

Saving behavior and wealth accumulation in a pure lifecycle model with income uncertainty

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Abstract

Several models of economic behavior currently compete for an explanation of individual wealth accumulation and savings patterns. In this paper we focus in particular upon the role of income uncertainty, and the role played by a retirement period, during which time-expected earnings are zero. We find that income uncertainty can alter savings patterns over the lifecycle significantly, with the greatest influence on the wealth of young individuals. However, its influence on the aggregate stock of wealth is less than earlier theoretical work indicates. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

The effect of uncertain income streams on savings and wealth accumulation, through a precautionary motive, has been the subject of numerous recent investigations. Much of this literature has been surveyed by Browning and Lusardi (1996) in their very extensive review. They conclude that our present understanding of the role and magnitude of precautionary savings is limited:

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while there are significant theoretical results supporting it, empirical support remains weak. For example, the theoretical work of Skinner (1988), Zeldes (1989) and Caballero (1991) implies that income uncertainty could account for as much as half of all private wealth. On the other hand, the empirical findings of Dynan (1993), Guiso et al. (1992), and Lusardi (1996) provide very little support for precautionary savings; an exception is Carroll and Samwick (1995).

Hubbard et al. (1995) argue that income uncertainty will have different effects on high- and low-(lifetime) income individuals: in the presence of asset-based means testing for social security, it may be optimal for households with low human capital to accumulate less wealth as income uncertainty increases. In contrast, households with high human capital should save more in response to income uncertainty.

The purpose of the present paper is to re-examine the class of theoretical stochastic income models which support a strong role for precautionary savings. We propose that if these are generalized to include a retirement period (at which point the income process changes), and an uncertain lifespan, different predictions can emerge. Viewed simply: if stochastic income models have no retirement phase then they may incorrectly attribute to a precautionary motive, savings which in reality are destined for a retirement period, or designed to protect individuals against extreme longevity.

Browning and Lusardi (1996, p. 1838) argue that, since the bulk of savings in the U.S. is undertaken by the wealthy and those near the end of the working phase of the lifecycle, and since these groups are less likely to be motivated by the fear of future income shocks, ‘... the precautionary motive has some role to play in explaining saving behavior, but it is unlikely to be as important as some studies suggest’.

This observation is important, for it suggests that income uncertainty can have different effects on savings behavior at different points in the lifecycle. We propose in this paper that the overall private wealth *stock* in the economy is less influenced by income uncertainty than the existing theoretical literature suggests. At the same time the savings and wealth accumulation *patterns* of individuals can be substantially altered by such uncertainty.

To illustrate this we develop a lifecycle model (with a retirement phase) where income and lifespan are uncertain. We introduce ‘impatience’, as suggested by Carroll (1992) and Laibson (1997), in order that the model replicate some stylized facts. The pure retirement motive is characterized by a target wealth level for the end of the working period. Within this framework we propose that it is the very specific choices of parameterization which have given rise to the belief that income uncertainty is responsible for a large part of the wealth stock.

A significant obstacle to lifecycle model building in the presence of income uncertainty is that closed-form solutions for consumption and saving equations can be derived only for very particular functional forms. Furthermore, the introduction of a retirement period means that the income generating process

must change at the point of retirement. We develop a model which still yields closed-form solutions for the consumption, saving and wealth accumulation equations in a fairly widely recognized framework – the maximization of expected utility over the lifecycle, where the instantaneous utility function is of the exponential type and the income stream is a random walk. Both the tractability characteristics and the peculiarities of this model are well known, e.g. Weil (1993), Van der Ploeg (1993), or Deaton (1992).

In the next section the basic model is developed and equations of motion are presented for consumption, saving and wealth. The derivations are relegated to the appendices. We also develop an exact measure of the risk premium, which provides a money metric of the utility cost of income uncertainty. This is then related to Kimball's (1990) precautionary premium. In Section 3 we simulate the model subject to a variety of assumptions and parameter values. This enables us to evaluate the relative importance of the different savings motives in explaining the aggregate wealth stock, and the flow of savings over the lifecycle. Conclusions are offered in Section 4.

2. The model

2.1. The setup

Consider an overlapping generations (OG) economy in which each agent can live for a maximum of $T + N$ periods. Individuals are identical at birth, with the same preferences and endowment (initial nonhuman wealth and future income process). These individuals may die at the end of any particular period with probability $1 - p$. Such an occurrence is termed an *accidental death*. Individuals who survive to period $T + N$ die of a *natural death* at the end of that period. Following the development of Caballero (1991), the population size is normalized at 1 for each period. Accordingly, the number of individuals dying accidentally in any period is $1 - p$, and the number of individuals having a natural death in any period is $[(1 - p)/(1 - p^{T+N})]p^{T+N}$, which is equal to the number of individuals who survive to the natural life span $T + N$. The number of births in each period is $(1 - p)/(1 - p^{T+N})$, which equals the sum of deaths from accidental and natural causes, so that the population size is maintained at 1.

The income process is

$$Y_t = \begin{cases} Y_0 + \xi_t, & t \leq T, \\ \xi_t, & t > T, \end{cases}$$

where $\{\xi_t\}_{t=0}^{T+N}$ is a random walk defined by

$$\xi_{t+1} = \xi_t + \varepsilon_{t+1}$$

with $\zeta_0 = 0$, and $\{\varepsilon_t\}_{t=1}^{T+N}$ is normally and independently distributed:

$$\varepsilon_t \sim N(0, \sigma_1^2) \quad \text{for } t \leq T,$$

$$\varepsilon_t \sim N(0, \sigma_2^2) \quad \text{for } t > T.$$

Thus, at time $t = 0$,

$$E(Y_t) = Y_0 \quad \text{for } t \leq T; \quad E(Y_t) = 0 \quad \text{for } t > T.$$

Each individual has the same tastes and these are defined by the exponential utility function which has constant absolute risk aversion (CARA). There are two stages in life: work and retirement. These are distinguished by a decline in expected income of Y_0 at the point of retirement and also a change in the income variance. A representative individual faces the following lifetime utility maximization problem:

$$\begin{aligned} V(A_0) = \max \quad & E_0 \sum_{t=1}^{T+N} -\frac{1}{\theta} e^{-\theta c_t} \left(\frac{p}{1+\delta} \right)^t \\ \text{s.t.} \quad & A_t = (1+r)A_{t-1} + Y_t - C_t, \\ & A_{T+N} \geq 0, \quad \text{given } A_0, \end{aligned} \quad (1)$$

where E_t is the expectation operator conditional on information available at time t , θ is the coefficient of absolute risk aversion, C_t is the consumption, Y_t is the income, A_t is the non-human wealth, r is the interest rate, and δ is the rate of time preference. The initial wealth A_0 is determined by the intergenerational equilibrium condition that the wealth stock of those dying in any period is passed on to those who are born in that period. We assume furthermore that such wealth is distributed evenly among the new born.¹

To facilitate the development of the results we use the following notation:

$$\alpha \equiv \frac{1}{1+r}, \quad \beta \equiv p \frac{1+r}{1+\delta}, \quad \Gamma_1^* \equiv \frac{1}{2} \theta \sigma_1^2 + \frac{1}{\theta} \ln \beta,$$

$$\Gamma_2^* \equiv \frac{1}{2} \theta \sigma_2^2 + \frac{1}{\theta} \ln \beta.$$

¹ The assumption that the inheritance is received at the beginning of the economic life can be relaxed without affecting the nature of the solutions. The assumption that bequests are equal is necessary, since this is a representative agent model. However, since we are interested primarily in the aggregate stock of wealth rather than its distribution within a given age cohort, this restriction is not too serious.

Denote

$$\bar{t}_p \equiv \frac{1-p}{1-p^{T+N}} \sum_{t=1}^{T+N} tp^{t-1} = \frac{1}{1-p} - \frac{(T+N)p^{T+N}}{1-p^{T+N}},$$

$$\bar{t}_\alpha \equiv \frac{1}{1-\alpha} - \frac{(T+N)\alpha^{T+N}}{1-\alpha^{T+N}}.$$

Given the survival rate p , \bar{t}_p is the average age of the whole population. Also denote

$$\text{pop}_p \equiv \frac{1-p}{1-p^{T+N}} \sum_{t=1}^T p^{t-1} = \frac{1-p^T}{1-p^{T+N}}, \quad \text{pop}_\alpha \equiv \frac{1-\alpha^T}{1-\alpha^{T+N}},$$

where pop_p is the size of the working population.

2.2. The optimal solution

Since the derivations and proofs are all quite long they are presented in the appendix. There we show that the maximization problem (1) gives the consumption function:²

$$C_t = Y_t - Y_0 + \frac{r}{1-\alpha^{T+N}} A_0 + \frac{1-\alpha^T}{1-\alpha^{T+N}} Y_0$$

$$+ \left(T - \frac{1-\alpha^T}{1-\alpha} \right) \frac{\Gamma_1^* - \Gamma_2^*}{1-\alpha^{T+N}}$$

$$+ (T - \bar{t}_\alpha)\Gamma_2^* + (t - T)\Gamma_1^*, \quad t \leq T,$$

$$C_t = Y_t + \frac{r}{1-\alpha^{T+N}} A_0 + \frac{1-\alpha^T}{1-\alpha^{T+N}} Y_0 + \left(T - \frac{1-\alpha^T}{1-\alpha} \right) \frac{\Gamma_1^* - \Gamma_2^*}{1-\alpha^{T+N}}$$

$$+ (T - \bar{t}_\alpha)\Gamma_2^* + (t - T)\Gamma_2^*, \quad t > T. \tag{2}$$

Let $S_t \equiv A_t - (1+r)A_{t-1}$ be saving. Then, by (2) and the budget equation, we have

$$S_t = Y_0 - \frac{r}{1-\alpha^{T+N}} A_0 - \frac{1-\alpha^T}{1-\alpha^{T+N}} Y_0 - \left(T - \frac{1-\alpha^T}{1-\alpha} \right) \frac{\Gamma_1^* - \Gamma_2^*}{1-\alpha^{T+N}}$$

$$- (T - \bar{t}_\alpha)\Gamma_2^* - (t - T)\Gamma_1^*, \quad t \leq T,$$

²We would like to thank a referee for pointing out that our formulation in an earlier version violated the Euler equation at the point of retirement. Our current solution satisfies the Euler equation and gives a smooth consumption path at any point of time.

$$S_t = -\frac{r}{1-\alpha^{T+N}}A_0 - \frac{1-\alpha^T}{1-\alpha^{T+N}}Y_0 - \left(T - \frac{1-\alpha^T}{1-\alpha}\right)\frac{\Gamma_1^* - \Gamma_2^*}{1-\alpha^{T+N}} \\ - (T - \bar{t}_\alpha)\Gamma_2^* - (t-T)\Gamma_2^*, \quad t > T. \quad (3)$$

Recursively using (3) gives the individual wealth profile:

$$A_t = \frac{1-\alpha^{T+N-t}}{1-\alpha^{T+N}}A_0 + \frac{(1-\alpha^N)(\alpha^{T-t}-\alpha^T)}{1-\alpha^{T+N}}\frac{Y_0}{r} \\ + \frac{\alpha^{T-t}-\alpha^T}{1-\alpha^{T+N}}\left(\frac{1-\alpha^N}{1-\alpha} - N\alpha^N\right)\frac{\Gamma_2^* - \Gamma_1^*}{r} \\ + \left[t - \frac{\alpha^{T+N-t}-\alpha^{T+N}}{1-\alpha^{T+N}}(T+N)\right]\frac{\Gamma_1^*}{r} \quad \text{for } t \leq T, \quad (4a)$$

$$A_t = \frac{1-\alpha^{T+N-t}}{1-\alpha^{T+N}}A_0 + \frac{(1-\alpha^T)(1-\alpha^{T+N-t})}{1-\alpha^{T+N}}\frac{Y_0}{r} \\ + \frac{1-\alpha^{T+N-t}}{1-\alpha^{T+N}}\left(\frac{1-\alpha^T}{1-\alpha} - T\right)\frac{\Gamma_2^* - \Gamma_1^*}{r} \\ + \left[t - \frac{\alpha^{T+N-t}-\alpha^{T+N}}{1-\alpha^{T+N}}(T+N)\right]\frac{\Gamma_2^*}{r} \quad \text{for } t \geq T. \quad (4b)$$

Finally, the maximum expected utility is

$$V(A_0) = -\frac{1-\alpha^{T+N}}{\theta r} \exp\left\{-\theta\left[\frac{r}{1-\alpha^{T+N}}A_0 + \frac{1-\alpha^T}{1-\alpha^{T+N}}Y_0\right.\right. \\ \left.\left.+ \left(T\alpha^{T+N} - \frac{1-\alpha^T}{1-\alpha}\right)\frac{\Gamma_1^* - \Gamma_2^*}{1-\alpha^{T+N}} - \bar{t}_\alpha\Gamma_2^*\right]\right\}. \quad (5)$$

2.3. Equilibrium

Aggregate wealth, W , is the sum of individual wealth holdings:

$$W \equiv \frac{1-p}{1-p^{T+N}} \sum_{t=1}^{T+N} p^{t-1}A_t. \quad (6)$$

Following our assumption on A_0 , the equilibrium condition on the transfer of wealth is

$$\frac{1-p}{1-p^{T+N}}A_0 = (1-p)W. \quad (7)$$

The right-hand side is the expected total wealth left from the newly dead, and the left-hand side is the total initial wealth endowment, with each new-born getting A_0 . Substituting (7) into (4) gives A_t , $t = 1, \dots, T + N$, depending on W ; then substituting these A_t in turn into (6) determines a unique aggregate wealth:

$$W^* = \frac{1 - \alpha^{T+N}}{\alpha^{T+N} - p^{T+N}} \left[(\text{pop}_p - \text{pop}_z) \frac{Y_0}{r} + (\bar{t}_z - \bar{t}_p) \frac{\Gamma_1^*}{r} \right] - N \frac{\Gamma_2^* - \Gamma_1^*}{(1 - p^{T+N})r} \\ + \left[\frac{\alpha^T - \alpha^{T+N}}{1 - \alpha} - \frac{(1 - \alpha^{T+N})(p^T - p^{T+N})}{(1 - p)(1 - p^{T+N})} \right] \frac{\Gamma_2^* - \Gamma_1^*}{(\alpha^{T+N} - p^{T+N})r}. \quad (8)$$

Then, from (7), we can find $A_0^* = (1 - p^{T+N})W^*$. By substituting W^* and A_0^* into (5), (4) and (3), we obtain V^* , A_t^* and S_t^* .

The effect of income uncertainty, or any other savings motive, on wealth accumulation and savings patterns can be ascertained by choosing limiting values for the parameters which define these motives in Eqs. (8) and (3). But first, some characteristics of the model should be noted.

Consumption is stochastic, yet saving is non-stochastic. This is because consumption adjusts fully to the (permanent) income shocks in each period. By (2), we can see that consumption is stochastically continuous in the sense that the expected consumption is continuous in t if we treat t as a continuous variable. Further, consumption growth is stochastically smooth during the working and retirement periods, but makes an adjustment at retirement – that is $\Delta C_t = \Gamma_1^* + \varepsilon_{t+1}$ for $t \leq T$, and $\Delta C_t = \Gamma_2^* + \varepsilon_{t+1}$ for $t > T$.³

Second, even though considerable stocks of wealth can be passed between generations, there is no bequest motive nor are there gifts *inter vivos*: inheritances are received because of an uncertain date of death, yet the *amount* of such inheritances will depend upon the degree of risk aversion, uncertainty about death and income, and the rate of return and the rate of time preference.

Third, we have adopted the standard assumptions on insurance in this literature: it is available neither for income nor lifespan uncertainty. Kotlikoff (1988) has argued that fair annuities are difficult to supply in practice because of agent heterogeneity and the self-selection which this implies.

Finally, it is important to distinguish between the effects of uncertainty on wealth accumulation and on utility (Kimball, 1990). We show in the appendix

³ It is also to be noted that since we cannot incorporate a trend in Y_t if we are to obtain explicit equations of motion, the parameterizations could be considered to be net of any actual trend observed in real processes. Thus, while savings are generated late in the working life by having a declining consumption stream, this consumption stream would not necessarily decline relative to an upwardly trending income process.

that the total cost of σ_1^2 and σ_2^2 , measured in terms of goods Y_0 , is

$$\begin{aligned} \text{cost}(\sigma_1, \sigma_2) &= \left(\frac{1}{1-\alpha} - \frac{T\alpha^{T+N}}{1-\alpha^T} \right) \frac{\theta}{2} \sigma_1^2 \\ &+ \frac{\alpha^T}{1-\alpha^T} \left(\frac{1-\alpha^N}{1-\alpha} - N\alpha^N \right) \frac{\theta}{2} \sigma_2^2. \end{aligned} \quad (9)$$

This is the risk premium, as defined in Pratt (1964). Since we use an exponential utility function, as indicated by the welfare expression (5), our risk premium is the same as the precautionary premium, as defined in Kimball (1990).⁴ The precautionary premium is how much the average income Y_0 has to change to counter the effect of the income risk on consumption, and the risk premium is how much the average income Y_0 has to change to counter the effect of the income risk on overall utility.

2.4. Parameterization

As a starting point, we choose a set of parameter values which enable the model to replicate some stylized facts on saving. In particular, if most household savings accrue later in the working life, then a Carroll (1992) type of ‘impatience’ will generate this: a rate of time preference which exceeds the interest rate will yield a declining desired consumption stream and thus increasing savings in the later part of the working life. But this is tempered by a precautionary motive which induces households to build up a stock of wealth early in order to protect against unfavorable income shocks. This serves the same function as a buffer stock and depends upon the degree of prudence.⁵

The first column of Table 2 defines our initial, or basic set of, parameters. The rate of time preference exceeds the interest rate by 2%. We normalize the income stream such that $Y_0 = 100$ in the certain lifetime case, and set the coefficient of absolute risk aversion θ at 3%. θ is also the measure of prudence. Since average consumption equals average income minus average savings, average consumption can either be greater or less than average income, depending upon the inheritance, A_0 . The coefficient of variation on the income process σ/Y_0 is set at 0.05 during the working period. This is consistent with the values suggested in income studies – MaCurdy (1982) proposes 0.10, although Guiso et al. (1991)

⁴ We would like to thank a referee for pointing this out to us.

⁵ Laibson (1997) and some others reviewed in Rabin (1998) argue that discount rates are higher for the near term than the distant future. The time inconsistency property of this formulation is ruled out here. Likewise, Becker and Mulligan (1997) propose that an element of the rate of time preference may be a choice variable for individuals.

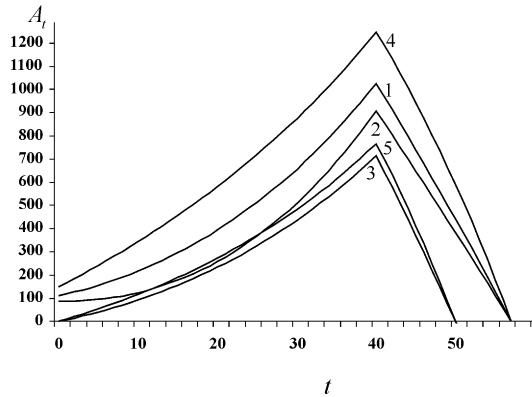


Fig. 1.

suggest a value as low as 0.02. We have reduced this value to 0.03 for the retirement period.

The expected length of the economic life of an agent is set equal to 50 years, with a working life of 40 and a retirement period of 10 years in the certain lifetime case. When the lifetime is uncertain there is an infinite number of combinations of p and T which will give the same expected lifetime. We choose $T + N = 57$ and $p = 0.99523$ as one such combination, simply because it is what Caballero (1991) chooses and therefore it provides us with a comparison point. This combination of values generates an aggregate wealth to income ratio in the neighborhood of 5, which is appropriate for developed economies. The corresponding asset profile is the solid line in Fig. 1.

3. Results

The primary objective is to investigate the effect of σ_1 and σ_2 on savings and wealth. But two types of income uncertainty go into our model, and the effects of these should be distinguished. Individuals face uncertainty in their earnings – working life, and also in the demands which may be placed on their resources in retirement – for example, unpredictable health conditions, better or poorer than the norm. The latter type of uncertainty might reasonably be viewed as being motivated by a retirement. But both types of uncertainty are incorporated into income uncertainty, and our results therefore form an upper bound on the influence of earnings uncertainty.⁶

⁶ As is well known, the variance of the random term increases with time, and our retirement phase is thus characterized by a higher variance than the working period, even though the mean is lower by the amount Y_0 and $\sigma_2^2 < \sigma_1^2$.

3.1. The flow of savings

Table 1 contains the expected values of S/Y_0 for different age groups and different assumed values of the income variance. The saving rate of the pre-retirement age quartiles is computed as the median saving rate in each 10-year bracket, defined by Eq. (3). Thus, $E(S_5/Y_0)$ is the value used for the first quartile, $E(S_{15}/Y_0)$ for the second, and so forth. With σ_1/Y_0 below 5% we still obtain savings patterns which conform generally to the observation that saving increases over the working lifecycle (Browning and Lusardi, 1996). However, with $\sigma_1/Y_0 = 0.07$ the savings profile is much too flat to conform with observed patterns, given the other parameter values of the model. In this instance individuals have an incentive to accumulate very early in the lifecycle.

While our consumption profile declines slightly over the lifecycle it is possible, even with impatience ($r < \delta$), for the desired consumption profile to slope upward if income uncertainty is strong enough.⁷

3.2. The stock of wealth

The parameter values for different specifications of the model are given in the top half of Table 2. The lower half of the table yields the values for the variables of interest: aggregate wealth W , the expected bequest A_0 , the maximal asset value over the life cycle A_{\max} , the expected value of lifetime utility U , and the cost of σ_1^2 and σ_2^2 – defined as the number of units of Y_0 which would be equivalent to abolishing income uncertainty. Columns 2–4 define the marginal effect on aggregate wealth accumulation of altering one motive of an agent's optimization. Column 5 contains the results for the case of no uncertainty of any kind and where the interest rate equals the rate of time preference. It can be considered analogous to the simplest type of lifecycle model. The final columns focus on the effects of the two types of income uncertainty.

Column 2 indicates that the absence of total lifetime income uncertainty would reduce W by 22%. Eliminating uncertainty regarding the time of death, while maintaining income uncertainty, has a considerably stronger effect, as illustrated in column 3 – W would be reduced by 37%.

A key issue for earnings uncertainty is the timing of its resolution: if there were no uncertain post-retirement needs then total income uncertainty would resolve itself long before the expected date of death. Thus, at the approach of retirement, as lifecycle earnings uncertainty diminishes, assets which may have been accumulated to protect against bad earnings shocks could be used to protect against extreme longevity. The implication of this is that the level of wealth at retirement, with earnings uncertainty in the working life, may not differ greatly

⁷A referee has also pointed out to us that the receipt of an inheritance will moderate the precautionary motive. In our simulations, the inheritance equals approximately one year's earnings.

Table 1
Expected saving patterns under income uncertainty (in %, $\sigma_2 = 3\%$)

	$\sigma_1/Y_0 = 1\%$	$\sigma_1/Y_0 = 3\%$	$\sigma_1/Y_0 = 5\%$	$\sigma_1/Y_0 = 7\%$
S^*/Y_0 for youngest age quartile	1.8	3.7	7.4	12.9
S^*/Y_0 for second age quartile	9.8	10.4	11.7	13.6
S^*/Y_0 for third age quartile	17.7	17.1	16.0	14.4
S^*/Y_0 for oldest age quartile	25.6	23.8	20.3	15.1

Table 2
Conditional effect of saving motives on wealth accumulation

	Complete model	Zero income uncertainty	Zero lifespan uncertainty	No inter-temporal substitution	Retirement as sole motive	Zero retirement-period income uncertainty	Zero working-period income uncertainty
	1	2	3	4	5	6	7
r	0.02	0.02	0.02	0.03	0.03	0.02	0.02
δ	0.04	0.04	0.04	0.03	0.03	0.04	0.04
p	0.99523	0.99523	1.0	0.99523	1.0	0.99523	0.99523
T	40	40	40	40	40	40	40
N	17	17	10	17	10	17	17
θ	0.03	0.03	0.03	0.03	0.03	0.03	0.03
σ_1/Y_0	0.05	0	0.05	0.05	0	0.05	0
σ_2/Y_0	0.03	0	0.03	0.03	0	0	0.03
W	463.5	363.2	292.5	624.7	327.9	455.5	371.2
A_0	110.6	86.6	0.0	149.0	0.0	108.7	88.6
A_{\max}	1026.0	906.7	713.5	1247.3	766.3	1011.8	920.8
$\text{cost}(\sigma_1, \sigma_2)$	10.5	0	9.1	9.1	0	10.3	0.3
V	-61.6	-52.0	-58.3	-66.1	-57.9	-59.1	-54.1

from the level of wealth without such uncertainty. This is confirmed by the results in column 7 of the table: wealth at the point of retirement is about 10% less with $\sigma_1 = 0$. However, earnings uncertainty still provides young workers with an incentive to save more and earlier – see Table 1 or Fig. 1. The consequence is that aggregate wealth in the economy, which is the sum over all age cohorts, is higher by 22%.

In contrast to this case, needs-related uncertainty in retirement, in combination with no earnings uncertainty (column 6), has minimal effects on the aggregate wealth of the economy relative to the absence of both types of income uncertainty (column 2). We surmise that the reason for this is that the effect of lifespan uncertainty is sufficiently strong that the additional uncertainty has just a marginal effect.

The importance of *lifespan* uncertainty arises from the fact that it is never resolved, and wealth stocks designed to protect against extreme longevity are therefore carried into later life: the fear of being poor in old age induces prudent individuals to accumulate significantly more wealth than if their date of death were known with certainty. Note that when we examine the effects of lifespan uncertainty there are two channels of influence: a change in the terminal survival date and a change in the per-period survival probability. We have experimented with each of these and find that the greater influence on the wealth stock is through the change in the survival date.

The degree of ‘patience’ is also important: if the rate of interest increases, relative to the rate of time preference (in column 4 $r = \delta$), consumption is postponed and more wealth accumulates, although it yields counterfactual savings profiles in the working life. In this context we have also examined the effect of reducing σ towards zero and have found that this reduction has a similar effect to the case where individuals are impatient ($r > \delta$). This provides one sensitivity test for the model in that the effects of income uncertainty are not heavily depend upon this particular combination of parameters.

In contrast, the effect of income uncertainty *is* heavily dependent upon the presence of a retirement period: with individuals working to the time of death we find that reducing σ_1 and σ_2 from 5% and 3%, respectively, to zero reduces the capital stock by 26%.

Finally if the only motive for saving is a pure retirement one (column 5), in a certain world with a preference for an even consumption stream ($r = \delta$), the wealth stock would be lower than our base case results in column 1.

The effect of income uncertainty on expected utility is also presented in the lower part of Table 2: an individual would be willing to sacrifice 10.5% of expected income every year in order to avoid earnings uncertainty.

4. Conclusions

Our primary objective has been to investigate why some theoretical savings models attribute such a significant role to income uncertainty, whereas the empirical literature provides much more qualified support. To this end we have modelled households as having a pure retirement phase in their lifecycle, as being impatient and prudent. We find that greater income uncertainty induces individuals to save greater amounts early in their lifecycle, but that there may be reversion in savings patterns later in the working life. As a result, earnings uncertainty has a significant impact upon the savings pattern over the lifecycle, in addition to its impact on the overall level of wealth in the economy.

The theoretical novelty of the paper – the modelling of a pure retirement phase in which the stochastic income process changes – accounts for why our results differ from those of Zeldes (1989) and Caballero (1991). In contrast to

these authors, who propose that as much as half of aggregate private wealth may be attributable to *earnings* uncertainty, our results indicate that total lifetime *income* uncertainty (which includes the needs-related shocks in retirement) accounts for no more than half of this value. Furthermore, of the total income uncertainty we have modelled, only a portion of that can be attributed to earnings uncertainty in the working life, and this is the more customary interpretation of income uncertainty.

While we have not modelled social security, as Hubbard et al. (1995) have done recently in an ingenious way, it is to be noted that their findings are fully consistent with the results we have presented here: they show that asset-based means tested social security payments in retirement effectively put a lower bound on bad shocks, and therefore reduce the uncertainty-induced incentive to accumulate for individuals with low lifetime incomes.

We have also developed results on the utility cost of uncertainty – as opposed to the savings effects of uncertainty, and are unaware of any other comparable findings in the literature.

Finally, we recognize the limitations of the kind of analytical model we have developed here. The characteristics of the exponential utility framework are well known – e.g. Deaton (1992) or Irvine and Wang (1994) or Weil (1993) or Van der Ploeg (1993). But our primary objective has been to reexamine the findings of models which have used similar, and more restricted, frameworks, and to show that the importance which has been attributed to income uncertainty in explaining aggregate wealth depends heavily on the restrictions implied by the modelling processes.

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Appendix

A.1. Euler equation

We now solve problem (1) for the Euler equation using backward induction. We will solve it for a general atemporal utility function $u(c)$.

Problem in the retirement stage: Given the initial wealth A_T , the consumer problem in the retirement stage is

$$V_T(A_T) \equiv \max \quad E_T \sum_{t=T+1}^{T+N} -\frac{1}{\theta} e^{-\theta c_t} \left(\frac{p}{1+\delta} \right)^t$$

$$\text{s.t.} \quad A_t = RA_{t-1} + Y_t - C_t$$

$$A_{T+N} \geq 0 \quad \text{given } A_T,$$

where $R \equiv 1 + r$. It is a standard recursive problem, and the Euler equation is well known as

$$u'(C_t^*) = R\rho E_{t-1}[u'(C_{t+1}^*)], \quad t \geq T, \quad (\text{A.1})$$

where $\rho \equiv p/(1+\delta)$. We will now continue backward induction from $t = T$.

Problem at period T:

$$V_{T-1}(A_{T-1}) \equiv \max_{C_T} \quad \rho^T u(C_T) + E_{T-1} V_T(A_T)$$

$$\text{s.t.} \quad A_T = RA_{T-1} + Y_T - C_T \quad \text{given } A_{T-1}.$$

It can be reduced to

$$V_{T-1}(A_{T-1}) \equiv \max_{C_T} \quad \rho^T u(C_T)$$

$$+ E_{T-1} V_T(RA_{T-1} + Y_T - C_T),$$

which gives the first-order condition:

$$\rho^T u'(C_T^*) = E_{T-1} V'_T(A_T^*).$$

The envelope theorem also implies

$$V'_{T-1}(A_{T-1}^*) = R E_{T-1} V'_T(A_T^*).$$

Problem at period T - 1:

$$V_{T-2}(A_{T-2}) \equiv \max_{C_{T-1}} \quad \rho^{T-1} u(C_{T-1}) + E_{T-2} V_{T-1}(A_{T-1})$$

$$\text{s.t.} \quad A_{T-1} = RA_{T-2} + Y_{T-1} - C_{T-1} \quad \text{given } A_{T-2}.$$

It can be reduced to

$$V_{T-2}(A_{T-2}) \equiv \max_{C_{T-1}} \rho^{T-1} u(C_{T-1}) + E_{T-2} V_{T-1}(RA_{T-2} + Y_{T-1} - C_{T-1}),$$

which gives the first-order condition:

$$\rho^{T-1} u'(C_{T-1}^*) = E_{T-2} V'_{T-1}(A_{T-1}^*).$$

The envelope theorem also implies

$$V'_{T-2}(A_{T-2}^*) = R E_{T-2} V'_{T-1}(A_{T-1}^*).$$

The solution: We can now see a clear pattern. We will generally have

$$\rho^t u'(C_t^*) = E_{t-1} V'_t(A_t^*) \tag{A.2}$$

and

$$V'_{t-1}(A_{t-1}^*) = R E_{t-1} [V'_t(A_t^*)] \tag{A.3}$$

for $t = 1, \dots, T$. (A.2) and (A.3) imply $R\rho^t u'(C_t^*) = V'_{t-1}(A_{t-1}^*)$. Using (A.2) again, we have

$$\rho^{t-1} u'(C_{t-1}^*) = E_{t-2} V'_{t-1}(A_{t-1}^*) = E_{t-2} [R\rho^t u'(C_t^*)].$$

Thus,

$$u'(C_t^*) = R\rho E_{t-1} [u'(C_{t+1}^*)], \quad t \leq T - 1.$$

Combining with (A.1), the Euler equation is thus

$$u'(C_t^*) = R\rho E_{t-1} [u'(C_{t+1}^*)] \quad \text{for all } t. \tag{A.4}$$

A.2. The optimal solution – Section 2.2

The difference equation for individual wealth: For our utility function $u(c) = -(1/\theta)e^{-\theta c}$, the Euler equation (A.4) becomes

$$e^{-\theta C_t} = \beta E_{t-1}(e^{-\theta C_{t+1}}), \quad 1 \leq t \leq T + N. \tag{A.5}$$

One can easily verify that the following is a solution for (A.5):

$$C_{t+1} - C_t = \begin{cases} \Gamma_1^* + \varepsilon_{t+1} & \text{for } t < T, \\ \Gamma_2^* + \varepsilon_{t+1} & \text{for } t \geq T. \end{cases} \tag{A.6}$$

By (A.6),

$$\begin{aligned} C_{t+1} - C_t &= \Gamma_1^* + Y_{t+1} - Y_t && \text{for } t < T, \\ C_{T+1} - C_T &= \Gamma_2^* + Y_0 + Y_{T+1} - Y_T && \text{for } t = T, \\ C_{t+1} - C_t &= \Gamma_2^* + Y_{t+1} - Y_t && \text{for } t > T. \end{aligned}$$

Then, by the budget constraint, for $t < T$, we have

$$\begin{aligned} \Gamma_1^* + Y_{t+1} - Y_t &= C_{t+1} - C_t \\ &= R(A_t - A_{t-1}) + Y_{t+1} - Y_t - (A_{t+1} - A_t) \end{aligned}$$

implying

$$A_t - A_{t-1} = \alpha(A_{t+1} - A_t) + \alpha\Gamma_1^*, \quad t < T. \quad (\text{A.7a})$$

Similarly, for $t > T$,

$$A_t - A_{t-1} = \alpha(A_{t+1} - A_t) + \alpha\Gamma_2^*, \quad t > T. \quad (\text{A.7b})$$

For $t = T$, we have

$$\begin{aligned} \Gamma_2^* + Y_{T+1} - Y_T &= C_{T+1} - C_T - Y_0 \\ &= R(A_T - A_{T-1}) + Y_{T+1} - Y_T - Y_0 \\ &\quad - (A_{T+1} - A_T). \end{aligned}$$

Then,

$$A_T - A_{T-1} = \alpha(A_{T+1} - A_T) + \alpha\Gamma_2^* + \alpha Y_0. \quad (\text{A.7c})$$

Individual wealth: Then, for $t > T$,

$$A_t - A_{t-1} = \alpha^{T+N-t}(A_{T+N} - A_{T+N-1}) + \Gamma_2^* \sum_{i=1}^{T+N-t} \alpha^i,$$

implying

$$\begin{aligned} A_{t-1} &= A_t - \frac{\alpha - \alpha^{T+N+1-t}}{1 - \alpha} \Gamma_2^* - \alpha^{T+N-t}(A_{T+N} - A_{T+N-1}) \\ &= A_{t+1} - \frac{\alpha - \alpha^{T+N-t}}{1 - \alpha} \Gamma_2^* - \alpha^{T+N-t-1}(A_{T+N} - A_{T+N-1}) \\ &\quad - \frac{\alpha - \alpha^{T+N+1-t}}{1 - \alpha} \Gamma_2^* - \alpha^{T+N-t}(A_{T+N} - A_{T+N-1}) \end{aligned}$$

$$\begin{aligned}
 &= \dots \\
 &= A_{T+N} - \Gamma_2^* \sum_{i=t}^{T+N} \frac{\alpha - \alpha^{T+N+1-i}}{1-\alpha} - (A_{T+N} - A_{T+N-1}) \sum_{i=t}^{T+N} \alpha^{T+N-i} \\
 &= A_{T+N} - \frac{\alpha \Gamma_2^*}{1-\alpha} \frac{(1-\alpha)(T+N) - \alpha - (1-\alpha)t + \alpha^{T+N+1-t}}{1-\alpha} \\
 &\quad - (A_{T+N} - A_{T+N-1}) \frac{1 - \alpha^{T+N+1-t}}{1-\alpha},
 \end{aligned}$$

implying

$$\begin{aligned}
 A_t &= A_{T+N} - (A_{T+N} - A_{T+N-1}) \frac{1 - \alpha^{T+N-t}}{1-\alpha} \\
 &\quad - \frac{(1-\alpha)(T+N-t-1) - \alpha + \alpha^{T+N-t}}{1-\alpha} \frac{\alpha \Gamma_2^*}{1-\alpha}.
 \end{aligned}$$

Since $A_{T+N}^* = 0$, we have

$$\begin{aligned}
 A_t &= \frac{1 - \alpha^{T+N-t}}{1-\alpha} A_{T+N-1} \\
 &\quad - \frac{(1-\alpha)(T+N-t-1) - \alpha + \alpha^{T+N-t}}{1-\alpha} \frac{\alpha \Gamma_2^*}{1-\alpha}, \quad t \geq T.
 \end{aligned}$$

In particular, for $t = T$,

$$A_T = \frac{1 - \alpha^N}{1-\alpha} A_{T+N-1} - \frac{(1-\alpha)(N-1) - \alpha + \alpha^N}{1-\alpha} \frac{\alpha \Gamma_2^*}{1-\alpha}.$$

The above two imply

$$\begin{aligned}
 &(1 - \alpha^N)A_t - (1 - \alpha^{T+N-t})A_T \\
 &= \frac{\alpha \Gamma_2^*}{(1-\alpha)^2} \{ (1 - \alpha^{T+N-t}) [(1-\alpha)(N-1) - \alpha + \alpha^N] \\
 &\quad - (1 - \alpha^N) [(1-\alpha)(T+N-t-1) - \alpha + \alpha^{T+N-t}] \} \\
 &= \frac{\alpha \Gamma_2^*}{1-\alpha} [(t-T)(1 - \alpha^N) + (\alpha^N - \alpha^{T+N-t})N].
 \end{aligned}$$

Since $1/r = \alpha/(1 - \alpha)$, we then have

$$A_t = \frac{1 - \alpha^{T+N-t}}{1 - \alpha^N} A_T + \left(t - T + \frac{\alpha^N - \alpha^{T+N-t}}{1 - \alpha^N} N \right) \frac{\Gamma_2^*}{r}, \quad t \geq T. \quad (\text{A.8a})$$

In particular,

$$A_{T+1} = \frac{1 - \alpha^{N-1}}{1 - \alpha^N} A_T + \left(1 + \frac{\alpha^N - \alpha^{N-1}}{1 - \alpha^N} N \right) \frac{\Gamma_2^*}{r}. \quad (\text{A.8b})$$

For $t \leq T - 1$, by (A.7a),

$$\begin{aligned} A_t - A_{t-1} &= \alpha^{T-t}(A_T - A_{T-1}) + \Gamma_1^* \sum_{i=1}^{T-t} \alpha^i \\ &= \alpha^{T-t}(A_T - A_{T-1}) + \frac{\alpha - \alpha^{T+1-t}}{1 - \alpha} \Gamma_1^*. \end{aligned}$$

Then,

$$\begin{aligned} A_{t-1} &= A_t - (1 - \alpha^{T-t}) \frac{\Gamma_1^*}{r} - \alpha^{T-t}(A_T - A_{T-1}) \\ &= A_{t+1} - (1 - \alpha^{T-t-1}) \frac{\Gamma_1^*}{r} - \alpha^{T-t-1}(A_T - A_{T-1}) \\ &\quad - (1 - \alpha^{T-t}) \frac{\Gamma_1^*}{r} - \alpha^{T-t}(A_T - A_{T-1}) \\ &= \dots \\ &= A_T - \frac{\Gamma_1^*}{r} \sum_{i=t}^T (1 - \alpha^{T-i}) - (A_T - A_{T-1}) \sum_{i=t}^T \alpha^{T-i} \\ &= A_T - \frac{\Gamma_1^*}{r} \frac{(1 - \alpha)T - \alpha - (1 - \alpha)t + \alpha^{T+1-t}}{1 - \alpha} \\ &\quad - (A_T - A_{T-1}) \frac{1 - \alpha^{T+1-t}}{1 - \alpha} \end{aligned}$$

implying

$$\begin{aligned} A_t &= A_T - (A_T - A_{T-1}) \frac{1 - \alpha^{T-t}}{1 - \alpha} \\ &\quad - \frac{(1 - \alpha)(T - t - 1) - \alpha + \alpha^{T-t}}{1 - \alpha} \frac{\Gamma_1^*}{r}. \end{aligned}$$

Then, by (A.7b),

$$A_t = A_T - \alpha(A_{T+1} - A_T) \frac{1 - \alpha^{T-t}}{1 - \alpha} - \frac{Y_0 + \Gamma_2^*}{r} (1 - \alpha^{T-t}) - \frac{(1 - \alpha)(T - t - 1) - \alpha + \alpha^{T-t}}{1 - \alpha} \frac{\Gamma_1^*}{r},$$

i.e.,

$$A_t = A_T - \alpha(A_{T+1} - A_T) \frac{1 - \alpha^{T-t}}{1 - \alpha} - \frac{Y_0 + \Gamma_2^*}{r} (1 - \alpha^{T-t}) + \left(t - T + \frac{1 - \alpha^{T-t}}{1 - \alpha} \right) \frac{\Gamma_1^*}{r}, \quad t \leq T.$$

Substituting (A.8b) into this yields

$$A_t = \frac{1 - \alpha^{T+N-t}}{1 - \alpha^N} A_T - (1 - \alpha^{T-t}) \frac{Y_0}{r} + \left(t - T + \frac{1 - \alpha^{T-t}}{1 - \alpha} \right) \frac{\Gamma_1^*}{r} + \left(\frac{\alpha^N - \alpha^{T+N-t}}{1 - \alpha^N} N - \frac{1 - \alpha^{T-t}}{1 - \alpha} \right) \frac{\Gamma_2^*}{r} \quad \text{for } t \leq T. \tag{A.8c}$$

In particular, for $t = 0$, (A.8c) becomes

$$A_0 = \frac{1 - \alpha^{T+N}}{1 - \alpha^N} A_T - (1 - \alpha^T) \frac{Y_0}{r} + \left(\frac{1 - \alpha^T}{1 - \alpha} - T \right) \frac{\Gamma_1^*}{r} + \left(\frac{1 - \alpha^T}{1 - \alpha^N} N \alpha^N - \frac{1 - \alpha^T}{1 - \alpha} \right) \frac{\Gamma_2^*}{r}. \tag{A.9}$$

Eqs. (A.8c) and (A.9) imply

$$(1 - \alpha^{T+N})A_t - (1 - \alpha^{T+N-t})A_0 = \frac{Y_0}{r} (\alpha^{T-t} - \alpha^T)(1 - \alpha^N) + [(1 - \alpha^{T+N})t - (\alpha^{T-t} - \alpha^T)(T + N)\alpha^N] \frac{\Gamma_1^*}{r} + \left(\frac{1 - \alpha^N}{1 - \alpha} - N\alpha^N \right) (\alpha^{T-t} - \alpha^T) \frac{\Gamma_2^* - \Gamma_1^*}{r}. \tag{A.10}$$

Thus,

$$\begin{aligned} \hat{A}_t = & \frac{1 - \alpha^{T+N-t}}{1 - \alpha^{T+N}} A_0 + \frac{(1 - \alpha^N)(\alpha^{T-t} - \alpha^T)}{1 - \alpha^{T+N}} \frac{Y_0}{r} \\ & + \frac{\alpha^{T-t} - \alpha^T}{1 - \alpha^{T+N}} \left(\frac{1 - \alpha^N}{1 - \alpha} - N\alpha^N \right) \frac{\Gamma_2^* - \Gamma_1^*}{r} \\ & + \left[t - \frac{\alpha^{T+N-t} - \alpha^{T+N}}{1 - \alpha^{T+N}} (T + N) \right] \frac{\Gamma_1^*}{r} \quad \text{for } t \leq T. \end{aligned} \quad (\text{A.11a})$$

As a special case of (A.11a), for $t = T$, we have

$$\begin{aligned} \hat{A}_T = & \frac{1 - \alpha^N}{1 - \alpha^{T+N}} A_0 + \frac{(1 - \alpha^N)(1 - \alpha^T)}{1 - \alpha^{T+N}} \frac{Y_0}{r} \\ & + \frac{1 - \alpha^T}{1 - \alpha^{T+N}} \left(\frac{1 - \alpha^N}{1 - \alpha} - N\alpha^N \right) \frac{\Gamma_2^* - \Gamma_1^*}{r} \\ & + \left[T - \frac{\alpha^N - \alpha^{T+N}}{1 - \alpha^{T+N}} (T + N) \right] \frac{\Gamma_1^*}{r}. \end{aligned} \quad (\text{A.11b})$$

Substituting (A.11b) into (A.8a) gives

$$\begin{aligned} \hat{A}_t = & \frac{1 - \alpha^{T+N-t}}{1 - \alpha^{T+N}} A_0 + \frac{(1 - \alpha^T)(1 - \alpha^{T+N-t})}{1 - \alpha^{T+N}} \frac{Y_0}{r} \\ & + \frac{1 - \alpha^{T+N-t}}{1 - \alpha^{T+N}} \left(\frac{1 - \alpha^T}{1 - \alpha} - T \right) \frac{\Gamma_2^* - \Gamma_1^*}{r} \\ & + \left[t - \frac{\alpha^{T+N-t} - \alpha^{T+N}}{1 - \alpha^{T+N}} (T + N) \right] \frac{\Gamma_2^*}{r} \quad \text{for } t \geq T. \end{aligned} \quad (\text{A.11c})$$

We can also verify that (A.11b) is a special case of (A.11c).

Saving and consumption: By (A.11a), the saving for $t \leq T$ is

$$\begin{aligned} \hat{S}_t = \hat{A}_t - R\hat{A}_{t-1} = & -\frac{r}{1 - \alpha^{T+N}} A_0 + \frac{\alpha^T - \alpha^{T+N}}{1 - \alpha^{T+N}} Y_0 \\ & + \frac{\alpha^T}{1 - \alpha^{T+N}} \left(\frac{1 - \alpha^N}{1 - \alpha} - N\alpha^N \right) \Gamma_2^* \\ & + \left[\frac{1}{1 - \alpha} - \frac{(T + N)\alpha^{T+N}}{1 - \alpha^{T+N}} - \frac{\alpha^T}{1 - \alpha^{T+N}} \left(\frac{1 - \alpha^N}{1 - \alpha} - N\alpha^N \right) - t \right] \Gamma_1^*. \end{aligned}$$

Thus,

$$\begin{aligned} \hat{S}_t &= -\frac{r}{1-\alpha^{T+N}} A_0 + \frac{\alpha^T - \alpha^{T+N}}{1-\alpha^{T+N}} Y_0 \\ &\quad + \alpha^T \left(\frac{1-\alpha^N}{1-\alpha} - N\alpha^N \right) \frac{\Gamma_2^* - \Gamma_1^*}{1-\alpha^{T+N}} \\ &\quad + (\bar{t}_\alpha - t) \Gamma_1^*, \quad t \leq T. \end{aligned} \tag{A.12a}$$

By (A.11c), the saving for $t > T$ is

$$\begin{aligned} \hat{S}_t &= \hat{A}_t - R\hat{A}_{t-1} = -\frac{r}{1-\alpha^{T+N}} A_0 - \frac{1-\alpha^T}{1-\alpha^{T+N}} Y_0 \\ &\quad - \frac{1}{1-\alpha^{T+N}} \left(T - \frac{1-\alpha^T}{1-\alpha} \right) \Gamma_1^* \\ &\quad + \left[\bar{t}_\alpha - t + \frac{1}{1-\alpha^{T+N}} \left(T - \frac{1-\alpha^T}{1-\alpha} \right) \right] \Gamma_2^*. \end{aligned}$$

Thus,

$$\begin{aligned} \hat{S}_t &= -\frac{r}{1-\alpha^{T+N}} A_0 - \frac{1-\alpha^T}{1-\alpha^{T+N}} Y_0 \\ &\quad + \left(T - \frac{1-\alpha^T}{1-\alpha} \right) \frac{\Gamma_2^* - \Gamma_1^*}{1-\alpha^{T+N}} + (\bar{t}_\alpha - t) \Gamma_2^*, \quad t > T. \end{aligned} \tag{A.12b}$$

By the budget constraint, the consumption is

$$\begin{aligned} \hat{C}_t &= Y_t - \hat{S}_t = Y_t + \frac{r}{1-\alpha^{T+N}} A_0 - \frac{\alpha^T - \alpha^{T+N}}{1-\alpha^{T+N}} Y_0 \\ &\quad - \alpha^T \left(\frac{1-\alpha^N}{1-\alpha} - N\alpha^N \right) \frac{\Gamma_2^* - \Gamma_1^*}{1-\alpha^{T+N}} + (t - \bar{t}_\alpha) \Gamma_1^*, \quad t \leq T, \end{aligned} \tag{A.13a}$$

$$\begin{aligned} \hat{C}_t &= Y_t - \hat{S}_t = Y_t + \frac{r}{1-\alpha^{T+N}} A_0 + \frac{1-\alpha^T}{1-\alpha^{T+N}} Y_0 \\ &\quad - \left(T - \frac{1-\alpha^T}{1-\alpha} \right) \frac{\Gamma_2^* - \Gamma_1^*}{1-\alpha^{T+N}} + (t - \bar{t}_\alpha) \Gamma_2^*, \quad t > T. \end{aligned} \tag{A.13b}$$

Since

$$\begin{aligned}
 t - \bar{t}_\alpha - \alpha^T \left(N\alpha^N - \frac{1 - \alpha^N}{1 - \alpha} \right) \frac{1}{1 - \alpha^{T+N}} \\
 &= t - T - \frac{1}{1 - \alpha^{T+N}} \left(\frac{1 - \alpha^T}{1 - \alpha} - T \right), \\
 \alpha^T \left(N\alpha^N - \frac{1 - \alpha^N}{1 - \alpha} \right) \frac{1}{1 - \alpha^{T+N}} \\
 &= \frac{1}{1 - \alpha^{T+N}} \left(\frac{1 - \alpha^T}{1 - \alpha} + N\alpha^{T+N} \right) - \frac{1}{1 - \alpha},
 \end{aligned}$$

we can further simplify (A.13a) and (A.13b) to (2). There is no consumption drop at retirement; in this case, a saving drop at retirement accommodates the income drop.

Welfare: Let us now find the maximum utility. The Euler equation is, for all t ,

$$e^{-\theta C_t} = \beta E_{t-1} e^{-\theta C_{t+1}}.$$

Then,

$$e^{-\theta C_1} = \beta E_1 e^{-\theta C_2} = \beta^2 E_1 e^{-\theta C_3} = \dots = \beta^{t-1} E_1 e^{-\theta C_t}.$$

We then have

$$\begin{aligned}
 V(A_0) &= - E_0 E_1 \sum_{t=1}^{T+N} \frac{1}{\theta} e^{-\theta C_t} (\alpha\beta)^t = - \frac{1}{\theta} E_0 \sum_{t=1}^{T+N} (\alpha\beta)^t \beta^{1-t} e^{-\theta C_t} \\
 &= - \frac{\alpha\beta}{\theta} \frac{1 - \alpha^{T+N}}{1 - \alpha} (E_0 e^{-\theta C_1}) = - \frac{\beta}{\theta r} (1 - \alpha^{T+N})(E_0 e^{-\theta C_1}). \quad (\text{A.14})
 \end{aligned}$$

By (A.13a),

$$\begin{aligned}
 C_1 &= Y_1 + \frac{r}{1 - \alpha^{T+N}} A_0 - \frac{\alpha^T - \alpha^{T+N}}{1 - \alpha^{T+N}} Y_0 \\
 &\quad - \alpha^T \left(\frac{1 - \alpha^N}{1 - \alpha} - N\alpha^N \right) \frac{\Gamma_2^* - \Gamma_1^*}{1 - \alpha^{T+N}} + (1 - \bar{t}_\alpha) \Gamma_1^*.
 \end{aligned}$$

Since $Y_1 = Y_0 + \zeta_1 = Y_0 + \varepsilon_1$, we have

$$C_1 = \varepsilon_1 + b,$$

where

$$\begin{aligned}
 b \equiv & \frac{r}{1 - \alpha^{T+N}} A_0 + \frac{1 - \alpha^T}{1 - \alpha^{T+N}} Y_0 \\
 & - \alpha^T \left(\frac{1 - \alpha^N}{1 - \alpha} - N\alpha^N \right) \frac{\Gamma_2^* - \Gamma_1^*}{1 - \alpha^{T+N}} + (1 - \bar{t}_x) \Gamma_1^*.
 \end{aligned}
 \tag{A.15}$$

By the definition of Γ_1^* , we have

$$E_0 e^{-\theta C_1} = E_0 e^{-\theta[\varepsilon_1 + b]} = e^{-\theta b} E_0 e^{-\theta \varepsilon_1} = e^{-\theta b} e^{1/2\theta^2 \sigma_1^2} = e^{\theta(1/2\theta \sigma_1^2 - b)}.$$

Thus,

$$V(A_0) = -\frac{\beta}{\theta r} (1 - \alpha^{T+N}) e^{\theta(1/2\theta \sigma_1^2 - b)} = -\frac{1}{\theta r} (1 - \alpha^{T+N}) e^{\theta(\Gamma_1^* - b)}.$$

Substituting (A.15) into this then gives

$$\begin{aligned}
 V(A_0) = & -\frac{1 - \alpha^{T+N}}{\theta r} \exp \left\{ -\theta \left[\frac{r}{1 - \alpha^{T+N}} A_0 + \frac{1 - \alpha^T}{1 - \alpha^{T+N}} Y_0 \right. \right. \\
 & \left. \left. - \alpha^T \left(\frac{1 - \alpha^N}{1 - \alpha} - N\alpha^N \right) \frac{\Gamma_2^* - \Gamma_1^*}{1 - \alpha^{T+N}} - \bar{t}_x \Gamma_1^* \right] \right\}.
 \end{aligned}
 \tag{A.16}$$

A.3. Equilibrium

By definition, we have

$$W = \frac{1 - p}{1 - p^{T+N}} \sum_{t=1}^{T+N} p^{t-1} A_t.
 \tag{A.17}$$

The equilibrium condition is

$$\frac{1 - p}{1 - p^{T+N}} A_0 = (1 - p)W.
 \tag{A.18}$$

Substituting (A.11a), (A.11c) and (A.18) into (A.17) gives

$$\begin{aligned}
 W = & (1 - p)W \sum_{t=1}^{T+N} p^{t-1} \frac{1 - \alpha^{T+N-t}}{1 - \alpha^{T+N}} + \frac{1 - p}{1 - p^{T+N}} \frac{Y_0}{r} \\
 & \left[\frac{1 - \alpha^N}{1 - \alpha^{T+N}} \sum_{t=1}^T p^{t-1} (\alpha^{T-t} - \alpha^T) \right]
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{1 - \alpha^T}{1 - \alpha^{T+N}} \left[\sum_{t=T+1}^{T+N} p^{t-1} (1 - \alpha^{T+N-t}) \right] \\
 & + \frac{1 - p}{1 - p^{T+N}} \left\{ \sum_{t=1}^T p^{t-1} \left[t - \frac{\alpha^{T-t} - \alpha^T}{1 - \alpha^{T+N}} \left(T\alpha^N + \frac{1 - \alpha^N}{1 - \alpha} \right) \right] \right\} \\
 & + \sum_{t=T+1}^{T+N} p^{t-1} \frac{1 - \alpha^{T+N-t}}{1 - \alpha^{T+N}} \left(T - \frac{1 - \alpha^T}{1 - \alpha} \right) \left\} \frac{\Gamma_1^*}{r} \\
 & + \frac{1 - p}{1 - p^{T+N}} \left\{ \sum_{t=1}^T p^{t-1} \frac{\alpha^{T-t} - \alpha^T}{1 - \alpha^{T+N}} \left(\frac{1 - \alpha^N}{1 - \alpha} - N\alpha^N \right) \right\} \\
 & + \sum_{t=T+1}^{T+N} p^{t-1} \left[t - T - N + \frac{1 - \alpha^{T+N-t}}{1 - \alpha^{T+N}} \left(N + \frac{1 - \alpha^T}{1 - \alpha} \right) \right] \left\} \frac{\Gamma_2^*}{r}.
 \end{aligned}$$

This equation can be simplified substantially (after a considerable effort) and then solved for the aggregate wealth:

$$\begin{aligned}
 W^* &= \frac{1 - \alpha^{T+N}}{\alpha^{T+N} - p^{T+N}} \left[(\text{pop}_p - \text{pop}_\alpha) \frac{Y_0}{r} + (\bar{t}_\alpha - \bar{t}_p) \frac{\Gamma_1^*}{r} \right] - N \frac{\Gamma_2^* - \Gamma_1^*}{(1 - p^{T+N})r} \\
 & + \left[\frac{\alpha^T - \alpha^{T+N}}{1 - \alpha} - \frac{(1 - \alpha^{T+N})(p^T - p^{T+N})}{(1 - p)(1 - p^{T+N})} \right] \frac{\Gamma_2^* - \Gamma_1^*}{(\alpha^{T+N} - p^{T+N})r}.
 \end{aligned} \tag{A.19}$$

Then, substituting $A_0^* = (1 - p^{T+N})W^*$ from (A.18) into (A.16) gives equilibrium welfare V^* .

A.4. Cost of income uncertainty

Let V^* be a function of (Y_0, σ_1) . We look for ΔY_0 such that

$$V^*(Y_0 + \Delta Y_0, \sigma_1) = V^*(Y_0, 0).$$

ΔY_0 , defined by the above equation, is thus the cost of σ in terms of goods. By (A.16), the above equation can be expanded as

$$\begin{aligned}
 & - \frac{1 - \alpha^{T+N}}{\theta r} \exp \left\{ -\theta \left[\frac{r}{1 - \alpha^{T+N}} A_0 + \frac{1 - \alpha^T}{1 - \alpha^{T+N}} (Y_0 + \Delta Y_0) \right. \right. \\
 & \quad \left. \left. - \alpha^T \left(\frac{1 - \alpha^N}{1 - \alpha} - N\alpha^N \right) \frac{\Gamma_2^* - \Gamma_1^*}{1 - \alpha^{T+N}} - \bar{t}_\alpha \Gamma_1^* \right] \right\}
 \end{aligned}$$

$$= -\frac{1 - \alpha^{T+N}}{\theta r} \exp \left\{ -\theta \left[\frac{r}{1 - \alpha^{T+N}} A_0 + \frac{1 - \alpha^T}{1 - \alpha^{T+N}} Y_0 - \left(\frac{1 - \alpha^N}{1 - \alpha} - N\alpha^N \right) \frac{\alpha^T}{1 - \alpha^{T+N}} \frac{\theta}{2} \sigma_2^2 - \bar{t}_\alpha \gamma \right] \right\},$$

which gives

$$\text{cost}(\sigma_1) \equiv \Delta Y_0 = \left(\frac{1}{1 - \alpha} - \frac{T\alpha^{T+N}}{1 - \alpha^T} \right) \frac{\theta}{2} \sigma_1^2.$$

Similarly, let V^* be a function of (Y_0, σ_2) and consider

$$V^*(Y_0 + \Delta Y_0, \sigma_2) = V^*(Y_0, 0),$$

implying

$$\begin{aligned} & -\frac{1 - \alpha^{T+N}}{\theta r} \exp \left\{ -\theta \left[\frac{r}{1 - \alpha^{T+N}} A_0 + \frac{1 - \alpha^T}{1 - \alpha^{T+N}} (Y_0 + \Delta Y_0) - \alpha^T \left(\frac{1 - \alpha^N}{1 - \alpha} - N\alpha^N \right) \frac{\Gamma_2^* - \Gamma_1^*}{1 - \alpha^{T+N}} - \bar{t}_\alpha \Gamma_1^* \right] \right\} \\ & = -\frac{1 - \alpha^{T+N}}{\theta r} \exp \left\{ -\theta \left[\frac{r}{1 - \alpha^{T+N}} A_0 + \frac{1 - \alpha^T}{1 - \alpha^{T+N}} Y_0 - \left(\frac{1 - \alpha^N}{1 - \alpha} - N\alpha^N \right) \frac{-\alpha^T}{1 - \alpha^{T+N}} \frac{\theta}{2} \sigma_1^2 - \bar{t}_\alpha \Gamma_1^* \right] \right\}, \end{aligned}$$

which gives

$$\text{cost}(\sigma_2) \equiv \Delta Y_0 = \frac{\alpha^T}{1 - \alpha^T} \left(\frac{1 - \alpha^N}{1 - \alpha} - N\alpha^N \right) \frac{\theta}{2} \sigma_2^2.$$

The total cost of income uncertainty is

$$\text{cost}(\sigma_1, \sigma_2) = \text{cost}(\sigma_1) + \text{cost}(\sigma_2).$$

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