Asymptotic Stability of a Jump-Diffusion Equation and its Numerical Approximation*

Graeme Chalmers[†] Desmond J. Higham[‡]

June 25, 2007

Abstract

Asymptotic linear stability is studied for stochastic differential equations (SDEs) that incorporate Poisson-driven jumps and their numerical simulation using Euler-type discretisations. The property is shown to have a simple explicit characterisation for the SDE, whereas for the discretisation a condition is found that is amenable to numerical evaluation. This allows us to evaluate the asymptotic stability behaviour of the methods. One surprising observation is that there exist problem parameters for which an explicit, forward Euler-based method has better stability than its trapezoidal and backward Euler counterparts. Other computational experiments indicate that all Euler-type methods reproduce the correct asymptotic stability for sufficiently small step sizes. By using a recent result of Appleby, Berkolaiko and Rodkina, we give a rigorous verification that both stability and instability are reproduced for small step sizes. This property is known not to hold for general, nonlinear problems.

keywords A-stability, almost sure, backward Euler, Euler-Maruyama, implicit, jump-diffusion, Poisson process, stochastic differential equation, theta method, trapezoidal rule

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*This manuscript appears as University of Strathclyde Mathematics Research Report 14 (2007).
[†]Department of Mathematics, University of Strathclyde, Glasgow G1 1XH, UK
[‡]Department of Mathematics, University of Strathclyde, Glasgow G1 1XH, UK

1 Introduction

Stability is an important property in any timestepping scenario. For stochastic differential equations (SDEs), two very natural, but distinct, concepts are mean-square and asymptotic stability. Mean-square stability is more amenable to analysis, and hence this property dominates in the literature [3, 13, 21]. Asymptotic stability has received some attention in the case of non-jump SDEs [2, 13, 16, 20]. However, in the jump-SDE context, which is becoming increasingly important in mathematical finance [4, 8, 6, 7, 11, 12, 17, 19, 22], we are only aware of mean-square results [14, 15]. This motivates the work in this article, where asymptotic stability is studied for jump-SDEs.

Our test model has the linear, scalar form

$$\mathrm{d}X(t) = \mu X(t^-)\,\mathrm{d}t + \sigma X(t^-)\,\mathrm{d}W(t) + \gamma X(t^-)\,\mathrm{d}N(t), \quad X(0) = X_0, \tag{1.1}$$

for $t > 0$, where $X_0 \neq 0$ with probability one. We use $X(t^-)$ to denote $\lim_{s\uparrow t^-} X(s)$. Here $W(t)$ is a scalar Brownian motion and $N(t)$ is a scalar Poisson process with jump intensity λ [4, 6]. In addition to λ, this model involves three other constants:

μ is the drift coefficient,

σ is the diffusion constant,

γ is the jump coefficient.

We assume throughout that $\lambda > 0$ and $\gamma \neq 0$ (otherwise the problem reduces to a non-jump SDE). We may view the problem (1.1) in terms of the exponentially distributed jump times of the Poisson process. Between each jump, the solution evolves according to the non-jump version, $\mathrm{d}X(t) = \mu X(t)\,\mathrm{d}t + \sigma X(t)\,\mathrm{d}W(t)$. At a jump time, the solution gets an instantaneous kick and $X(t)$ is replaced by $(1+\gamma)X(t)$. For $\gamma > 0$ or $\gamma < -2$ this has the effect of increasing the solution size, and for $-2 < \gamma < 0$ the solution size is decreased.

The class (1.1) is important in its own right as a model in mathematical finance [4, 6, 19], but here we are using it as a natural extension to the linear test problem that has proved valuable in the analysis of numerical methods for ODEs [10] and SDEs [2, 3, 13, 20, 21]. It is known that (1.1) has the solution

$$X(t) = X_0\,(1+\gamma)^{N(t)} \exp\left[\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W(t)\right]; \tag{1.2}$$

see, for example,[4, 5, 6].

2 Model Stability

Following the standard definition for non-jump SDEs [18], given parameters μ, σ, γ and λ, we will say that the problem (1.1) is asymptotically stable (sometimes called asymptotically stable in the large) if

$$\lim_{t\to\infty} |X(t)| = 0, \quad \text{with probability } 1, \tag{2.1}$$

for any X_0.

We now give a lemma that characterises asymptotic stability in terms of the problem parameters.

Lemma 2.1 *Suppose $\gamma \neq -1$ in (1.1), then*

$$\lim_{t\to\infty} |X(t)| = 0, \quad \text{with prob. } 1 \quad \Longleftrightarrow \quad \mu - \frac{1}{2}\sigma^2 + \lambda \log|1+\gamma| < 0. \tag{2.2}$$

Proof:
Taking logarithms in (1.2) gives

$$\log|X(t)| = \log|X_0| + \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W(t) + N(t)\log|1+\gamma|. \tag{2.3}$$

We know that

$$\lim_{t\to\infty} \frac{W(t)}{t} = 0, \quad \text{and} \quad \lim_{t\to\infty} \frac{N(t)}{t} = \lambda,$$

with probability one, by the Law of the Iterated Logarithm [18] and the Strong Law of Large Numbers [9]. Hence,

$$\lim_{t\to\infty} \frac{1}{t} \log|X(t)| = \mu - \frac{1}{2}\sigma^2 + \lambda\log|1+\gamma|, \quad \text{with prob. one.} \tag{2.4}$$

We consider separately the cases where $\mu - \frac{1}{2}\sigma^2 + \lambda\log|1+\gamma|$ is positive, negative and zero.

Case 1: For $\mu - \frac{1}{2}\sigma^2 + \lambda\log|1+\gamma| < 0$, it follows from (2.4) that we can find a random variable $\xi \equiv \xi(X_0, \varepsilon)$, where $0 < \varepsilon < \frac{1}{2}\sigma^2 - \mu - \lambda\log|1+\gamma|$, such that

$$|X(t)| \leq \xi \exp\left(\left(\mu - \frac{1}{2}\sigma^2 + \lambda\log|1+\gamma| + \varepsilon\right) t\right), \quad t \geq 0,$$

and hence $\lim_{t\to\infty} |X(t)| = 0$, with probability one.

Case 2: Similarly, for $\mu - \frac{1}{2}\sigma^2 + \lambda \log|1+\gamma| > 0$, we can find a random variable $\xi = \xi(X_0, \varepsilon)$, where $0 < \varepsilon < \mu - \frac{1}{2}\sigma^2 + \lambda \log|1+\gamma|$, such that

$$|X(t)| \geq \xi \exp\left(\left(\mu - \frac{1}{2}\sigma^2 + \lambda \log|1+\gamma| + \varepsilon\right)t\right), \quad t \geq 0,$$

and hence $|X(t)| \to \infty$ as $t \to \infty$, with probability one.

Case 3: For $\mu - \frac{1}{2}\sigma^2 + \lambda \log|1+\gamma| = 0$, we return to equation (2.3) and introduce the compensated Poisson process $\widetilde{N}(t) := N(t) - \lambda t$, so that (2.3) simplifies to

$$\log|X(t)| = \log|X_0| + \sigma W(t) + \widetilde{N}(t) \log|1+\gamma|.$$

We note that $W(t)$ and $\widetilde{N}(t)$ are independent and that $\mathbb{E}\left[\sigma W(t) + \widetilde{N}(t)\log|1+\gamma|\right] = 0$, but

$$\operatorname{Var}\left[\sigma W(t) + \widetilde{N}(t)\log|1+\gamma|\right] = \left(\sigma^2 + \lambda\big(\log|1+\gamma|\big)^2\right)t \to \infty, \qquad \text{as } t \to \infty.$$

So, $|X(t)|$ certainly does not converge to zero in this case. □

In the exceptional case where $\gamma = -1$, a jump kills the solution, so we have

$$X(t) = X_0 \exp\left[\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W(t)\right] \cdot \mathbf{1}_{\{N(t)=0\}}, \quad t \geq 0,$$

where $\mathbf{1}_A$ denotes the indicator function for A. So $\mathbb{P}[X(t) = 0] \geq 1 - e^{-\lambda t}$ and we conclude that, for any μ, σ and λ, $\lim_{t\to\infty}|X(t)| = 0$, with probability one. We note that the condition (2.2) in Lemma 2.1 could be regarded as applying in the $\gamma = -1$ case if we view $\log(0)$ as $-\infty$.

We also note that the jump coefficient γ appears in (2.2) in the form $|1+\gamma|$, a term which is symmetric about $\gamma = -1$. This follows from the fact that the stability definition (2.1) involves only the modulus of the solution, and, in this sense, the effect of a jump with $\gamma = -1 + a$ is the same as for a jump with $\gamma = -1 - a$.

The stability characterisation $\mu - \frac{1}{2}\sigma^2 + \lambda \log|1+\gamma| < 0$ involves four parameters, and hence is difficult to visualize. In Figure 1 we focus on the effect of the jump parameters, λ and γ. Here, we have contoured the function $\lambda \log|1+\gamma|$. So, for a fixed drift μ and diffusion σ, the contour at height $\frac{1}{2}\sigma^2 - \mu$ gives the boundary between asymptotic stability and instability. The broad features of the plot are intuitively reasonable. For $\gamma > 0$, increasing either the jump coefficient γ or the jump intensity λ makes the problem less stable. On the other hand, for $-1 < \gamma < 0$, where a jump reduces the solution magnitude,

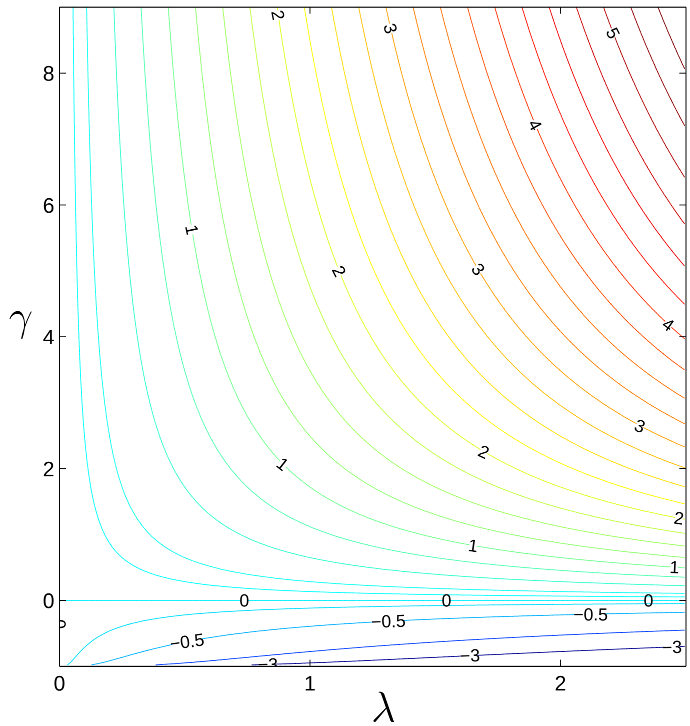

Figure 1: *Contour plot of* $\lambda \log|1+\gamma|$ *illustrating asymptotic stability of solutions of (1.1).*

increasing the jump frequency λ makes the problem more stable. For $\gamma = 0$ we revert to the condition $\mu - \frac{1}{2}\sigma^2 < 0$ for the non-jump SDE. Figure 1 only shows the case $\gamma \geq -1$, because of the underlying symmetry that we mentioned earlier.

3 Theta Method Stability

A generalisation of the theta method to jump-SDEs was introduced in [15] and studied in terms of strong convergence and linear mean-square stability, with further results for nonlinear problems appearing in [14]. Applied to the test equation (1.1) the method takes the form

$$Y_{n+1} = Y_n + (1-\theta)\mu Y_n \,\Delta t + \theta \mu Y_{n+1} \,\Delta t + \sigma Y_n \,\Delta W_n + \gamma Y_n \,\Delta N_n, \qquad (3.1)$$

with $Y_0 = X_0$. Here $Y_n \approx X(t_n)$, with $t_n = n\Delta t$, $\Delta W_n = W(t_{n+1}) - W(t_n)$ is the Brownian increment, $\Delta N_n = N(t_{n+1}) - N(t_n)$ is the Poisson increment and $\theta \in [0,1]$ is a parameter.

We suppose that the stepsize Δt is fixed. For the implicit case, $\theta > 0$, we require $\theta \, \mu \, \Delta t \neq 1$ in order for the method to be well defined. Given θ and Δt, we may write the recurrence (3.1) in the form

$$(1 - \theta\mu\Delta t)Y_{n+1} = \big(1 + (1-\theta)\mu \, \Delta t + \sigma \sqrt{\Delta t} \, \xi_n + \gamma \, \Delta N_n\big)Y_n, \tag{3.2}$$

where the ξ_n are independent standard Normal random variables and the ΔN_n are independent Poisson random variables with mean $\lambda\Delta t$ and variance $\lambda\Delta t$.

By analogy with the SDE definition (2.1), given parameters μ, σ, λ and γ and values for θ and Δt, we say that the theta method is asymptotically stable if

$$\lim_{n\to\infty} |Y_n| = 0, \quad \text{with probability 1}, \tag{3.3}$$

for any X_0.

Lemma 2.1 characterises those parameters in the underlying continuous problem that give asymptotic stability/instability and our aim is therefore to study whether the discrete approximation can produce the same long time behaviour.

The following lemma will be useful.

Lemma 3.1 (Higham [13])
Given a sequence of real-valued, non-negative, independent and identically distributed random variables $\{\zeta_k\}_{k\geq 0}$, consider the sequence of random variables $\{\eta_k\}_{k\geq 1}$ defined by

$$\eta_k = \left(\prod_{i=0}^{k-1} \zeta_i\right)\eta_0,$$

where $\eta_0 \geq 0$ and $\eta_0 \neq 1$ with probability 1. Suppose that the random variables $\log(\zeta_i)$ are square integrable. Then

$$\lim_{k\to\infty} \eta_k = 0, \text{ with probability 1} \quad \Longleftrightarrow \quad \mathbb{E}\big[\log(\zeta_i)\big] < 0.$$

Proof: See [13]. $\square$

In order to apply Lemma 3.1 to (3.2), we take

$$\eta_k := |Y_k| \quad \text{and} \quad \zeta_i := \left|\frac{1}{1-\theta\mu\,\Delta t}\Big(1 + (1-\theta)\mu\,\Delta t + \sigma\sqrt{\Delta t}\,\xi_i + \gamma\,\Delta N_i\Big)\right|.$$

A necessary and sufficient condition for asymptotic stability of the numerical method is thus

$$\mathbb{E}\left[\log\left|\frac{1}{1-\theta\mu\,\Delta t}\Big(1 + (1-\theta)\mu\,\Delta t + \sigma\sqrt{\Delta t}\,\xi_i + \gamma\,\Delta N_i\Big)\right|\right] < 0. \tag{3.4}$$

Hence, the stability issue involves the expected value of the logarithm of a linear combination of independent normal and Poisson random variables. We are not aware of any useful analytical expression for this quantity.

To gain some computational insight, we may rearrange (3.4) into the form

$$\mathbb{E}\Big[\log\big|1+(1-\theta)\mu\,\Delta t+\sigma\sqrt{\Delta t}\,\xi+\gamma\,\Delta N\big|\Big]-\log\big|1-\theta\mu\,\Delta t\big|$$

and expand over the possible values of ΔN to get

$$\begin{aligned}
&\mathbb{E}\Big[\log\big|1+(1-\theta)\mu\,\Delta t+\sigma\sqrt{\Delta t}\,\xi+\gamma\,\Delta N\big|\Big]\\
&\qquad=\sum_{k=0}^{\infty}\mathbb{P}\big(\Delta N_i=k\big)\mathbb{E}\Big[\log\big|1+(1-\theta)\mu\,\Delta t+\sigma\sqrt{\Delta t}\,\xi+\gamma k\big|\Big]\\
&\qquad=\frac{e^{-\lambda\Delta t}}{\sqrt{2\pi}}\sum_{k=0}^{\infty}\frac{(\lambda\Delta t)^k}{k!}\int_{\mathbb{R}}\log\big|1+(1-\theta)\mu\,\Delta t+\sigma\sqrt{\Delta t}\,x+\gamma k\big|e^{-x^2/2}\,\mathrm{d}x\\
&\qquad\simeq\frac{e^{-\lambda\Delta t}}{\sqrt{2\pi}}\sum_{k=0}^{K}\frac{(\lambda\Delta t)^k}{k!}\int_{-R}^{R}\log\big|1+(1-\theta)\mu\,\Delta t+\sigma\sqrt{\Delta t}\,x+\gamma k\big|e^{-x^2/2}\,\mathrm{d}x\\
&\qquad\simeq\frac{e^{-\lambda\Delta t}}{\sqrt{2\pi}}\Delta x\sum_{k=0}^{K}\frac{(\lambda\Delta t)^k}{k!}\Bigg(\sum_{j=0}^{J}\log\big|1+(1-\theta)\mu\,\Delta t+\sigma\sqrt{\Delta t}\,x_j+\gamma k\big|\exp\Big(-x_j^2/2\Big)\Bigg).
\end{aligned}$$

Here, we truncated the infinite sum to the range $0\le k\le K$, truncated each infinite integral to the range $-R\le x\le R$, and then applied a simple quadrature approximation to each integral, using a spacing Δx, with $J=\frac{2R}{\Delta x}-1$, $x_0=-R$ and $x_{j+1}=x_j+\Delta x$.

The plots in Figure 2 were produced with $K=10$, $R=10$ and $\Delta x=0.0004$. In each case, for fixed values of $\mu=0.25$ and $\sigma=0.5$, we show the range of γ and λ values for which the theta method is stable. Computations are given for $\theta=0,\ 0.25,\ 0.5,\ 0.75$ and 1. For reference the contour for the underlying test problem (as given in Figure 1) is also shown. The three pictures correspond to stepsizes $\Delta t=0.1,\ 0.01$ and 0.001. The pictures suggest that varying theta has little effect on the asymptotic stability properties, and also that all theta methods will reproduce the correct asymptotic stability for sufficiently small Δt. In section 4 we give a rigorous proof of the latter property.

The surface plot in Figure 3 gives another view, showing the expected value on the left hand side of (3.4) for the fixed values $\mu=1$, $\sigma=2$, $\lambda=1.5$ and $\gamma=0.25$, as a function of θ and Δt. Here, $\mu-\frac{1}{2}\sigma^2+\lambda\log|1+\gamma|=-0.66$, so, by Lemma 2.1, the problem is stable. The black contour line, highlighted underneath the surface, shows where the expected value in (3.4) is zero. This is the critical value where the method moves from instability to

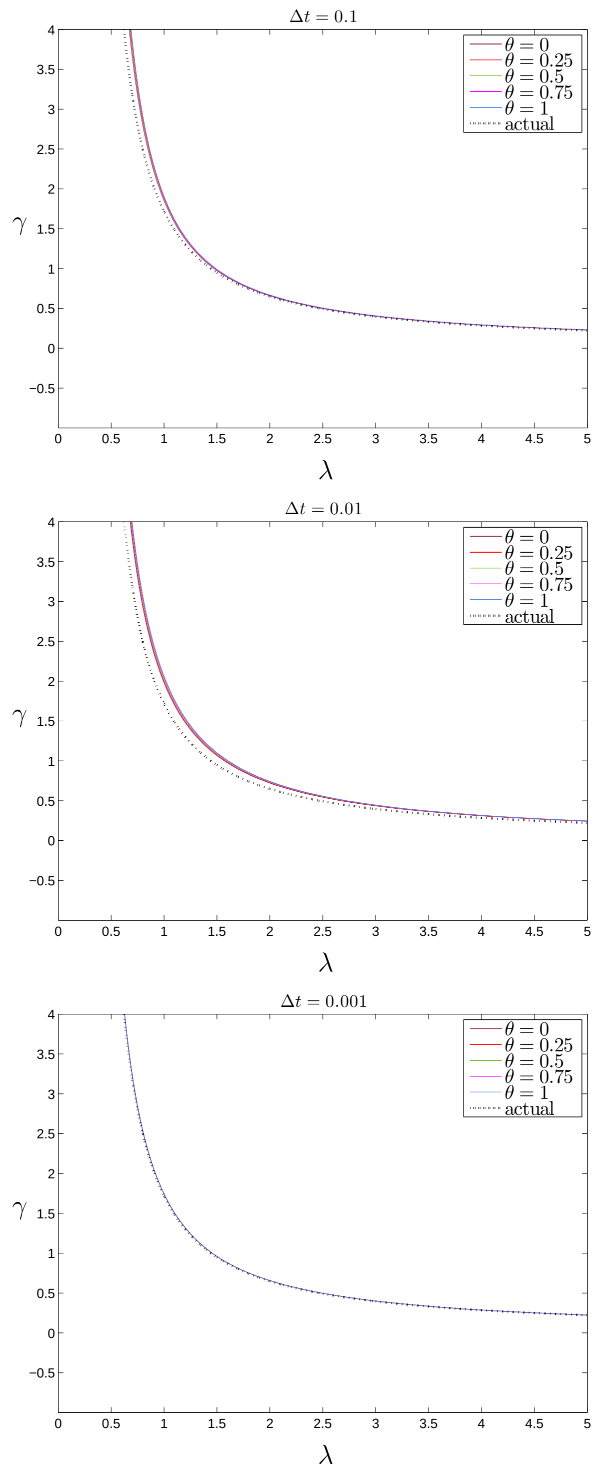

Figure 2: *Asymptotic stability boundaries for the theta methods and the underlying jump-SDE, with* $\mu = 0.25$ *and* $\sigma = 0.5$.

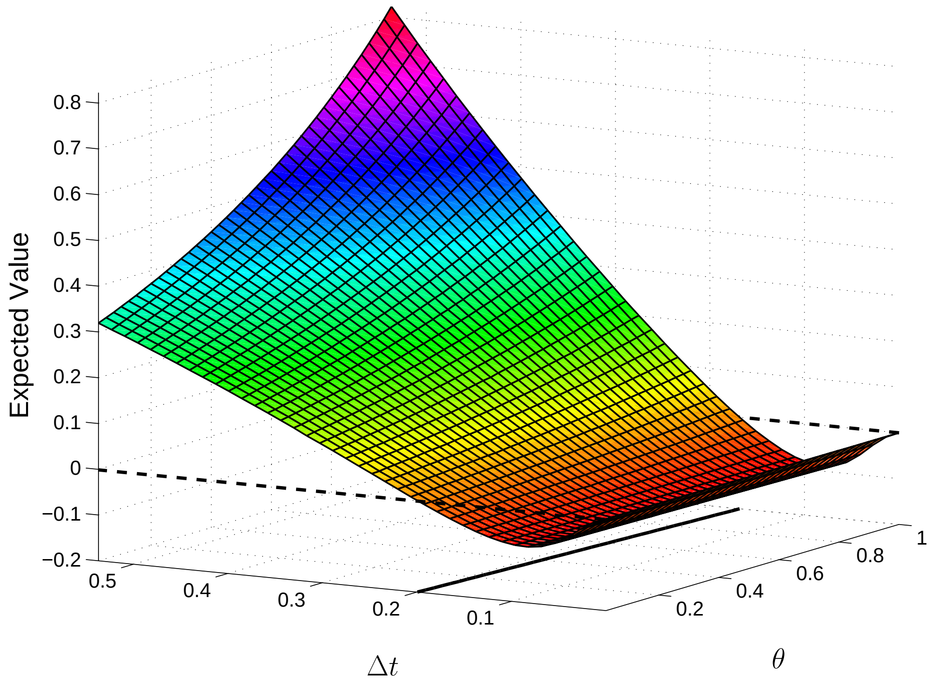

Figure 3: *Left hand side of (3.4) as a function of θ and Δt, illustrating conditions for asymptotic stability of the theta-method (3.1).*

stability. The contour indicates that, for these problem parameters, the stability behaviour, measured as the range of Δt values that reproduce asymptotic stability, is best for $\theta = 0$ and gets uniformly worse as θ increases. This effect is at odds with the behaviour seen for deterministic problems [10] and for mean-square stability on SDEs and jump-SDEs [13, 15, 21]. To confirm this visual observation, Table 1 computes the expected value in (3.4) two different ways, one by the quadrature technique and the other by Monte Carlo (with 95% confidence intervals shown), for $\theta = 0,\ 0.5$ and 1 with $\Delta t = 0.18$. We see that the expected value increases with θ, and that $\theta = 0$ is stable whereas $\theta = 1$ is unstable. As a final check, Figure 4 show one path for each of the three methods, with the vertical axis scaled logarithmically. The behaviour for $\theta = 0$ and $\theta = 0.5$ is clearly consistent with asymptotic stability. For $\theta = 1$, the lower picture, which covers a longer time scale, reveals the asymptotic instability.

$\Delta t = 0.18$	$\theta = 0$	$\theta = 0.5$	$\theta = 1$
Quadrature	-0.0203	-0.0043	0.0188
Monte Carlo	-0.0156 ± 0.0082	-0.0027 ± 0.0086	0.0163 ± 0.0090

Table 1: *Comparison of expected value approximations computed by quadrature and Monte Carlo simulation*

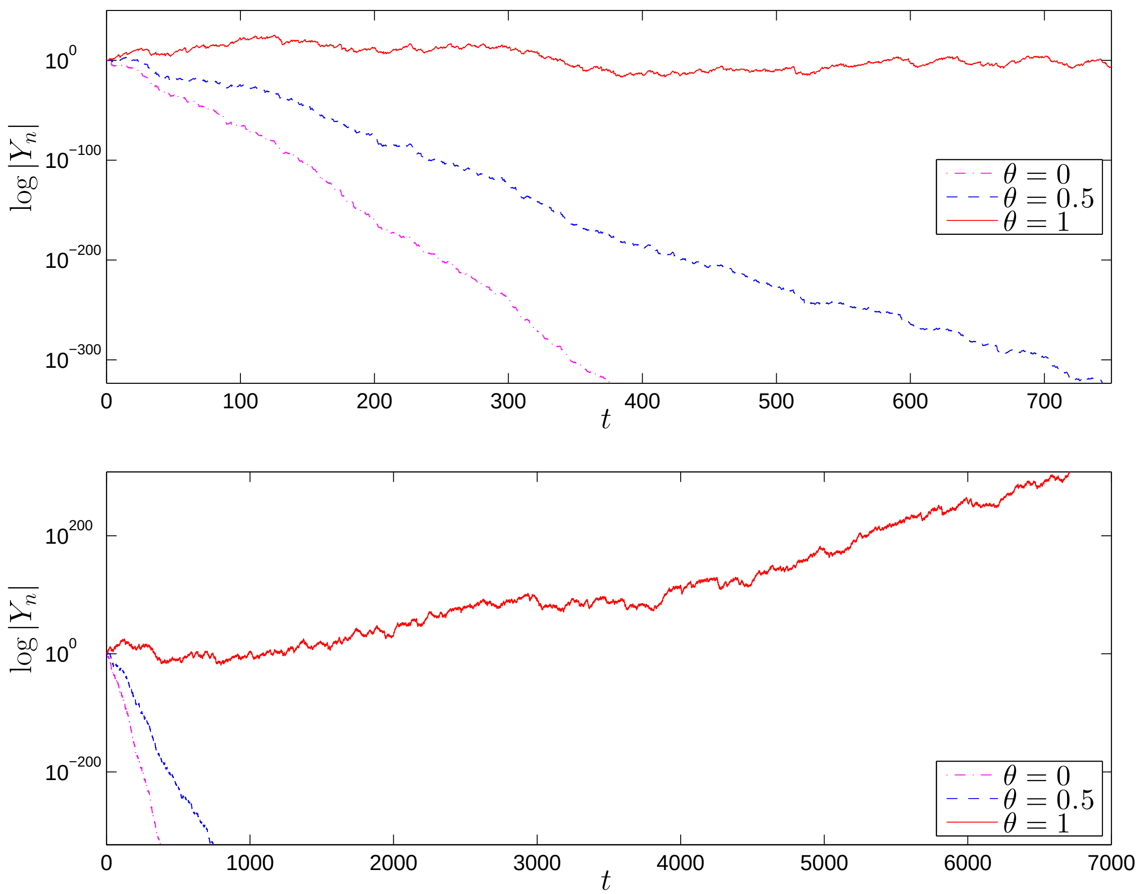

Figure 4: *Medium (upper) and long (lower) time trajectories with fixed* $\Delta t = 0.18$ *showing asymptotic stability for* $\theta = 0$ *and* 0.5*, and instability for* $\theta = 1$

4 Euler-Maruyama for Small Step Size

In [16] it was shown that on the nonlinear SDE $\mathrm{d}X(t) = (X(t) - X(t)^3)\,\mathrm{d}t + 2X(t)\,\mathrm{d}W(t)$, the basic Euler–Maruyama method does not preserve asymptotic stability for any $\Delta t > 0$. This motivated a study of small step size asymptotic stability. It was shown in [16] that on linear, scalar, SDEs, the theta method will preserve asymptotic stability for all sufficiently small Δt. In this section we extend this result to the case of the jump-SDE (1.1). Further, we simultaneously cover both the stable and unstable regimes, obtaining positive results in both cases. The analysis makes use of a recent result by Appleby, Berkolaiko and Rodkina [1].

For convenience, we focus on the $\theta = 0$ or extended Euler–Maruyama method for jump-SDEs. As we show in Corollary 5.1, the result then extends readily to general θ.

With $\theta = 0$ the recurrence (3.1) reduces to

$$Y_{n+1} = Y_n(1 + \mu\Delta t + \sigma\sqrt{\Delta t}\,\xi_n + \gamma\Delta N_n). \tag{4.1}$$

Lemma 3.1 then gives a necessary and sufficient condition for asymptotic stability of the form

$$\mathbb{E}\Big[\log|1 + \mu\Delta t + \sigma\sqrt{\Delta t}\,\xi + \gamma\Delta N|\Big] < 0, \tag{4.2}$$

where ξ is standard normal and ΔN is Poisson with parameter $\lambda\Delta t$, respectively.

Theorem 4.1 *Given μ, σ, γ and λ such that $\mu - \frac{1}{2}\sigma^2 + \lambda\log|1+\gamma| < 0$, so that, by Lemma 2.1, the jump-SDE (1.1) is asymptotically stable, there exists a $\Delta t^\star = \Delta t^\star(\mu,\sigma,\gamma,\lambda)$ such that the Euler–Maruyama method (4.1) is asymptotically stable for all $0 < \Delta t < \Delta t^\star$.*

Conversely, given μ, σ, γ and λ such that $\mu - \frac{1}{2}\sigma^2 + \lambda\log|1+\gamma| > 0$, so that, by Lemma 2.1, the jump-SDE (1.1) is not asymptotically stable, there exists a $\Delta t^\star = \Delta t^\star(\mu,\sigma,\gamma,\lambda)$ such that the Euler–Maruyama method (4.1) is not asymptotically stable for any $0 < \Delta t < \Delta t^\star$.

Proof:

Multiplying the expected value in (4.2) by $e^{\lambda\Delta t}$ for convenience, and expanding, we get

$$\begin{aligned} e^{\lambda\Delta t}\,\mathbb{E}\Big[\log\big|1+\mu\Delta t+\sigma\sqrt{\Delta t}\xi+\gamma\Delta N\big|\Big] &= \sum_{k=0}^{\infty}\frac{(\lambda\Delta t)^k}{k!}\mathbb{E}\Big[\log\big|1+\gamma k+\mu\Delta t+\sigma\sqrt{\Delta t}\,\xi\big|\Big] \\ &= \mathbb{E}\Big[\log\big|1+\mu\Delta t+\sigma\sqrt{\Delta t}\,\xi\big|\Big] \\ &\quad +\lambda\Delta t\,\mathbb{E}\Big[\log\big|1+\gamma+\mu\Delta t+\sigma\sqrt{\Delta t}\,\xi\big|\Big] \\ &\quad +\sum_{k=2}^{\infty}\frac{(\lambda\Delta t)^k}{k!}\mathbb{E}\Big[\log\big|1+\gamma k+\mu\Delta t+\sigma\sqrt{\Delta t}\,\xi\big|\Big]. \end{aligned} \tag{4.3}$$

We now consider three distinct cases, depending on the value of γ.

Case 1: $\gamma \neq -1/k$**:**
First, we deal with the generic case where $\gamma \neq -1/k$ for any integer $k \geq 1$. In this case, we may write (4.3) as

$$\begin{aligned} e^{\lambda\Delta t}\,\mathbb{E}\Big[\log|1+\mu\Delta t+\sigma\sqrt{\Delta t}\,\xi+\gamma\Delta N|\Big] &= \mathbb{E}\Big[\log|1+\mu\Delta t+\sigma\sqrt{\Delta t}\,\xi|\Big] \\ &\quad +\lambda\Delta t\bigg(\log|1+\gamma|+\mathbb{E}\Big[\log|1+\hat{\mu}\Delta t+\hat{\sigma}\sqrt{\Delta t}\,\xi|\Big]\bigg) \\ &\quad +\sum_{k=2}^{\infty}\frac{(\lambda\Delta t)^k}{k!}\log|1+\mu\Delta t+\gamma k| \\ &\quad +\sum_{k=2}^{\infty}\frac{(\lambda\Delta t)^k}{k!}\,\mathbb{E}\Big[\log|1+r_k\,\xi|\Big], \end{aligned} \tag{4.4}$$

where $\hat{\mu}=\frac{\mu}{1+\gamma}$, $\hat{\sigma}=\frac{\sigma}{1+\gamma}$ and $r_k=\frac{\sigma\sqrt{\Delta t}}{1+\mu\Delta t+\gamma k}$, for $k=2,3,\ldots$, and, for sufficiently small Δt, there is no issue of 'division by zero' or 'log of zero'.

Now, using [1, Theorem 5] with $\psi(\cdot)\equiv\log(\cdot)$, we find that

$$\mathbb{E}\Big[\log|1+\mu\Delta t+\sigma\sqrt{\Delta t}\,\xi|\Big]=\bigg(\mu-\frac{1}{2}\sigma^2\bigg)\Delta t+o(\Delta t) \tag{4.5}$$

and

$$\lambda\Delta t\bigg(\log|1+\gamma|+\mathbb{E}\Big[\log|1+\hat{\mu}\Delta t+\hat{\sigma}\sqrt{\Delta t}\,\xi|\Big]\bigg)=\lambda\Delta t\log|1+\gamma|+O(\Delta t^2). \tag{4.6}$$

Now, restricting Δt to, say, $\Delta t \leq 1/2$, we may choose a constant K_1 such that $|\gamma K_1| \geq 1+\mu\Delta t$, and hence $|1+\mu\Delta t+\gamma k| \leq |2\gamma k K_1|$. Then $\log|1+\mu\Delta t+\gamma k| \leq \log|2\gamma k K_1| =$

$\log|2\gamma K_1|+\log k$, for $k \geq 2$. Furthermore, there exists some $\widehat{k} \geq 2$ such that $|1+\mu\Delta t+\gamma k| > 1$ for $k > \widehat{k}$. We then have

$$\begin{aligned}
&\left|\sum_{k=2}^{\infty} \frac{(\lambda\Delta t)^k}{k!} \log|1+\mu\Delta t+\gamma k|\right| \\
\leq\ & \left|\sum_{k=2}^{\widehat{k}} \frac{(\lambda\Delta t)^k}{k!} \log|1+\mu\Delta t+\gamma k|\right| + \left|\sum_{k=\widehat{k}+1}^{\infty} \frac{(\lambda\Delta t)^k}{k!} \log|1+\mu\Delta t+\gamma k|\right| \\
\leq\ & \sum_{k=2}^{\widehat{k}} \frac{(\lambda\Delta t)^k}{k!} \Big|\log|1+\mu\Delta t+\gamma k|\Big| + \sum_{k=\widehat{k}+1}^{\infty} \frac{(\lambda\Delta t)^k}{k!} \log|1+\mu\Delta t+\gamma k| \\
\leq\ & (\lambda\Delta t)^2 \Bigg(\sum_{k=2}^{\widehat{k}} \frac{(\lambda\Delta t)^{k-2}}{k!} \Big|\log|1+\mu\Delta t+\gamma k|\Big| \\
& \qquad\qquad + \sum_{k=\widehat{k}+1}^{\infty} \frac{(\lambda\Delta t)^{k-2}}{k!} \log|1+\mu\Delta t+\gamma k| \Bigg) \\
=\ & (\lambda\Delta t)^2 \Bigg(K_2\widehat{k} + \sum_{k=\widehat{k}+1}^{\infty} \frac{(\lambda\Delta t)^{k-2}}{k!} \Big(\log|2\gamma K_1| + \log k\Big) \Bigg) \\
\leq\ & \lambda^2\Delta t^2 \Bigg(K_2\widehat{k} + \log|2\gamma K_1| \sum_{k=\widehat{k}+1}^{\infty} \frac{(\lambda\Delta t)^{k-2}}{k!} + \sum_{k=\widehat{k}+1}^{\infty} \frac{(\lambda\Delta t)^{k-2}}{k!} \log k \Bigg) \qquad (4.7) \\
=\ & \Big(K_2\widehat{k} + \log|2\gamma K_1| K_3 + K_4 \Big) \lambda^2\Delta t^2, \\
=\ & O(\Delta t^2). \qquad (4.8)
\end{aligned}$$

Here, $K_2 = \max_{\Delta t \leq \frac{1}{2},\ 2 \leq k \leq \widehat{k}} \Big|\log|1+\mu\Delta t+\gamma k|\Big| (\lambda\Delta t)^{k-2}/(k!)$, and, taking Δt to satisfy $\lambda\Delta t < 1$, constants K_3, K_4 are bounds (uniform in Δt) for the two convergent infinite series in (4.7).

To bound the final term in (4.4), we note that

$$\begin{aligned}
\left|\sum_{k=2}^{\infty} \frac{(\lambda\Delta t)^k}{k!} \mathbb{E}\Big[\log|1+r_k\,\xi|\Big]\right| &= \left|\sum_{k=2}^{\infty} \frac{(\lambda\Delta t)^k}{k!} \cdot \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \log|1+r_k\,x|\, e^{-x^2/2}\,\mathrm{d}x\right| \\
&= \frac{1}{\sqrt{2\pi}} (\lambda\Delta t)^2 \left|\sum_{k=2}^{\infty} \frac{(\lambda\Delta t)^{k-2}}{k!} F(r_k)\right|, \qquad (4.9)
\end{aligned}$$

where $F(r_k) = \int_{\mathbb{R}} \log|1+r_k\,x|\, e^{-x^2/2}\,\mathrm{d}x$. Making the substitution $r_{k+1}\,x = r_k\,y$, we have

$$F(r_{k+1}) = \int_{\mathbb{R}} \log|1+r_k\,y| \exp\left(-\left(\frac{r_k}{r_{k+1}}\right)^2 \frac{y^2}{2}\right) \cdot \frac{r_k}{r_{k+1}}\,\mathrm{d}y.$$

Noting that $r_k/r_{k+1} > 1$ and taking absolute values, we find

$$\begin{aligned}
|F(r_{k+1})| &= \left|\frac{r_k}{r_{k+1}}\right| \left|\int_{\mathbb{R}} \log|1 + r_k\, y| \exp\left(-\left(\frac{r_k}{r_{k+1}}\right)^2 \frac{y^2}{2}\right)\, \mathrm{d}y\right| \\
&\le \left|\frac{r_k}{r_{k+1}}\right| \left|\int_{\mathbb{R}} \log|1 + r_k\, y| \exp\left(-\frac{y^2}{2}\right)\, \mathrm{d}y\right| \\
&= \left|\frac{r_k}{r_{k+1}}\right| |F(r_k)|.
\end{aligned}$$

Hence,

$$\frac{|F(r_{k+1})|}{|F(r_k)|} \le \left|\frac{r_k}{r_{k+1}}\right|.$$

We can now examine the convergence of the infinite series in equation (4.9). If we set,

$$a_k = \left|\frac{(\lambda\Delta t)^{k-2} F(r_k)}{k!}\right|,$$

then

$$\begin{aligned}
\frac{a_{k+1}}{a_k} &= \left|\frac{\lambda\Delta t\, F(r_{k+1})}{(k+1)F(r_k)}\right| \\
&\le \left|\frac{\lambda\Delta t}{k+1} \cdot \frac{r_k}{r_{k+1}}\right| \\
&= \left|\frac{\lambda\Delta t\,(1 + \mu\Delta t + \gamma(k+1))}{(k+1)(1+\mu\Delta t + \gamma k)}\right| \quad \to 0 \qquad \text{as } k \to \infty.
\end{aligned}$$

Hence, the series in (4.9) is absolutely convergent, and we have

$$\left|\sum_{k=2}^{\infty} \frac{(\lambda\Delta t)^k}{k!} \mathbb{E}\Big[\log|1 + r_k\,\xi|\Big]\right| = O(\Delta t^2). \tag{4.10}$$

Using (4.5), (4.6), (4.8) and (4.10) in (4.4) gives

$$e^{\lambda\Delta t}\, \mathbb{E}\Big[\log|1 + \mu\Delta t + \sigma\sqrt{\Delta t}\,\xi + \gamma\Delta N|\Big] = \Big(\mu - \frac{1}{2}\sigma^2 + \lambda\log|1+\gamma|\Big)\Delta t + o(\Delta t).$$

It follows that for sufficiently small Δt and $\mu - \frac{1}{2}\sigma^2 + \lambda\log|1+\gamma| \neq 0$, the sign of $\mathbb{E}\Big[\log|1 + \mu\Delta t + \sigma\sqrt{\Delta t}\,\xi + \gamma\Delta N|\Big]$ matches the sign of $\mu - \frac{1}{2}\sigma^2 + \lambda\log|1+\gamma|$; so by Lemma 2.1 and (4.2) the result follows.

Case 2: $\gamma = -1$:
When $\gamma = -1$, we know that the problem (1.1) is asymptotically stable for all values of μ,

σ and λ. Hence, we must show that the numerical method has the same property for all sufficiently small Δt.

In this case, (4.3) becomes

$$\begin{aligned} e^{\lambda\Delta t}\,\mathbb{E}\big[\log|1+\mu\Delta t+\sigma\sqrt{\Delta t}\,\xi-\Delta N|\big] \;&=\; \mathbb{E}\big[\log|1+\mu\Delta t+\sigma\sqrt{\Delta t}\,\xi|\big] \\ &\quad+\lambda\Delta t\,\mathbb{E}\big[\log|\mu\Delta t+\sigma\sqrt{\Delta t}\,\xi|\big] \\ &\quad+\sum_{k=2}^{\infty}\frac{(\lambda\Delta t)^k}{k!}\mathbb{E}\big[\log|1-k+\mu\Delta t+\sigma\sqrt{\Delta t}\,\xi|\big]. \end{aligned} \tag{4.11}$$

To analyse the second term in the expansion of (4.11), we write

$$\mathbb{E}\big[\log|\mu\Delta t+\sigma\sqrt{\Delta t}\xi|\big]=\log(\sqrt{\Delta t})+\mathbb{E}\big[\log|\mu\sqrt{\Delta t}+\sigma\xi|\big],$$

and so

$$\mathbb{E}\big[\log|\mu\Delta t+\sigma\sqrt{\Delta t}\xi|\big]-\frac{1}{2}\log\Delta t=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\log|\mu\sqrt{\Delta t}+\sigma x|e^{-x^2/2}\,\mathrm{d}x. \tag{4.12}$$

Now choosing some constant $K_\delta=\sigma(1+\delta),\ 0<\delta<1$, we have $\log|K_\delta\,x|\geq\log|\mu\sqrt{\Delta t}+\sigma x|$ for $x\in\big(-\infty,c_1\sqrt{\Delta t}\big]\cup\big[c_2\sqrt{\Delta t},\infty\big)$, where

$$(c_1,c_2)=\begin{cases}\Big(-\mu/(\sigma-K_\delta),-\mu/(\sigma+K_\delta)\Big), & \mu<0\\ \Big(-\mu/(\sigma+K_\delta),-\mu/(\sigma-K_\delta)\Big), & \mu>0.\end{cases}$$

Note that as $K_\delta>\sigma$, we have $c_1\leq 0, c_2\geq 0,\ \forall\mu\in\mathbb{R}$. So, splitting the integral up in the natural way, taking absolute values and applying the triangle inequality, we have

$$\begin{aligned} \left|\int_{-\infty}^{\infty}\log|\mu\sqrt{\Delta t}+\sigma x|e^{-x^2/2}\,\mathrm{d}x\right| &\leq \left|\int_{-\infty}^{c_1\sqrt{\Delta t}}\log|K_\delta\,x|e^{-x^2/2}\,\mathrm{d}x\right|+\left|\int_{c_2\sqrt{\Delta t}}^{\infty}\log|K_\delta\,x|e^{-x^2/2}\,\mathrm{d}x\right| \\ &\quad+\left|\int_{c_1\sqrt{\Delta t}}^{c_2\sqrt{\Delta t}}\log|\mu\sqrt{\Delta t}+\sigma x|e^{-x^2/2}\,\mathrm{d}x\right|. \end{aligned} \tag{4.13}$$

We deal with the first two integrals in (4.13) in the same manner. Using the triangle inequality we have

$$\left|\int_{-\infty}^{c_1\sqrt{\Delta t}}\log|K_\delta\,x|e^{-x^2/2}\,\mathrm{d}x\right|\leq\left|\int_{-\infty}^{0}\log|K_\delta\,x|e^{-x^2/2}\,\mathrm{d}x\right|+\left|\int_{c_1\sqrt{\Delta t}}^{0}\log|K_\delta\,x|e^{-x^2/2}\,\mathrm{d}x\right|.$$

The first term on the right-hand side has an analytical expression. For the second term, we use $e^{-x^2/2} \le 1$, so that

$$\begin{aligned}
\left|\int_{c_1\sqrt{\Delta t}}^{0} \log|K_\delta\, x| e^{-x^2/2}\,\mathrm{d}x\right| &\le \left|\int_{c_1\sqrt{\Delta t}}^{0} \log|K_\delta\, x|\,\mathrm{d}x\right| \\
&= \left|\int_{c_1\sqrt{\Delta t}}^{0} \log(-K_\delta\, x)\,\mathrm{d}x\right| \\
&= \left|c_1\sqrt{\Delta t}\big(1-\log(-K_\delta\, c_1\sqrt{\Delta t})\big)\right| \\
&\le \sqrt{\Delta t}|c_1|\big(1+|\log K_\delta|+|\log(-c_1)|+\frac{1}{2}|\log\Delta t|\big).
\end{aligned}$$

So we have,

$$\begin{aligned}
\left|\int_{-\infty}^{c_1\sqrt{\Delta t}} \log|K_\delta\, x| e^{-x^2/2}\,\mathrm{d}x\right| &\le \frac{\sqrt{2\pi}}{4}\Big(\epsilon+|\log\frac{2}{K_\delta^2}|\Big) \\
&\quad + \sqrt{\Delta t}|c_1|\big(1+|\log K_\delta|+|\log(-c_1)|+\frac{1}{2}|\log\Delta t|\big),
\end{aligned}$$

where $\epsilon = -\int_0^\infty e^{-t}\log t\,\mathrm{d}t = \lim_{n\to\infty}\big(\sum_{k=1}^{n}\frac{1}{k}-\log n\big)$ is Euler's constant. Similarly,

$$\begin{aligned}
\left|\int_{c_2\sqrt{\Delta t}}^{\infty} \log|K_\delta\, x| e^{-x^2/2}\,\mathrm{d}x\right| &\le \frac{\sqrt{2\pi}}{4}\Big(\epsilon+|\log\frac{2}{K_\delta^2}|\Big) \\
&\quad + \sqrt{\Delta t}\, c_2\big(1+|\log K_\delta|+|\log c_2|+\frac{1}{2}|\log\Delta t|\big).
\end{aligned}$$

Taking $c_3 = \max(|c_1|, c_2)$, both integrals may therefore bounded by

$$\max\left(\left|\int_{-\infty}^{c_1\sqrt{\Delta t}} \log|K_\delta\, x| e^{-x^2/2}\,\mathrm{d}x\right|, \left|\int_{c_2\sqrt{\Delta t}}^{\infty} \log|K_\delta\, x| e^{-x^2/2}\,\mathrm{d}x\right|\right) \le$$
$$\frac{\sqrt{2\pi}}{4}\Big(\epsilon+|\log\frac{2}{K_\delta^2}|\Big) + \sqrt{\Delta t}\, c_3\big(1+|\log K_\delta|+|\log c_3|+\frac{1}{2}|\log\Delta t|\big). \tag{4.14}$$

For the third component of (4.13), we note that our choice of K_δ means we avoid a "log of zero" over the interval $[c_1\sqrt{\Delta t}, c_2\sqrt{\Delta t}]$ and therefore we may bound this definite integral in modulus as

$$\begin{aligned}
\left|\int_{c_1\sqrt{\Delta t}}^{c_2\sqrt{\Delta t}} \log|\mu\sqrt{\Delta t}+\sigma x| e^{-x^2/2}\,\mathrm{d}x\right| &\le \left|\int_{c_1\sqrt{\Delta t}}^{c_2\sqrt{\Delta t}} \log|\mu\sqrt{\Delta t}+\sigma x|\,\mathrm{d}x\right| \\
&= \left|K_5\sqrt{\Delta t}+K_6\sqrt{\Delta t}\log\Delta t\right|,
\end{aligned}$$

where

$$\begin{aligned}
K_5 &= \frac{1}{\sigma}\Big((\mu+\sigma c_2)\big(\log|\mu+\sigma c_2|-1\big)-(\mu+\sigma c_1)\big(\log|\mu+\sigma c_1|-1\big)\Big) \\
K_6 &= -\frac{K_\delta|\mu|}{\sigma^2-K_\delta^2},
\end{aligned}$$

independent of Δt.

Since $\Delta t < 1$, we have $|\log \Delta t| = -\log \Delta t$ and so, using the bounds (4.14) in (4.13) and (4.12) we find that

$$\left|\mathbb{E}\big[\log|\mu\Delta t + \sigma\sqrt{\Delta t}\xi|\big] - \frac{1}{2}\log\Delta t\right| \leq K_7,$$

for some constant K_7 independent of Δt. Now the first term on the right-hand side of (4.11) was shown to be $O(\Delta t)$ in (4.5) and the third term can be shown to be $O(\Delta t^2)$ using the same technique that we used for the infinite series in Case 1. Hence, we conclude that for all small Δt, $\left|e^{\lambda\Delta t}\mathbb{E}[\log|1+\mu\Delta t+\sigma\sqrt{\Delta t}\xi - \Delta N|] - \frac{1}{2}\log\Delta t\right|$ is uniformly bounded, showing that $\mathbb{E}[1+\log|\mu\Delta t+\sigma\sqrt{\Delta t}\xi - \delta N|]$ is negative for small Δt, as required.

Case 3: $\gamma = -1/k^\star$**, for** $1 < k^\star \in \mathbb{N}$

In this third case, (4.3) can be expanded as

$$\begin{aligned}
e^{\lambda\Delta t}\,\mathbb{E}\Big[\log\big|1+\mu\Delta t+\sigma\sqrt{\Delta t}\,\xi - \frac{\Delta N}{k^\star}\big|\Big] \quad = \quad & \mathbb{E}\big[\log|1+\mu\Delta t+\sigma\sqrt{\Delta t}\,\xi|\big] \\
& + \lambda\Delta t\,\mathbb{E}\Big[\log\big|1-\frac{1}{k^\star}+\mu\Delta t+\sigma\sqrt{\Delta t}\,\xi\big|\Big] \\
& + \frac{(\lambda\Delta t)^{k^\star}}{(k^\star)!}\mathbb{E}\big[\log|\mu\Delta t+\sigma\sqrt{\Delta t}\,\xi|\big] \\
& + \sum_{k\neq k^\star}\frac{(\lambda\Delta t)^k}{k!}\mathbb{E}\Big[\log\big|1-\frac{k}{k^\star}+\mu\Delta t+\sigma\sqrt{\Delta t}\,\xi\big|\Big].
\end{aligned}$$

The first term on the right-hand side is dealt with by (4.5). The remaining terms can be analysed using the arguments developed for Cases 1 and 2 in order to show that

$$e^{\lambda\Delta t}\,\mathbb{E}\Big[\log|1+\mu\Delta t+\sigma\sqrt{\Delta t}\,\xi - \frac{\Delta N}{k^\star}|\Big] = \Big(\mu - \frac{1}{2}\sigma^2 + \lambda\log\big|1-\frac{1}{k^\star}\big|\Big)\Delta t + o(\Delta t),$$

and so the asymptotic stability result follows from Lemma 2.1 and (4.2). □

5 Theta Method for Small Step Size

Using an idea from [16, section 4.3], we may extend Theorem 4.1 to the case of the general theta method.

Corollary 5.1 *The statements in Theorem 4.1 for the Euler–Maruyama method (4.1) also apply to the general theta method (3.1).*

Proof:

The result follows from Theorem 4.1 when we observe that the theta method (3.1) is equivalent to the Euler–Maruyama method (4.1) applied to the perturbed problem

$$\mathrm{d}X(t) = \frac{\mu}{1-\theta\mu\Delta t} X(t^-)\,\mathrm{d}t + \frac{\sigma}{1-\theta\mu\Delta t} X(t^-)\,\mathrm{d}W(t) + \frac{\gamma}{1-\theta\mu\Delta t} X(t^-)\,\mathrm{d}N(t), \quad X(0) = X_0.$$

$\square$

6 Discussion

The main conclusions of this work are that (a) a standard theta method discretisation for jump-SDEs will correctly preserve asymptotic stability for sufficiently small stepsizes, but (b) in general there is no benefit to using implicitness. This raises the open question of whether new methods can be devised that guarantee Δt-independent stability preservation, and hence offer efficiency gains on stiff problems.

Acknowledgement We thank Gregory Berkolaiko for bringing [1, Theorem 5] to our attention.

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