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Constrained min-max predictive control: a polynomial-time approach

T. Alamo, D. Muñoz de la Peña, D. Limon and E.F. Camacho

Departamento de Ingeniería de Sistemas y Automática, Universidad de Sevilla
Escuela Superior de Ingenieros, Camino de los Descubrimientos s/n. 41092 Sevilla, SPAIN
Telephone: +34 954487347 Fax: +34 954487340
email: {alamo,davidmps,limon,eduardo}@cartuja.us.es

***Abstract*— In this work, an efficient way of implementing a constrained min-max predictive controller is presented. The new approach modifies the objective function in such a way that the resulting min-max problem can be solved in polynomial time. Different modifications are proposed. The main contribution of the paper is to provide a robust constrained min-max predictive controller that can be implemented in real time. The new controller stabilizes the uncertain system. Key words: Predictive control of linear systems, Robust control, Optimization algorithms, min-max.**

I. Introduction

Predictive control is a popular strategy and algorithms that handle constraints and guarantee closed loop stability are given [6], [10]. In this paper, the min-max predictive control is addressed. This formulation takes into account the disturbances, optimizing a control profile over all possible disturbances.

In general, solving a min-max problem subject to constraints and disturbances is computationally too demanding for practical implementation. The complexity of min-max MPC can be addressed by the use of multi-parametric programming [3], [7], [12]. The number of regions needed to characterize the solution to the problem may grow in an exponential way with the prediction horizon. Fortunately, the computation of the explicit solution is made off-line and the evaluation of the controller can be made by means of a binary tree.

Due to the fact that the computation of the explicit solution can be realized off-line it it possible to manage a In this paper, an efficient way of implementing a constrained quadratic min-max predictive controller is presented. The new approach relies on a slight modification of the objective function. This modification allows us to compute the max function in polynomial time.

The implementation of the proposed controller is based on a cutting plane scheme, in which it is necessary to compute the max function at each step of the algorithm. Taking into account that the number of steps is polynomial on the number of decision variables and that the maximum of the modified objective can be obtained in polynomial time, we conclude that the proposed min-max problem can be solved in polynomial time. The proposed controller inherits the stability and robustness of the standard min max controller. The system state is ultimately bounded under the new controller.

The paper is organized as follows: In section 2 the min-max problem for bounded uncertainties is stated. The computational complexity of the standard min max controller is analyzed in section 3. In section 4 we show that there are instances in which solving the max problem can be made in polynomial time. A slight modification of the functional is proposed in section 5. This modification allows us to compute the control action corresponding to the min-max problem in polynomial time. The stability of the proposed controller is addressed in section 6. The paper draws to a close with a section of conclusions.

II. Min-Max MPC with global uncertainties

Consider the discrete-time linear system with bounded uncertainties:

$$x_{k+1} = Ax_k + Bu_k + Dw_k \tag{1}$$

where $x_k \in \mathbb{R}^{n_x}$ is the state, $u_k \in \mathbb{R}^{n_u}$ is the control input, and $w_k \in W$ is the uncertainty, that is supposed to be bounded by a hypercube, that is, $W = \{ w \in \mathbb{R}^q : \|w\|_\infty \leq \varepsilon \}$.

In the following, it will be assumed that the control input is given by $u_k = Kx_k + v_k$, where K is chosen in order to achieve some desired property for the non constrained problem. In this way, and without increasing the complexity of the problem, some amount of feedback is provided to the predictions [8], [9]. The MPC controller will compute the sequence of correction control signals along the control horizon $\{v_0, \dots, v_{Nc-1}\}$. Defining $A_K = (A+BK)$, the dynamics of the system can be rewritten as

$$x_{k+1} = A_K x_k + Bv_k + Dw_k$$

In what follows, W_N will denote the set of possible disturbance sequences of length N. The objective function is:

$$V_N(x, \mathbf{v}, \mathbf{w}) = \sum_{j=0}^{N-1} [x_j^T Q x_j + u_j^T R u_j] + x_N^T P x_N$$

where $\mathbf{v} = \{v_0, v_1, \dots, v_{N_c-1}\}$ represents the sequence of N_c correction control inputs,

$$\mathbf{w} = \{w_0, w_1, \dots, w_{N-1}\} \in W_N$$

represents a possible sequence of input disturbances to the system. On the other hand, x_j and u_j are the predicted state and control input respectively for a given $\mathbf{w} \in W_N$:

$$x_j = A_K^j x + \sum_{i=1}^{j} A_K^{i-1} B v_{j-i} + \sum_{i=1}^{j} A_K^{i-1} D w_{j-i}$$
$$u_j = K x_j + v_j$$
$$v_j = 0, \; j \geq N_c$$

We consider linear constraints in state and input, $x_k \in X$, $u_k \in U$. In order to achieve stability, a polytopic terminal region constraint ($x_N \in \Omega$) will also be added [10]. Terminal region Ω should be chosen to be a robust admissible invariant set for the system. That is, $\Omega \subseteq X$ must satisfy the following constraints:

- **C1**: If $x \in \Omega$ then $A_K x + Dw \in \Omega$, for every $\omega \in W$.
- **C2**: If $x \in \Omega$ then $Kx \in U$.

In order to achieve convergence to a bounded set containing the origin, the terminal cost $x^\top P x$ will be assumed to satisfy:

- **C3**: $P - A_K^\top P A_K > Q + K^\top R K$

It is important to note that the stability of $A_K = A + BK$ guarantees the existence of a finite matrix P satisfying C3.

In this way, the min-max constrained predictive controller results in the solution of the following min-max optimization problem (denoted $P_N(x)$):

$$J_N^*(x) = \min_{\mathbf{v}} \max_{\mathbf{w} \in W_N} V_N(x, \mathbf{v}, \mathbf{w})$$

$$s.t. \begin{cases} x_j \in X, & \forall \mathbf{w} \in W_N, \quad j = 0, \ldots, N-1 \\ x_N \in \Omega, & \forall \mathbf{w} \in W_N \\ u_j \in U, & \forall \mathbf{w} \in W_N, \quad j = 0, \ldots, N-1 \end{cases}$$

This optimization problem is solved at each sample instant. An optimal vector of control correction signals $\mathbf{v}^*$ is obtained and the control input $u_0 = Kx + v_0^* = K_{MPC}(x)$ is applied.

III. Computational complexity of the standard min-max controller

Under the proposed assumptions, it is possible to obtain a set of linear constraints that do not depend on the uncertainty vector $\mathbf{w}$ and that guarantee the robust fulfillment of the constraints.

The objective function $V_N(x, \mathbf{v}, \mathbf{w})$ is a quadratic function of x, $\mathbf{v}$ and $\mathbf{w}$. That is, matrices H_x, H_v and H_w can be found in such a way that $V_N(x, \mathbf{v}, \mathbf{w}) = \|H_x x + H_v \mathbf{v} + H_\omega \mathbf{w}\|_2^2$. We conclude that the min-max problem $P_N(x)$ can be rewritten as the following:

$$J_N^*(x) = \min_{\mathbf{v}} \max_{\mathbf{w} \in W_N} \|H_x x + H_v \mathbf{v} + H_\omega \mathbf{w}\|_2^2$$
$$\text{s.t.} \quad G_x x + G_v \mathbf{v} \leq d$$

Denote the max-function $V_N^*(x, \mathbf{v})$ as:

$$V_N^*(x, \mathbf{v}) = \max_{\mathbf{w} \in W_N} V_N(x, \mathbf{v}, \mathbf{w})$$

Taking into account the convex nature of $V_N(x, \mathbf{v}, \mathbf{w})$, $V_N^*(x, \mathbf{v})$ can be obtained evaluating $V_N(x, \mathbf{v}, \mathbf{w})$ at the 2^{qN} vertices of the hypercube W_N. This implies a computational time that grows exponentially with qN, rending the computation of the optimal control sequence $\mathbf{v}^*$ a very hard task that extremely complicates the implementation of the min-max constrained predictive control in a real application.

IV. Approximating the max function

Given x and $\mathbf{v}$, $V_N(x, \mathbf{v}, \mathbf{w})$ can be rewritten as a quadratic function on $\mathbf{w}$. That is:

$$V_N(x, \mathbf{v}, \mathbf{w}) = \|H_x x + H_v \mathbf{v} + H_\omega \mathbf{w}\|_2^2 =$$
$$= \mathbf{w}^\top M \mathbf{w} + q(x, \mathbf{v})^\top \mathbf{w} + V_N(x, \mathbf{v}, 0)$$

where $M = H_w^\top H_w$ and $q(x, \mathbf{v}) = 2 H_w^\top (H_x x + H_v \mathbf{v})$. Therefore, the computation of $\tilde{V}_N^*(x, \mathbf{v})$ belongs to the following class of maximization problem:

$$\mu^* = \max_{\|\mathbf{w}\|_\infty \leq \varepsilon} \mathbf{w}^\top \tilde{M} \mathbf{w} + q^\top \mathbf{w}$$

In principle, computing μ^* is a NP-problem. However, there are some instances in which μ^* can be calculated in polynomial time. The complexity of the computation can be dramatically reduced if matrix $\tilde{M} \in \mathbb{R}^{n \times n}$, where $n = Nq$, belongs to one of the following categories:

1) $\tilde{M}$ **is a positive definite diagonal matrix**: In this case, $\mu^* = \varepsilon^2 \text{ tr } \tilde{M} + \varepsilon \|q\|_1$ and the maximum is attained at $\mathbf{w}^* = \varepsilon \text{ sign } (q)$.
2) $\tilde{M}$ **is semidefinite negative**: In this case,
$$\mu^* = -\min_{\|\mathbf{w}\|_\infty \leq \varepsilon} (\mathbf{w}^\top (-\tilde{M}) \mathbf{w} - q^\top \mathbf{w})$$
It results that $-\tilde{M}$ is semidefinite positive. Therefore μ^* can be computed solving a quadratic convex problem. This can be accomplished in polynomial time.
3) **All the elements of $\tilde{M}$ are non negative**: In this case, it is well known (see [11], [2]) that the maximization problem can be posed as a min cut graph problem. This graph problem can be solved in polynomial time $O(n^3)$.
4) $\tilde{M}$ **is a positive definite band matrix**: We say that $\tilde{M}$ is a L-band matrix if $|i - j| \geq L$, implies $\tilde{M}_{ij} = 0$. It has been recently shown (see [1]) that under the assumption of band structure, the maximization problem can be solved in polynomial time $O(n^2 2^L)$.

The next figure compares the computational burden of the standard max problem (denoted full matrix in the figure) with the computational burden required when matrix $\tilde{M}$ belongs to one of the four categories considered in this section. (Note that the figure is in a logarithmic scale).

V. Modification of the functional

In this section, we propose a modification of the functional that allows us to solve the min max problem in polynomial time while preserving the stability and robustness properties of the standard approach. The new functional is an upper bound of the original one and will be denoted $\tilde{V}_N(x, \mathbf{v}, \mathbf{w})$. The proposed functional differs only in a quadratic term on $\mathbf{w}$ and a constant:

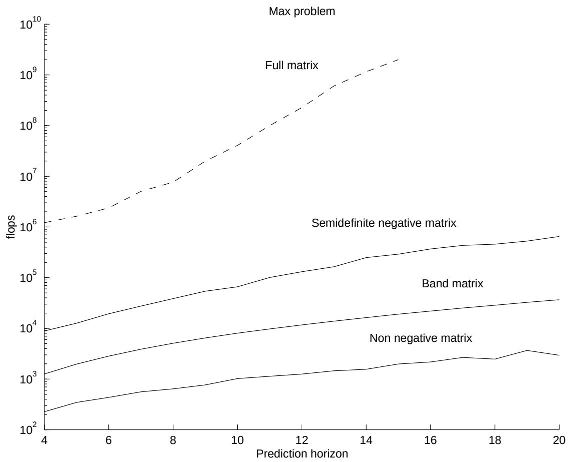

Fig. 1. Computational burden of the max problem

$$\tilde{V}_N(x,\mathbf{v},\mathbf{w}) = V_N(x,\mathbf{v},\mathbf{w}) + \mathbf{w}^\top F\mathbf{w} + c\varepsilon^2 =$$

$$= \mathbf{w}^\top (M+F)\mathbf{w} + q(x,\mathbf{v})^\top \mathbf{w} + V_N(x,\mathbf{v},0) + c\varepsilon^2$$

The modified max function will be denoted:

$$\tilde{V}_N^*(x,\mathbf{v}) = \max_{\|\mathbf{w}\|_\infty \leq \varepsilon} \tilde{V}_N(x,\mathbf{v},\mathbf{w})$$

Choosing conveniently F and c the following assumption will be satisfied:

- **C4**: $V_N^*(x,\mathbf{v}) \leq \tilde{V}_N^*(x,\mathbf{v}) \leq V_N^*(x,\mathbf{v}) + \sigma\varepsilon^2$

Note that $\sigma\varepsilon^2$ bounds the difference between the modified functional and the original one. We will show how to compute F and c in such a way that C4 is satisfied and σ minimized. For that purpose, it is important to introduce the following lemma:

Lemma 1: Let us suppose that $T \geq 0$ is a diagonal matrix and that at least one of the following hypotheses is satisfied:

1) $0 \leq F \leq T$ and $c = 0$.
2) $0 \geq F \geq -T$ and $c = \text{ tr } T$.

Then making $\sigma = \text{ tr } T$, $C4$ is satisfied.

Proof:

Let us suppose that hypothesis (1) holds. Then from $F \geq 0$ it is inferred that $\mathbf{w}^\top F\mathbf{w} \geq 0$ and therefore $V_N^*(x,\mathbf{v}) \leq \tilde{V}_N^*(x,\mathbf{v})$. From $F \leq T$ and the diagonal nature of T it is inferred that

$$\mathbf{w}^\top F\mathbf{w} \leq \mathbf{w}^\top T\mathbf{w} \leq (\text{ tr } T)\varepsilon^2$$

Thus, $\tilde{V}_N^*(x,\mathbf{v}) \leq V_N^*(x,\mathbf{v}) + (\text{ tr } T)\varepsilon^2$.

Let us now suppose that hypothesis (2) holds:

$$\mathbf{w}^\top F\mathbf{w} \geq -\mathbf{w}^\top T\mathbf{w} \geq -(\text{ tr } T)\varepsilon^2$$

From this inequality and the fact that $c = \text{ tr } T$ it is inferred that $\mathbf{w}^\top F\mathbf{w} + c\varepsilon^2 \geq 0$. Thus, $V_N^*(x,\mathbf{v}) \leq \tilde{V}_N^*(x,\mathbf{v})$. On the other hand, $0 \geq F$ implies that $\mathbf{w}^\top F\mathbf{w} \leq 0$ and therefore: $\tilde{V}_N^*(x,\mathbf{v}) \leq V_N^*(x,\mathbf{v}) + (\text{ tr } T)\varepsilon^2$. ■

In what follows, and using previous lemma, we will show how to choose F and c in such a way that $\tilde{M} = M + F$ belongs to one of the categories for which the max problem can be solved in polynomial time.

1) $\tilde{M} = M + F$ **is positive definite and diagonal:**
Let us suppose that S and T are diagonal matrices that minimize the following LMI problem:

$$\min \ \text{tr } T$$

$$\text{s.t.} \quad 0 \leq S - M \leq T$$

Then, making $F = S - M$, $c = 0$ and $\sigma = \text{ tr } T$ it results that $\tilde{M} = M + F = T$ is a positive definite diagonal matrix and $C4$ is satisfied.

2) $\tilde{M} = M + F$ **is semidefinite negative:**
Obtain matrix F and a diagonal matrix T such that the following LMI problem is solved:

$$\min \ \text{tr } T$$

$$\text{s.t.} \quad \begin{cases} 0 \geq F \geq -T \\ M + F \leq 0 \end{cases}$$

Then, making $c = \sigma = \text{ tr } T$ it results that $\tilde{M} = M + F$ is semidefinite negative and C4 holds.

3) $\tilde{M} = M + F$ **has non negative elements**
Note that there are systems for which M has non negative elements (see [2]). In this case no approximation is required and $F = 0$, $c = \sigma = 0$. If M does have negative elements, compute F and diagonal matrix T that solve the following LMI problem:

$$\min \ \text{tr } T$$

$$\text{s.t.} \quad \begin{cases} 0 \leq F \leq T \\ M_{i,j} + F_{i,j} \geq 0 \quad \forall i, \forall j \end{cases}$$

Then making $c = 0$ and $\sigma = \text{ tr } T$ it results that $\tilde{M} = M + F$ has nonnegative elements and C4 holds.

4) $\tilde{M} = M + F$ **is a semipositive definite Band matrix:**
Let us suppose that F and the diagonal matrix T solve the following LMI problem:

$$\min \ \text{tr } T$$

$$\text{s.t.} \quad \begin{cases} 0 \leq F \leq T \\ M_{i,j} + F_{i,j} = 0 \quad \forall |i-j| \geq L \end{cases}$$

Then making $c = 0$ and $\sigma = \text{ tr } T$ it results that $\tilde{M} = M + F$ is a positive definite L-band matrix and C4 holds.

Note that the control input to the system $u_k = Kx_k + v_k$ is chosen in such a way that $A_K = A + BK$ is stable. This implies that the elements M_{ij} of matrix M vanish with the absolute value of $|i-j|$. Thus, the original matrix M can be approximated by an L-band and therefore the value of σ, that measures the difference between $V_N^*(x,\mathbf{v})$ and $\tilde{V}_N^*(x,\mathbf{v})$, decreases in an exponential way with the width of the band matrix.

VI. PROPOSED MIN-MAX PREDICTIVE CONTROL: STABILITY

The new min-max problem (that will be denoted $\tilde{P}_N(x)$) is stated as

$$\tilde{J}_N^*(x) = \min_{\mathbf{v}} \tilde{V}_N^*(x, \mathbf{v})$$
$$\text{s.t.} \quad G_x x + G_v \mathbf{v} \leq d$$

where $\tilde{V}_N(x, \mathbf{v})$ denotes one of the approximations of the max function proposed in the last section. Note that the feasibility region of $\tilde{P}_N(x)$ equals the one of the standard min-max problem.

This optimization problem can be solved using the ellipsoid method or any other cutting plane algorithm [4]. The number of evaluations of $\tilde{V}_N^*(x, \mathbf{v})$ needed to obtain a solution to the problem with a given accuracy is a polynomial on the number of decision variables ($N_c n_u$). Thus, taking into account that the number of operations needed to compute $\tilde{V}_N^*(x, \mathbf{v})$ is a polynomial on the prediction horizon, the overall complexity of the modified min max problem is polynomial on the number of decision variables and the prediction horizon.

The following property, which is proved in appendix A, plays an important role when analyzing the stability of the proposed controllers.

Property 1: Denote

$$\Gamma(x, w) = \|x\|_P^2 - \|A_K x + Dw\|_P^2 - x^\top Q^* x + \gamma \varepsilon^2$$

where $Q^* = Q + K^\top RK$. If the system $x_{k+1} = A_K x_k$ is asymptotically stable then there exists a positive definite matrix P and a positive scalar γ such that $C3$ is satisfied and

$$\Gamma(x, w) \geq 0, \ \forall x \in \mathbb{R}^n, \ \forall w \in W \qquad (2)$$

Moreover, given $\gamma > 0$, the problem of determining if there exists $P > 0$ such that equation (2) holds can be formulated as a LMI problem.

As it is stated in the following theorem, the new min-max controller guarantees that the uncertain system evolves to a bounded set that contains the origin. The size of this set depends on the size of the uncertainty.

Theorem 1: Let us suppose that

1) Assumptions C1, C2, C3 and C4 are satisfied.
2) $\mathbf{v}^* = \{v_0^*, v_1^*, \ldots, v_{N_c-1}^*\}$ is the optimal solution to problem $\tilde{P}_N(x)$.

Then the min-max controller ($\tilde{K}_{MPC}(x) = Kx + v_0^*$) guarantees that the state system is ultimately bounded and:

1) $\mathbf{v}_s = \{v_1^*, v_2^*, \ldots, v_{N_c-1}^*, 0\}$ is a feasible solution to problem $\tilde{P}_N(Ax + B\tilde{K}_{MPC}(x) + Dw)$, $\forall w \in W$.
2) $\tilde{J}_N^*(x) - \tilde{J}_N^*(Ax + B\tilde{K}_{MPC}(x) + Dw) > x^\top Qx - (\gamma + \sigma)\varepsilon^2$, $\forall w \in W$.

This theorem is proved in appendix B. It is important to note that γ and σ do not depend on the size of the uncertainty. We conclude that the proposed modification of the functional allows us to implement the min-max controller in polynomial time while preserving the stability of the standard min max MPC controller.

VII. CONCLUSIONS

In this paper we presented a new formulation of the min-max predictive control. The new controller is based on a modification of the functional that allows us to compute the max function in polynomial time. This allows us the implementation of the min-max predictive control in real applications. It has been shown that the proposed controller guarantees the robust satisfaction of the constraints and the convergence to a bounded set that contains the origin.

The complexity of the new approach depends on the prediction horizon and the class of modified functional obtained. The next figure shows the dramatic reduction in complexity for each of the proposed modifications (note that the figure is in a logarithmic scale).

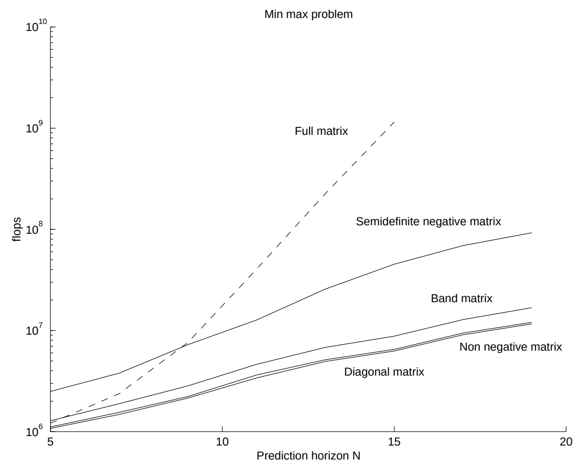

Fig. 2. Complexity of the proposed min-max controllers

VIII. ACKNOWLEDGEMENTS

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APPENDIX

Proof of property 1

Denote $S = P - A_K^\top P A_K - Q^*$ and

$$\Gamma(x,w) = x^\top P x - \|A_K x + Dw\|_P^2 - x^\top Q^* x + \gamma\varepsilon^2 =$$
$$= x^\top S x - 2x^\top A_K^\top P D w - w^\top D^\top P D w + \gamma\varepsilon^2$$

From the hypothesis C3, it is clear that S is definite positive. Therefore, $\Gamma(x,w)$ is a convex function on x. As a matter of fact, it can be easily shown that the minimum of $\Gamma(x,w)$ is attained at $x^* = S^{-1} A_K^\top P D w$. Thus,

$$\Gamma(x,w) \geq -w^\top D^\top (P A_K S^{-1} A_K^\top P + P) D w + \gamma\varepsilon^2$$

Taking into account that the term on the right is a concave function on w, it is concluded that $\Gamma(x,w) \geq 0$, if and only if

$$\gamma\varepsilon^2 \geq \max_{\omega \in \text{vert}\{W\}} w^\top D^\top (P A_K S^{-1} A_K^\top P + P) D w$$

where vert$\{W\}$ denotes the vertexes of W. Note that S is definite positive. This implies that there is a finite value of γ that satisfies previous inequality. Dividing last inequality by ε^2, the following equivalent inequality is obtained:

$$\gamma \geq \max_{\vartheta \in \text{vert}\{B_1^q\}} \vartheta^\top D^\top (P A_K S^{-1} A_K^\top P + P) D \vartheta$$

where vert$\{B_1^q\}$ denotes the vertexes of the unit hypercube in $\mathbb{R}^q$. Using Schur's complement [5], this is satisfied if and only if for every $\vartheta \in \text{vert}\{B_1^q\}$:

$$\begin{bmatrix} \gamma - \vartheta^\top D^\top P D \vartheta & \vartheta^\top D^\top P A_K \\ A_K^\top P D \vartheta & P - A_K^\top P A_K - Q^* \end{bmatrix} > 0 \qquad (3)$$

■

Stability of the proposed min-max controller

(1) Let us suppose that given x_k, $\mathbf{v} = \{v_0, v_1, \ldots, v_{N_c-1}\}$ satisfies the constraints of problem $\tilde{P}_N(x_k)$. In what follows, it will be shown that $\mathbf{v}_s = \{v_1, v_2, \ldots, v_{N_c-1}, 0\}$ satisfies the constraints of the problem at sample time $k+1$. Note that $x_{k+1} = A_K x_k + B v_0 + D w_k$, where $w_k \in W$. This means that given x_{k+1} and $\mathbf{w}_a \in W_N$, there is $\mathbf{w}_b \in W_N$ such that for every $0 \leq j \leq N-1$:

$$x_j(x_{k+1}, \mathbf{v_s}, \mathbf{w}_a) = x_{j+1}(x_k, \mathbf{v}, \mathbf{w}_b) \in X \qquad (4)$$
$$u_j(x_{k+1}, \mathbf{v_s}, \mathbf{w}_a) = u_{j+1}(x_k, \mathbf{v}, \mathbf{w}_b) \in U \qquad (5)$$

Thus, it only remains to show that $x_N(x_{k+1}, \mathbf{v}_s, \mathbf{w}) \in \Omega$, $\forall \mathbf{w} \in W_N$. Note that using the previous equalities, it can be affirmed that $x_{N-1}(x_{k+1}, \mathbf{v}_s, \mathbf{w}) \in \Omega$, $\forall \mathbf{w} \in W_N$. Taking into account the assumptions on Ω, it is inferred that $x_N(x_{k+1}, \mathbf{v}_s, \mathbf{w})$ will also be contained in Ω in spite of the uncertainty.

(2) From equations (4) and (5) it is inferred that given x_{k+1} and $\mathbf{w}_a$, it is always possible to obtain $\mathbf{w}_b$ such that

$$V_N(x_{k+1}, \mathbf{v}_s, \mathbf{w}_a) - V_N(x_k, \mathbf{v}, \mathbf{w}_b) =$$
$$= \|x_{N-1}(x_{k+1}, \mathbf{v}_s, \mathbf{w}_a)\|_{Q^*}^2 +$$
$$+ \|x_N(x_{k+1}, \mathbf{v}_s, \mathbf{w}_a)\|_P^2$$
$$-x_k^\top Q x_k - u_k^\top R u_k -$$
$$- \|x_{N-1}(x_{k+1}, \mathbf{v}_s, \mathbf{w}_a)\|_P^2$$

It is clear that there is $\hat{w} \in W$ such that

$$x_N(x_{k+1}, \mathbf{v}_s, \mathbf{w}_a) =$$
$$A_K x_{N-1}(x_{k+1}, \mathbf{v}_s, \mathbf{w}_a) + D\hat{w}$$

Thus, applying property 2, it is readily obtained that

$$V_N(x_{k+1}, \mathbf{v}_s, \mathbf{w}_a) - V_N(x_k, \mathbf{v}, \mathbf{w}_b) <$$
$$-x_k^\top Q x_k + \gamma\varepsilon^2$$

From this inequality it is easily inferred that

$$V_N^*(x_{k+1}, \mathbf{v}_s) - V_N^*(x_k, \mathbf{v}) < -x_k^\top Q x_k + \gamma\varepsilon^2$$

Taking into account that $\mathbf{v}_s$ is a feasible solution to problem $\tilde{P}_N(A_K x + B v_0^* + Dw)$:

$$\tilde{J}_N(x_{k+1}) \leq \tilde{V}_N^*(x_{k+1}, \mathbf{v}_s) \leq V_N^*(x_{k+1}, \mathbf{v}_s) + \sigma\varepsilon^2$$
$$< V_N^*(x_k, \mathbf{v}^*) - x_k^\top Q x_k + (\gamma + \sigma)\varepsilon^2$$
$$\leq \tilde{V}_N^*(x_k, \mathbf{v}^*) - x_k^\top Q x_k + (\gamma + \sigma)\varepsilon^2 =$$
$$= \tilde{J}_N^*(x_k) - x_k^\top Q x_k + (\gamma + \sigma)\varepsilon^2$$

Define

$$\tilde{\Phi}_\varepsilon = \{x \in \mathbb{R}^n \ : \ \tilde{P}_N(x) \text{ is feasible}$$
$$\text{and } x^\top Q x \leq (\gamma + \sigma)\varepsilon^2\}$$

Then, the system evolves to set $\Omega_\alpha = \{\, x \in \mathbb{R}^n : \tilde{J}_N^*(x) \leq \alpha(\varepsilon) \,\}$ where $\alpha(\varepsilon) = \max_{x \in \tilde{\Phi}_\varepsilon} \tilde{J}_N^*(x) + (\gamma + \sigma)\varepsilon^2$

■