints (7) imposed on δ are automatically satisfied by the parameterization (13). Recall that constraints (10) are imposed so as to obtain a Car dynamics of the state variable under the risk-neutral measure. These constraints could be relaxed, but at the cost of losing the analytical tractability in the bond pricing (as will be shown below).[23] Even if we impose a Car risk-neutral dynamics, we still have a large number of degrees of freedom in the specification of the s.d.f. since Φ^*, $\mu^*(z_t, z_{t-1})$ and the π^*_{ij}'s are then chosen arbitrarily. However, we may wish to parameterize more parsimoniously the s.d.f. and, therefore, impose stronger constraints on the risk-neutral dynamics. Let us illustrate this point by a simple bivariate example.

The historical dynamics is defined by:

$$\begin{bmatrix} y_{1,t} \\ y_{2,t} \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2' z_t \end{bmatrix} + \begin{bmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{bmatrix} \begin{bmatrix} y_{1,t-1} \\ y_{2,t-1} \end{bmatrix} + \begin{bmatrix} \sigma_1 \varepsilon_{1,t} \\ (\sigma_2' z_t)\, \varepsilon_{2,t} \end{bmatrix}$$

and by some $\pi_{ij,t}$'s. Moreover, let us assume that we impose an additive risk-sensitivity vector ν:

$$\nu(z_t, z_{t-1}, y_{t-1}) = \begin{pmatrix} b_1' y_{t-1} + \nu_1' z_t \\ b_2' y_{t-1} + \nu_2' z_t \end{pmatrix}.$$

We get:

$$\Omega(z_t, z_{t-1}) \nu(z_t, z_{t-1}, y_{t-1}) = \begin{bmatrix} \sigma_1 b_1' y_{t-1} + \sigma_1 \nu_1' z_t \\ \sigma_2' z_t \,(b_2' y_{t-1} + \nu_2' z_t) \end{bmatrix},$$

which must be additive of the form $\Phi^* y_{t-1} + \mu^*(z_t, z_{t-1})$. It is only possible if $b_2 = 0$ and in this case we get:

$$\Phi^* = \begin{bmatrix} \sigma_1 b_1' \\ 0 \end{bmatrix} \text{ and } \mu^*(z_t, z_{t-1}) = \begin{bmatrix} \sigma_1 \nu_1' z_t \\ (\sigma_2 \odot \nu_2)' z_t \end{bmatrix},$$

where $\odot$ denotes the Hadamard (element by element) product. In other words $\Phi^* = \begin{bmatrix} \varphi_1^{*\prime} \\ 0 \end{bmatrix}$, $\mu^*(z_t, z_{t-1}) = \begin{bmatrix} \mu_1^{*\prime} z_t \\ \mu_2^{*\prime} z_t \end{bmatrix}$ where φ_1^*, μ_1^* and μ_2^* are arbitrary. Finally, the risk-neutral dynamics is given by:

$$\begin{bmatrix} y_{1,t} \\ y_{2,t} \end{bmatrix} = \begin{bmatrix} \tilde{\mu}'_1 z_t \\ \tilde{\mu}_2' z_t \end{bmatrix} + \begin{bmatrix} \tilde{\varphi}_{11} & \tilde{\varphi}_{12} \\ \varphi_{21} & \varphi_{22} \end{bmatrix} \begin{bmatrix} y_{1,t-1} \\ y_{2,t-1} \end{bmatrix} + \begin{bmatrix} \sigma_1 \varepsilon^*_{1,t} \\ (\sigma_2' z_t)\, \varepsilon^*_{2,t} \end{bmatrix}$$

and by $\{\pi^*_{ij}\}$ where $\tilde{\varphi}_{11}$, $\tilde{\varphi}_{12}$, $\tilde{\mu}_1$, $\tilde{\mu}_2$ and the π^*_{ij}'s are arbitrary, but the autoregressive coefficients of the second equations are the same as in the historical dynamics.

3.4. The specific case of no instantaneous causality between z_t and y_t

The general framework nests the case where there is no instantaneous causality between z_t and y_t in the historical dynamics (as in Dai, Singleton and Yang, 2007 [44]), that is:

$$\begin{aligned} \mu(z_t, z_{t-1}) &= \mu(z_{t-1}) = c_\mu z_{t-1} \text{ (say)} \\ \Omega(z_t, z_{t-1}) &= \Omega(z_{t-1}) \text{ (say).} \end{aligned}$$

[23] Boudoukh et al. (1999) [25] develop a model with regime-transition probabilities that are state-dependent under both physical and risk-neutral measures. However, to make bond pricing tractable, Bakoukh et al. have to resort to approximations. Specifically, they consider that there are a finite number of states per regimes (11 states per regime).

Let us assume moreover that:

$$\nu(z_t, z_{t-1}, y_{t-1}) = \nu(z_{t-1}, y_{t-1}) \text{ (say)}$$

and, therefore, $\mu^*(z_t, z_{t-1}) = \mu^*(z_{t-1}) = c_\mu^* z_{t-1}$ (say).

In this case, the historical and risk-neutral Laplace transforms become:

$$\begin{aligned}
\varphi_{t-1}(u,v) &= \exp\Bigg(v'(\Phi y_{t-1} + c_\mu z_{t-1}) + \frac{1}{2}\left[v'\Omega(e_1)v, \ldots, v'\Omega(e_J)v\right] z_{t-1} + \\
&\quad + \left[\log \sum_{j=1}^{J} \pi_{1j,t-1}\exp(u_1), \ldots, \log \sum_{j=1}^{J} \pi_{Jj,t-1}\exp(u_J)\right] z_{t-1} \Bigg) \quad \text{and} \\
\varphi_{t-1}^{\mathbb{Q}}(u,v) &= \exp\Bigg(v'\left[(\Phi + \Phi^*) y_{t-1} + \left(c_\mu + c_\mu^*\right) z_{t-1}\right] + \frac{1}{2}\left[v'\Omega(e_1)v, \ldots, v'\Omega(e_J)v\right] z_{t-1} + \\
&\quad + \left[\log \sum_{j=1}^{J} \pi_{1j}^* \exp(u_1), \ldots, \log \sum_{j=1}^{J} \pi_{Jj}^* \exp(u_J)\right] z_{t-1} \Bigg).
\end{aligned}$$

So the risk-neutral dynamics is defined by $y_t = \left(c_\mu + c_\mu^*\right)' z_{t-1} + (\Phi + \Phi^*) y_{t-1} + \Omega(z_{t-1})\varepsilon_t^*$ where z_t is an homogenous Markov chain defined by the transition matrix$\{\pi_{ij}^*\}$ and ε_t^* is $IIN(0, I)$. In particular, there is no instantaneous causality in the risk-neutral world either.

4. Pricing

4.1. Pricing of riskless zero-coupon bonds

It is well-known that the existence of a positive stochastic discount factor is equivalent to the absence of arbitrage opportunities (see Hansen and Richard, 1987 [86] and Berholon, Monfort and Pegoraro, 2007 [18]) and that the price at t of a zero-coupon bond with residual maturity h is given by:

$$B(t,h) = E_t^{\mathbb{Q}}\left[\exp(-r_{t+1} - \ldots - r_{t+h})\right], \tag{14}$$

where $r_{t+i} = a_1' z_{t+i-1} + b_1' y_{t+i-1}$, $i = 1, \ldots, h$. Since (z_t, y_t) is Car(1) under $\mathbb{Q}$, $B(t,h)$ is easily computed using the following lemma:

Lemma 2. *Let us consider a multivariate Car(1) process Z_t and its conditional Laplace transform given by $\exp[a'(s)Z_t + b(s)]$. Let us further denote by $L_{t,h}(\omega)$ its multi-horizon Laplace transform given by:*

$$L_{t,h}(\omega) = E_t\left[\exp\left(\omega_{H-h+1}' Z_{t+1} + \ldots + \omega_H' Z_{t+h}\right)\right], \; t = 1, \ldots, T, \; h = 1, \ldots, H,$$

where $\omega = (\omega_1', \ldots, \omega_H')$ is a given sequence of vectors. We have, for any t,

$$L_{t,h}(\omega) = \exp\left(A_h' Z_t + B_h\right), \; h = 1, \ldots, H,$$

where the sequences A_h, B_h, $h = 1, \ldots, H$ are obtained recursively by:

$$
\begin{aligned}
A_h &= a(\omega_{H-h+1} + A_{h-1}) \\
B_h &= b(\omega_{H-h+1} + A_{h-1}) + B_{h-1},
\end{aligned}
$$

with the initial conditions $A_0 = 0$ *and* $B_0 = 0$.

Proof. See Appendix B. □

From Equation (11) we know that (z_t, y_t) is risk-neutral Car(1) and that its conditional Laplace transform is based on the functions:

$$
\begin{aligned}
a'(u,v) &= \left[(A_1^*(u,v), \ldots, A_J^*(u,v)), v'(\Phi + \Phi^*)\right] \text{ and} \\
b(u,v) &= 0.
\end{aligned}
$$

so we have the following proposition:

Proposition 3. *We have:*

$$
B\,(t,h) = \exp\left(-a_h^{'} z_t - b_h^{'} y_t\right), \tag{15}
$$

and the yield of residual maturity h, $R(t,h)$ *is given by:*

$$
R(t,h) = \frac{1}{h}\left(a_h z_t + b_h^{'} y_t\right), \tag{16}
$$

where a_h *and* b_h *are computed recursively, for* $h = 1, \ldots, H$, *by (with* $a_0 = a_1$ *and* $b_0 = b_1$*):*

$$
(a_h', b_h') = (a_1', b_1') - a'\left(\omega_{H-h+1} - \left(a_{h-1}' - a_1', b_{h-1}' - b_1'\right)'\right),
$$

where the sequence ω_h, $h = 1, \ldots, H$ *is defined by* $\omega_H = 0$, $\omega_1 = \omega_2 = \ldots = \omega_{H-1} = (-a_1', -b_1')'$ *and where* $a'(u,v) = [(A_1^*(u,v), \ldots, A_J^*(u,v)), v'(\Phi + \Phi^*)]$.

Proof. We have:

$$
B(t,h) = \exp\left(-a_1' z_t - b_1' y_t\right) E_t^{\mathbb{Q}}\left(-a_1' z_{t+1} - b_1' y_{t+1} - \ldots - a_1' z_{t+h-1} - b_1' y_{t+h-1}\right).
$$

Using Lemma 2 with $\omega_H = 0$, $\omega_h' = (-a_1', -b_1')$ for $h = 1, \ldots, H-1$, we get:

$$
B(t,h) = \exp\left(-a_1' z_t - b_1' y_t + \tilde{a}_h' z_t + \tilde{b}_h' y_t\right),
$$

where $(\tilde{a}_h', \tilde{b}_h') = a'(\omega_{H-h+1} + (\tilde{a}_{h-1}', \tilde{b}_{h-1}'))$, $\tilde{a}_0 = 0$ and $\tilde{b}_0 = 0$.

Taking $a_h = a_1 - \tilde{a}_h$, $b_h = b_1 - \tilde{b}_h$, with $(a_h', b_h') = (a_1', b_1') - a'\left(\omega_{H-h+1} - \left(a_{h-1}' - a_1', b_{h-1}' - b_1'\right)'\right)$, we get $B\,(t,h) = \exp\left(-a_h^{'} z_{t-1} - b_h^{'} y_{t-1}\right)$. □

4.2. Pricing of (zero-recovery-rate) defaultable bonds

A defaultable zero-coupon bond providing one money unit at $t+h$ if entity n is still alive at $t+h$ and zero otherwise has a price at t given by:

$$B_n^D(t,h) = E_t^{\mathbb{Q}}\left[\exp\left(-r_{t+1} - \ldots - r_{t+h}\right)\mathbb{I}_{\{d_{n,t+h}=0\}}\right] \tag{17}$$

if $d_{n,t}=0$ and 0 otherwise.

Proposition 4. *The price of a zero-recovery-rate zero-coupon defaultable bond issued by debtor n is such that:*

$$\begin{aligned} B_n^D(t,h) &= E_t^{\mathbb{Q}}\left[\exp\left(-r_{t+1} - \ldots - r_{t+h} - \alpha_n' z_{t+1} - \beta_n' y_{t+1} - \gamma_n' x_{n,t+1} - \ldots \right.\right. \\ &\qquad \left.\left. - \ldots - \alpha_n' z_{t+h} - \beta_n' y_{t+h} - \gamma_n' x_{n,t+h}\right)\right]. \end{aligned} \tag{18}$$

Proof. Equation (17) can be rewritten:

$$\begin{aligned} B_n^D(t,h) &= E_t^{\mathbb{Q}}\left[E^{\mathbb{Q}}\left(\exp\left(-r_{t+1} - \ldots - r_{t+h}\right)\mathbb{I}_{\{d_{n,t+h}=0\}} \mid \underline{z}_{t+h}, \underline{y}_{t+h}, \underline{x}_{n,t+h}, d_{n,t}=0\right)\right] \\ &= E_t^{\mathbb{Q}}\left[\exp\left(-r_{t+1} - \ldots - r_{t+h}\right)\mathbb{Q}\left(d_{n,t+h}=0 \mid \underline{z}_{t+h}, \underline{y}_{t+h}, \underline{x}_{n,t+h}, d_{n,t}=0\right)\right]. \end{aligned}$$

Moreover,

$$\begin{aligned} & \mathbb{Q}\left(d_{n,t+h}=0 \mid \underline{z}_{t+h}, \underline{y}_{t+h}, \underline{x}_{n,t+h}, d_{n,t}=0\right) \\ =\; & \textstyle\prod_{i=1}^{h} \mathbb{Q}\left(d_{n,t+i}=0 \mid \underline{z}_{t+h}, \underline{y}_{t+h}, \underline{x}_{n,t+h}, d_{n,t+i-1}=0\right) \end{aligned}$$

and, since $d_{n,t}$ does not cause $(z_t, y_t, x_{n,t})$ in the Granger's or Sims' sense (see Appendix E), we have:

$$\begin{aligned} & \mathbb{Q}\left(d_{n,t+i}=0 \mid \underline{z}_{t+h}, \underline{y}_{t+h}, \underline{x}_{n,t+h}, d_{n,t+i-1}=0\right) \\ =\; & \mathbb{Q}\left(d_{n,t+i}=0 \mid \underline{z}_{t+i}, \underline{y}_{t+i}, \underline{x}_{n,t+i}, d_{n,t+i-1}=0\right) \\ =\; & \exp\left(-\lambda_{n,t+i}\right). \end{aligned}$$

where the last equality comes from the fact that the conditional historical and risk-neutral distributions of $d_{n,t}$ are the same (see Subsection 3.2.2). □

It can be shown (see Appendix C) that $(z_t, y_t, x_{n,t})$ is Car(1) under $\mathbb{Q}$, with a conditional Laplace transform of the type $\exp[\overline{a}'(u,v,w)(z_t', y_t', x_{n,t}')]$ where $\overline{a}(u,v,w) = [(\tilde{A}_1, \ldots, \tilde{A}_J), (v' + w'Q_{2n})(\Phi + \Phi^*) + w'Q_{3n}, w'Q_{4n}]$, where

$$\begin{aligned} \tilde{A}_i(u,v,w) &= \log(\sum_{j=1}^{J} \pi_{ij}^* \exp\{u_j + (v' + w'Q_{2n})\left[\mu\left(e_j,e_i\right) + \mu^*\left(e_j,e_i\right)\right] + w'q_{1n}\left(e_j,e_i\right) + \\ &\quad \frac{1}{2}(v' + w'Q_{2n})\Sigma\left(e_j,e_i\right)\left(v + Q_{2n}'w\right) + \frac{1}{2}w'Q_{5n}\left(e_j,e_i\right)Q_{5n}'\left(e_j,e_i\right)w\}). \end{aligned}$$

Therefore we have the following result:

Proposition 5. *The price of a zero-recovery-rate zero-coupon defaultable bond issued by debtor n is given by:*

$$B_n^D(t,h) = \exp\left(-c_{n,h}^{'} z_t - f_{n,h}^{'} y_t - g_{n,h}^{'} x_{n,t}\right) \tag{19}$$

and the defaultable yields are:

$$R_n^D(t,h) = \frac{1}{h}\left(c_{n,h}^{'} z_t + f_{n,h}^{'} y_t + g_{n,h}^{'} x_{n,t}\right), \tag{20}$$

where $(c_{n,h}', f_{n,h}', g_{n,h}')$ is computed recursively by:

$$\left(c_{n,h}', f_{n,h}', g_{n,h}'\right) = \left(a_1', b_1', 0\right) - \overline{a}\left(\omega_{H-h+1} - \left(c_{n,h-1}' - a_1', f_{n,h-1}' - b_1', -g_{n,h-1}'\right)'\right)$$

where the sequence ω_h, $h = 1, \ldots, H$ is defined by $\omega_H = (-\alpha_n', -\beta_n', -\gamma_n')$ and $\omega_h = (-\alpha_n' - a_1', -\beta_n' - b_1', -\gamma_n')$ for $h = 1, \ldots, H-1$, with $c_{n,0} = a_1$, $f_{n,0} = b_1$, $g_{n,0} = 0$.

Proof. From Proposition 4, we have:

$$\begin{aligned} B_n^D(t,h) &= \exp\left(-a_1' z_t - b_1' y_t\right) E_t^{\mathbb{Q}}\left(-a_1' z_{t+1} - b_1' y_{t+1} - \alpha_n' z_{t+1} - \beta_n' y_{t+1} - \gamma_n' x_{n,t+1} - \ldots\right. \\ &\quad -a_1' z_{t+h-1} - b_1' y_{t+h-1} - \alpha_n' z_{t+h-1} - \beta_n' y_{t+h-1} - \gamma_n' x_{n,t+h-1}) \\ &\quad \left.-\alpha_n' z_{t+h} - \beta_n' y_{t+h} - \gamma_n' x_{n,t+h}\right). \end{aligned}$$

Using Lemma 2 with $\omega_H = (-\alpha_n', -\beta_n', -\gamma_n')$ and $\omega_h = (-\alpha_n' - a_1', -\beta_n' - b_1', -\gamma_n')$ for $h = 1, \ldots, H-1$, we get:

$$B_n^D(t,h) = \exp\left(-a_1^{'} z_t - b_1^{'} y_t + \tilde{c}_{n,h}^{'} z_t + \tilde{f}_{n,h}^{'} y_t + \tilde{g}_{n,h}^{'} x_{n,t}\right),$$

where $(\tilde{c}_{n,h}', \tilde{f}_{n,h}', \tilde{g}_{n,h}') = \overline{a}'\left(w_{H-h+1} + (\tilde{c}_{n,h-1}', \tilde{f}_{n,h-1}', \tilde{g}_{n,h-1}')\right)$ and $\tilde{c}_{n,0} = 0$, $\tilde{f}_{n,0} = 0$ and $\tilde{g}_{n,0} = 0$.

Taking $c_{n,h} = a_1 - \tilde{c}_{n,h}$, $f_{n,h} = b_1 - \tilde{f}_{n,h}$ and $g_{n,h} = -\tilde{g}_{n,h}$, with $\left(c_{n,h}', f_{n,h}', g_{n,h}'\right) = (a_1', b_1', 0) - \overline{a}\left(\omega_{H-h+1} - \left(c_{n,h-1}' - a_1', f_{n,h-1}' - b_1', g_{n,h-1}'\right)'\right)$ and $c_{n,0} = a_1$, $f_{n,0} = b_1$, $g_{n,0} = 0$, we get $B_n^D(t,h) = \exp(-c_{n,h}^{'} z_t - f_{n,h}^{'} y_t - g_{n,h}^{'} x_{n,t})$. □

In this setting, credit spreads are given by:

$$\begin{aligned} s_n(t,h) &= R_n^D(t,h) - R_n(t,h) \\ &= \frac{1}{h}\left[\left(a_h - c_{n,h}\right)' z_t + \left(b_h - f_{n,h}\right)' y_t + g_h^{'} x_{n,t}\right]. \end{aligned} \tag{21}$$

In particular, the spread of maturity one is:

$$s_n(t,1) = r_{n,t+1}^D - r_{t+1},$$

with:

$$\begin{aligned} r^D_{n,t+1} &= -\log\left(E^{\mathbb{Q}}_t\left[\exp\left(-r_{t+1}-\lambda_{n,t+1}\right)\right]\right) \\ &= r_{t+1}-\log\left(E^{\mathbb{Q}}_t\left[\exp\left(-\lambda_{n,t+1}\right)\right]\right). \end{aligned}$$

So

$$\begin{aligned} s_n(t,1) &= -\log\left(E^{\mathbb{Q}}_t\left[\exp\left(-\lambda_{n,t+1}\right)\right]\right) \\ &= -\log\left(E^{\mathbb{Q}}_t\left[\exp\left(-\alpha^{'}_n z_{t+1}-\beta^{'}_n y_{t+1}-\gamma^{'}_n x_{n,t+1}\right)\right]\right), \end{aligned}$$

which can be easily computed and we get (using Appendix C):

$$s_n(t,1) = \left\{\beta'_n\left(\Phi+\Phi^*\right)+\gamma'_n Q_{3n}\right\} y_t + \gamma'_n Q_{4n} x_{n,t} - \left(\begin{array}{ccc} \tilde{A}_1 & \dots & \tilde{A}_J \end{array}\right) z_t,$$

with

$$\begin{aligned} \tilde{A}_i &= \log(\sum_{j=1}^{J}\pi^*_{ij}\exp\{-\alpha'_{n,j}-(\beta'_n+\gamma'_n Q_{2,n})\left[\mu\left(e_j,e_i\right)+\mu^*\left(e_j,e_i\right)\right]-\gamma'_n q_{1n}\left(e_j,e_i\right)+ \\ &\quad \frac{1}{2}(\beta'_n+\gamma'_n Q_{2,n})\Sigma\left(e_j,e_i\right)(\beta'_n+\gamma'_n Q_{2,n})'+\frac{1}{2}\gamma'_n Q_{5n}\left(e_j,e_i\right)Q'_{5n}\left(e_j,e_i\right)\gamma_n\}). \end{aligned}$$

4.3. Pricing of non-zero-recovery-rate defaultable bonds

Formula (18), which can read

$$B^D_n\left(t,h\right) = E^{\mathbb{Q}}_t\left[\exp\left(-r_{t+1}-\ldots-r_{t+h}-\lambda_{n,t+1}-\ldots-\lambda_{n,t+h}\right)\right], \quad (22)$$

has been obtained under the assumption of zero recovery rate. This formula can be extended to the case with non-zero recovery rates, providing that the $\lambda_{n,t}$'s are interpreted as risk-neutral "recovery-adjusted" default intensities. More precisely, we have the following result (dropping the subscript n for the sake of clarity):

Proposition 6. *If, for any bond issued by debtor n before t, the recovery payoff –that is assumed to be paid at time t in case of default between $t-1$ and t of debtor n– is equal to the product of a function $\zeta_{n,t}$ of the information available at time t by the survival-contingent market value of the bond at t, the price at t of a bond with residual maturity h is:*

$$B^{DR}_n(t,h) = E^{\mathbb{Q}}_t\left[\exp(-r_{t+h}-\ldots-r_{t+h}-\tilde{\lambda}_{n,t+1}-\ldots-\tilde{\lambda}_{n,t+h})\right], \quad (23)$$

where $\tilde{\lambda}_{n,s}$ is defined by (for any s):

$$\exp(-\tilde{\lambda}_{n,s}) = \exp(-\lambda_{n,s}) + (1-\exp(-\lambda_{n,s}))\,\zeta_{n,s}.$$

Proof. See Appendix D. □

The assumption of Proposition 6 is similar to the "Recovery of Market Value" assumption made by Duffie and Singleton (1999) [60] except that, in their discrete-time approach, they assume that ζ_t is known at time $t-1$, and that conditionally to the information at $t-1$, $d_{n,t}$ is independent of the recovery payoff at t.

5. Internal consistency (IC) conditions

5.1. IC conditions based on riskless yields

If the short rate r_{t+1} is a component of y_t, for instance the first one, we have to impose an internal consistency condition implying that $r_{t+1} = a_1^{'} z_t + b_1^{'} y_t$ is equal to the first component of y_t, that is:

$$a_1 = 0,\, b_1 = \widetilde{e}_1,$$

where $\widetilde{e}_i$ is the vector selecting the i^{th} component of y_t.

Moreover, if another component of y_t, for instance the second one, is equal to a riskless yield of maturity h_0 –ie $R(t, h_0)$– we have to impose that $(1/h_0)\left(a_{h_0}^{'} z_t + b_{h_0}^{'} y_t\right)$ is equal to the second component of y_t, that is

$$\begin{cases} a_{h_0} & = 0 \\ b_{h_0} & = h_0 \tilde{e}_2. \end{cases}$$

5.2. IC conditions based on defaultable yields

Similarly, if the first component of $x_{n,t}$ is a defaultable yield with residual maturity h_0, equation (19) implies that we have to impose:

$$\begin{cases} c_{n,h_0} & = 0 \\ f_{n,h_0} & = 0 \\ g_{n,h_0} & = h_0 \hat{e}_1. \end{cases}$$

where $\hat{e}_i$ denotes the vector selecting the i^{th} component of $x_{n,t}$.

5.3. IC conditions based on asset returns

If the first component of y_t is the geometric return of a market index, we have to impose

$$\exp\left(-r_{t+1}\right) E_t^{\mathbb{Q}} \left(\exp\left(y_{1,t+1}\right)\right) = 1.$$

Using equation (11), this gives

$$\begin{pmatrix} A_{1,0}^* & \dots & A_{J,0}^* \end{pmatrix} z_t + \left(\Phi_1 + \Phi_1^*\right) y_t = a_1^{'} z_t + b_1^{'} y_t,$$

with $A_{i,0}^* = \log\left\{\sum_{j=1}^{J} \pi_{ij}^* \exp\left[\mu_1\left(e_j, e_i\right) + \mu_1^*\left(e_j, e_i\right) + \frac{1}{2}\sigma_1^2\left(e_j, e_i\right)\right]\right\}$, μ_1 and μ_1^* being the first components of μ and μ^* respectively, σ_1^2 being the $(1,1)$ entry of Σ and Φ_1 and Φ_1^* the first rows of Φ and Φ^* respectively. Then we get

$$\begin{cases} a_1 & = \begin{pmatrix} A_{1,0}^* & \dots & A_{J,0}^* \end{pmatrix}^{'} \\ b_1 & = \left(\Phi_1 + \Phi_1^*\right)^{'}. \end{cases}$$

Similarly, if the first component of $x_{n,t}$ is the return of a stock attached to entity n, we must have:

$$\exp\left(-r_{t+1}\right) E_t^{\mathbb{Q}} \left(\exp\left(x_{1,n,t+1}\right)\right) = 1$$

or

$$r_{t+1} = \log\left[E_t^{\mathbb{Q}}\left(\exp\left(x_{1,n,t+1}\right)\right)\right].$$

Using the fact that $(z_t, y_t, x_{n,t})$ is Car(1) under $\mathbb{Q}$ (see Appendix C), it is readily seen that $\log\left[E_t^{\mathbb{Q}}\left(\exp\left(x_{1,n,t+1}\right)\right)\right]$ is linear in $z_t, y_t, x_{n,t}$ and the IC constraint follows.

6. Inference

6.1. Observability

We assume that z_t, y_t and the $x_{n,t}$'s are partitioned into $z_t = (z_{1t}^{'}, z_{2t}^{'})'$, $y_t = (y_{1t}^{'}, y_{2t}^{'})'$ and $x_t = (x_{1,n,t}^{'}, x_{2,n,t}^{'})'$, that $z_{1t}, y_{1t}, x_{1,n,t}$ are observed by the econometrician and z_{2t}, y_{2t} and $x_{2,n,t}$ are not. Typically, $z_{1,t}$ and $z_{2,t}$ will be two regime processes valued respectively in $E_1 = \{e_1, \dots, e_{J_1}\}$ and $E_2 = \{e_1, \dots, e_{J_2}\}$ so z_t will be equal to $z_{1,t} \otimes z_{2,t}$, where $\otimes$ denotes the Kronecker product operator. The implementation of the following estimation strategy requires that the transition probabilities do not depend on the unobserved vectors $y_{2,t-1}$.[24] Moreover, we assume that we observe at each date t a vector of risk-free yields denoted by R_t and, for each obligor n, a vector of defaultable yields denoted by $R_{n,t}^D$. Note that if some yields are included in the vectors y_t or $x_{n,t}$, they do not enter the vectors R_t and $R_{n,t}^D$ (see Section 5). The period of observation is $\{1, \dots, T\}$.

6.2. Decomposition of the joint p.d.f. and estimation strategy

Let us denote by θ^{zy} the vector of parameters defining the historical dynamics of (z_t, y_t), by θ_n^x the vector of parameters defining the conditional p.d.f. of $x_{n,t}$ given $\underline{z}_t, \underline{y}_t, \underline{x}_{n,t-1}$ and by θ_n^d the vector of parameters defining the conditional p.d.f. of $d_{n,t}$ given $\underline{z}_t, \underline{y}_t, \underline{x}_{n,t}, \underline{d}_{n,t-1}$.

The joint p.d.f. of $\underline{w}_t$ is:

$$
\begin{aligned}
f\left(\underline{w}_t, \theta\right) = & \prod_{t=1}^{T} f\left(z_t, y_t \mid \underline{z}_{t-1}, \underline{y}_{t-1}; \theta^{zy}\right) \\
& \times \prod_{n=1}^{N} \prod_{t=1}^{T} f\left(x_{n,t} \mid \underline{z}_t, \underline{y}_t, \underline{x}_{n,t-1}; \theta_n^x\right) \\
& \times \prod_{n=1}^{N} \prod_{t=1}^{T} f\left(d_{n,t} \mid \underline{z}_t, \underline{y}_t, \underline{x}_{n,t}, \underline{d}_{n,t-1}; \theta_n^d\right).
\end{aligned}
$$

The parameters appearing in $M_{t-1,t}$ are denoted by θ^*. The theoretical values of R_t and R_{tn}^D given by the model are denoted by $R_t\left(\theta^{zy}, \theta^*\right)$ and $R_{nt}^D\left(\theta^{zy}, \theta_n^x, \theta_n^d, \theta^*\right)$ respectively. A sequential strategy of estimation is the following:

1. Estimate θ^{zy} and θ^* from the osbervations of y_{1t}, z_{1t}, R_t, $t = 1, \dots, T$.
2. Estimate the θ_n^x's and the θ_n^d's from the observations of $x_{1n,t}$ and $R_{n,t}^D$, $t = 1, \dots, T$, taking as given the values of θ^{zy} and θ^*, and the values of $y_{2,t}$ and $z_{2,t}$ being fixed at the approximated values obtained from step 1.

The remaining of the current section details these two steps. The methodology that is proposed builds on the so-called inversion technique developed by Chen and Scott (1993) [33]. This technique is adapted in order to accomodate regime switching. Naturally, many different estimation strategies could be implemented. For instance, in our application (Section 9), the estimation of the model parameters does not rely on inversion techniques but resorts to state-space modeling and Kalman filtering.

[24] Formally, with the notation of Equation (1), $p\left(z_t \mid \underline{z}_{t-1}, \underline{y}_{t-1}\right)$ has to be equal to $p\left(z_t \mid \underline{z}_{t-1}, \underline{y}_{1,t-1}\right)$.

6.3. Estimation of the parameters (θ^{zy}, θ^*)

Using equation (16), we have, with obvious notations:

$$R_t(\theta^{zy}, \theta^*) = Az_t + B_1 y_{1,t} + B_2 y_{2,t}.$$

If m is the dimension of y_{2t}, let us partition R_t in $\left(R_{1,t}^{'}, R_{2,t}^{'}\right)^{'}$ where $R_{2,t}$ is of dimension m. With obvious notations, we get:

$$R_{2,t}(\theta^{zy}, \theta^*) = A_2 z_t + B_{21} y_{1,t} + B_{22} y_{2,t},$$

and denoting $\left(y_{1,t}^{'}, R_{2,t}^{'}\right)^{'}$ by $\tilde{y}_t$ we get:

$$\tilde{y}_t = \begin{pmatrix} I & 0 \\ B_{21} & B_{22} \end{pmatrix} y_t + \begin{pmatrix} 0 \\ A_2 \end{pmatrix} z_t$$

or

$$\tilde{y}_t = \tilde{B} y_t + \tilde{A} z_t$$

and

$$y_t = \tilde{B}^{-1}\left(\tilde{y}_t - \tilde{A} z_t\right)$$

and from equation (2) we get:

$$\tilde{B}^{-1}\left(\tilde{y}_t - \tilde{A} z_t\right) = \mu(z_t, z_{t-1}) + \Phi\left[\tilde{B}^{-1}\left(\tilde{y}_{t-1} - \tilde{A} z_{t-1}\right)\right] + \Omega(z_t, z_{t-1})\varepsilon_t$$

or

$$\tilde{y}_t = \tilde{A} z_t + \tilde{B}\mu(z_t, z_{t-1}) + \tilde{B}\Phi\left[\tilde{B}^{-1}\left(\tilde{y}_{t-1} - \tilde{A} z_{t-1}\right)\right] + \tilde{B}\Omega(z_t, z_{t-1})\varepsilon_t$$

or

$$\tilde{y}_t = \tilde{\mu}(z_t, z_{t-1}) + \tilde{\Phi}\tilde{y}_{t-1} + \tilde{\Omega}(z_t, z_{t-1})\varepsilon_t, \tag{24}$$

with

$$\begin{cases} \tilde{\mu}(z_t, z_{t-1}) & = \tilde{A} z_t + \tilde{B}\mu(z_t, z_{t-1}) - \tilde{B}\Phi\tilde{B}^{-1}\tilde{A} z_{t-1} \\ \tilde{\Phi} & = \tilde{B}\Phi\tilde{B}^{-1} \\ \tilde{\Omega}(z_t, z_{t-1}) & = \tilde{B}\Omega(z_t, z_{t-1}). \end{cases}$$

The conditional distribution of $\tilde{y}_t$ given $\underline{z}_t, \underline{\tilde{y}}_{t-1}$, is similar to that of y_t given $\underline{z}_t, \tilde{y}_{t-1}$, and in particular is Gaussian, the difference being that $\tilde{y}_t$ is fully observable. Assuming moreover that the $R_{1,t}$ are observed with Gaussian errors we get, with obvious notations:

$$\begin{aligned} R_{1,t} &= A_1 z_t + B_{11} y_{1,t} + B_{12} y_{2,t} + \xi_t \\ &= A_1 z_t + B_{11} y_{1,t} \\ &\quad + B_{12} B_{22}^{-1}(R_{2t} - A_2 z_t - B_{21} y_{1,t}) + \xi_t, \end{aligned} \tag{25}$$

with $\xi_t \sim IIN\left(0, \sigma^2 I\right)$.

Putting equations (24),(25) and (1) together, we have a dynamic model in which the only latent variables are $z_{2,t}$ and which can be estimated by the maximum likelihood methods using Hamilton's approach (see Appendix F).[25] At this stage, IC constraints on (θ^{zy}, θ^*) must be taken into account.

[25] Note that this algorithm can handle time-varying transition probabilities (which is required in the case where the π_{ij}'s depend on $y_{1,t-1}$).

6.4. Estimation of $\left(\theta_n^x, \theta_n^d\right)$

From the inversion method of 6.3, we can get approximations of the $y_{2,t}$'s.[26] Then using equation (20), we get:

$$R_{t,n}^D = C_1^n z_{1,t} + C_2^n z_{2,t} + D_1^n y_{1,t} + D_2^n y_{2,t} + F_1^n x_{1,n,t} + F_2^n x_{2,n,t}. \quad (26)$$

and using equations (2), (3) and (26) and replacing $y_{2,t}$ and $z_{2,t}$ by their approximations, we get a system in which the only latent variables are the $x_{2,n,t}$. Taking θ^{zy} and θ^* as given, the parameters θ_n^x and θ_n^d can be estimated either by an inversion technique or by Kalman filtering, taking into account IC conditions.

Note that in this strategy, the observable variables $d_{n,t}$'s have not been used. If the recovery rate was effectively zero, $\lambda_{n,t}$ would be the default intensity and the conditonal p.d.f. of $d_{n,t}$ given $\underline{z}_t, \underline{y}_t, \underline{x}_{n,t}, d_{n,t-1}$ would be:

$$d_{n,t} d_{n,t-1} + (1 - d_{n,t-1}) \exp\left[-\left(1 - d_{n,t-1}\right) \lambda_{n,t}\right] \times \left[1 - \exp\left(-\lambda_{n,t}\right)\right]^{d_{n,t}}.$$

This p.d.f. could be incorporated in the likelihood function. However, in the more realistic case of non-zero recovery rate, we have seen that (see Subsection 4.3) the $\lambda_{n,t}$'s must be interpreted as risk-neutral "recovery adjusted" default intensities and, therefore, they cannot be used for describing the historical dynamics of the $d_{n,t}$'s.

6.5. Possible adaptations of the estimation strategy

Mainly for the sake of presentation clarity, the first step of the sequential strategy presented above involves only observations of macroeconomic factors and riskless yields. In particular, no credit-spread data are used in the estimation of θ^*, the parameters appearing in the s.d.f. $M_{t-1,t}$ as well as in the estimation of the unobserved factors $y_{2,t}$ and of the unobserved regimes $z_{2,t}$. However, spread data may contain useful information for the estimation of θ^{zy} and of θ^*. In that case, the strategy should be adapted in order to include credit-spread data in the first step of the estimation. It can be seen that the main lines of the estimation strategy are not affected when the vector R_t and $y_{1,t}$ considered in the first step are respectively augmented with observed defaultable-bond yields and with observable specific factor $x_{1,n,t}$ (that are associated with the additional yields).[27]

Another adaptation of the strategy would be the following. The first step presented above implies a nesting of recursive computations of the theoretical formulas giving riskless (or risky) rates and recursive computation of the Kitagawa-Hamilton algorithms, which could be time-consuming. In order to alleviate the computational cost it is possible, for instance, to estimate first system (24) –or an analogue system including risky rates– with unconstrained parameters, using standard Kitagawa-Hamilton filter, and then to compute smoothed estimates values of the z_t's. The latter values of z_t would further be considered as observations and the remaining steps would estimate all the parameters (except the ones appearing in the $\pi_{ij,t}$'s) using either inversion techniques or the Kalman filter.

[26] Note that in the inversion method, the z_{2t} are replaced by those states presenting the highest smoothed probabilities.

[27] Naturally, the dimension of R_{2t} should still be equal to the number of unobserved macro-factors y_{2t}.

7. Liquidity risk

There is compelling evidence that yields and spreads contain components that are closely linked to liquidity.[28] In addition, empirical evidence points to the existence of commonality amongst the liquidity components of prices of different bonds (see e.g. Fontaine and Garcia, 2009 [72]).

The estimation of the liquidity premium is of concern for several reasons. For instance, gauging the liquidity-risk premium provides policy makers –central bankers in particular– with insights on the valuation of liquidity by the markets (see Taylor and Williams, 2008 [126], Wu, 2008 [132] or Michaud and Upper, 2008 [112]). Furthermore, if one wants to extract default probabilities from market data, one has to distinguish between what is related to default and what is caused by the liquidity of the considered instruments (see, e.g., Longstaff, Mithal and Neis, 2005 [103] or Bühler and Trapp 2008 [27]).

However, the identification of the liquidity premium, that is, distinguishing between the default-related and the liquidity-related components of yield spreads, remains a challenging task. Basically, the identification of the liquidity component relies on the ability to exhibit risk factors that reflects liquidity valuation. In Liu, Longstaff and Mandell (2006) [101] or Feldhütter and Lando (2008) [69], the liquidity factor is latent and the identification is based on assumptions regarding the relative liquidity of different interest-rate instruments.[29] Alternatively, as mentioned in Subection 1.1, the liquidity factor could be proxied by some observable factors.[30] However, according to Wang (2009) [129], usual liquidity proxies are able to explain only a minor part of the liquidity component. One may resort to intermediate –or mixed– approach, where part of the liquidity-factor dynamics is observable (through observed proxies) and part of it is latent.

Let us come back to our modeling framework. We have seen in section 4 that incorporating default risk in the pricing methodology implies to replace the short rate r_{t+1} by a "default-adjusted" short-rate $r_{t+1} + \lambda_{n,t+1}$. Besides, in order to take into account recovery-rate effects, $\lambda_{n,t+1}$ can be seen as a "recovery adjusted" default intensity between t and $t+1$ (see Appendix D). So the price at t of a defaultable asset providing the payoff $g\left(\underline{w}_{t+h}\right)$ at $t+h$, in case of absence of default, is:

$$E_t^{\mathbb{Q}}\left[\exp\left(-r_{t+1} - \lambda_{n,t+1} - \ldots - r_{t+h} - \lambda_{n,t+h}\right) g\left(\underline{w}_{t+h}\right)\right].$$

As suggested by Duffie and Singleton (1999) [60], intensity-based model can also account for liquidity effects by introducing a stochastic process that is interpreted as the carrying cost of non-liquid defaultable securities. This process then appears alongside the default intensity in the spread between the "pure" –i.e. default and liquidity-adjusted– short rate and the short rate associated with a defaultable bond. Accordingly, let us introduce an

[28] The influence of liquidity effects on bond pricing has been investigated, amongst others, by Longstaff (2004) [102], Jong and Driessen (2007) [53], Van Landschoot (2004) [100], Chen, Lesmond and Wei (2007) [32], Covitz and Downing (2007) [41], Acharya and Pedersen (2005) [1] or Eisenschmidt and Tapking (2009) [62].

[29] In both studies, the liquidity factor that is estimated corresponds to the so-called "convenience yield", that can be seen as a premium that one is willing to pay when holding Treasuries. This premium stems from various features of Treasury securities, such as repo specialness (see Feldhütter and Lando, 2008).

[30] Amongst the many liquidity proxies idnetified in the literature stand: bid-ask spreads, market-depth measures, bond supply, spread between bonds of the same maturity but with different ages or spread between off-the-run and on-the -run Treasuries (see Longstaff, 2004[102], Beber, Brandt and Kavajecz, 2009 [16], Fontaine and Garcia, 2009 [72] or Wang, 2009 [129]). More generally, for credit spread determinants, see e.g. Duffie and Singleton (1997) [59], Elton (2001) [64], Collin-Dufresne, Goldstein and Martin (2001) [39], Elton et al. (2004) [65], Covitz and Downing (2007) [41] and Davies (2008) [50].

"illiquidty intensity" between t and $t+1$, denoted with $\lambda^L_{n,t+1}$. If $\lambda_{n,t+1}$ and $\lambda^L_{n,t+1}$ are specified in an affine way,

$$\begin{cases} \lambda_{n,t+1} & = \alpha_n^{'} z_{t+1} + \beta_n^{'} y_{t+1} + \gamma_n^{'} x_{n,t+1} \\ \lambda^L_{n,t+1} & = \alpha_n^{L'} z_{t+1} + \beta_n^{L'} y_{t+1} + \gamma_n^{L'} x_{n,t+1}, \end{cases}$$

we could price not only riskless bonds $B_n(t,h)$ and defaultable bonds $B_n^D(t,h)$ as above, but also bonds facing liquidity risk $B_n^L(t,h)$ and bonds facing both default and liquidity risk $B_n^{DL}(t,h)$. We would have:

$$\begin{cases} B(t,h) & = E_t^{\mathbb{Q}}\left[\exp\left(-r_{t+1} - \ldots - r_{t+h}\right)\right] \\ B_n^D(t,h) & = E_t^{\mathbb{Q}}\left[\exp\left(-r_{t+1} - \lambda_{n,t+1} - \ldots - r_{t+h} - \lambda_{n,t+h}\right)\right] \\ B_n^L(t,h) & = E_t^{\mathbb{Q}}\left[\exp\left(-r_{t+1} - \lambda^L_{n,t+1} - \ldots - r_{t+h} - \lambda^L_{n,t+h}\right)\right] \\ B_n^{DL}(t,h) & = E_t^{\mathbb{Q}}\left[\exp\left(-r_{t+1} - \lambda_{n,t+1} - \lambda^L_{n,t+1} - \ldots - r_{t+h} - \lambda_{n,t+h} - \lambda^L_{n,t+h}\right)\right]. \end{cases}$$

In the context of a Car(1) risk-neutral dynamics of $(z_t, y_t, x_{n,t})$, these prices are exponential linear in $(z_t, y_t, x_{n,t})$ and the corresponding yields are linear in $(z_t, y_t, x_{n,t})$.

If the obligors issue only bonds facing both default and liquidity risks, and if the same factors affect both kinds of intensities, it is not possible to distinguish between the two of them. In order to operate –or to gain some insights on– a decomposition between the default intensity on the one hand and the liquidity intensity on the other, one has to rely on additional assumptions. For instance, these assumptions may reflect some priors about the relative effects of the risk factors on the different obligors. This is illustrated in the application (Section 9).

8. Model extensions

8.1. Multi-lag dynamics for y_t and $x_{n,t}$ processes

The model can easily be extended to allow for y_t and $x_{n,t}$ dynamics that include several lags. In particular, when observed data are used in the estimation process –the $y_{1,t}$ and $x_{1,n,t}$ defined in Section 6–, preliminary analysis of the data could point to the need of taking different lags into account to model the historical dynamics of these variables. The flexibility in the choice of the lag structure constitutes an advantage of working in discrete-time over most continuous-time models (see, e.g., Monfort and Pegoraro, 2007 [113] or Gourieroux, Monfort and Polimenis, 2006 [78]).

Equations (2) and (3) imply that the multivariate factors y_t and x_t follow auto-regressive process of order one. However, to the extent that a VAR(p) amounts to a VAR(1) once the last p lags of the endogenous variable are stacked in the same vector, the pricing techniques of the bonds –namely equations (16) and (20)– are not affected if y_t and x_t follow VAR(p). However, in order to make the estimation strategy presented in Section 6 still effective –in particular regarding inversion techniques–, the unobserved vector variables $y_{2,t}$ and $x_{2,n,t}$ should not enter equations (2) and (3) with lags larger than one. To the extent that this restriction only applies to the unobserved factors –for which insights on the appropriate distributions are a priori not readily available– such a constraint is not really restrictive.

8.2. Interpretation of a regime as the default state of an entity

In this subsection, we consider the specific case where one the Markov chain included in z_t corresponds to the default state of a given entity.[31] The specificity of that situation lies in the fact that the default of this entity then enters the s.d.f.. Therefore, we leave the framework described in Subsection 3.1 where all defaultable entities were small enough not to have any impact at the macroeconomic level. As a consequence, the "zero" entity may represent a whole industry or a very big institution. This could be extended to a few major entities but one has to bear in mind that increasing their number results in an exponential growth in the dimension of z_t.

The fact that this default enter the s.d.f. results in a new component in bond prices: a compensation for investors risk-aversion towards the default event of entity zero. As pointed out by Yu (2002) [133] and Jarrow, Lando and Yu (2005) [91], such components arise only when the default-event risk is not diversifiable. [32]

In addition, as mentioned in introduction (Subsection 1.2.3), this interpretation is also linked with previous studies attempting to introduce contagion effects in affine term-structure models. Indeed, the default of entity zero may lead to a simultaneous increase in the default intensities of any other debtor.

For sake of simplicity, let us assume that such a crisis variable is the only regime captured by z_t, which can be observable or not. In this case, assuming that the state $e_2 = (0,1)'$ is the crisis state, we have:

$$\begin{aligned} \pi\left(e_2 \mid e_2, y_{t-1}\right) &= 1 \\ \pi\left(e_1 \mid e_2, y_{t-1}\right) &= 0. \end{aligned}$$

Moreover, we could specify:

$$\pi\left(e_1 \mid e_1, y_{t-1}\right) = \exp\left(-\lambda_{0,t-1}\right),$$

with $\lambda_{0,t-1} = \alpha_0 + \beta_0' y_{1,t-1}$. In this case, $\lambda_{0,t-1}$ can be interpreted as a systemic-risk intensity. Conditions (7) and (10) $\{\pi\left(e_j \mid e_i, y_{t-1}\right) \exp\left[\delta_j\left(e_i, y_{t-1}\right)\right] = \pi^*_{ij}\}$ imply the followings:

- $\pi^*_{21} = 0$, $\pi^*_{22}=1$, $\delta_1\left(e_2, y_{t-1}\right)$ is undefined, $\delta_2\left(e_2, y_{t-1}\right) = 0$ and, therefore, $\delta'\left(e_2, y_{t-1}\right) z_t = 0$.
- $\exp\left[\delta_1\left(e_1, y_{t-1}\right)\right] = \pi^*_{11} \exp\left(\lambda_{0,t-1}\right)$ or $\delta_1\left(e_1, y_{t-1}\right) = \log \pi^*_{11} + \alpha_0 + \beta_0' y_{t-1}$.
- $\exp\left[\delta_2\left(e_1, y_{t-1}\right)\right] = \left(1 - \pi^*_{11}\right)\left[1 - \exp\left(-\lambda_{0,t-1}\right)\right]^{-1}$, or $\delta_2\left(e_1, y_{t-1}\right) = \log\left(1 - \pi^*_{11}\right) - \log\left[1 - \exp\left(-\alpha_0 - \beta_0' y_{t-1}\right)\right]$.

Denoting $\pi^*_{11} = \exp\left(-\lambda^*_0\right)$, λ^*_0 being the systemic-risk intensity in the risk-neutral world, we get:

$$\begin{aligned} \delta_1\left(e_1, y_{t-1}\right) &= \lambda_{0,t-1} - \lambda^*_0 \\ \delta_2\left(e_1, y_{t-1}\right) &= \log\left[1 - \exp\left(-\lambda^*_0\right)\right] - \log\left[1 - \exp\left(-\lambda_{0,t-1}\right)\right] \\ &\simeq \log\left(\lambda^*_0\right) - \log\left(\lambda_{0,t-1}\right) \text{ if } \lambda^*_0, \lambda_{0,t-1} \text{ are small.} \end{aligned}$$

In particular, the risk-neutral intensity λ^*_0 and the historical intensity$\lambda_{0,t-1}$ are different functions, contrary to what happended in the previous sections. Both the riskless yields:

$$R\left(t,h\right) = \frac{1}{h}\left(a_h' z_t + b_h' y_t\right)$$

[31] We assume here that the vector z_t is a Kronecker product of several Markov chains.

[32] Using bond price data for 104 U.S. firms and historical default rates, Driessen (2005) [55] was not able to estimate this kind of default-event risk premia with significant statistical precision.

and the defaultable yields:

$$R_n^D(t,h) = \frac{1}{h}\left(c'_{n,h}z_t + f'_{n,h}y_t + g'_{n,h}x_{n,t}\right)$$

will be different functions of y_t (and of x_{nt} for $R_n^D(t,h)$) before and after the systemic crisis. The term structure of the impact of the systemic crisis will be:

$$\begin{cases} a_{2,h} - a_{1,h} & \text{for the riskless yield of residual maturity } h, \\ c_{2,n,h} - c_{1,n,h} & \text{for the defaultable yield of residual maturity } h. \end{cases}$$

8.3. modeling credit-rating transitions

Subsection 1.3 discusses why it may be desirable to model credit-rating migration and provides a brief review of the literature dealing with rating-migration modeling in term-structure frameworks. In the present subsection, we show how our framework can be adapted in order to account explicitly for rating migration. The structure, building on Lando's (1998) [98] approach (see also Feldhütter and Lando, 2008 [69]), accomodates a time-varying rating-migration matrix while allowing different ratings to respond in a correlated yet different fashion to the same change in the general economic conditions.

While most of the previous framework is still valid, some changes regard the modeling of the default intensity. Specifically, the historical dynamics of $(z_t, y_t, x_{n,t})$, as well as the s.d.f. specifications are still given by equations (1), (2), (3) and (6). However, in this adapted framework, each firm n is also characterized by a credit-rating process, denoted by $\tau_{n,t}$. For any firm n and period t, $\tau_{n,t}$ can take one of K values: the first $K-1$ values correspond to credit ratings and the K^{th} correspond to the default state.[33] Like the $d_{n,t}$'s, the $\tau_{n,t}$'s, $n = 1, \dots, N$, are independent conditionally to$(z_t, y_t, x_t, \underline{w}_{t-1})$. In addition, we assume that the rating transition probabilities, for firm n and from period $t-1$ to period t, is a function of $(z_t, y_t, x_{n,t})$. Accordingly, this transition matrix is denoted with $\Pi(z_t, y_t, x_{n,t})$ and we have:

$$P(\tau_{n,t} = j \mid \tau_{n,t-1} = i) = \Pi_{i,j}(z_t, y_t, x_{n,t}),$$

where $\Pi_{i,j}(z_t, y_t, x_{n,t})$, the (i,j) entry of $\Pi(z_t, y_t, x_{n,t})$, represents the actual probability of going from state i to state j in one time step. Each of these entries must be in $[0,1]$ and for each line, the sum of the entries must sum to one. In other words, $\left[\begin{array}{ccc} 1 & \cdots & 1 \end{array}\right]'$ is an eigenvector of $\Pi(z_t, y_t, x_{n,t})$ associated with the eigenvalue 1. In addition, the default state being absorbing, the bottom row of $\Pi(z_t, y_t, x_{n,t})$ is equal to $\left[\begin{array}{cccc} 0 & \cdots & 0 & 1 \end{array}\right]$.

In this context, a defaultable zero-coupon bond providing one money unit at $t+h$ if entity n is still alive in $t+h$ and zero otherwise has a price, in period t, that is given by (assuming that entity n has not defaulted before t):

$$B_n^D(t,h) = E_t^{\mathbb{Q}}\left[\exp\left(-r_{t+1} - \ldots - r_{t+h}\right)\mathbb{I}_{\left\{\tau_{n,t+h} < K\right\}}\right]. \tag{27}$$

In order to keep a quasi-explicit formula for defaultable zero-coupon bonds, we assume that $\Pi(z_t, y_t, x_{n,t})$ admits the diagonal representation:

$$\Pi(z_t, y_t, x_{n,t}) = V.\Psi(z_t, y_t, x_{n,t}).V^{-1},$$

where the columns of V are the eigenvectors of $\Pi(z_t, y_t, x_{n,t})$ and constitute a basis in $\mathbb{R}^K$ and $\Psi(z_t, y_t, x_{n,t})$ is a diagonal matrix of eigenvalues that are positive and smaller than

[33] For instance, rating 1 can be the highest (Aaa in Moody's rankings) and $K-1$ can be the lowest (C in Moody's rankings). In addition, we have, $d_{n,t} = \mathbb{I}\left(\tau_{n,t+h} = K\right)$.

one.[34] Given that 1 is an eigenvalue of $\Pi(z_t, y_t, x_{n,t})$, $\Psi(z_t, y_t, x_{n,t})$ can be written in the following manner:

$$\Psi(z_t, y_t, x_{n,t}) = \begin{bmatrix} \exp\left[-\psi_1\left(w_t\right)\right] & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \exp\left[-\psi_{K-1}\left(w_t\right)\right] & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix},$$

with, for any $i < K$, $\psi_i\left(w_t\right) \geq 0$. Then, it is easily seen that, conditionally on $\left(\underline{z}_{t+h}, \underline{y}_{t+h}, \underline{x}_{n,t+h}, \tau_{n,t} = i\right)$ the probability of defaulting before $t+h$ corresponds to the entry (i, K) of the matrix that is given by:

$$V.\Psi(z_{t+1}, y_{t+1}, x_{n,t+1}) \dots \Psi(z_{t+h}, y_{t+h}, x_{n,t+h}).V^{-1}.$$

This probability is therefore given by:

$$P(\tau_{n,t+h} = K \mid \underline{z}_{t+h}, \underline{y}_{t+h}, \underline{x}_{n,t+h}, \tau_{n,t} = i) = \sum_{j=1}^{K} V_{i,j} V^{-1}_{j,K} \exp\left[-\sum_{p=1}^{h} \psi_j\left(w_{t+p}\right)\right],$$

where $V_{i,j}$ and $V^{-1}_{i,j}$ are the entries (i, j) of , respectively,V and V^{-1}. Since $V_{i,K} V^{-1}_{K,K} = 1$ (see Appendix G) using $\psi_K \equiv 0$, we get:

$$P(\tau_{n,t+h} < K \mid \underline{z}_{t+h}, \underline{y}_{t+h}, \underline{x}_{n,t+h}, \tau_{n,t} = i) = -\sum_{j=1}^{K-1} V_{i,j} V^{-1}_{j,K} \exp\left[-\sum_{p=1}^{h} \psi_j\left(w_{t+p}\right)\right]. \quad (28)$$

If the eigenvalues ψ_j are some linear combinations of $(z_t, y_t, x_{n,t})$, Equations (27) and (28) implies that the price of a bond is a sum of $K-1$ multi-horizon Laplace tranforms. As a consequence, the bond prices can be obtained using the algorithm presented in Appendix B. However, it should be noted that in this context, the prices are no longer exponential affine in the factors, which implies in particular that the Kalman filter has to be adapted so as to accomodate the nonlinearity of the state-space measurement equations. In such a context, Feldhütter and Lando (2008) [69] use the extended Kalman filter. As an alternative, the unscented Kalman filter could be implemented (see e.g. Christoffersen et al., 2009 [35] for an application of the unscented Kalman filter on yields data).

9. Application

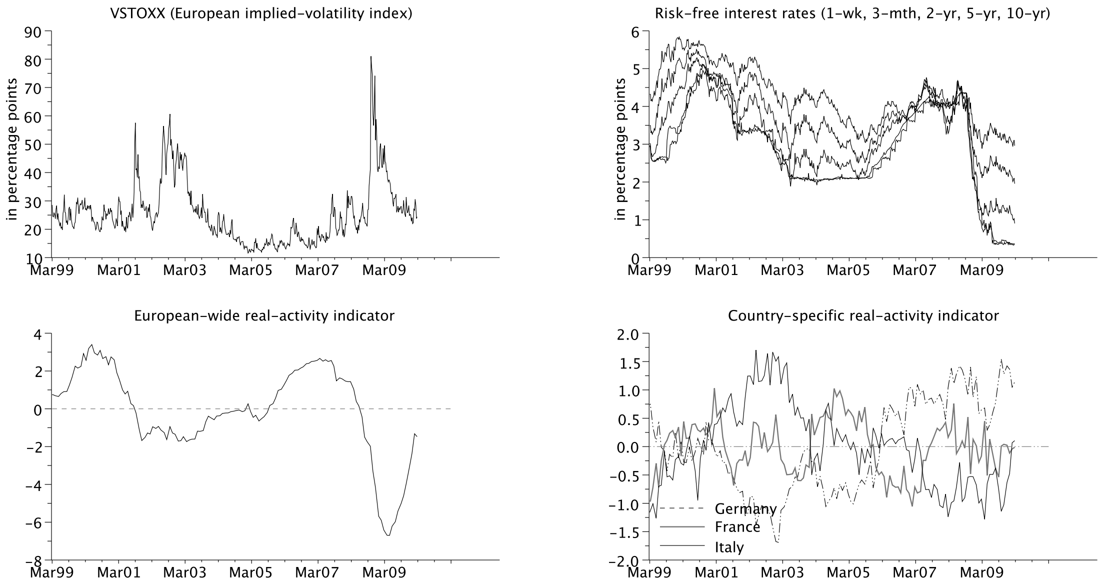
In this section, building on the framework described above, we investigate the joint dynamics of euro-area sovereign yield curves. We consider 10 euro-area countries: Austria, Belgium, Finland, France, Germany, Greece, Italy, the Netherlands, Portugal and Spain.[35] All these countries are considered as "risky" debtors, in the sense that each country presents a hazard rate $\lambda_{n,t}$. This hazard rate is not a pure default rate since it is adjusted to (a) the presence of a non-zero recovery rate (see Subsection 4.3) and to (b) liquidity effects (see Section

[34]The fact that the eigenvalues have a modulus smaller than one is necessary in the case of time-homogenous Markov chain processes.

[35]Ireland is not part of the sample given the non-availability of data to construct a business-cycle indicator: the European-Commission qualitative survey data used to construct these indicators (see Subsection 9.1) stop in 2008 for Ireland.

Figure 2: **Estimation data**

Notes: The upper-right plot shows the zero-coupon risk-free rates: the yields include, from the lowest to the highest (with occasional cross-overs when the yield curve is inverted): 1-week, 3-month, 2-year, 5-year and 10-year zero-coupon yields (the zero-coupon yield curve is bootstrapped from the swap yield curve). The euro-area business-cycle indicator (in the lower left chart) is the first principal component of a set of six series (the year-on-year growth rate of the industrial production and 5 European Commission short-term qualitative surveys). The lower-right plot shows the $x_{BC,n,t}$, for selected countries i.e. the parts of the country-specific business-cycle indicators that are orthogonal to the eurozone one (obtained by OLS regressions of the countries' business-cycle indicators on the euro-area one).



7). The model includes three regimes and three latent factors. Whereas one latent factor intervenes in the dynamics of the risk-free yields, two affect the hazard rates only. Among the latter, one is expected to reflect liquidity pricing, whereas the other is assumed to be default-related. The identifiation of the liquidity-related factor is based on the use of bond prices issued by KfW, a German agency. The bonds issued by KfW are less liquid than the sovereign counterparts, but present the same credit quality for they are guaranteed by the Federal Republic of Germany: accordingly, the bonds issued by KfW and those issed by the German government are equally exposed to the default factor but not not to the liquidity-related factor.

9.1. Data

Subection 9.1.1 briefly presents the data used in the estimation. The next subsection (9.1.2) discusses our choices regarding the risk-free yield curve. The following (Subsection 9.1.3) introduces the KfW-*Bund* spread that we will exploit to identify the liquidity-related latent factor. Then, in 9.1.4, we provide a preliminary analysis of euro-area yield differentials.

9.1.1. Overview

The data are weekly and cover the period from February 1999 through February 2010.

The vector y_t contains three observable factors: a business-cycle indicator, a volatility index and the short-term (one-week) risk-free yield. The business-cycle indicator is represented by the first principal component of a set of 6 series: 5 business-confidence indicators corresponding to quanta of European Commission short-term qualitative surveys (indus-

Table 1: **Descriptive statistics of observed macroeconomic factors and selected yields**

Notes: The table reports summary statistics for different variables that are used in the estimation. The data are weekly and cover the period from February 1999 to February 2010. Two auto-correlations are shown (the 1-week and the 1-year auto-correlations). EA bus.-cycle refers to the euro-area business-cycle indicator based on European-Commission short-term qualitative surveys and industrial production data (the indicator is the first principal component of six series). The yields are continuously compounded and are in percentage annual terms. The lower panel of the table presents the covariances and the correlations (in italics) of the series.

	VSTOXX	EA bus.-cycle	Risk-free yds			German yds		Italian yds		Portuguese yds	
			1-week	2-year	10-year	2-year	10-year	2-year	10-year	2-year	10-year
Mean	25.6	-0.089	2.928	3.229	4.306	3.194	4.251	3.341	4.569	3.351	4.476
Median	23.86	-0.078	2.985	3.357	4.232	3.307	4.205	3.417	4.474	3.404	4.41
Standard deviation	10.849	2.326	1.176	1.055	0.762	1.003	0.635	0.962	0.62	0.941	0.624
Skewness	1.637	-0.973	-0.358	-0.241	0.199	-0.182	0.177	0.011	0.177	0.042	0.16
Kurtosis	6.608	3.76	2.612	2.259	2.103	2.158	2.022	2.024	2.247	2.142	2.319
Auto-correlation (order 1)	0.945	0.999	0.997	0.995	0.992	0.993	0.988	0.995	0.989	0.99	0.984
Auto-correlation (order 52)	0.226	0.245	0.182	0.306	0.573	0.319	0.578	0.326	0.6	0.337	0.588
Covariances / *Correlations*											
VSTOXX	117.508	-12.429	0.045	-1.734	0.688	-1.484	0.784	-0.414	1.966	-0.127	1.918
Euro-area bus.-cycle	*-0.493*	5.401	1.835	1.897	0.964	1.751	0.663	1.512	0.324	1.461	0.315
Risk-free 1-week yield	*0.004*	*0.672*	1.38	1.134	0.642	1.076	0.469	1.026	0.401	1.016	0.362
Risk-free 2-year yield	*-0.152*	*0.774*	*0.916*	1.111	0.67	1.053	0.509	0.991	0.431	0.963	0.405
Risk-free 10-year yield	*0.083*	*0.545*	*0.718*	*0.835*	0.579	0.638	0.475	0.609	0.426	0.6	0.424
German 2-year yield	*-0.137*	*0.752*	*0.914*	*0.998*	*0.836*	1.004	0.486	0.946	0.42	0.919	0.394
German 10-year yield	*0.114*	*0.449*	*0.629*	*0.761*	*0.984*	*0.765*	0.403	0.464	0.36	0.459	0.365
Italian 2-year yield	*-0.04*	*0.677*	*0.909*	*0.978*	*0.833*	*0.983*	*0.76*	0.923	0.438	0.895	0.407
Italian 10-year yield	*0.293*	*0.225*	*0.55*	*0.66*	*0.903*	*0.676*	*0.916*	*0.735*	0.384	0.431	0.379
Portuguese 2-year yield	*-0.013*	*0.669*	*0.92*	*0.972*	*0.838*	*0.976*	*0.769*	*0.991*	*0.739*	0.884	0.404
Portuguese 10-year yield	*0.284*	*0.217*	*0.494*	*0.616*	*0.893*	*0.63*	*0.923*	*0.68*	*0.98*	*0.688*	0.389

trial confidence, construction confidence, retail trade confidence, service confidence and consumer confidence) and the year-on-year growth rate of industrial production.[36] These monthly data are seasonally adjusted and converted into weekly data using a simple linear interpolation.[37] Our measure of perceived european-market volatility is the VSTOXX, which can be seen as an European analogue to the American VIX. Specifically, the VSTOXX index is constructed using implied option prices written on the DJ Euro STOXX 50 index.

At the country level, we also compute business-cycle indicators, using the same methodology as for the euro-area as a whole. The first principal components explain between 59% (Italy) and 86% (Spain) of the variances of the six variables considered for each country.

Sovereign zero-coupon yield curves are based on government-bond prices that are extracted from Datastream. Since governments issue essentially coupon bonds, we first convert observed bond quotes into zero-coupon yields. For each period and each country, we use a parametric form –Svensson or Nelson-Siegel– that minimizes the squared deviations between observed prices and modeled prices. More details about the methodology are given in Appendix H.

Zero-coupon risk-free yields are bootstrapped from swap rates (see next subsection for a discussion about the original swap yields that are used). All yields used in the estimation are continuously compounded.

Figure 2 shows some of the data used in the estimation and Table 1 reports descriptive statistics for some of the data.

9.1.2. The risk-free yield curve

What is a relevant proxy for euro-area risk-free yields over the last decade? A first solution consists in chosing a reference country, say Germany, and then in considering that its associated default intensity is null. However, this arbitrary choice would be debatable and would notably prevents us from modeling credit and liquidity premium for the reference country. Following, amongst others, Grinblatt (2001) [80], Blanco, Brennan and Marsh (2005) [23] or McCauley (2002) [110], we resort to an alternative approach that consists in proxying risk-free rates with swap rates. One might object that a swap rate is a derivative product whose payments are based on yields faced by banks, and therefore include a credit-risk component.[38] However, it has to be noted that the maturity of the underlying floating yield is short. Therefore, the swap rate reflects only the future *refreshed* default probabilities of prime banks over short horizon (see, e.g., Sun, Sundaresan, and Wang, 1993 [124] or Collin-Dufresne and Solnik, 2001 [40]). Heuristically, the shorter the maturity, the smaller the credit-risk component is expected to be. This is illustrated by Feldhütter and Lando (2008) [69] who find an average spread between the 3-month LIBOR rate and an estimated riskless yield of 3 basis points.[39]

Nevertheless, the financial crisis initiated in 2007 has shown that, in extreme cases, even short-term interbank lending is not risk-free: as soon as 2007, significant credit-risk premia emerged in interbank rates, even for short-term horizons of 6 months or even 3 months (see

[36]The first principal component explains 82% of the variances of the variables.

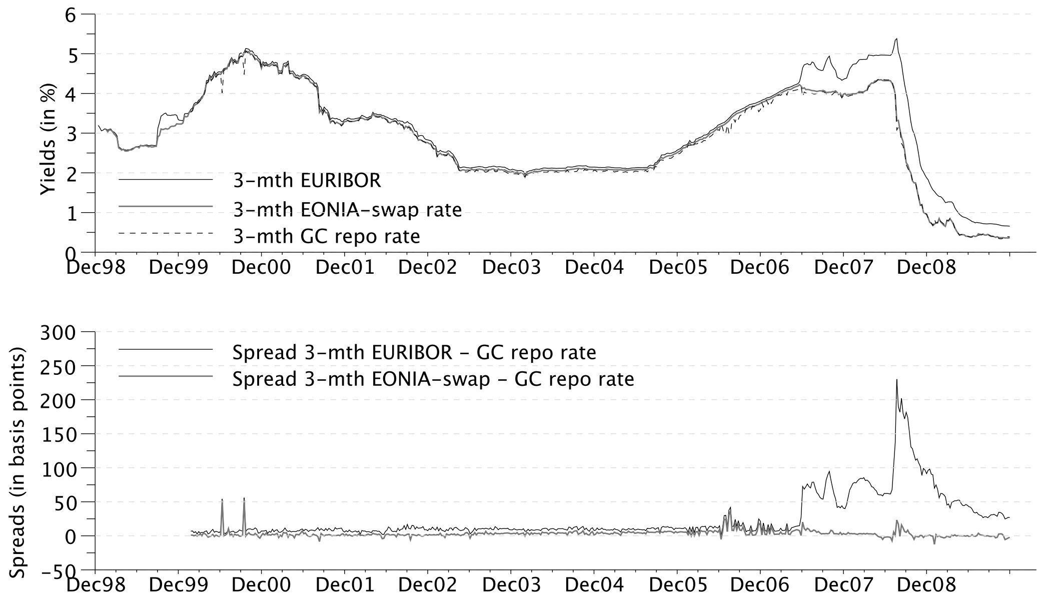
[37]Seasonal adjustements are carried out using the Census-X12 procedure.

[38]Note that we do not consider the swap counterparty risk. Indeed, this risk is very limited in the case of a swap contract (see, e.g., Bomfim, 2003 [24] who shows that even during turmoils in the financial markets, the swap-contract counterparty risk remains very small due to netting and credit enhancement mechanisms, including call margins).

[39]Feldhütter and Lando use U.S. data, for the period from 1996 to 2005. The riskless short rate is unobservable: the spread between this rate and the Treasury short-term yield corresponds to a convenience yield (from holding Treasuries). The Kalman filter is used to estimate the six factors of their model.

Figure 3: **Money market rates**

Notes: The upper plots shows tthree different money-market interest rates: the 3-month EURIBOR, the 3-month EONIA swap rate and the general-collateral (GC) repo rate. The lower plot displays the differentials between the first two rates and the GC repo rate. The data are collected from Bloomberg.



e.g. Taylor and Williams, 2008 [126]). Therefore, using EURIBOR swap rates as risk-free rates over the last three years would be misleading. Also, for the more recent period, we use EONIA (euro overnight index average) swaps instead of EURIBOR swaps to derive the risk-free yield curve.[40] Since the floating-rate payment of an EONIA swap rate is based on overnight rates, the credit-risk component is far lower than for several-month IBOR rates in periods of financial-market stress. As an alternative, we could use repo rates as a measure of the riskless rates (as in Longstaff, 2000 [102] or Liu, Longstaff and Mandell, 2006 [101] or Eisenschmidt and Tapking, 2009 [62]). Indeed, insofar as repo loans are overcollaterlalized by default-free Treasury bonds, they can be considered as riskless loans. However, a drawback of this approach is that only short-term maturities are available for the risk-free yield curve.

Figure 3 reports some of the yields that have just been mentionned. First, one can observe that, contrary to EURIBOR rates, EONIA swap rates have remained very close to the repo rate over the last couple of years. This suggests that using repo rates or EONIA swap rates is not really different (but we choose EONIA-swaps because longer maturities are available for this latter instrument). When long-term EONIA swaps are not available (before 2005), we use EURIBOR swap rates. As is shown in Figure 3, over this period, EURIBOR was very close to the repo rate, which validates this replacement over this first period.

[40]It is not possible to use Eonia swap rates over the whole period since these rates are only available from 2005 onwards for longer maturities. For an in-depth presentation of EONIA swaps, see Barclays (2008) [13]. For a comparison between Euribor and Eonia swaps, see e.g. Pilegaard, Durre and Evjen (2003) [121].

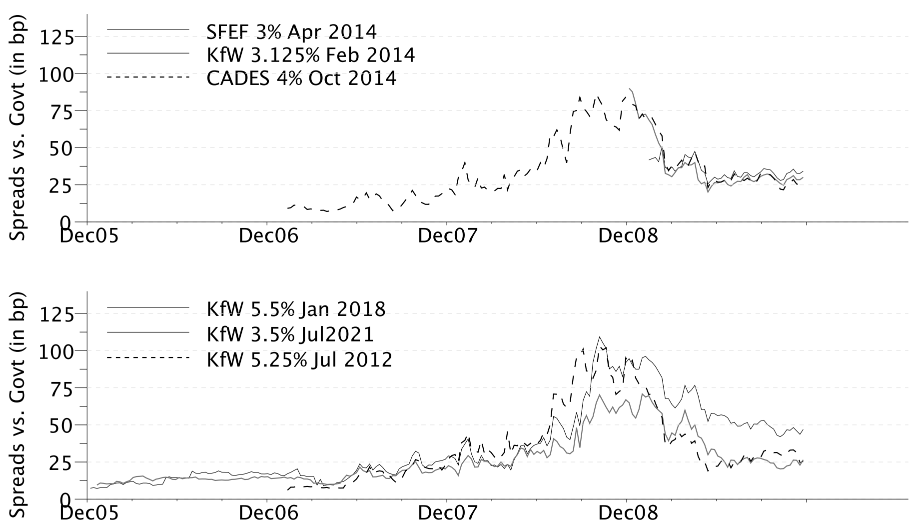
9.1.3. The KfW-*Bund* spread

Our identification of a liquidity-related latent factor is partly based on the yield spread between German federal bonds and KfW agency bonds (see the lower plot of Figure 4). The latter are less liquid than the sovereign counterparts, but are explicitly and fully guaranteed against default by the German federal government.[41] Consequently, the spread between these two kinds of bonds can be seen as a measure of the German government bond-market liquidity premium demanded by investors. In the same spirit, Longstaff (2004) [102] computes liquidity premia based on the spread between U.S. Treasuries and bonds issued by Refcorp, that are guaranteed by the Treasury.

In order to check that this liquidity-pricing measure is not purely specific to Germany, we can look at comparable spreads in alternative countries. To that respect, let us consider two debtors whose issuances are guaranteed by the French government, namely the CADES (Caisse d'amortissement de la dette sociale) and the SFEF (Société de financement de l'économie française).[42] The upper plot of Figure 4 shows that, over the recent period –when the French spreads are available–, the KfW-*Bund* spread shares most of its fluctuations with to the spread between SFEF bonds and French Treasury bonds (OATs), as well as with the CADES-OAT spread.

Figure 4: **Differentials between government and government-guaranted bonds**

Notes: The upper plot shows the yield differentials between European bonds that are government guaranteed (SFEF and CADES by the French government; KfW by the German government) and government bonds of approximately the same maturity (French government bonds for SFEF and CADES, German bonds for KfW). The coupons and maturities of the bonds are reported in the legends. The lower plot displays yield differentials between KfW and German-government bonds at different maturities. The yields are provided by Barclays capital.



[41] An understanding between the European Commission and the German Federal Ministry of Finance (1 March 2002) stated that the guarantee of the Federal Republic of Germany will continue to be available to KfW. The three main rating agencies –Fitch, Standard and Poor's and Moody's– have assigned a triple-A rating to KfW (see KfW website www.kfw.de/EN_Home/Investor_Relations/Rating.jsp). In addition, as the German federal bonds, KfW's bonds are zero-weighted under the Basle capital rules. The relevance of the KfW-*Bund* spread as a liquidity proxy is also pointed out by McCauley (1999) [109] and exploited by Schwarz (2009) [123].

[42] Note that contrary to the ones issued by the later, those issued by the former (CADES) do not benefit from the explicit –but only implicit– guarantee from the French government. However, both issuer are triple-A rated, as the French government.

Table 2: **Correlations and principal components analysis of euro-area yield differentials**

Notes: Panel A reports the covariances and correlations (in italics) of 10-year swap spreads (the swap spreads are defined as the differentials between zero-coupon government yields and zero-coupon swap yields of the same maturity) across nine different euro-area countries (Greek yields are not used since they are not available before mid-2001). Panel B presents results of principal-component analyses carried out on swap spreads. There are three analyses that correspond respectively to three maturities: 2 years, 5 years and 10 years. For each analysis, Panel B reports the eigenvalues of the covariance matrices and the propotions of variance explained by the corresponding component (denoted by "Prop. of var." in Panel B).

Panel A: Covariance and *correlations* of 10-year swap spreads

	Germ.	France	Italy	Spain	Austr.	Belg.	Finl.	Port.	Neth.
Germany	0.032	0.032	0.038	0.037	0.033	0.033	0.031	0.045	0.035
France	*0.882*	0.042	0.06	0.055	0.05	0.049	0.042	0.064	0.045
Italy	*0.645*	*0.878*	0.111	0.094	0.088	0.088	0.064	0.108	0.068
Spain	*0.701*	*0.899*	*0.953*	0.088	0.079	0.078	0.061	0.098	0.062
Austria	*0.671*	*0.888*	*0.958*	*0.967*	0.076	0.073	0.055	0.089	0.056
Belgium	*0.678*	*0.884*	*0.969*	*0.967*	*0.969*	0.074	0.054	0.089	0.057
Finland	*0.774*	*0.908*	*0.851*	*0.913*	*0.893*	*0.877*	0.05	0.069	0.048
Portugal	*0.732*	*0.9*	*0.935*	*0.958*	*0.937*	*0.943*	*0.883*	0.12	0.072
Netherlands	*0.865*	*0.96*	*0.893*	*0.921*	*0.901*	*0.912*	*0.933*	*0.91*	0.052

Panel B: Principal components

Component	1	2	3	4	5	6	7	8	9
2-year spread									
Eigenvalue	6.68	1.04	0.50	0.30	0.19	0.10	0.09	0.06	0.04
Prop. of var.	0.74	0.12	0.06	0.03	0.02	0.01	0.01	0.01	0.00
Cumul. prop.	0.74	0.86	0.92	0.95	0.97	0.98	0.99	1.00	1.00
5-year spread									
Eigenvalue	7.20	0.95	0.46	0.15	0.10	0.05	0.04	0.03	0.02
Prop. of var.	0.80	0.11	0.05	0.02	0.01	0.01	0.00	0.00	0.00
Cumul. prop.	0.80	0.91	0.96	0.98	0.99	1.00	1.00	1.00	1.00
10-year spread									
Eigenvalue	8.07	0.56	0.14	0.07	0.05	0.04	0.03	0.02	0.02
Prop. of var.	0.90	0.06	0.02	0.01	0.01	0.00	0.00	0.00	0.00
Cumul. prop.	0.90	0.96	0.98	0.99	1.00	1.00	1.00	1.00	1.00

9.1.4. Euro-area government yields

Table 1 suggests that euro-area government yields are highly correlated across countries and across maturities. In addition, government yields appear to be strongly correlated with risk-free yields. Table 2 reports the correlations between the swap spreads for different countries over the sample periods (the swap spreads are defined as the differentials between zero-coupon government yields and zero-coupon swap yields of the same maturity) and present a principal-component analysis of these spreads across countries. The correlations suggest that swap spreads largely comove across countries. The principal-component analysis (see lower part of Table 2) indicates that, for different maturities (2, 5 and 10 years), the first two principal components roughly explain 90% of the swap-spread variances across countries. This suggests that a model with a limited number of common factors may be able to explain the bulk of euro-area yield-differential fluctuations.

9.2. Model specifications

We consider six macroeconomic factors y_t. Three of them are observable (the euro-area business-cycle index $y_{BC,t}$, the volatility index $y_{VX,t}$, and the short rate r_{t+1}) and three of them are latent ($y_{f,t}$, $y_{c,t}$ and $y_{\ell,t}$). Moreover, there is a latent Markov chain z_t with three states. The interpretations of the latent variables and of the regimes are given below.

9.2.1. The historical dynamics

For the sake of parsimony and estimation tractability, the macroeconomic latent factors ($y_{f,t}$, $y_{c,t}$ and $y_{\ell,t}$) follow auto-regressive processes of order one in the historical world. Moreover, their dynamics are independent from the Markov chain and from the dynamics of the observable factors (under the historical measure). This allows us to estimate the historical dynamics of the observable factors (and the regime variables z_t) independently from the latent factors. The historical dynamics of the three observable factors is given by:

$$\begin{bmatrix} y_{VX,t} \\ y_{BC,t} \\ r_{t+1} \end{bmatrix} = \begin{bmatrix} \mu'_{VX} z_t \\ \mu_{BC} \\ \mu_r \end{bmatrix} + \Phi(L) \begin{bmatrix} y_{VX,t-1} \\ y_{BC,t-1} \\ r_t \end{bmatrix} + \begin{bmatrix} \sigma'_{VX} z_t & 0 & 0 \\ 0 & \sigma_{BC} & 0 \\ 0 & 0 & \sigma_r \end{bmatrix} \varepsilon_t. \tag{29}$$

As is shown in the previous equation, the regimes affect parameters that are primarily linked to the VSTOXX index. This is done to facilitate the interpretation of the regimes in terms of market stress. More precisely, two out of the three regimes –regimes 2 and 3, say– are expected to correspond to such periods. In these two regimes, the innovations that affect the VSTOXX may be more volatile than in the first regime (which is reflected by the entries of σ_{VX}). In addition, the VSTOXX drifts μ_{VX} may differ across the three regimes. The third regime is expected to correspond to extreme conditions reflected by a large one-off jump in the level of the VSTOXX (captured by a strong third entry in the drift μ_{VX}). Such a regime is supposed to be only one-period long and necessarily takes place amidst financial-distress conditions, that is, the regime prevailing before and after an occurence of the third regime is necessarily the second one. Furthermore, the probability of staying in the financial-distress regime (regime 2) may depend on the VSTOXX level. Formally, the transition matrix driving the Markov chain reads:

$$\{\pi_{ij,t-1}\}_{i,j} = \begin{bmatrix} \pi_{11} & (1-\pi_{11}) & 0 \\ [1-\pi_{22}(y_{VX,t-1})-\pi_{23}] & \pi_{22}(y_{VX,t-1}) & \pi_{23} \\ 0 & 1 & 0 \end{bmatrix} \tag{30}$$

where $\pi_{22}(y_{VX,t-1})$ is a logit function of $y_{VX,t-1}$.

Each country n is further described by an observable business-cycle index $bc_{n,t}$, which is the sum of two orthogonal components: the first one is linked to the euro-area business-cycle $y_{BC,t}$ and the second one, denoted by $x_{n,BC,t}$, is assumed to follow an AR(1) process. Formally, using the notations introduced in Equation (3):

$$\begin{aligned} bc_{n,t} &= \chi_n \times y_{BC,t} + x_{n,BC,t} \qquad \text{where} \\ x_{n,BC,t} &= Q_{4,n,BC} \times x_{n,BC,t-1} + Q_{5,n,BC}\eta_{n,t}. \end{aligned}$$

An additional country-specific factor is related to the regime variable z_t. This factor is aimed at reproducing potential long-lasting effects of some regimes on the default intensities $\lambda_{n,t}$'s (the effect being still felt after the lifetime of a regime, but decaying at a constant rate). It is denoted by $x_{n,z,t}$ and follows:

$$x_{n,z,t} = (1-Q_{4,n,z}) \times \left(q'_{1,n,z} z_t\right) + Q_{4,n,z} x_{n,z,t-1}.$$

Finally, the default intensities are given by:

$$\lambda_{n,t} = \alpha_n' z_t + \beta_n' y_t + \gamma_{n\cdot}' \left[\begin{array}{c} x_{n,BC,t} \\ x_{n,z,t} \end{array} \right]. \tag{31}$$

9.2.2. The risk-neutral dynamics

Under the historical measure, the three latent factors follow independent auto-regressive processes. Obviously, to have an impact on bond pricing, the latent factors have to affect the risk-free short rate r_t and/or the the hazard rates $\lambda_{n,t}$. We assume that the factors $y_{c,t}$ and $y_{\ell,t}$ intervene in the hazard rates but are independent from the risk-free short rate (under both the historical and the risk-neutral measures). As a result, the latter two factors do not contribute to the dynamics of risk-free yields.[43] (As will be shown below, this will be exploited in the estimation procedure.)

In order to facilitate the estimation (by reducing the number of parameters to estimate) and for identification purpose, additional assumptions are made regarding the risk-neutral dynamics of the latent factors. First, we assume that all three latent factors follow zero-mean auto-regressive processes of order one under the risk-neutral measure. Second, in the risk-neutral world, (a) only the last lag of the latent factor $y_{f,t}$ can affect the observable macroeconomic variables and (b) this latent factor is not affected by the lags of the observed macroeconomic factors.

Accordingly, the risk-neutral dynamics of the vector y_t (that is equal to $[y_{VX,t}, y_{BC,t}, r_{t+1}, y_{f,t}, y_{c,t}, y_{\ell,t}]'$) is given by:

$$y_t = \tilde{\mu}(z_t) + \tilde{\Phi}(L) y_{t-1} + \Omega(z_t)\varepsilon_t^*,$$

where (the $\times_{n \times n}$'s locating some $n \times n$ matrices to estimate):

$$\begin{aligned}
\tilde{\mu}(z_t) &= \left[\begin{array}{c} \times_{3\times3} z_t \\ 0_{3\times3} \end{array} \right], \\
\tilde{\Phi}(L) &= \left[\begin{array}{cccc} \times_{3\times3} & \times_{3\times1} & 0 & 0 \\ 0 & \times_{1\times1} & 0 & 0 \\ 0 & 0 & \times_{1\times1} & 0 \\ 0 & 0 & 0 & \times_{1\times1} \end{array} \right] + \left[\begin{array}{cccc} \times_{3\times3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] L + \dots \\
& \dots + \left[\begin{array}{cccc} \times_{3\times3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] L^p \qquad \text{and} \\
\Omega(z_t) &= \left[\begin{array}{cccccc} \sigma'_{VX} z_t & 0 & \cdots & & \cdots & 0 \\ 0 & \sigma_{BC} & \ddots & & & \vdots \\ \vdots & \ddots & \sigma_r & & & \\ & & & 1 & \ddots & \\ \vdots & & & \ddots & 1 & \vdots \\ 0 & \cdots & & \cdots & 0 & 1 \end{array} \right].
\end{aligned}$$

[43]This is easily seen from Equation (14).

9.3. Estimation

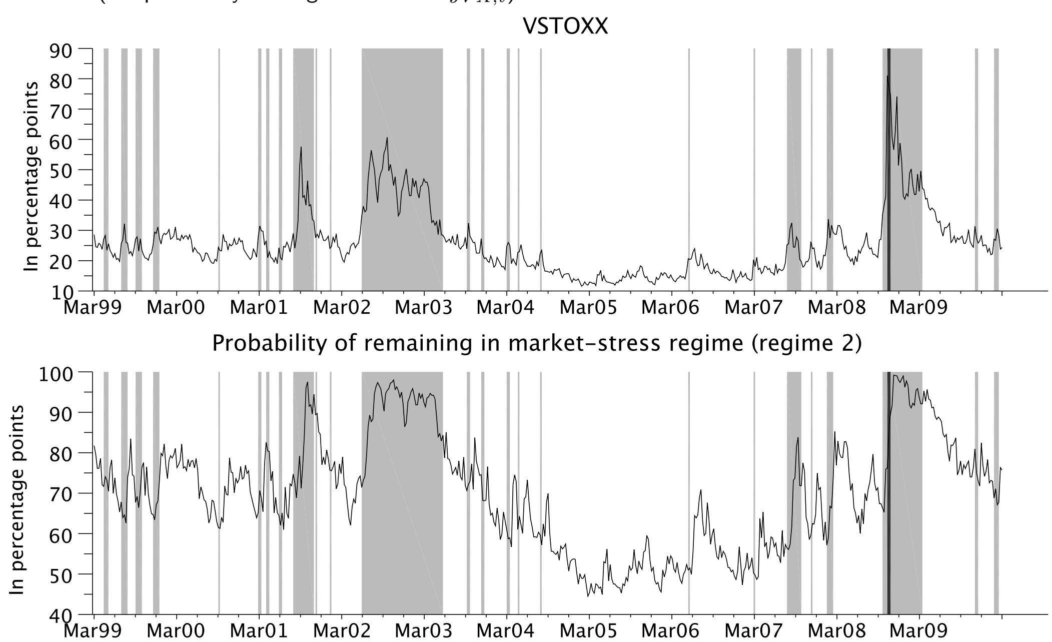
The estimation of the model involves several steps. In the first one, we estimate the historical dynamics of the three macroeconomic variables. At the end of this first step, some estimates of the regime variables z_t are computed and are taken as fixed in the next steps. The second step regards the dynamics of the risk-free yields. As mentionned in the previous subsection, only one latent factor ($y_{f,t}$) is concerned with this step. Once estimated, the latent factor $y_{f,t}$ is fixed for the third step, in which we estimate hazard rates of the different countries. The latent factors $y_{c,t}$ and $y_{\ell,t}$ are estimated in this third step.

9.3.1. Step 1: Historical dynamics of the observable macroeconomic factors

The parameters estimated in this step are those entering Equations (29) and (30). Whereas all parameters –those defining μ, Φ, Σ and the transition probabilities π_{ij}'s– could be estimated in a single step by maximising the log-likelihood, we resort to a faster sequential approach. First, an estimate of $\Phi(L)$ is obtained by estimating a simple VAR using OLS regressions. As suggested by the Akaïke criteria and by an iterative likelihood-ratio test, seven lags are included in the VAR.[44] The estimation of the three regimes is based on the residuals of the VAR.[45]

Figure 5: **Estimated regimes**

Notes: The VSTOXX is plotted in the upper panel, together with the estimated regimes: the grey-shaded areas correspond to the second regime (market-stress periods). The vertical dark-grey bar locates the third regime (strong upward shift of the VSTOXX). The lower panel presents the probability of remaining in the second regime. The latter probability depends on the VSTOXX (the probability is a logit function of $y_{VX,t}$).



[44]The iterative LR-test is carried out as follows: when considering lag l, we test the hypothesis that the coefficients on lag $l+1$ are jointly zero. We increment l until we can not reject the null hypothesis at the 5% critical value.

[45]The means and variances of the VAR residuals can shift across regimes following the specifications presented in 9.3.1. The estimates are obtained by MLE. All numerical optimizations carried out in this paper are based on the consecutive uses of three algorithms provided in the Scilab software (namely the bundle method, the quasi-Newton and Nelder-Mead methods).

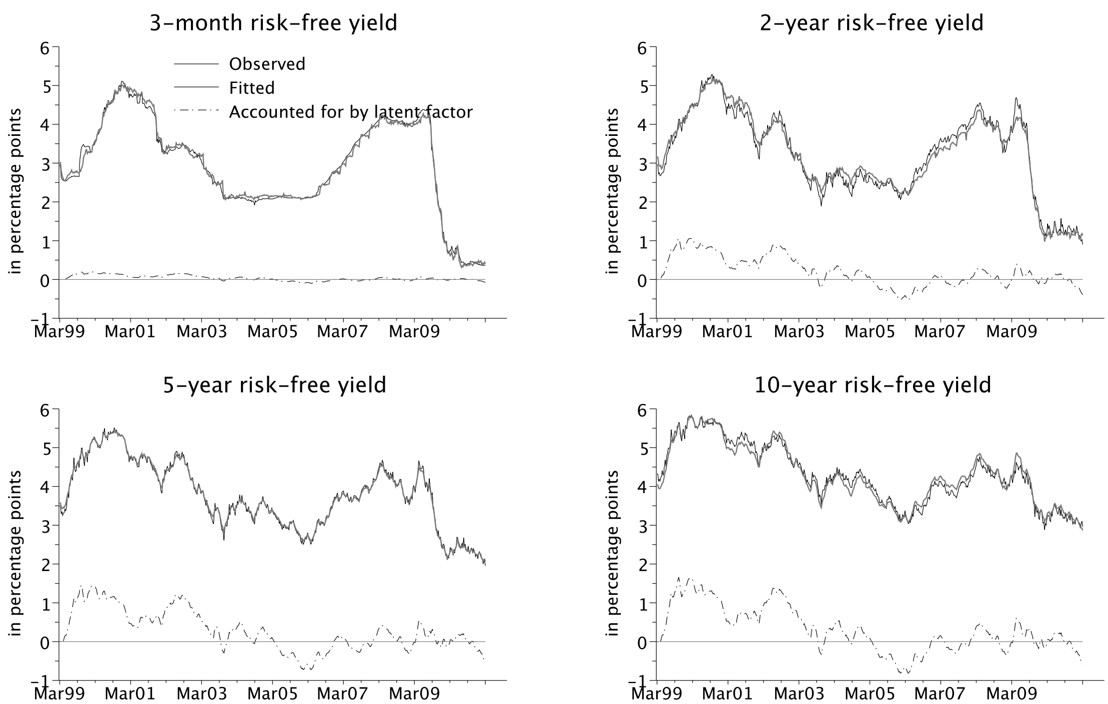
Figure 5 presents the resulting regime estimates.[46] As expected, in the second regime, the VSTOXX tends to be higher and is affected by more volatile shocks. The third regime –that is, the crisis regime– is detected for only one period in the sample, in the weeks following the Lehman Brothers bankruptcy. The lower plot of Figure 5 shows the probability of staying in state 2 (that is a logit function of $y_{VX,t-1}$). It appears that this probability positively depends on the VSTOXX. This implies notably that the life expectancy of the second regime is higher when the VSTOXX is higher, which can be checked in Figure 5.

9.3.2. Step 2: Estimation of the dynamics of the risk-free yields (including the estimation of the latent factor $y_{f,t}$)

The estimation is based on the risk-free yields described in Subsection 9.1.2. Four maturities are used in the estimation: 3 months, 2 years, 5 years and 10 years. The regime variables z_t that have been estimated previously (see 9.3.1) are taken as given. Given the sizeable number of parameters that remain to be estimated –there are 87 of them–, the choice of relevant starting values is key to deal with the numerical optimization of the log-likelihood.[47]

Figure 6: **Modeled and estimated risk-free yields**

Notes: Each plot shows the observed as well as the model-implied zero-coupon risk-free yield (for different maturities). The model includes four variables. Three are observable (the VSTOXX index $y_{VX,t}$, the eurozone business-cycle indicator $y_{BC,t}$ and the one-week risk-free yield r_{t+1}) and one is latent ($y_{f,t}$).



[46] For each period, the prevailing regime is assumed to be the one that corresponds to the largest smoothed probability (obtained by using Kim's (1994) [96] algorithm). As a rule, the regimes are clearly identified, in the sense that two of the smoothed probabilities (out of the three) are close to zero at each period.

[47] The starting value is obtained by applying a two-step approach. First, we estimate a model without latent factor, looking for parameters (in $\tilde{\Phi}$, $\tilde{\mu}$ and $\{\pi^*\}$) that minimize the squared pricing errors between modeled and observed yields. Second, these pre-estimated parameters are then taken as fixed and the additional parameters –that is, those that are related the latent factor $y_{s,t}$– are estimated by maximizing the log-likelihood computed by the Kalman filter.

Figure 6 displays modeled yields together with observed yields. The standard deviation of the pricing errors is of 12 basis points.[48] Figure 6 also shows the part of the modeled yields that is accounted for by the latent factor $y_{f,t}$. This is obtained as the difference between model-based yields and counterfactual yields that are obtained when the innovations $\varepsilon_{f,t}$ of the latent factor $y_{f,t}$ are set to 0. It turns out that the latent factor explains most of the fluctuations of long-term yields.

9.3.3. Step 3: Estimation of the hazard rates (including the estimation of the latent factors $y_{c,t}$ and $y_{\ell,t}$)

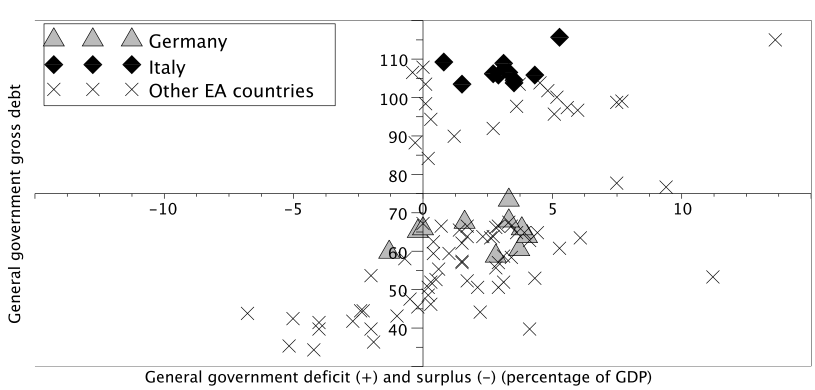
In this final step, we estimate the countries' hazard rates specified in Equation (31). These hazard rates depend on the country-specific business-cycle indicators (the $x_{n,t}$'s), on the regimes and on the macroeconomic variables y_t. The latter include the latent factors $y_{c,t}$ and $y_{\ell,t}$, that are estimated in the current step. Our objective is to make these two euro-area-wide factors interpretable. Specifically, whereas the factor $y_{c,t}$ is aimed at capturing default components of the spread, the factor $y_{\ell,t}$ is expected to integrate liquidity –or non-default– components. The identification of the liquidity components is based on the KfW-*Bund* spread (see 9.1.3 for details about this spread). More precisely, we assume that the sensitivity of KfW's hazard rate to...

1. ... the default factor $y_{c,t}$ is the same as the one of the Federal Republic of Germany (Germany hereinafter);
2. ... the liquidity factor $y_{\ell,t}$ differs from the one of Germany.

Note however that, to the extent that observed variables may also account for the KfW-*Bund* spread, the liquidity intensity is not solely explained by the factor $y_{\ell,t}$. Put differently, the liquidity factor $y_{\ell,t}$ is expected to capture only euro-area-wide unobserved components of the liquidity premia (beyond what can be explained by the observed factors that we consider).

Figure 7: **Government debts and deficits of euro-area countries**

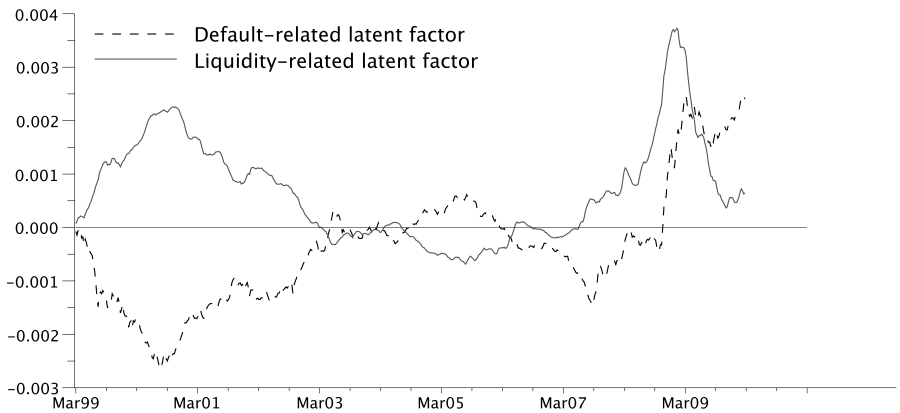
Notes: The figure displays general government deficits and debts (in percentage of GDP). One point corresponds to (a) one country of the 12-member euro area and (b) one year (2000 to 2009). The positions of Italy and Germany are highlighted.



[48]This standard deviation of the measurement error is assumed to be the same for the four maturities considered in the estimation.

Figure 8: **Estimated latent factors $y_{c,t}$ and $y_{\ell,t}$**

Notes: The figure shows the estimates of the latent factors $y_{c,t}$ and $y_{\ell,t}$. The latent factors are estimated by the Kalman filter.



The latent factors $y_{c,t}$ and $y_{\ell,t}$ are estimated by the Kalman filter. Potentially, a large state-space model, including observed yields of all countries, could be written and estimated. Nonetheless, the estimation would be highly time consuming for a standard computer.[49] Alternatively, we resort to a sequential approach. First, we estimate a smaller-scale state-space model by MLE, using yields of a subset of debtors: Germany, KfW and Italy, which provides us with some estimates of the latent factors $y_{c,t}$ and $y_{\ell,t}$ (see Figure 8).[50] The rationale for including Italy in the subset is to diversify the profile of the debtors considered in this step. In particular, as shown in Figure 7, Italy's debt-to-GDP ratio is one the highest among the EU countries, contrary to Germany. This is key since we want $y_{c,t}$ and $y_{\ell,t}$ to be euro-area-wide factors. Second, the hazard rates of the remaining countries are estimated one by one by performing non-linear least-squares on the swap spreads of the considered country.

9.3.4. Estimation results and interpretation

The estimation results are shown in Table 3. It appears that the effects of the considered observed variables on the hazard rates are statistically significant in many cases. For all countries but Italy, the business-cycle indicators enter the hazard rates with a negative sign. When statistically significant, the VSTOXX loading coefficients are positive, which suggests that swap spreads tend to increase when market volatility is high.[51] Moreover, most hazard rates react positively to a rise in risk-free short-term yields.

For some countries only (including Austria, Finland, France, Portugal and Spain), the regime variables $x_{n,z,t}$'s have a statistically significant impact on the hazard rates. Furthermore, when statistically significant, this effect is positive for market-stress regimes and is persistent.

[49] Each evaluation of the log-likelihood requires to apply the recursive formulas given in Propositions 3 and 5 for each obligor. For a weekly frequency and a maximum maturity of 10 year, this implies to run

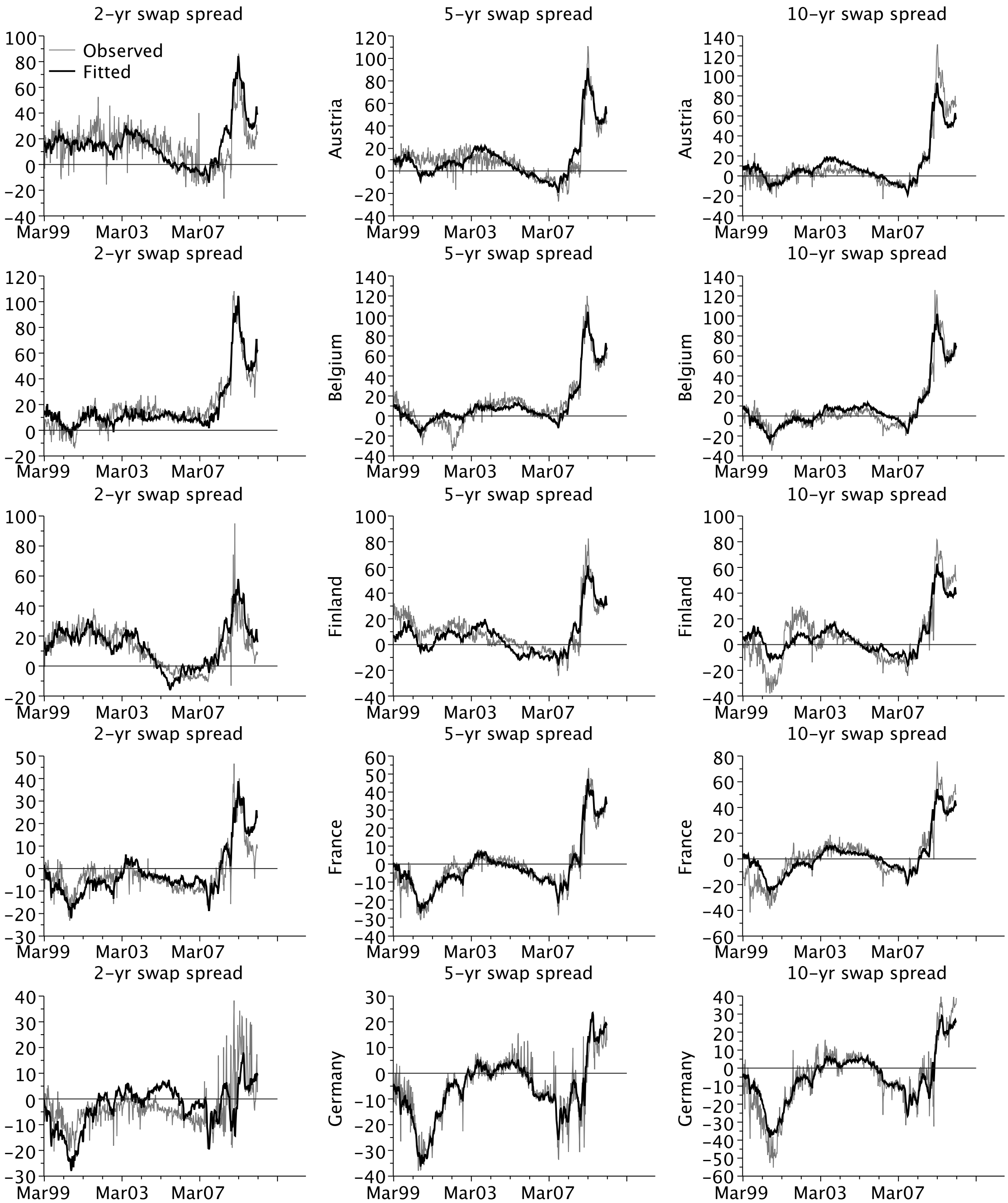
Table 3: **Estimation results**

Notes: The table reports estimation results of the debtors' hazard rates. The estimation data are weekly and span the period from February 1999 to February 2010. First, a state-space model involving the two unobserved latent factors ($y_{c,t}$ and $y_{\ell,t}$) and swap spreads –i.e. differential between zero-coupon government yields and swap yields– of Germany, Italy and KfW is estimated by maximizing the log-likelihood calculated by the Kalman filter (coefficient estimates are shown in the first three columns of Panel A, B and C). Second, non-linear least squares (NLS) are performed to obtain the hazard-rate parameterizations of remaining countries. Standard errors are reported in parenthesis below the coefficient estimates: for Germany, KfW and Italy, their computation is based on the information matrix computed using Engle and Watson's (1981) [66] formula, for other countries, the Newey-West (1987) [118] method is employed to correct for possible autocorrelation and heteroskedasticity of the residuals. In Panel C, the information provided is not the same for the first three debtors (Germany, Italy and KfW) as for the remaining ones. For the former, Panel C reports the MLE of the standard deviations of the pricing errors (and their standard errors of the estimator in parenthesis); for the latter, Panel C gives the emprirical standard deviations of the swap spreads (underlined figures) and the empirical standard deviations of the NLS residuals (in italics).

	Germ.	KfW	Italy	Austria	Belgium	Finland	France	Greece	Netherl.	Portug.	Spain
Panel A: Hazard rates ($\lambda_{n,t} = \alpha_n + \beta_{VX,n} y_{VX,t} + \beta_{BC,n}(\chi_n \times y_{BC,t} + x_{n,BC,t}) + \beta_{r,n} r_{t+1} + x_{n,z,t} + \beta_{c,n} y_{c,t} + \beta_{\ell,n} y_{\ell,t}$)											
Constant (α)	-0.252	-0.093	0.652	-0.135	-0.107	-0.166	-0.112	-0.121	-0.056	-0.059	-0.073
	(2.937)	(0.25)	(0.609)	(0.028)	(0.05)	(0.02)	(0.012)	(0.399)	(0.038)	(0.042)	(0.015)
VSTOXX (β_{VX})	0.153	0.143	0.015	-0.008	-0.05	0.113	0.062	-0.165	0.1	0.119	0.131
	(0.028)	(0.022)	(0.029)	(0.067)	(0.052)	(0.051)	(0.035)	(0.178)	(0.049)	(0.066)	(0.035)
Business-cycle (β_{BC})	-1.077	-0.752	0.422	-1.81	-1.233	-2.124	-0.893	-1.094	-1.368	-0.166	0.414
	(0.092)	(0.059)	(0.078)	(0.292)	(0.182)	(0.297)	(0.192)	(0.554)	(0.215)	(0.192)	(0.078)
Short-term rate (β_r)	2.045	-0.023	1.069	10.655	8.51	8.979	4.587	15.966	5.797	0.216	-1.745
	(0.966)	(0.779)	(0.995)	(2.759)	(2.184)	(2.319)	(1.675)	(5.291)	(1.954)	(2.035)	(1.282)
y_c latent factor (β_c)	3.826	3.826	6.316	4.766	4.95	4.098	3.995	14.758	4.567	6.367	6.103
	(0.224)	(0.224)	(0.38)	(0.596)	(0.216)	(0.396)	(0.246)	(1.342)	(0.415)	(0.446)	(0.431)
y_ℓ latent factor (β_ℓ)	-1.219	2.178	3.313	1.847	2.224	0.654	0.284	5.405	0.551	2.499	2.49
(liquidity-related)	(0.433)	(0.385)	(0.628)	(0.224)	(0.139)	(0.27)	(0.086)	(0.527)	(0.154)	(0.174)	(0.127)
Panel B: Dynamics of the $x_{n,z,t}$'s ($x_{n,z,t} = (1 - Q_{4,n,z}) \times (q'_{1,n,z} z_t) + Q_{4,n,z} x_{n,z,t-1}$)											
$Q_{4,z}$	0.18	0.511	0.504	0.995	0.848	0.994	0.996	0.861	0.921	0.938	0.992
	(1.45)	(0.739)	(0.144)	(0.002)	(0.119)	(0.002)	(0.002)	(0.124)	(0.03)	(0.033)	(0.004)
$q_{1,z}(2)$	-0.533	-0.066	1.461	0.002	0.022	0.002	0.001	0.177	0.035	0.018	0.001
	(10.211)	(0.529)	(1.547)	(0.000)	(0.041)	(0.000)	(0.000)	(0.371)	(0.024)	(0.019)	(0.000)
$q_{1,z}(3)$	-0.5	0.746	0.07	0.029	0.605	0.029	0.013	1.52	0.106	0.136	-0.008
	(3.751)	(1.629)	(1.102)	(0.008)	(0.469)	(0.008)	(0.005)	(1.212)	(0.043)	(0.06)	(0.008)
Panel C: Standard deviations of the pricing errors											
2-year	10.556	10.556	10.556	*12.069*	*7.529*	*8.583*	*5.473*	*41.627*	*9.294*	*14.343*	*8.15*
	(0.184)	(0.184)	(0.184)	15.7	20.463	13.678	9.771	73.293	13.778	26.309	19.046
5-year	4.614	4.614	4.614	*8.629*	*8.294*	*8.15*	*4.078*	*33.142*	*6.965*	*11.44*	*7.459*
	(0.092)	(0.092)	(0.092)	20.123	25.525	16.172	14.896	76.954	14.012	30.145	22.585
10-year	7.048	7.048	7.048	*9.4*	*7.575*	*10.724*	*8.07*	*23.708*	*8.911*	*12.653*	*9.52*
	(0.126)	(0.126)	(0.126)	27.527	27.232	22.465	20.517	70.398	22.74	34.653	29.658

Figure 9: **Model-implied vs. observed swap spreads (part 1)**

Notes: The figure plots together observed and model-implied swap spreads (the swap spreads are defined as the differentials between zero-coupon government yields and swap yields). Three maturities are considered: 2 years, 5 years and 10 years (these are the maturities used in the estimation).



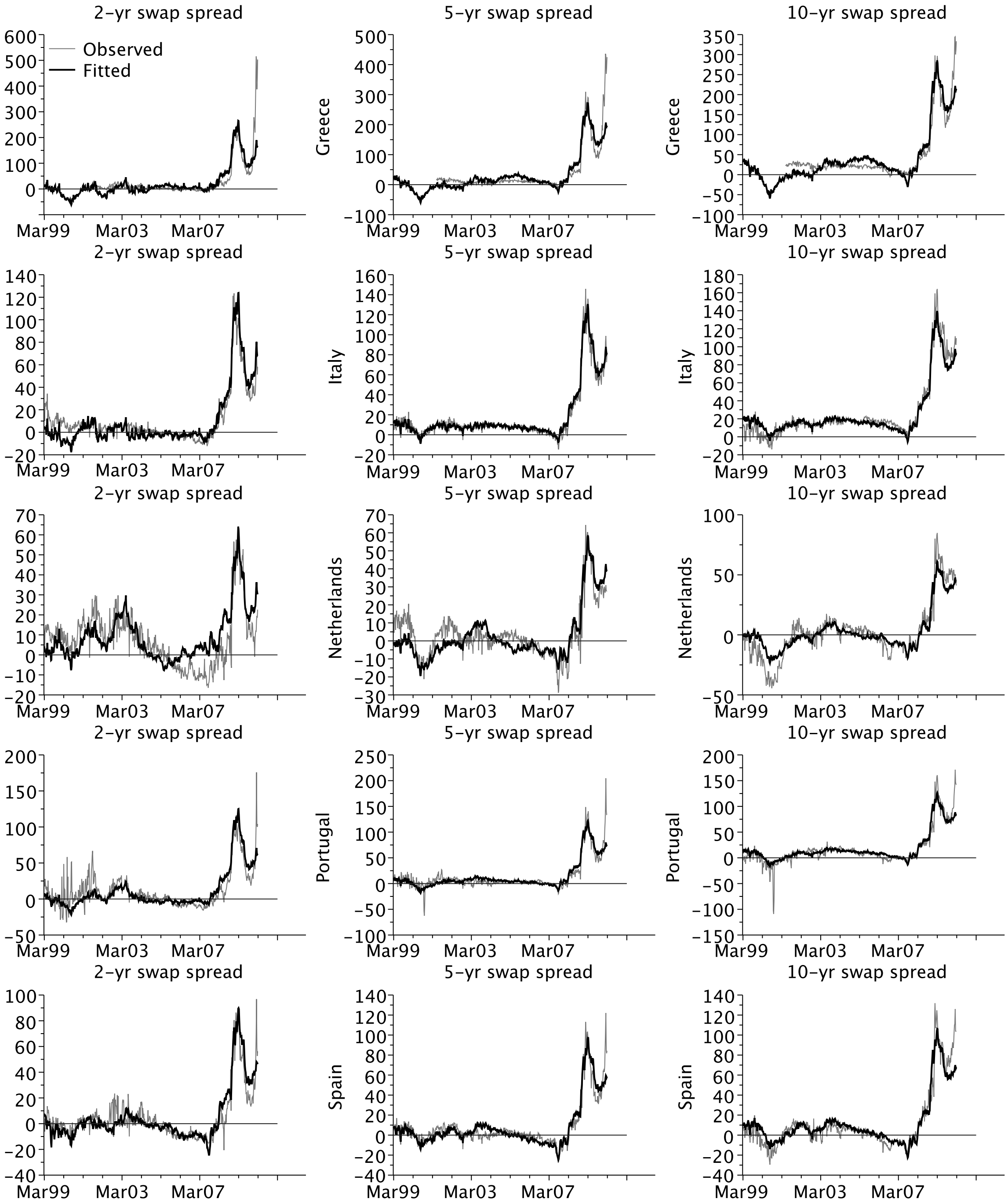
520-iteration loops for each country.

[50] Note that the latent factor $y_{f,t}$ is still taken as fixed (keeping its estimated values obtained in 9.3.2).

[51] Such effects are also observed by Colin-Dufresne et al. (2001) [39] and Amato and Luisi (2006) [4] (on U.S. corporate bonds). The former use regression analysis, the latter use an affine term-struture framework.

Figure 10: **Model-implied vs. observed swap spreads (part 2)**

Notes: The figure plots together observed and model-implied swap spreads (the swap spreads are defined as the differentials between zero-coupon government yields and swap yields). Three maturities are considered: 2 years, 5 years and 10 years (these are the maturities used in the estimation).



Overall, as shown in the lowest panel of Table 3, important shares of the swap spreads are explained by the model (across countries and maturities, roughly 75% of the variance is accounted for by the model). This is also illustrated by Figures 9 and 10, comparing observed swap spreads with their model-implied counterparts. However, it can be noted that the model does not capture the end-of-sample rise in the swap spreads for Greece, Portugal and Spain (from January 2010 onwards).[52]

Let us turn to the latent factors $y_{c,t}$ and $y_{\ell,t}$. These factors are plotted in Figure 8. The liquidity-related factor $y_{\ell,t}$ presents two humps: one in the late 90s (early 2000s) and one between Autumn 2008 and Summer 2009, following the Lehman Brothers' bankruptcy. The rise in liquidity premia in the early 2000s –concomitant with the collapse of the Internet bubble– is also found in U.S. data by Fontaine and Garcia (2009) [72], Longstaff (2004) [102] or Feldhütter and Lando (2008) [69]. Furthermore, Kempf, Korn and Uhrig-Homburg (2010) [95] look for liquidity factors on euro bond markets and exhibit a long-term illiquidity premium that also presents a hump in the early 2000s. The fact that the liquidity factor is particularly high during crises periods (burst of the dotcom bubble and post-Lehman periods) is consistent with the findings of Beber, Brandt and Kavajecz (2009) [16] who pinpoint that investors primarily chase liquidity during market-stress periods.[53] Figure 8 also shows that the correlation between the two latent factors dramatically changed in late 2007: whereas the two latent factors present opposite fluctuations between the late 90s and 2007, they tend to comove since then.

Figure 12 shows a scatter plot of the countries, whose coordinates correspond to the sensitivity of their hazard rates to the latent factors (i.e. the coordinates of country n are $(\beta_{n,c}, \beta_{\ell,c})$). The chart suggests that German government debt is the least exposed to the default-related factor as well as to the liquidity factor. This is consistent with the fact that, among sovereign euro-area bonds, the German *Bunds* are perceived to be the "safest haven" both in terms of credit quality and liquidity.[54] Greece occupies the opposite position. In addition, the chart shows that Austrian or Belgian government debts are only slightly more exposed to the risk factor $y_{c,t}$ than German bonds but are more exposed to the liquidity factor, which was expected. Surprisingly, in spite of the large size of the tradable debt issued by the Italian government, Italy's hazard rate appears to be particularly affected by the liquidity factor (among the countries considered in our subset, only Greece is more exposed than Italy to the liquidity factor).

Recall that the two latent factors do not account for all the model-based spreads but that observed factors also contribute to the estimated spreads dynamics. To that respect, Figure 11 presents the decompositions of the yield differentials between selected countries and Germany into three components: (a) the contribution of the liquidity-related latent factor, (b) the contribution of the default-related latent factor and (c) the contribution of observed (and regime) variables. The plots indicate that large shares of the spreads are liquidity-driven. In particular, the most parts of the French-German or Austrian-German spreads are explained by the liquidity-related factor $y_{\ell,t}$.

[52] This could be addressed by including an additional Markov chain that would affect the hazard rates of the different countries. However, in order to keep the possibility to estimate the model in several steps –as is done here–, the s.d.f. has to be independent from this additional Markov chain (otherwise, the estimation of the risk-free yield-curve dynamics has to take these additional regimes into account).

[53] Such a result is generated in a theoretical framework by Vayanos (2004) [127].

[54] The German bond market is the only one in Europe that has a liquid futures market, which boosts demand for German Bunds compared to other euro area debt (see e.g. Pagano and von Thadden, 2004 [120], Ejsing and Sihoven, 2009 [63] or Barrios et al., 2009 [14]).

Figure 11: **Decomposition of model-based yield differentials vs. Germany**

Notes: The left column of charts compares model-based and observed yield differentials between selected governments and Germany. The right column presents decompositions of the model-based spreads in three components: the first corresponds to the effect the latent factor $y_{c,t}$, the second corresponds to the effect of the latent factor $y_{\ell,t}$ and the third gives the influence of others variables (including observed macroeconomic variables and regime variables).

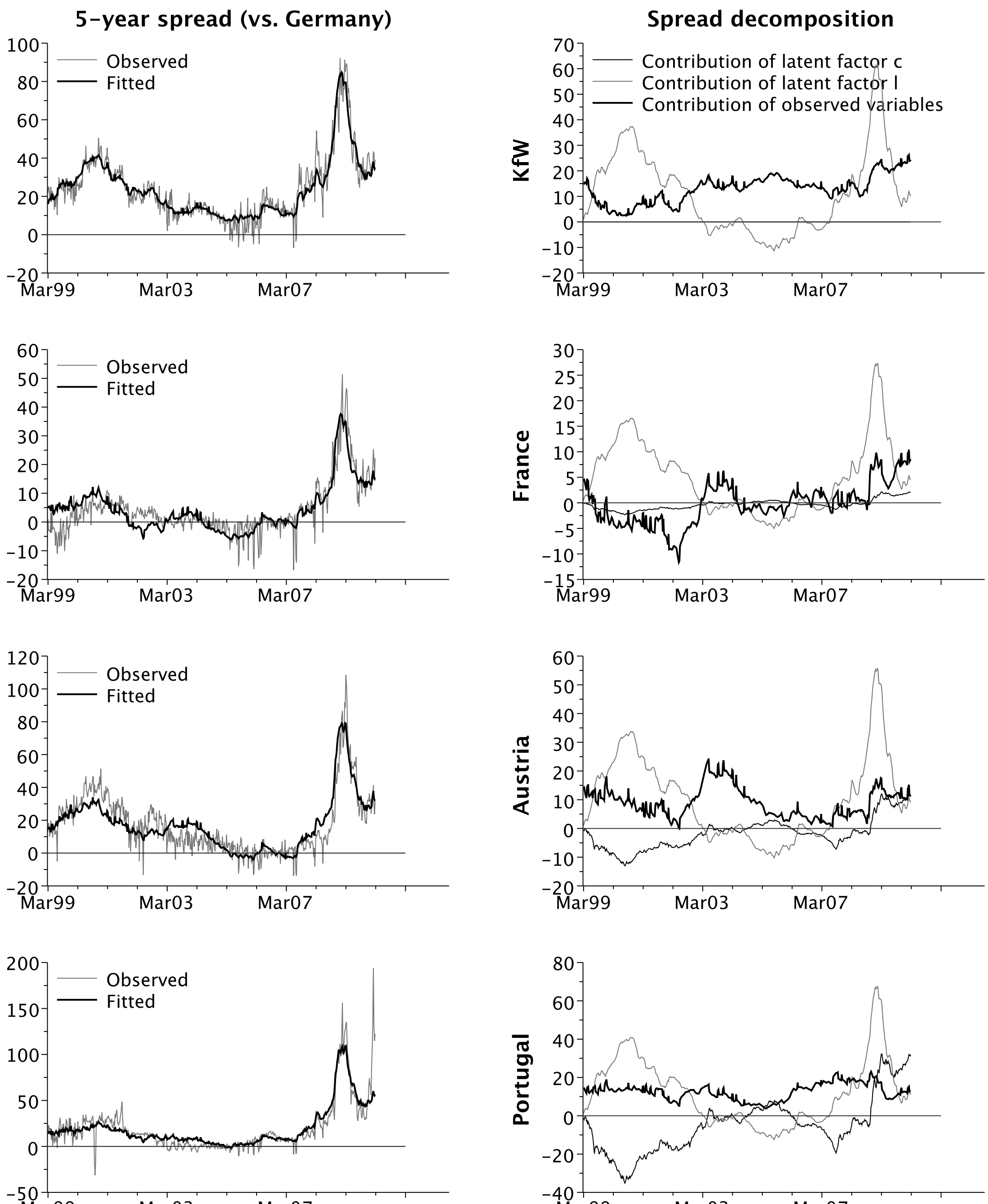
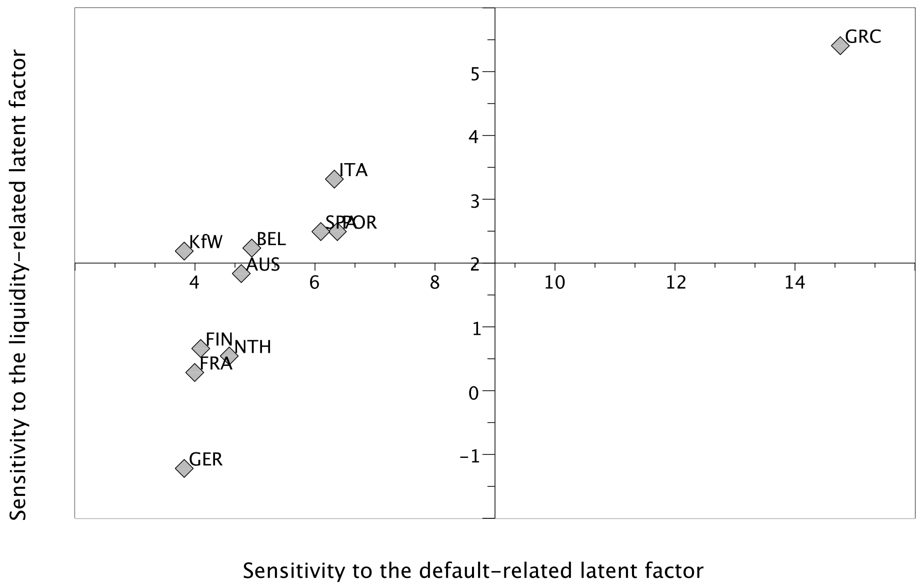


Figure 12: **Sensitivities of the countries' hazard rates to the latent factors**

Notes: The figure displays the sensitivityof the countries' hazard rates to the default-related factor $y_{c,t}$ and the liquidity-related factor $y_{\ell,t}$. The coordinates of the points are given in Table 3 (lines β_c and β_ℓ).



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A. P.d.f. under the risk-neutral world

Let us consider a couple (X, Y) of multivariate random vectors. Let denote with $f^{\mathbb{H}}(X, Y)$ and $f^{\mathbb{Q}}(X, Y)$ their respective joint p.d.f. under the probability measure $\mathbb{H}$ and $\mathbb{Q}$ and assume that the Radon-Nikodym derivative thate relates $\mathbb{H}$ and $\mathbb{Q}$ depends on X only and is proportional to $M(X)$. We have:

$$\begin{aligned} f^{\mathbb{Q}}(X,Y) &= \frac{f^{\mathbb{H}}(X,Y)M(X)}{\int f^{\mathbb{H}}(X,Y)M(X)dXdY} \\ &= \frac{f^{\mathbb{H}}(X)f^{\mathbb{H}}(Y \mid X)M(X)}{\iint f^{\mathbb{H}}(X)f^{\mathbb{H}}(Y \mid X)M(X)dXdY} \\ &= \frac{f^{\mathbb{H}}(X)f^{\mathbb{H}}(Y \mid X)M(X)}{\int f^{\mathbb{H}}(X)M(X)\left[\int f^{\mathbb{H}}(Y \mid X)dY\right]dX} \\ &= \frac{f^{\mathbb{H}}(X)M(X)}{\int f^{\mathbb{H}}(X)M(X)dX} f^{\mathbb{H}}(Y \mid X) \\ &= f^{\mathbb{Q}}(X)f^{\mathbb{H}}(Y \mid X). \end{aligned}$$

B. Proof of Lemma 2

The formula is true for $h = 1$ since:

$$L_{t,1}(\omega) = E_t\left(\omega'_H Z_{t+1}\right) = \exp\left[a'(\omega_H)Z_t + b(\omega_H)\right]$$

and therefore $A_1(\omega) = a(\omega_h)$ and $B_1(\omega) = b(\omega_h)$.

if it is true for $h - 1$, we get:

$$\begin{aligned} L_{t,h}(\omega) &= E_t\left[\exp\left(\omega'_{H-h+1}Z_{t+1}\right)E_{t+1}\left(\omega'_{H-h+2}Z_{t+2} + \ldots + \omega'_H Z_{t+H}\right)\right] \\ &= E_t\left[\exp\left(\omega'_{H-h+1}Z_{t+1}\right)L_{t+1,h-1}(\omega)\right] \\ &= E_t\left[\exp\left(\omega'_{H-h+1}Z_{t+1} + A_{h-1}(\omega)Z_{t+1} + B_{h-1}(\omega)\right)\right] \\ &= \exp\left[a(\omega'_{H-h+1} + A_{h-1}(\omega))Z_t + b(\omega'_{H-h+1} + A_{h-1}(\omega)) + B_{h-1}(\omega)\right] \end{aligned}$$

and the result follows.

C. The risk-neutral Laplace transform of $(z_t, y_t, x_{n,t})$

In this appendix, we compute $E^{\mathbb{Q}}_{t-1}\left(\exp\left[u'z_t + v'y_t + w'x_{n,t}\right]\right)$ and show that it is exponential affine in $(z_{t-1}, y_{t-1}, x_{n,t-1})$, that is, we show that $(z_t, y_t, x_{n,t})$ is Car(1) (see Darolles, Gourieroux and Jasiak, 2006 [45]).

$$
\begin{aligned}
E_{t-1}^{\mathbb{Q}}\left(\exp\left[u'z_t+v'y_t+w'x_{n,t}\right]\right) &= E_{t-1}^{\mathbb{Q}}\left(\exp\left[u'z_t+v'y_t+w'\left(q_{1n}\left(z_t,z_{t-1}\right)+\right.\right.\right.\\
&\quad \left.\left.\left. Q_{2n}y_t+Q_{3n}y_{t-1}+Q_{4n}x_{n,t-1}+Q_{5n}\left(z_t,z_{t-1}\right)\eta_{n,t}\right)\right]\right)\\
&= \exp\left(w'Q_{3n}y_{t-1}+w'Q_{4n}x_{n,t-1}\right)\times\\
&\quad E_{t-1}^{\mathbb{Q}}\left(\exp\left[u'z_t+(v'+w'Q_{2n})y_t+\right.\right.\\
&\quad \left.\left. w'q_{1n}\left(z_t,z_{t-1}\right)+w'Q_{5n}\left(z_t,z_{t-1}\right)\eta_{n,t}\right]\right)\\
&= \exp\left(w'Q_{3n}y_{t-1}+w'Q_{4n}x_{n,t-1}\right)\times\\
&\quad E_{t-1}^{\mathbb{Q}}\left(\exp\left[u'z_t+w'q_{1n}\left(z_t,z_{t-1}\right)+w'Q_{5n}\left(z_t,z_{t-1}\right)\eta_{n,t}+\right.\right.\\
&\quad \left.\left.(v'+w'Q_{2n})\left((\mu_t+\mu_t^*)+(\Phi+\Phi^*)\,y_{t-1}+\Omega_t\varepsilon_t^*\right)\right]\right)\\
&= \exp\left[\left\{(v'+w'Q_{2n})\left(\Phi+\Phi^*\right)+w'Q_{3n}\right\}y_{t-1}+\right.\\
&\quad \left. w'Q_{4n}x_{n,t-1}+\left(\begin{array}{ccc}\tilde{A}_1(u,v,w) & \ldots & \tilde{A}_J(u,v,w)\end{array}\right)z_{t-1}\right]
\end{aligned}
$$

with

$$
\begin{aligned}
\tilde{A}_i(u,v,w) &= \log\Big(\sum_{j=1}^{J}\pi_{ij}^*\exp\{u_j+(v'+w'Q_{2n})\left[\mu\left(e_j,e_i\right)+\mu^*\left(e_j,e_i\right)\right]+w'q_{1n}\left(e_j,e_i\right)+\\
&\quad \frac{1}{2}(v'+w'Q_{2n})\Sigma\left(e_j,e_i\right)\left(v+Q_{2n}'w\right)+\frac{1}{2}w'Q_{5n}\left(e_j,e_i\right)Q_{5n}'\left(e_j,e_i\right)w\}\Big).
\end{aligned}
$$

The fact that$(z_t, y_t, x_{n,t}, d_{n,t})$ is not Car(1) is obtained by noting that (for $d_{n,t-1} = 0$):

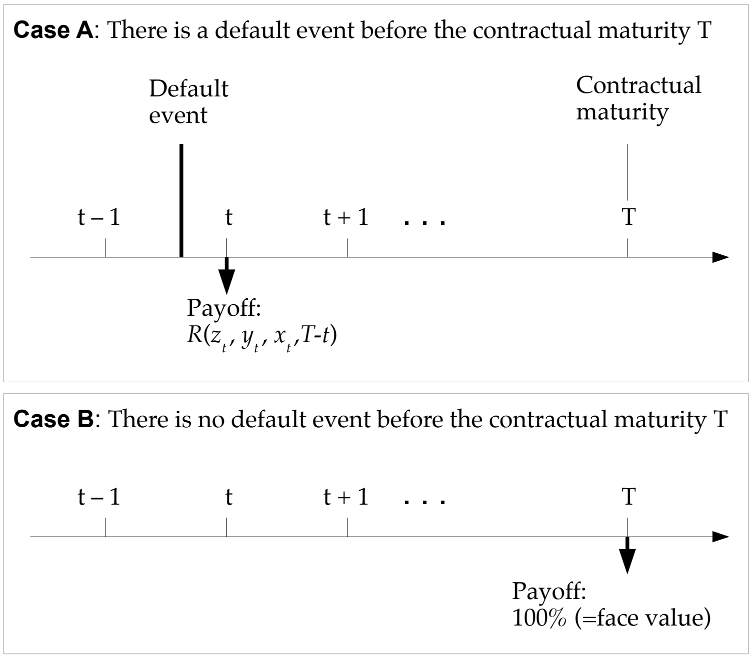
$$
\begin{aligned}
E_{t-1}^{\mathbb{Q}}\left(\exp\left[u'z_t+v'y_t+w'x_{n,t}+sd_{n,t}\right]\right) &=\\
E_{t-1}^{\mathbb{Q}}\left(E\left(\exp\left[u'z_t+v'y_t+w'x_{n,t}+sd_{n,t}\right]\mid z_t,y_t,x_{n,t},d_{n,t-1}=0\right)\right) &=\\
E_{t-1}^{\mathbb{Q}}\left(\exp\left[u'z_t+v'y_t+w'x_{n,t}\right]E\left(\exp\left[sd_{n,t}\right]\mid z_t,y_t,x_{n,t},d_{n,t-1}=0\right)\right) &=\\
E_{t-1}^{\mathbb{Q}}\left(\exp\left[u'z_t+v'y_t+w'x_{n,t}\right]\left(\exp\left(-\lambda_{n,t}\right)+\left[1-\exp\left(-\lambda_{n,t}\right)\right]\exp(s)\right)\right)
\end{aligned}
$$

This shows that $E_{t-1}^{\mathbb{Q}}\left(\exp\left[u'z_t+v'y_t+w'x_{n,t}+sd_{n,t}\right]\right)$ will be only a *sum* of two terms that are exponential affine in $(z_{t-1}, y_{t-1}, x_{n,t-1}, d_{n,t-1})$. Consequently, $(z_t, y_t, x_{n,t}, d_{n,t})$ is not Car(1).

D. Pricing of defaultable bonds with nonzero recovery rates

Section 4 gives quasi-explicit formulas for the pricing of bonds with zero recovery rates. In the current appendix, we present conditions under which one can derive formulas for nonzero-recovery-rate bond pricing. Figure 13 presents the payoff schedule considered here. As shown in this figure, if a debtor n defaults between $t-1$ and t (with $t < T$, where T denotes the contractual maturity of a bond issued by this debtor), recovery is assumed to take place at time t. In addition, we assume that the recovery payoff –i.e. one minus the loss-given-default– depends on (z_t, y_t, x_t). This recovery payoff is denoted by $R_{n,t}^{T-t} := R(z_t, y_t, x_t, T-t)$.

Figure 13: **Payoffs stemming from a defaultable bond (issued before $t-1$)**



Let us consider the price $B_n^{DR}(T-1,1)$, in period $T-1$, of a one-period nonzero-recovery-rate bond issued by a given debtor (before $T-1$). We distinguish three cases:

1. The debtor had defaulted before $T-2$, then: $B_n^{DR}(T-1,1)=0$.
2. The debtor defaulted between $T-2$ and $T-1$, then: $B_n^{DR}(T-1,1)=R^1_{n,T-1}$.
3. The debtor has not defaulted before $T-1$, then:

$$\begin{aligned}
B_n^{DR}(T-1,1) &= \exp(-r_T)E^{\mathbb{Q}}\left[\mathbb{I}_{\{d_{n,T}=0\}}+\mathbb{I}_{\{d_{n,T}=1\}}R^0_{n,T} \mid \underline{z}_{T-1},\underline{y}_{T-1},\underline{x}_{n,T-1},d_{n,T-1}=0\right] \\
&= \exp(-r_T)E^{\mathbb{Q}}\left[E^{\mathbb{Q}}\left(\mathbb{I}_{\{d_{n,T}=0\}}+\mathbb{I}_{\{d_{n,T}=1\}}R^0_{n,T} \mid \underline{z}_{T},\underline{y}_{T},\underline{x}_{n,T},d_{n,T-1}=0\right)\right. \\
&\qquad \left. \mid \underline{z}_{T-1},\underline{y}_{T-1},\underline{x}_{n,T-1},d_{n,T-1}=0\right] \\
&= \exp(-r_T)E^{\mathbb{Q}}\left[\exp(-\lambda_{n,T})+(1-\exp(-\lambda_{n,T}))\,R^0_{n,T} \mid \underline{z}_{T-1},\underline{y}_{T-1},\underline{x}_{n,T-1},d_{n,T-1}=0\right] \\
&= \exp(-r_T)E^{\mathbb{Q}}\left[\exp(-\lambda_{n,T})+(1-\exp(-\lambda_{n,T}))\,R^0_{n,T} \mid \underline{z}_{T-1},\underline{y}_{T-1},\underline{x}_{n,T-1}\right]
\end{aligned}$$

and, defining the random variable $\tilde{\lambda}^0_{n,T}$ by $\exp(-\tilde{\lambda}^0_{n,T})=\exp(-\lambda_{n,T})+(1-\exp(-\lambda_{n,T}))\,R^0_{n,T}$, we have (still in case 3):

$$B_n^{DR}(T-1,1)=E^{\mathbb{Q}}\left[\exp(-r_T-\tilde{\lambda}^0_{n,T}) \mid \underline{z}_{T-1},\underline{y}_{T-1},\underline{x}_{n,T-1}\right].$$

Further, let us consider the price of the same bond in period $T-2$. Assuming that there was no default before $T-2$:

$$\begin{aligned}
B_n^{DR}(T-2,2) &= \exp(-r_{T-1})E^{\mathbb{Q}}\left[\mathbb{I}_{\{d_{n,T-1}=0\}}\left(E^{\mathbb{Q}}\left[\exp(-r_T-\tilde{\lambda}^0_{n,T}) \mid \underline{z}_{T-1},\underline{y}_{T-1},\underline{x}_{n,T-1}\right]\right)\right. \qquad (32) \\
&\qquad \left. +\mathbb{I}_{\{d_{n,T-1}=1\}}R^1_{n,T-1} \mid \underline{z}_{T-2},\underline{y}_{T-2},\underline{x}_{n,T-2},d_{n,T-2}=0\right]
\end{aligned}$$

Let us introduce a random variable $\zeta^1_{n,T-1}$ that is defined through:

$$R^1_{n,T-1} = \zeta^1_{n,T-1} E^{\mathbb{Q}} \left[\exp(-r_T - \tilde{\lambda}^0_{n,T}) \mid \underline{z}_{T-1}, \underline{y}_{T-1}, \underline{x}_{n,T-1} \right].$$

With this notation, Equation (32) reads:

$$\begin{aligned} B^{DR}_n(T-2,2) &= E^{\mathbb{Q}} \left[\exp(-r_{T-1} - r_T - \tilde{\lambda}^0_{n,T}) \left(\mathbb{I}_{\{d_{n,T-1}=0\}} + \zeta^1_{n,T-1} \mathbb{I}_{\{d_{n,T-1}=1\}} \right) \right. \\ & \qquad \left. \mid \underline{z}_{T-2}, \underline{y}_{T-2}, \underline{x}_{n,T-2}, d_{T-2} = 0 \right] \\ &= E^{\mathbb{Q}} \left[E^{\mathbb{Q}} \left\{ \exp(-r_{T-1} - r_T - \tilde{\lambda}^0_{n,T}) \left(\mathbb{I}_{\{d_{n,T-1}=0\}} + \zeta^1_{n,T-1} \mathbb{I}_{\{d_{n,T-1}=1\}} \right) \right. \right. \\ & \qquad \left. \left. \mid \underline{z}_{T-1}, \underline{y}_{T-1}, \underline{x}_{n,T-1}, d_{n,T-2} = 0 \right\} \mid \underline{z}_{T-2}, \underline{y}_{T-2}, \underline{x}_{n,T-2}, d_{n,T-2} = 0 \right] \\ &= E^{\mathbb{Q}} \left[\exp(-r_{T-1} - r_T - \tilde{\lambda}^0_{n,T}) \left(\exp(-\lambda_{n,T-1}) + \zeta^1_{n,T-1} \left(1 - \exp(-\lambda_{n,T-1}) \right) \right) \right. \\ & \qquad \left. \mid \underline{z}_{T-2}, \underline{y}_{T-2}, \underline{x}_{n,T-2} \right]. \end{aligned}$$

Then, defining the random variable $\tilde{\lambda}^1_{n,T-1}$ with:

$$\exp(-\tilde{\lambda}^1_{n,T-1}) = \exp(-\lambda_{n,T-1}) + (1 - \exp(-\lambda_{n,T-1})) \, \zeta^1_{n,T-1},$$

we get (conditionally on $d_{n,T-2} = 0$):

$$B^{DR}_n(T-2,2) = E^{\mathbb{Q}} \left[\exp(-r_T - r_{T-1} - \tilde{\lambda}^0_{n,T} - \tilde{\lambda}^1_{n,T-1}) \mid \underline{z}_{T-2}, \underline{y}_{T-2}, \underline{x}_{n,T-2} \right].$$

Applying this methodology recursively, it is easily seen that the price of a nonzero-recovery-rate defaultable bond of maturity h is given by (assuming no default before t, i.e. conditionally on $d_{n,t} = 0$):

$$B^{DR}_n(t,h) = E^{\mathbb{Q}} \left[\exp(-r_{t+h} - \ldots - r_{t+1} - \tilde{\lambda}^0_{n,t+h} - \ldots - \tilde{\lambda}^{h-1}_{n,t+1}) \mid \underline{z}_t, \underline{y}_t, \underline{x}_{n,t} \right] \qquad (33)$$

where the $\tilde{\lambda}^{h-i}_{n,t+i}$'s are defined recursively in i by the backward equation:

$$\exp(-\tilde{\lambda}^{h-i}_{n,t+i}) = \exp(-\lambda_{n,t+i}) + (1 - \exp(-\lambda_{n,t+i})) \, \zeta^{h-i}_{n,t+i}$$

where

$$\zeta^{h-i}_{n,t+i} = \begin{cases} \dfrac{R^{h-i}_{n,t+i}}{E^{\mathbb{Q}}\left[\exp(-r_{t+h} - \ldots - r_{t+i+1} - \tilde{\lambda}^0_{n,t+h} - \ldots - \tilde{\lambda}^{h-i-1}_{n,t+i+1}) \mid \underline{z}_{t+i}, \underline{y}_{t+i}, \underline{x}_{n,t+i}\right]} & \text{if } i < h \\ R_{t+h,0} & \text{if } i = h. \end{cases}$$

Looking at Equation (33), it is tempting to interpret the $\tilde{\lambda}^{h-i}_{n,t+i}$'s as "recovery-adjusted" hazard rates for debtor n. However, the dependency of these intensities on the maturity h of the considered bond is problematic. Indeed, by analogy with the standard default intensities $\lambda_{n,t}$, one would like to have, at each period, only one adjusted intensity by debtor (and not a collection of adjusted intensities associated with the different bonds that have been issued by the considered debtor). To that end, Duffie and Singleton (1999) [60] propose a "recovery of market value" assumption. Under this assumption, the variable $R^m_{n,s}$ –that is, the recovery at time s of a bond with residual maturity m, in the event of default between $s-1$ and s– is equal to the product of a factor *common to all maturities* with the survival-contingent market value at time s. In the same spirit, let us assume that the $\zeta_{n,s}^m$'s do no longer depend on m. Then, the $\tilde{\lambda}_{n,s}^m$ do not depend on the maturity any longer and are simply given by:

$$\exp(-\tilde{\lambda}_{n,s}) = \exp(-\lambda_{n,s}) + (1 - \exp(-\lambda_{n,s}))\, \zeta_{n,s}.$$

Actually, this formulation is more general than the one considered by Duffie and Singleton (1999) when they expose a discrete-time motivation. Indeed, in the latter case, they assume that $\zeta_{n,s}$ is known at time $s-1$, which is not necessarily the case in the framework described above.

E. Equivalence of Granger and Sims non-causality

The stochastic process X_t does not cause the stochastic process Y_t in Granger's sense iff, for any t, Y_t is independent of $(X_{t-1}, \dots, X_1)$ conditionally on $(Y_{t-1}, \dots, Y_1)$.

The non-causality in Sims' sense is defined as follows: X_t does not cause the stochastic process Y_t in Sims' sense iff X_t is independent from$(Y_{t+1}, Y_{t+2}, \dots, Y_T)$ conditionally on $(Y_t, X_{t-1}, Y_{t-1}, \dots, X_1, Y_1)$.

Let us decompose the p.d.f. of (X_t, Y_t), denoted by $f(X, Y)$:

$$f(X,Y) = \prod_{t=1}^{T} f(X_t \mid X_{t-1}, \dots X_1, Y_T, \dots, Y_1) \prod_{t=1}^{T} f(Y_t \mid Y_{t-1}, \dots, Y_1)$$

Another decomposition is given by:

$$f(X,Y) = \prod_{t=1}^{T} f(X_t \mid Y_t, X_{t-1}, Y_{t-1}, \dots, X_1, Y_1) \prod_{t=1}^{T} f(Y_t \mid X_{t-1}, Y_{t-1}, \dots, X_1, Y_1).$$

If X_t does not cause the stochastic process Y_t in Granger's sense, then the second terms of the two equations above are equal. Consequently, the first two terms are equal. It can then be shown recursively that these two products are equal term by term. This implies that for any t, X_t is independent from$(Y_{t+1}, Y_{t+2}, \dots, Y_T)$ conditionally on$(Y_t, X_{t-1}, Y_{t-1}, \dots, X_1, Y_1)$, that is, X_t does not cause the stochastic process Y_t in Sims' sense. The reciprocal is shown in the same way.

F. Kitagawa-Hamilton algorithm for partially-hidden Markov chains

In this appendix, we describe how to use the Hamilton's (1990) [84] algorithm within the estimation strategy presented in Section 6, when the Markov chain is partially observed. As noted by Hamilton (1994) [85], while the algorithm was originally presented in a model with fixed transition probabilities, it readily generalizes to processes in which transition probabilities depend on a vector of observed variables.[55]

Let us denote with $\hat{y}_t$ the vector of observed variables $(\tilde{y}_t', R_{1t}, z_{1t}')'$. The Hamilton's algorithm consists in computing recursively the probabilities $p(z_{2t} \mid \underline{\hat{y}}_t)$. As a by product,

[55] See e.g. Filardo (1994) [71] ord Diebold, Lee and Weinbach (1993) [54] for implementation examples of Hamilton's algorithm in models with time-varying transition probabilities. For introductions to regime-switching models, see Hamilton (1994) [85] or Kim and Nelson (1999) [116].

the algorithm provides the conditional densities $f(\hat{y}_t \mid \underline{\hat{y}}_{t-1})$, which makes it possible to estimate the model parameters by maximization of the log-likelihood. The algorithm is based on the iterative implementation of the following steps (the input being $p(z_{2t-1} \mid \underline{\hat{y}}_{t-1})$):

1. The joint probability $p\left(z_{2t}, z_{2t-1} \mid \hat{y}_{t-1}\right)$ is computed using:

$$p\left(z_{2t}, z_{2t-1} \mid \underline{\hat{y}}_{t-1}\right) = p\left(z_{2t} \mid z_{2t-1}, \underline{\hat{y}}_{t-1}\right) \times p\left(z_{2t-1} \mid \underline{\hat{y}}_{t-1}\right)$$

   where the first term of the right-hand side is a sum of entries of the transition matrix $\{\pi_{ij,t-1}\}$ and the second term is the input.

2. The joint conditional density $f(\hat{y}_t, z_{2t}, z_{2t-1} \mid \underline{\hat{y}}_{t-1})$ is then given by:

$$f(\hat{y}_t, z_{2t}, z_{2t-1} \mid \underline{\hat{y}}_{t-1}) = f(\hat{y}_t \mid z_{2t}, z_{2t-1}, \underline{\hat{y}}_{t-1}) \times p\left(z_{2t}, z_{2t-1} \mid \underline{\hat{y}}_{t-1}\right)$$

   where

$$\begin{aligned} f(\hat{y}_t \mid z_{2t}, z_{2t-1}, \underline{\hat{y}}_{t-1}) &= f(\tilde{y}_t, R_{1t}, z_{1t} \mid z_{2t}, z_{2t-1}, \underline{\hat{y}}_{t-1}) \\ &= f(\tilde{y}_t, R_{1t} \mid z_{1t}, z_{2t}, z_{2t-1}, \underline{\hat{y}}_{t-1}) \times p(z_{1t} \mid z_{2t}, z_{2t-1}, \underline{\hat{y}}_{t-1}) \end{aligned}$$

   with

$$p(z_{1t} \mid z_{2t}, z_{2t-1}, \underline{\hat{y}}_{t-1}) = \frac{p(z_{1t}, z_{2t} \mid z_{2t-1}, \underline{\hat{y}}_{t-1})}{p(z_{2t} \mid z_{2t-1}, \underline{\hat{y}}_{t-1})}$$

   and all the terms can be computed.

3. The conditional density$f(\hat{y}_t \mid \underline{\hat{y}}_{t-1})$ is given by:

$$f(\hat{y}_t \mid \underline{\hat{y}}_{t-1}) = \sum_{z_{2,t}} \sum_{z_{2,t-1}} f(\hat{y}_t, z_{2t}, z_{2t-1} \mid \underline{\hat{y}}_{t-1}).$$

4. The joint density $p\left(z_{2t}, z_{2t-1} \mid \underline{\hat{y}}_t\right)$ comes from:

$$p\left(z_{2t}, z_{2t-1} \mid \underline{\hat{y}}_t\right) = \frac{f(\hat{y}_t, z_{2t}, z_{2t-1} \mid \underline{\hat{y}}_{t-1})}{f(\hat{y}_t \mid \underline{\hat{y}}_{t-1})}.$$

5. And eventually:

$$p\left(z_{2t} \mid \underline{\hat{y}}_t\right) = \sum_{z_{2,t-1}} p\left(z_{2t}, z_{2t-1} \mid \underline{\hat{y}}_t\right).$$

G. About the eigenvectors of the rating-migration matrix Π

In this appendix, using the notations presented in Subsection 8.3, we outline some properties of matrices Π and V. For notational simplicity, we drop arguments and time subscripts associated with these matrices.

- As the sum of the entries of each line of Π is equal to 1, the vector $\left[\begin{array}{ccc} 1 & \cdots & 1 \end{array}\right]'$ is an eigenvector of Π associated with the eigenvalue 1. Consequently, since this eigenvalue is supposed to be the last one appearing in Ψ, the last column of V –that collects the eigevectors of Π– is proportional to $\left[\begin{array}{ccc} 1 & \cdots & 1 \end{array}\right]'$.

- The fact that default is an absorbing state implies that the last row of Π is $\left[\begin{array}{cccc} 0 & \cdots & 0 & 1 \end{array}\right]$. Since we have $\Pi V = V\Psi$, it comes (considering the last line of this equation):

$$\forall j \quad V_{K,j} = V_{K,j} \exp\left(-\psi_j\right),$$

which implies: $\forall j < k \; V_{K,j} = 0$.

- The two previous points imply that the matrix V admits the following form:

$$V = \left[\begin{array}{cccc} V_{1,1} & \cdots & V_{1,K-1} & \gamma \\ \vdots & \ddots & \vdots & \vdots \\ V_{K-1,1} & \cdots & V_{K-1,K-1} & \gamma \\ 0 & \cdots & 0 & \gamma \end{array}\right]$$

Since $VV^{-1} = I$, we have (considering the last line)

$$\left[\begin{array}{cccc} V^{-1}_{K,1} & \cdots & V^{-1}_{K,,K-1} & V^{-1}_{K,K} \end{array}\right] = \left[\begin{array}{cccc} 0 & \cdots & 0 & \frac{1}{\gamma} \end{array}\right]$$

and, therefore, for $i = 1, \ldots, K$, we have $V_{i,K}V^{-1}_{K,K} = 1$.

- We have $V^{-1}\Pi = \Psi \; V^{-1}$. By multiplying both sides of the last equality by $\left[\begin{array}{ccc} 1 & \cdots & 1 \end{array}\right]'$, one gets:

$$\forall i < k \quad \sum_{j=1}^{K} V^{-1}_{i,j} = \exp\left(-\psi_i\right) \sum_{j=1}^{K} V^{-1}_{i,j}.$$

For this to be satisfied for any $\exp\left(-\psi_i\right)$ (if ψ_i is time-varying), we need to have $\forall j < K \; \sum_{j=1}^{K} V^{-1}_{i,j} = 0$.

H. Yield data

The estimation of our model requires zero-coupon yields. However, governments usually issue coupon-bearing bonds. This appendix details the methodology implemented to compute zero-coupon yield curves (out of coupon-bearing yields to maturity). Note that this appendix does not use systematically the same notations as in the rest of the paper. For an overview of the different methodologies used to perform such yield-curve conversions, see BIS, 2005[21].

H.1. Parametric forms

As Gurkaynak, Sack and Wright (2005) [81], we resort to a parametric approach which relies on the functional forms proposed by Nelson and Siegel (1987) [117] and extended by Svensson (1994) [125]. In the latter case, the yield of a zero-coupon bond with a time to maturity m for a point in time t is given by:[56]

$$\begin{aligned} y^m_t(\theta) \quad = \quad & \beta_0 + \beta_1 \left(-\frac{\tau_1}{m}\right)\left(1 - \exp(-\frac{m}{\tau_1})\right) + \beta_2 \left[\left(\frac{\tau_1}{m}\right)\left(1 - \exp(-\frac{m}{\tau_1})\right) - \exp(-\frac{m}{\tau_1})\right] \\ & + \beta_3 \left[\left(\frac{\tau_2}{m}\right)\left(1 - \exp(-\frac{m}{\tau_2})\right) - \exp(-\frac{m}{\tau_2})\right] \end{aligned}$$

where θ is the vector of parameters $[\beta_0, \beta_1, \beta_2, \beta_3, \tau_1, \tau_2]'$.

[56] The Svensson's model boils down to the Nelson and Siegel's one when $\beta_3 = 0$.

H.2. Estimation of the parameters

Formally, assume that for a given country and a given date t, we dispose of observed prices of N coupon-bearing bonds (with fixed coupon), denoted by $P_{1,t}, P_{2,t}, \ldots, P_{N,t}$. Let us denote by $CF_{k,i,t}$ the i^{th} (on n_k) cash flows that will be paid by the k^{th} bond at the date $\tau_{k,i}$. We can use the zero-coupon yields $\{y_t^m(\theta)\}_{m\geq 0}$ to compute a modeled (dirty) price $\hat{P}_{k,t}$ for this k^{th} bond:

$$\hat{P}_{k,t}(\theta) = \sum_{i=1}^{n_k} CF_{k,i,t} \exp\left(-\tau_{k,i} y_t^{\tau_{k,i}-t}(\theta)\right).$$

The approach then consists in looking for the vector θ that minimizes the distance between the N observed prices and modeled bond prices. Specifically, the vector θ_t is given by:

$$\theta_t = \arg\min_{\theta} \sum_{k=1}^{N} \omega_k (P_{k,t} - \hat{P}_{k,t}(\theta))^2$$

where the ω_k's are some weights that are chosen with respect to the preferences that one may have regarding the fit of different parts of the yield curve. Intuitively, taking the same value for all the ω_k's would lead to large yield errors for financial instruments with relatively short remaining time to maturity. This is linked to the concept of duration (i.e. the elasticity of the price with respect to one plus the yield): a given change in the yield corresponds to a small/large change in the price of a bond with a short/long term to maturity or duration. Since we do not want to favour a particular segment of the yield-curve fit, we weight the price error of each bond by the inverse of the remaining time to maturity.[57]

H.3. Data and preliminary filters

Table 4 gives some details about the data and methods that are used to compute the zero-coupon yield curves for each country. It appears that the number of bonds used widely differ among the countries (from 19 bonds for the Netherlands to 175 bonds for Germany).

As in Gurkaynak et al. (2005) [81], different filters are applied in order to remove those prices that would obviously bias the obtained yields. The prices of bonds that were issued before 1990 or that have atypical coupons (below 1% or above 10%) are excluded, as well as prices that are far from par (above 140 or below 80). In addition, the prices of bonds that have a time to maturity lower than 1 month are excluded.[58] When the time to maturity is comprised between 1 month and 12 months, the price is also excluded if the observed yield is 100 bp above or below the 3-month general-collateral repo rate.

Afterwards, additional exclusions of potential outliers are based on an ad-hoc filtering approach: at each date, a least-square cubic-spline fitting algorithm is applied on the available yields-to-maturity (with breakpoints set at maturities 0, 10, 20 and 30 years). The standard error of the deviations between the spline and the observed yields is then computed and the yields to maturity that are further than 1.5 standard deviations away from the spline are excluded.

Finally, in order to deal with the lack of data at the short end of the yield curve for some countries (reported in the last column of Table 4), we include the 3-month general-collateral

[57] Using remaining time to maturity instead of duration has not a large effect on estimated yields as long as we are not concerned with the very long end of the yield curve.

[58] The condition on the remaining time to maturity stems from the fact that the trading volume of a bond usually decreases considerably when it approaches its maturity date.

Table 4: **Zero-coupon yield curves: data and method used by country**

Notes: The paametric form that is fitted for each date is given in the second column for each country. The total number of bonds available by country is given in the third column; the number of bonds used at each date is obviously smaller; note that number of observations available for each date vary considerably over time. The fourth column indicates whether the use of repo rates is allowed in the estimation process.

	fitted parametric form	Number of bonds	Use of GC repo rates
Austria	Nelson-Siegel	38	
Germany	Svensson	175	
France	Svensson	120	
Italy	Svensson	168	
Netherlands	Nelson-Siegel	19	X
Spain	Nelson-Siegel	22	X
Portugal	Nelson-Siegel	36	X
Greece	Svensson	105	X
Belgium	Svensson	162	
Ireland	Nelson-Siegel	30	X
Finland	Svensson	53	

repo rate among the observed yields when less than 3 shorter-term bond prices (with time to maturity lower than 2 years) are observed at a given date.