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ON THE ZERO-DIVISOR GRAPHS OF COMMUTATIVE SEMIGROUPS

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Abstract. For a commutative semigroup S with 0, the zero-divisor graph of S denoted by $\Gamma(S)$ is the graph whose vertices are nonzero zero-divisor of S, and two vertices x, y are adjacent in case $xy = 0$ in S. In this paper we study the case where the graph $\Gamma(S)$ is complete r-partite for a positive integer r. Also we study the commutative semigroups which are finitely colorable.

1. Introduction

In [6] Beck introduced the concept of a zero-divisor graph $\mathrm{G}(R)$ of a commutative ring R. However, he lets all elements of R be vertices of the graph and his work was mostly concerned with coloring of rings. Later, D. F. Anderson and Livingston in [4] studied the subgraph $\Gamma(R)$ of $\mathrm{G}(R)$ whose vertices are the nonzero zero-divisors of R. The zero-divisor graph of a commutative ring has been studied extensively by several authors, e.g. [1], [5], [8], [12], and etc.

This notion has also been extended to (commutative) semigroups with zero, e.g. [9], [10], [13], and [14]. Throughout S denotes a commutative semigroup with 0. According to [10], the zero-divisor graph, $\Gamma(S)$, is an undirected graph with vertices $\mathrm{Z}(S)^* = \mathrm{Z}(S) \setminus \{0\}$, the set of nonzero zero-divisors of S, where for distinct $x, y \in \mathrm{Z}(S)^*$, the vertices x and y are adjacent if and only if $xy = 0$. In this paper we compare the algebraic structure of commutative semigroup S with the graphical structure of $\Gamma(S)$.

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For the sake of completeness, we state some definitions and notions used throughout to keep this paper as self contained as possible.

For a graph G, the set of vertices of G is denoted by $\mathrm{V}(G)$. The *degree* of a vertex v in G is the number of edges of G incident with v. An *r-partite* graph is a graph whose vertex set can be partitioned into r subsets so that no edge has both ends in any one subset. A *complete r-partite* graph is one in which each vertex is joined to every vertex that is not in the same subset as the given vertex. The *complete bipartite* (i.e., complete 2-partite) graph is denoted by $K_{m,n}$ where the set of partition has sizes m and n. We define a *coloring* of a graph G to be an assignment of colors (elements of some set) to the vertices of G, one color to each vertex, so that adjacent vertices are assigned distinct colors. If n colors are used, then the coloring is referred to as an *n-coloring*. If there exists an n-coloring of a graph G, then G is called n-colorable. The minimum n for which a graph G is n-colorable is called the *chromatic number* of G, and is denoted by $\chi(G)$. A *clique* of a graph is a maximal complete subgraph and the number of vertices in the largest clique of graph G, denoted by $\omega(G)$, is called the *clique number* of G. Obviously $\chi(G) \geq \omega(G)$ for general graph G (see [7, page 289]).

A non-empty subset I of S is called *ideal* if $xS \subseteq I$ for any $x \in I$. An ideal $\mathfrak{p}$ of a commutative semigroup is called a *prime ideal* of S if for any two element $x, y \in S$, $xy \in \mathfrak{p}$ implies $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$. Let $\mathrm{Z}(S)$ be its set of zero-divisors of S. In order that $\Gamma(S)$ be non empty, we usually assume S always contains at least one nonzero zero divisor. In [10] DeMeyer, McKenzie, and Schneider show that the number of minimal ideals of S gives a lower bound to the clique number of S. In [15] Zue and Wu studied a graph $\overline{\Gamma}(S)$ where the vertex set of this graph is $\mathrm{Z}(S)^* = \mathrm{Z}(S) \setminus \{0\}$ and for distinct elements $x, y \in \mathrm{Z}(S)^*$, if $xSy = 0$, then there is an edge connecting x and y. Note that $\Gamma(S)$ is a subgraph of $\overline{\Gamma}(S)$. Recently, F. DeMeyer and L. DeMeyer studied further the graph $\Gamma(S)$ and its extension to a simplicial complex, cf. [9]. Clearly for any prime ideal $\mathfrak{p}$ if x and y are adjacent in $\Gamma(S)$, then $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$. So for every prime ideal $\mathfrak{p}$ and every edge e, one of the end points of e belongs to $\mathfrak{p}$.

One may address three major problems in this area: characterization of the resulting graphs, characterization of the commutative semigroups with isomorphic graphs, and realization of the connections between the structures of a commutative semigroup and the corresponding graph. In this paper we focus on the third problem.

The organization of this paper is as follows:

In Section 2, we study the commutative semigroups whose zero divisor graphs are complete r-partite. It is shown that for a reduced commutative semigroup S

if $\Gamma(S)$ is a complete r-partite graph, with parts $V_1, V_2, ..., V_r$, then $V_t \cup \{0\}$ is an ideal and $\mathfrak{p}_t = Z(S) \setminus V_t$ is a prime ideal for any $1 \leq t \leq r$.

In Section 3, we study the commutative semigroups of finite chromatic number. We show that for a commutative semigroup S the following conditions are equivalent: (1) $\chi(S) < \infty$, (2) $\omega(S) < \infty$, and (3) the zero ideal is a finite intersection of prime ideals, where $\chi(S) = \chi(\Gamma(S))$ and $\omega(S) = \omega(\Gamma(S))$, see Theorem 3.3. As a corollary we show that $\chi(S) = \omega(S) = n$ if S is a reduced commutative semigroup and $0 = \cap_{i=1}^{n} \mathfrak{p}_i$ is a minimal prime decomposition of 0 (i.e. for any $i \neq j$, $\mathfrak{p}_i \neq \mathfrak{p}_j$ and for any $1 \leq t \leq n$, $0 \neq \cap_{i \neq t} \mathfrak{p}_i$). In addition, it is shown that for $n \leq 2$, $\chi(S) = n$ if and only if $\omega(S) = n$. It is shown that this result is not valid for $n = 3$. We give a finite commutative semigroup S with $\chi(S) = 4$ and $\omega(S) = 3$.

We follow standard notation and terminology from graph theory [7] and semigroup theory [11].

2. Complete r-partite graph

Let R be an infinite ring and let the zero-divisor graph of R, $\Gamma(R)$, be a complete r-partite with parts $V_1, V_2, \cdots, V_r$ and $r \geq 3$. In [1, Theorem 3.5] it is shown that for any integer $1 \leq t \leq r$ and for any $x \in V_t$, $Rx \subseteq V_t \cup \{0\}$, and $\cup_{i \neq t} V_i \cup \{0\}$ is a prime ideal. In the following we give a commutative semigroup version of this result.

Theorem 2.1. *Let S be a commutative and reduced semigroup and let $\Gamma(S)$ be a complete r-partite graph with parts $V_1, V_2, ..., V_r$. Then $V_t \cup \{0\}$ is an ideal and $\mathfrak{p}_t = Z(S) \setminus V_t$ is a prime ideal for any $1 \leq t \leq r$.*

Proof. For an arbitrary integer $1 \leq t \leq r$ choose $x \in V_t$ and $r \in S$ such that $rx \neq 0$. For any $i \neq t$, there exists $x_i \in V_i$ with $x_i x = 0$. Then $x_i(rx) = 0$. Since S is reduced we have $x_i \neq rx$ for all $i \neq t$ and hence $rx \in V_t$. Therefore $V_t \cup \{0\}$ is an ideal. By the same argument $\mathfrak{p}_t$ is an ideal. Now suppose that $xy \in \mathfrak{p}_t$, and $s_1 \in V_t$. Then $xs_1y \in \mathfrak{p}_t$, and so $xs_1y = 0$. If $xs_1 \neq 0$, then xs_1—y and $y \notin V_t$; otherwise x—s_1 and $x \notin V_t$. Therefore $x \in \mathfrak{p}_t$ or $s_1 y = 0$. That implies $x \in \mathfrak{p}_t$ or $y \in \mathfrak{p}_t$. Thus $\mathfrak{p}_t$ is a prime ideal. □

Remark. (a) It is easy to see that we can replace the condition "reduced" with the condition "for every $x \in S \setminus 0$, $x^2 \neq 0$" in the Theorem 2.1.

(b) In Theorem 2.1 if $\Gamma(S)$ is bipartite (i.e. $r = 2$), then $\Gamma(S)$ is guaranteed to be a complete bipartite graph.

The following examples show that the condition "reduced" is not redundant in the Theorem 2.1.

Example 2.2. *Let* $S = \{0, a, b, c, d\}$ *with* $b^2 = ab = bc = cd = 0$, $ac = c^2 = a^2 = c$, $d^2 = d$, $ad = bd = b$. *Then* $\Gamma(S)$ *is a bipartite graph as shown in the following diagram:*

$$a—b—c—d,$$

where the two parts of $\Gamma(S)$ *are* $V_1 = \{a, c\}$ *and* $V_2 = \{b, d\}$. *It is easy to see that* $\{a, c, 0\}$ *is not an ideal.*

Example 2.3. *Let* $S = \{0, x, y, z\}$ *with* $z^2 = yz = xz = 0$, $yx = x$, $x^2 = x$, $y^2 = y$. *Then* $\Gamma(S)$ *is a bipartite graph. In this case* $\{0, x, y\}$ *is an ideal but it is not a prime ideal.*

The condition "reduced" in the statement of Theorem 2.1 may be replaced by the condition "$|V_i| > 1$ for all i" as we outline below.

Theorem 2.4. *Suppose that* $\Gamma(S)$ *is complete* r*-partite graph with parts* $V_1, V_2, \cdots, V_r$ *such that for any* i, $|V_i| > 1$. *Then* S *is reduced.*

Proof. Let $x \in V_i$ and $r \in S$ such that $rx \neq 0$. Since for any $i \neq t$ $|V_i| > 1$, there exists $x_i \in V_i$ such that $rx \neq x_i$ but $rxx_i = 0$. Thus $rx \in V_t$. By the same argument as Theorem2.1 it is easy to show that $\mathfrak{p}_i = \mathrm{Z}(S) \setminus V_i$ is a prime ideal. Now suppose that $x \in V_i$ and $x^n = 0$ and $x^{n-1} \neq 0$. Since $V_i \bigcup \{0\}$ is an ideal of S we have that $x^{n-1} \in V_i$. But $x^n = x^{n-1}x = 0$ and so $x^2 = 0$. We show that each part V_i contains at most one nilpotent element. Let $x \neq y \in V_i$ are two nilpotent elements. Then $xy \neq 0, y^2 = x^2 = 0$. Therefore xy is adjacent to x, which is a contradiction (note that $xy \in V_i$). Now the assertion holds. Let $0 \neq x \in S$ be a nilpotent element. By part (b), $x^2 = 0$. There exists $1 \leq t \leq r$ such that $x \in V_t$ and so $0 = x^2 \in \mathfrak{p}_t$. Since $\mathfrak{p}_t$ is a prime ideal we have that $x \in \mathfrak{p}_t$ and so $x = 0$. This is a contradiction. □

In [1, Theorem 3.5], it is shown that for an infinite ring R, if $\Gamma(R)$ is a complete r-partite graph with $r \geq 3$ then r is a power of a prime integer. The following example shows that this is not true for commutative semigroups. First we recall a notion that we use in this example. Let $S_1, S_2, \cdots$ be commutative semigroups with a zero element and $S_i \cap S_j = \{0\}$ whenever $i \neq j$, the 0-*orthogonal union* of $S_1, S_2, \cdots$ is the commutative semigroup $S = S_1 \cup S_2 \cup \cdots$ in which every S_i is a subsemigroup and $S_iS_j = 0$ whenever $i \neq j$.

Example 2.5. *Let* S *be the 0-orthogonal union of* $S_1, S_2, \cdots$. *Let* $|S_i| > 2$ *for all* $i = 1, 2, \cdots, r$. *Then* $\Gamma(S)$ *is a complete* r*-partite graph if and only if* $Z(S)$

is a 0-orthogonal union of commutative semigroups without nonzero zero-divisors (namely, the commutative semigroups $S_i = V_i \cup \{0\}$*).*

Remark. Note that in Example 2.7 the condition $|S_i| > 2$ is necessary. For example consider $S = \{0, a, b, c\}$ with $ab = ac = a^2 = 0$, $bc = c^2 = b^2 = a$. In this case $\Gamma(S)$ is complete bipartite and S is not a 0-orthogonal union of non-zero commutative semigroups.

3. Commutative semigroups of finite chromatic number

In this section, we begin to characterize the commutative semigroups of finite chromatic number. Note that Beck in [6] and D. D. Anderson and Naseer in [3] let all elements of R be vertices of the graph $\Gamma(R)$) but we just consider the nonzero zero-divisors. This is the reason why the chromatic number (resp. clique number) of $\Gamma(R)$, in this paper, is one less than the chromatic number (resp. clique number) of $\Gamma(R)$ in [6] and [3].

A commutative semigroup is called *reduced* if for any $x \in S$, $x^n = 0$ implies $x = 0$. The annihilator of $x \in S$ is denoted by $\operatorname{Ann}(x)$ and it is defined as

$$\operatorname{Ann}(x) = \{a \in S | ax = 0\}.$$

In the following we bring a necessarily condition for a commutative and reduced semigroup to satisfying the a.c.c on annihilators.

Proposition 3.1. *Let S be a commutative and reduced semigroup in which $\Gamma(S)$ does not contain an infinite clique. Then S satisfies the a.c.c on annihilators.*

Proof. Suppose that $\operatorname{Ann} x_1 < \operatorname{Ann} x_2 < \cdots$ be an increasing chain of ideals. For each $i \geq 2$, choose $a_i \in \operatorname{Ann} x_i \setminus \operatorname{Ann} x_{i-1}$. Then each $y_n = x_{n-1}a_n$ is nonzero, for $n = 2, 3, \cdots$. Also $y_i y_j = 0$ for any $i \neq j$. Since S is a commutative and reduced semigroup, we have $y_i \neq y_j$ when $i \neq j$. Therefore we have an infinite clique in S. This is a contradiction and so the assertion holds. □

Lemma 3.2. *Let S be a commutative semigroup and let $Ann\, a$ be a maximal element of $\{Ann\, x : 0 \neq x \in S\}$. Then $Ann\, a$ is a prime ideal.*

Proof. Let $xy \in \operatorname{Ann} a$, and $x, y \notin \operatorname{Ann} a$. Then $xxy \in \operatorname{Ann} a$, and so $x^2ya = 0$. Since $ya \neq 0$ and $\operatorname{Ann} a \subset \operatorname{Ann} ya$, we have $\operatorname{Ann} a = \operatorname{Ann} ya$. Thus $x^2 \in \operatorname{Ann} a$ and hence $x \in \operatorname{Ann} xa = \operatorname{Ann} a$. This is a contradiction. □

Given an ideal I of S we define the radical of I to be $\sqrt{I} = \{s \in S | s^n \in I$ for some positive integer $n\}$. Clearly $\sqrt{I}$ is an ideal containing I. An ideal is said to be a radical ideal if $\sqrt{I} = I$. It is known that if I is an ideal of a commutative semigroup S, then $\sqrt{I}$ is the intersection of the prime ideals minimal over I, cf. [2, Theorem 3.3].

Now let S be a commutative reduced semigroup. Then the zero ideal $\{0\}$, which is a radical ideal, is the intersection of the prime ideals minimal over $\{0\}$. In the following we give a graph-theoretical characterization of the case where the zero ideal is a finite intersection of prime ideals.

Theorem 3.3. *For a commutative and reduced semigroup S the following are equivalent:*

(1) $\chi(S)$ *is finite.*
(2) $\omega(S)$ *is finite (i.e. $\Gamma(S)$ does not contain an infinite clique).*
(3) *The zero ideal in S is a finite intersection of prime ideals.*

Proof. Since clique$\,(S) \leq \chi(S)$, the implications (1)$\Rightarrow$(2) and is evident.

Now we prove (3)$\Rightarrow$(1). Let $0 = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \cdots \cap \mathfrak{p}_k$ where for any i, $\mathfrak{p}_i$ is a prime ideal. For any $0 \neq x \in \mathrm{Z}(S)$, there exists minimum j, such that $x \notin \mathfrak{p}_j$. Color x with j. Now suppose that x, y are colored to color j. If $xy = 0$, then $xy \in \mathfrak{p}_j$. Since $\mathfrak{p}_j$ is a prime ideal, then $x \in \mathfrak{p}_j$ or $y \in \mathfrak{p}_j$, which is contradiction. So we have a k-coloring. Thus $\chi(S) \leq k$.

It is now sufficient to show (2)$\Rightarrow$(3). By Proposition 3.1, S satisfies the a.c.c. on annihilators. Let $T = \{\mathrm{Ann}\, x_i | i \in I\}$ be the set of maximal members of the family $\{\mathrm{Ann}\, a | a \neq 0\}$. By Lemma 3.2 every element of T is a prime ideal. we show that T is a finite set. Let $\mathfrak{p} = \mathrm{Ann}\, x$, $\mathfrak{q} = \mathrm{Ann}\, y$ be two distinct elements of T. We can assume that there exists $r \in \mathfrak{p} \setminus \mathfrak{q}$. Then $0 = rx \in \mathfrak{q}$. Since $\mathfrak{q}$ is a prime ideal we have that $x \in \mathfrak{q}$ and hence $xy = 0$. Now let $L = \{x_i | \mathrm{Ann}\, x_i \in T\}$. By the above argument L is a clique of $\Gamma(S)$. Since $\omega(S)$ is finite, we have that L is finite and hence T is a finite set. Consider $0 \neq x \in S$. Then $\mathrm{Ann}\, x \subseteq \mathrm{Ann}\, x_i$ for some $i \in I$. If $xx_i = 0$, then $x_i \in \mathrm{Ann}\, x \subseteq \mathrm{Ann}\, x_i$, and so $x_i^2 = 0$. Since S is a commutative and reduced semigroup, then $x_i = 0$, which is a contradiction. Therefore $xx_i \neq 0$, and then $x \notin \mathrm{Ann}\, x_i$. Thus $\cap_{i \in I} \mathrm{Ann}\, x_i = 0$. □

It is known that $\chi(G) \geq \omega(G)$ for general graph G (see [7, page 289]). Beck showed that if R is a finite direct product of reduced coloring and principal ideal rings then $\chi(\Gamma(R)) = \omega(\Gamma(R))$. In the following result the equality $\chi(S) = \omega(S)$ is shown for some special case.

Corollary 3.4. *Suppose S be a commutative and reduced semigroup. Suppose $0 = \cap_{i=1}^{n} \mathfrak{p}_i$ is a minimal prime decomposition of 0 (i.e. for any $i \neq j$, $\mathfrak{p}_i \neq \mathfrak{p}_j$ and for any $1 \leq t \leq n$, $0 \neq \cap_{i \neq t} \mathfrak{p}_i$). Then $\chi(S) = \omega(S) = n$.*

Proof. By the proof of Theorem 3.3, we have $\chi(S) \leq n$. Let $x_i \in \cap_{i \neq t} \mathfrak{p}_i \setminus \mathfrak{p}_t$. Then $x_1, x_2, \cdots, x_n$ is a clique and so $\omega(S) \geq n$. Now we have $n \leq \omega(S) \leq \chi(S) \leq n$, and hence $\omega(S) = \chi(S) = n$. □

Example 3.5. *Let X be a n-set. We know that $(\mathcal{P}(X), \cap)$ is a commutative and reduced semigroup, where $\mathcal{P}(X)$ is the power set of X. For any $x \in X$, set $B_x = X - \{x\}$. Clearly, for any $x \in X$, $(\mathcal{P}(B_x), \cap)$ is a prime ideal, and $\cap_{x \in X} \mathcal{P}(B_x) = \{\varnothing\}$. Thus $\chi(\mathcal{P}(X)) = \omega(\mathcal{P}(X)) = n$.*

Beck showed that for $n \leq 3$, $\chi(\Gamma(R)) = n$ if and only if $\omega(\Gamma(R)) = n$. Now we are ready to show that for $n \leq 2$, $\chi(\Gamma(S)) = n$ if and only if $\omega(\Gamma(S) = n$.

Theorem 3.6. *Let S be a commutative semigroup. Then for $n \leq 2$, $\chi(\Gamma(S)) = n$ if and only if $\omega(\Gamma(S) = n$.*

Proof. The case $n = 1$ is clear. If $\chi(S) = 2$, then $\Gamma(S)$ has at least two vertices and so $\omega(S) \geq 2$. On the other hand $\omega(S) \leq \chi(S) = 2$. Thus $\omega(S) = 2$.

Conversely, let $\omega(S) = 2$. If $\chi(S) > 2$, then $\Gamma(S)$ is not bipartite and so has a cycle of odd length. Let C be the odd cycle of minimal length. Since $\omega(S) = 2$, the length of C is at least five (otherwise, the length of C is 3 and so $\omega(S) = 3$ that is a contradiction). Set

$$C\colon\ x_1 — x_2 — \cdots — x_n — x_1,$$

where $n \geq 5$ is an odd integer. If $x_1 x_3 = 0$, then $\Gamma(S)$ has a cycle of length 3, which is a contradiction. Thus $x_1 x_3 \neq 0$. Since all vertices in the cycle C has degree 2 and $x_1 x_3$ has degree 3, we have $x_1 x_3 \neq x_i$ for any $1 \leq i \leq n$. Now consider the following cycle:

$$C'\colon\ x_1x_3 — x_4 — x_5 — \cdots — x_{n-1} — x_n — x_1x_3.$$

It is easy to see that the length of C' is $n - 2$, which is a contradiction. Thus $\Gamma(S)$ has no odd cycle. Therefore $\Gamma(S)$ is bipartite and so $\chi(S) = 2$. □

Beck conjectured that $\chi(\Gamma(R)) = \omega(\Gamma(R))$ in general. In [3], D. D. Anderson and Naseer have given an example of a finite local ring with $\chi(\Gamma(R)) = 5$ and $\omega(\Gamma(R)) = 4$ thus giving a counterexample to Beck's conjecture. For $n = 1$ or 2, $\chi(S) = n$ if and only if $\omega(S) = n$. Now by giving an example we show that this result is not true for $n = 3$.

Example 3.7. Let $S = \{0, a, b, c, d, e, f\}$ with $fx = x^2 = 0$ for all $x \in S$. Also $ab = bc = cd = de = ae = 0$, and $ac = ad = bd = be = ce = f$. Then $\chi(S) = 4$ and $\omega(S) = 3$

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