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TWO SHEAF CONSTRUCTIONS FOR NONCOMMUTATIVE RINGS

Jonathan S. Golan

0. Introduction and background. There are two different methods of associating a sheaf of rings with a general noncommutative ring. The first of these, introduced by Pierce [8], has since been extended by Burgess and Stephenson [3,4] and by Bergman [2]. In [3], Burgess and Stephenson describe these sheaves for various classes of noncommutative rings, particularly for rings regular in the sense of Von Neumann. In [4] they make use of these sheaves to develop a structure theory for rings and apply this theory to various rings the properties of which are expressible in arithmetic, as opposed to module-theoretic, terms. The second method, first introduced in [5], involves the construction of a monopresheaf over the set of all prime torsion theories on the category of left modules over the ring and then the embedding of this monopresheaf into its associated sheaf. This was used for constructing the foundations of a representation theory for general noncommutative rings, the consideration of which has been extended in [6,7,11]. In particular, the results in [6] are used to lay the basis for an "algebraic geometry" over noncommutative rings. The purpose of this paper is to show the relation between these two constructions.

Throughout the following, R will denote an associative (but not necessarily commutative) ring with unit element 1. All modules and module homomorphisms will be taken from the category R-mod of unitary left R-modules. The complete brouwerian lattice of all (hereditary) torsion theories on R-mod will be denoted by R-tors. Notation and terminology concerning R-tors will follow [5]. In particular, if M is a left R-module then $\xi(M)$ will denote the meet of all torsion theories in R-tors relative to which M is torsion and $\chi(M)$ will denote the join of all torsion theories in R-tors relative to which M is torsionfree. The unique minimal element of R-tors is then $\xi = \xi(0)$ and the unique maximal element of R-tors is then $\chi = \chi(0)$. If $\tau \in$ R-tors then we denote by $T_\tau(M)$ the τ-torsion submodule of a left R-module M and by R_τ the quotient ring of R at τ. A left R-module N is said to be *τ-cocritical* if and only if N is τ-torsionfree but every proper homomorphic image of N is τ-torsion. Elements of R-tors of the form $\chi(N)$, where N is a nonzero left R-module which is cocritical with respect to some element of R-tors, are said to be *prime*. The family of all prime elements of R-tors is called the *left spectrum* of R and is denoted by R-sp.

1. **Skeletal torsion theories.** If $(L, \vee, \wedge)$ is an arbitrary complete brouwerian lattice then every element a of L has a *meet pseudocomplement*. That is to say, there exists a unique element a* of L satisfying the conditions

(1) $a \wedge a^* = 0_L$;

(2) If b is an element of L satisfying $a \wedge b = 0_L$ then $b \leqslant a^*$.

Elements of L of the form a* for some $a \in L$ are said to be *skeletal* (some sources use the much-overworked term *regular*) and the set of all skeletal elements of L is called the *skeleton* of L. It is easily checked that an element a of L is skeletal if and only if $a^{**} = a$. If L is a complete brouwerian lattice with skeleton S then we can define a new binary operation $\underline{\vee}$ on S by setting $a \underline{\vee} b = (a \vee b)^{**}$. By Glivenko's Theorem [1, page 157] the lattice $(S, \underline{\vee}, \wedge)$ is boolean. Moreover, the map $L \to S$ given by $a \mapsto a^{**}$ is an epimorphism in the category of distributive lattices [1, page 158].

Since the lattice R-tors is brouwerian [9], every (hereditary) torsion theory $\tau \in$ R-tors has a meet pseudocomplement which, in order to conform to the notation of [5], we will denote by $\tau^{\perp}$. This meet pseudocomplement can be characterized as follows:

PROPOSITION 1. [10] *If* $\tau \in$ R-tors *then* $\tau^{\perp} = \wedge\{\chi(M) | M$ *is a τ-torsion simple left* R-*module*$\}$.

In particular, we note that, the meet-pseudocomplement of any torsion theory in R-tors is semiprime (which is to say that it is the meet of prime torsion theories) or equals χ. Using Proposition 1, it is straightforward to characterize the skeleton of R-tors.

PROPOSITION 2. *A torsion theory* $\tau \in$ R-tors *is skeletal if and only if* $\tau = \wedge\{\chi(M) | M \in A\}$ *for some set* A *of simple left* R-*modules.*

PROOF. Let R-simp be a full set of representatives of isomorphism classes of simple left R-modules. By Proposition 1, we see that every skeletal torsion theory on

R-mod is of the desired form. Conversely, let A be a subset of R-simp and let $\tau = \wedge \{\chi(M) | M \in A\}$. Set B = R-simp\A. We claim that $B = \{M' \in$ R-simp$| M'$ is τ-torsion$\}$. Indeed, if $M' \in B$ then by Proposition 12.2 of [5] we see that M' is $\chi(M)$-torsion for each $M \in A$ and so M' is τ-torsion. Conversely, if M' is τ-torsion then surely $M' \in B$. Therefore, using Proposition 1, we have $\tau^{\perp} = \wedge \{\chi(M') | M' \in B\}$ and, using the same argument again, we get $\tau^{\perp\perp} = \wedge \{\chi(M) | M \in A\} = \tau$. Therefore τ is skeletal.

Let us denote the skeleton of R-tors by R-skel. Then, using Glivenko's Theorem as previously cited, we see that R-skel can be turned into a boolean lattice by defining on it a new join operation $\veebar$ defined by $\tau_1 \veebar \tau_2 = (\tau_1 \vee \tau_2)^{\perp\perp}$.

2. **Central idempotents.** Let B(R) denote the set of all central idempotents of the ring R. Then we can turn B(R) into a boolean lattice $(B(R), \oplus, \otimes)$ with lattice operations defined by

$$e \oplus f = e + f - ef$$

and

$$e \otimes f = ef.$$

If $e \in B(R)$ then Re is an idempotent ideal of R and so e defines a centrally-splitting torsion theory $\tau_e \in$ R-tors characterized by the condition that a left R-module M is τ_e-torsion if and only if $eM = 0$. Note that $\tau_0 = \chi$, $\tau_1 = \xi$, and that, for every $e \in B(R)$, we have $\tau_e = \xi(R/Re) = \xi(R(1-e))$.

PROPOSITION 4. *If* $e \in B(R)$ *then* $T_{\tau_e}(R) = R(1-e)$ *and* $R_{\tau_e} = Re$.

PROOF. Clearly $R(1-e)$ is τ_e-torsion. Moreover, if $r \in T_{\tau_e}(R)$ then $re = 0$ so $r = r - re = r(1-e) \in R(1-e)$. Thus

$$Re = R/R(1-e) = R/T_{\tau_e}(R) \subseteq R_{\tau_e}.$$

To show that we in fact have equality it suffices to show that Re is τ_e-injective, namely that it is injective relative to every embedding $0 \to {}_R I \to R$, where I is a left ideal of R satisfying the condition that R/I is τ_e-torsion. Indeed, if I is such a left ideal of R then $I = Ie \oplus I(1-e)$. Moreover, $e(R/I) = 0$ so $Re \subseteq I$ whence $Re \subseteq Ie$. Therefore $Ie = Re$ and so $I = Re \oplus I(1-e)$, where $I(1-e)$ is a τ_e-torsion left R-module. Since Re is τ_e-trosionfree, this means that the restriction of any R-homomorphism $\alpha: I \to Re$ to $I(1-e)$ must be the 0-map and, thus, any such α can be extended to an R-homomorphism from R to Re. This is just what we wanted to prove.

We now establish a canonical embedding of B(R) into the boolean lattice R-skel.

PROPOSITION 5. *If* R *is a ring then the function*

$$\theta\colon (B(R),\oplus,\otimes) \to (R\text{-skel},\underline{\vee},\wedge)$$

defined by $\theta\colon e \mapsto \tau_e^{\perp}$ *is an embedding in the category of boolean lattices.*

PROOF. If $e \in B(R)$ then we claim that $\tau_e^{\perp} = \tau_{1-e}$ in R-tors. Indeed, if M is a left R-module which is $(\tau_e \wedge \tau_{1-e})$-torsion and if $m \in M$ then $em = 0 = (1-e)m$, so $m = 0$. Thus $\tau_e \wedge \tau_{1-e} = \xi$, which implies that $\tau_e^{\perp} \geqslant \tau_{1-e}$. Assume that this inequality is strict. There then exists a nonzero left R-module M which is $\tau_e^{\perp}$-torsion and τ_{1-e}-torsionfree. If $0 \neq m \in M$ then $(1-e)m \neq 0$. But $e(1-e)m = 0$ and so $(1-e)m \in T_{\tau_e}(M)$, contradicting the fact that M is $\tau_e^{\perp}$-torsion and hence τ_e-torsionfree. Therefore we must have $\tau_e^{\perp} = \tau_{1-e}$. In particular, this implies that $\tau_e^{\perp\perp} = \tau_{1-(1-e)} = \tau_e$ and so $\tau_e \in$ R-skel for each $e \in B(R)$.

If $e,f \in B(R)$, we claim that $\tau_e \vee \tau_f = \tau_{ef}$. Indeed, if M is a τ_e-torsion left R-module then $eM = 0$ and so $efM = 0$. Therefore $\tau_e \leqslant \tau_{ef}$ and, similarly, $\tau_f \leqslant \tau_{ef}$. This implies that $\tau_e \vee \tau_f \leqslant \tau_{ef}$. To show the reverse inequality, let M be a left R-module which is $(\tau_e \vee \tau_f)$-torsionfree. If $0 \neq m \in M$ then $em \neq 0$ and so $f(em) \neq 0$. Therefore $efm \neq 0$ for all $0 \neq m \in M$, which implies that M is τ_{ef}-torsionfree. Therefore $\tau_{ef} \leqslant \tau_e \vee \tau_f$ and so we have equality. Since $ef \in B(R)$, this implies that $\tau_e \vee \tau_f$ is skeletal, and thus $\tau_e \vee \tau_f = \tau_e \,\underline{\vee}\, \tau_f$ for all $e,f \in B(R)$.

Next, we claim that, if $e,f \in B(R)$, then $\tau_e \wedge \tau_f = \tau_{e\oplus f}$. Indeed, if M is a $(\tau_e \wedge \tau_f)$-torsion left R-module and if $m \in M$ then $em = 0 = fm$, and so $(e \oplus f)m = 0$. Thus $\tau_e \wedge \tau_f \leqslant \tau_{e\oplus f}$. Conversely, if M is a $\tau_{e\oplus f}$-torsion left R-module and if $m \in M$ then $0 = (e \oplus f)em = em$ and $0 = (e \oplus f)fm = fm$, and so M is $(\tau_e \wedge \tau_f)$-torsion. Thus $\tau_{e\oplus f} = \tau_e \wedge \tau_f$.

Therefore, making use of the above claims, we see that if $e,f \in B(R)$ we have

(1) $\theta(e \oplus f) = \tau_{e\oplus f}^{\perp} = \tau_{1-(e\oplus f)} = \tau_{(1-e)(1-f)} = \tau_{1-e} \vee \tau_{1-f} = \theta(e) \vee \theta(f)$;

(2) $\theta(e \otimes f) = \tau_{ef}^{\perp} = \tau_{1-ef} = \tau_{(1-e)\oplus(1-f)} = \tau_{1-e} \wedge \tau_{1-f} = \theta(e) \wedge \theta(f)$;

(3) $\theta(1-e) = \tau_{1-e}^{\perp} = \tau_e = \theta(e)^{\perp}$.

Thus θ is a morphism in the category of boolean lattices which is clearly an embedding.

3. The Pierce spectrum. A subset P of the boolean lattice B(R) is said to be a *prime ideal* of B(R) if and only if the following conditions are satisfied:

(1) If $e,f \in P$ then $e \oplus f \in P$;

(2) If $e \in P$ and $f \in B(R)$ then $ef \in P$;

(3) If $e,f \in B(R)\backslash P$ then $ef \in B(R)\backslash P$.

The set of all prime ideals of B(R) is called the *Pierce spectrum* of R and will be denoted by Pspec(R). Every prime torsion theory on R-mod defines a prime ideal of B(R), as the following result shows.

PROPOSITION 6. *If* $\pi \in$ R-sp *then* $W_\pi = \{e \in B(R) | \theta(e) \leqslant \pi\}$ *is an element of* Pspec(R).

PROOF. If $e,f \in W_\pi$ then $\theta(e \oplus f) = \theta(e) \vee \theta(f) \leqslant \pi$ and so $e \oplus f \in W_\pi$. If $e \in W_\pi$ and if $f \in B(R)$ then $\theta(ef) = \theta(e) \wedge \theta(f) \leqslant \pi$ and so $ef \in W_\pi$. If $e,f \in B(R)\backslash W_\pi$ then $\theta(e) \nleqslant \pi$ and $\theta(f) \nleqslant \pi$. By Proposition 19.11 of [5] this implies that

$$\theta(ef) = \theta(e) \wedge \theta(f) \nleqslant \pi$$

and so $ef \in B(R)\backslash W_\pi$.

For any $e \in B(R)$, let $V(e) = \{P \in \text{Pspec}(R) | e \notin P\} = \{P \in \text{Pspec}(R) | 1-e \in P\}$. Then the function $e \mapsto V(e)$ is an embedding of B(R) into P(Pspec(R)) in the category of boolean lattices. Moreover, one can define a topology on Pspec(R), called the *Stone topology*, by taking as a subbase of open sets the family $\{V(e) | e \in B(R)\}$. See [1] for details.

For any $\tau \in$ R-tors, let $\text{pgen}(\tau) = \{\pi \in \text{R-sp} | \pi \geqslant \tau\}$. Then one can define a topology on R-sp, called the *basic order topology*, by taking as a base of open sets the family $\{\text{pgen}(\xi(R/I)) | I \text{ a left ideal of } R\}$. See [5] for details.

PROPOSITION 7. *If* R-sp *is topologized with the basic order topology, and* Pspec(R) *is topologized with the Stone topology, then the function* ω: R-sp $\to$ Pspec(R) *defined by* $\omega: \pi \mapsto W_\pi$ *is continuous.*

PROOF. If $e \in B(R)$ then

$$\begin{aligned}\omega^{-1}(V(e)) &= \{\pi \in \text{R-sp} | W_\pi \in V(e)\} \\ &= \{\pi \in \text{R-sp} | e \notin W_\pi\} \\ &= \{\pi \in \text{R-sp} | \theta(e) \nleqslant \pi\}\end{aligned}$$

$$= \{\pi \in \text{R-sp} \mid \theta(1\text{-}e) \leqslant \pi\}$$

$$= \text{pgen}(\tau_e) = \text{pgen}(\xi(R/Re))$$

and this is an open subset of R-sp.

We can define a sheaf of rings on the space Pspec(R), topologized with the Stone topology, by assigning to each open set of the form V(e) the ring Re. (This uniquely defines the sheaf since the family of all sets of the form V(e) is a subbase for the Stone topology on Pspec(R).) This construction was defined by Pierce [8] and later extended in [2,3,4]. We will call this sheaf the *Pierce Sheaf* of R and denote it by S(_). We can also define a monopresheaf of rings $\bar{Q}(\_,R)$ on the space R-sp, topologized with the basic-order topology, by setting $\bar{Q}(U,R) = R_{\wedge U}$ for any open subset U of R-sp. This construction was introduced in [5]. We say that a ring R is *left semidefinite* if and only if $\tau = \wedge\, \text{pgen}(\tau)$ for every $\tau \in$ R-tors. The class of all left semidefinite rings is quite large and includes, for example, all rings having left Gabriel dimension (these are the *left seminoetherian* rings). See Section 20 of [5].

The Pierce sheaf over a left semidefinite ring may be far from trivial. For example, consider the ring R defined in the following manner: for each prime integer $p \geqslant 2$, let $\mathbf{Z}_p$ be the field of integers modulo p. Let $S = \Pi_{p\geqslant 2}\mathbf{Z}_p$ and let R be the smallest subring of S containing $A = \Sigma_{p\geqslant 2}\mathbf{Z}_p$ and the multiplicative identity 1_S of S. Then R is a ring having infinitely-many central idempotents and hence a nontrivial Pierce sheaf. Moreover, the elements of R are of the form $n1_S + a$ for $n \in \mathbf{Z}$ and $a \in A$. We claim that R is left semidefinite and, to prove this, it suffices to prove that every nonzero left R-module has a cocritical submodule [5]. Indeed, let Rx be a nonzero cyclic left R-module. If $Ax \neq 0$ then there exists a prime integer p for which $\mathbf{Z}_p \cap (0{:}x) = 0$ and so $\mathbf{Z}_p$ is isomorphic to a submodule of Rx. But $\mathbf{Z}_p$ is simple as a left R-module and so is surely cocritical. If $Ax = 0$ then Rx is isomorphic to a homomorphic image of R/A and, hence, to $\mathbf{Z}$ or to $\mathbf{Z}_n$, for some integer n. But $\mathbf{Z}$ is clearly cocritical as a left $\mathbf{Z}$-module and hence as a left R-module, while $\mathbf{Z}_n$ has a simple (and hence cocritical) submodule. This proves that R is left semidefinite.

If R is a left semidefinite ring then we note that

$$\bar{Q}(\text{pgen}(\tau),R) = R_\tau \text{ for every } \chi \neq \tau \in \text{R-tors}.$$

Since $\bar{Q}(\_,R)$ is a monopresheaf, it can be embedded in a sheaf of rings $\tilde{Q}(\_,R)$. If R is sufficiently nice, then $\tilde{Q}(\_,R)$ has the property that its stalk at $\pi \in R\text{-sp}$ is just R_π. See [6,7,11] for details.

We now use the above discussion to characterize the sheaf $S(\_)$ in terms of the monopresheaf $\bar{Q}(\_,R)$.

PROPOSITION 8. *If* R *is a left semidefinite ring then* $S(\_) = \omega_* \bar{Q}(\_,R)$, *where* $\omega_* \bar{Q}(\_,R)$ *is the direct image of* $\bar{Q}(\_,R)$ *defined by the continuous map* ω.

PROOF. Recall that the direct image $\omega_* \bar{Q}(\_,R)$ defined by ω is the presheaf on Pspec(R) defined by the condition that for any open subset W of Pspec(R) we have

$$\omega_* \bar{Q}(W,R) = \bar{Q}(\omega^{-1}(W),R).$$

In particular, for any $e \in B(R)$ we then have

$$\begin{aligned} \omega_* \bar{Q}(V(e),R) &= \bar{Q}(\omega^{-1}(V(e)),R) \\ &= \bar{Q}(\mathrm{pgen}(\tau_e),R) \\ &= R_{\wedge \mathrm{pgen}(\tau_e)} \\ &= R_{\tau_e} = Re = S(V(e)). \end{aligned}$$

Since two monopresheaves which agree on a subbase are equal, this implies that $\omega_* \bar{Q}(\_,R) = S(\_)$.

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Indiana University
Bloomington, Indiana 47401
and
University of Haifa
Haifa, Israel

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