

The Gaussian Multiple Access Diamond Channel

Wei Kang

Nan Liu

School of Information Science and Engineering
Southeast University, Nanjing, Jiangsu, P. R. China, 210096
wkang@seu.edu.cn nanliu@seu.edu.cn

Abstract—In this paper, we study the capacity of the diamond channel. We focus on the special case where the channel between the source node and the two relay nodes are two separate links of finite capacity and the link from the two relay nodes to the destination node is a Gaussian multiple access channel. We call this model the Gaussian multiple access diamond channel. We propose both an upper bound and a lower bound on the capacity. Since the upper and lower bounds take on similar forms, it is expected that they coincide for certain channel parameters. To show this, we further focus on the symmetric case where the separate links to the relays are of the same capacity and the power constraints of the two relays are the same. For the symmetric case, we give necessary and sufficient conditions that the upper and lower bounds meet. Thus, for a Gaussian multiple access diamond channel that satisfies these conditions, we have found its capacity.

I. INTRODUCTION

The diamond channel was first introduced by Schein in 2001 [1]. It models the communication from a source node to a destination node with the help of two relay nodes. The channel between the source node and the two relays form a broadcast channel as the first stage and the channel between the two relays and the destination node form a multiple access channel as the second stage. The capacity of the diamond channel in its most general form is open. Achievability results were proposed in [1], while for the general diamond channel, the best known converse results is still the cut-set bound [2]. Capacity has been found for some special classes of diamond channels in [3], [4].

The problem of sending correlated codes through a multiple access channel was studied in [5]. This channel model can be regarded as a special class of the diamond channel where the broadcast channel between the source node and the two relay nodes are two separate links of finite capacity. We call this the multiple access diamond channel. Achievability results for the discrete multiple access diamond channel were proposed in [5], [6].

In this paper, we consider the multiple access diamond channel where the multiple access channel from the two relay nodes to the destination node is Gaussian, see Figure 1. We call this channel model the Gaussian multiple access diamond channel. We first propose an upper bound on the capacity which is tighter than the cut-set bound. The main technique

we use in the upper bound derivation is the introduction of an imaginary random variable used to bound the correlation between the two relay signals. This technique has also been used in solving the Gaussian multiple description problem [7]. We then propose a lower bound on the capacity where the relays send correlated codewords into the channel. Comparing the upper and lower bounds, we find that they are of similar forms and therefore, when the channel parameters satisfy certain conditions, they would coincide, yielding the capacity. To illustrate this, we focus our attention on the symmetric case, where the power constraints of the relay nodes are the same and the links from the source node to the two relay nodes are of the same capacity. For the symmetric case, we give necessary and sufficient conditions that the upper and lower bounds meet. Thus, for a symmetric Gaussian multiple access diamond channel that satisfies these conditions, we have found its capacity.

II. SYSTEM MODEL

We consider the Gaussian multiple access diamond channel, see Figure 1. The capacity of the link from the source node to Relay k is R_k , $k = 1, 2$. The received signal at the destination node is

$$Y = X_1 + X_2 + U$$

where X_1 and X_2 are the input signals from Relay 1 and Relay 2, respectively, and U is a zero-mean unit-variance Gaussian random variable. It is assumed that U is independent to (X_1, X_2) .

Let W be a message that the source node would like to transmit to the destination node. Assume that W is uniformly distributed on $\{1, 2, \dots, M\}$. An (M, n, ϵ_n) code consists of an encoding function at the source node

$$f^n : \{1, 2, \dots, M\} \rightarrow \{1, 2, \dots, 2^{nR_1}\} \times \{1, 2, \dots, 2^{nR_2}\},$$

two encoding functions at the relays

$$f_k^n : \{1, 2, \dots, 2^{nR_k}\} \rightarrow \mathbb{R}^n, \quad k = 1, 2$$

which satisfy the average power constraint: for any x_k^n that Relay k input into the Gaussian multiple access channel, it satisfies

$$\frac{1}{n} \sum_{i=1}^n x_{ki}^2 \leq P_k, \quad k = 1, 2$$

This work was supported by the National Natural Science Foundation of China under Grants 60902008 and 60972027, International Science and Technology Cooperation Program under Grant 2008DFA12090, and Sino-Europe cooperation project S2010GR0902.

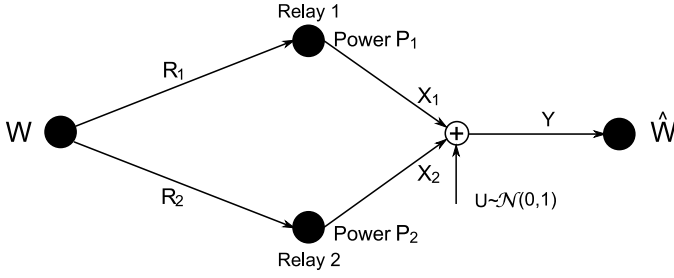


Fig. 1. The Gaussian multiple access diamond channel

and one decoding function at the destination node

$$g^n : \mathbb{R}^n \rightarrow \{1, 2, \dots, M\}$$

The probability of error is defined as

$$\epsilon_n = \sum_{w=1}^M \frac{1}{M} \Pr[g^n(Y^n) \neq w | W = w]$$

Rate R is said to be achievable if there exists a sequence of $(2^{nR}, n, \epsilon_n)$ codes such that $\epsilon_n \rightarrow 0$ as $n \rightarrow \infty$. The capacity of the Gaussian multiple access diamond channel is supremum of all achievable rates.

We would like to characterize the capacity of the Gaussian multiple access diamond channel in terms of the channel parameters R_1, R_2, P_1 and P_2 .

III. AN UPPER BOUND

Theorem 1 An upper bound on the capacity of the Gaussian multiple access diamond channel is

$$\max(T_1, T_2)$$

where

$$T_1 = \max_{0 \leq \rho \leq \rho^*} \min \left\{ \begin{array}{l} R_1 + \frac{1}{2} \log[1 + (1 - \rho^2)P_2] \\ R_2 + \frac{1}{2} \log[1 + (1 - \rho^2)P_1] \\ \frac{1}{2} \log[1 + P_1 + P_2 + 2\rho\sqrt{P_1P_2}] \\ R_1 + R_2 - \frac{1}{2} \log \frac{1}{1 - \rho^2} \end{array} \right\} \quad (1)$$

$$T_2 = \max_{\rho^* \leq \rho \leq 1} \min \left\{ \begin{array}{l} R_1 + \frac{1}{2} \log[1 + (1 - \rho^2)P_2] \\ R_2 + \frac{1}{2} \log[1 + (1 - \rho^2)P_1] \\ \frac{1}{2} \log[1 + P_1 + P_2 + 2\rho\sqrt{P_1P_2}] \\ R_1 + R_2 \end{array} \right\} \quad (2)$$

and

$$\rho^* = \sqrt{1 + \frac{1}{4P_1P_2} - \frac{1}{2\sqrt{P_1P_2}}}$$

Remark: The cut-set bound for the Gaussian multiple access diamond channel is

$$\max_{0 \leq \rho \leq 1} \min \left\{ \begin{array}{l} R_1 + \frac{1}{2} \log[1 + (1 - \rho^2)P_2] \\ R_2 + \frac{1}{2} \log[1 + (1 - \rho^2)P_1] \\ \frac{1}{2} \log[1 + P_1 + P_2 + 2\rho\sqrt{P_1P_2}] \\ R_1 + R_2 \end{array} \right\}$$

Hence, our upper bound is tighter than the cut-set bound.

Proof: From the cut-set bound, we always have

$$R \leq R_1 + R_2 \quad (3)$$

We also have

$$nR = H(W) \quad (4)$$

$$= H(X_1^n, X_2^n) + H(W | X_1^n, X_2^n) \quad (5)$$

$$\leq H(X_1^n, X_2^n) + H(W | Y^n) \quad (6)$$

$$\leq H(X_1^n, X_2^n) + n\epsilon_n \quad (7)$$

$$\leq I(X_1^n, X_2^n; Y^n) + H(X_1^n, X_2^n | Y^n) + n\epsilon_n \quad (8)$$

$$\leq I(X_1^n, X_2^n; Y^n) + 2n\epsilon_n \quad (8)$$

$$= h(Y^n) - h(Y^n | X_1^n, X_2^n) + 2n\epsilon_n$$

$$= h(Y^n) - \frac{n}{2} \log(2\pi e) + 2n\epsilon_n \quad (9)$$

where (5) is because without loss of generality, we may consider deterministic encoders, i.e., (X_1^n, X_2^n) is a deterministic function of W , (6) is because of Markov chain $W \rightarrow (X_1^n, X_2^n) \rightarrow Y^n$, and (7) and (8) both follow from Fano's inequality. We further have

$$nR = H(W) \quad (10)$$

$$\geq H(X_1^n, X_2^n) \quad (11)$$

$$\geq I(X_1^n, X_2^n; Y^n)$$

$$= h(Y^n) - \frac{n}{2} \log(2\pi e) \quad (12)$$

(11) is follows from the same reasoning as (5). Now, we upper bound $h(Y^n)$ as

$$\begin{aligned} h(Y^n) &\leq \sum_{i=1}^n h(Y_i) \\ &\leq \sum_{i=1}^n \frac{1}{2} \log(2\pi e) \left(P_{1i} + P_{2i} + 2\rho_i \sqrt{P_{1i}P_{2i}} + 1 \right) \end{aligned} \quad (13)$$

where in (13), we have defined $P_{ki} \triangleq E[X_{ki}^2]$, $k = 1, 2$ and $\rho_i = \frac{E[X_{1i}X_{2i}]}{\sqrt{P_{1i}P_{2i}}}$, and used the fact that given power constraint, the Gaussian distribution maximizes the differential entropy.

Another upper bound on R is

$$nR \leq H(X_1^n, X_2^n) + n\epsilon_n \quad (14)$$

$$= H(X_1^n | X_2^n) + H(X_2^n) + n\epsilon_n$$

$$\leq H(X_1^n | X_2^n) + nR_2 + n\epsilon_n$$

$$= I(X_1^n; Y^n | X_2^n) + H(X_1^n | Y^n, X_2^n) + nR_2 + n\epsilon_n$$

$$\leq I(X_1^n; Y^n | X_2^n) + nR_2 + 2n\epsilon_n$$

$$\leq \sum_{i=1}^n I(X_{1i}; Y_i | X_{2i}) + nR_2 + 2n\epsilon_n$$

$$\leq \sum_{i=1}^n \frac{1}{2} \log [(1 - \rho_i^2)P_{1i} + 1] + nR_2 + 2n\epsilon_n \quad (15)$$

where (14) is because of (7) and (15) follows from the same

reasoning as (13). Similarly,

$$nR \leq \sum_{i=1}^n \frac{1}{2} \log [(1 - \rho_i^2)P_{2i} + 1] + nR_1 + 2n\epsilon_n$$

Since the inputs from Relay 1 must satisfy the average power constraint P_1 , we have

$$0 \leq \frac{1}{n} \sum_{i=1}^n \rho_i^2 P_{1i} \leq \frac{1}{n} \sum_{i=1}^n P_{1i} \leq P_1$$

Therefore, there exists a $\rho_a \in [0, 1]$ such that

$$\rho_a^2 P_1 = \frac{1}{n} \sum_{i=1}^n \rho_i^2 P_{1i}$$

Due to the concavity of the logarithm function, we have

$$\begin{aligned} R &\leq \frac{1}{n} \sum_{i=1}^n \frac{1}{2} \log [(1 - \rho_i^2)P_{1i} + 1] + R_2 + 2\epsilon_n \\ &\leq \frac{1}{2} \log \left(\frac{1}{n} \sum_{i=1}^n [(1 - \rho_i^2)P_{1i} + 1] \right) + R_2 + 2\epsilon_n \\ &\leq \frac{1}{2} \log \left(\frac{1}{n} \sum_{i=1}^n P_{1i} - \frac{1}{n} \sum_{i=1}^n \rho_i^2 P_{1i} + 1 \right) + R_2 + 2\epsilon_n \\ &\leq \frac{1}{2} \log [(1 - \rho_a^2)P_1 + 1] + R_2 + 2\epsilon_n \end{aligned} \quad (16)$$

By a similar argument, from (13), we have

$$\begin{aligned} \frac{1}{n} h(Y^n) &\leq \frac{1}{2} \log(2\pi e) \left(\frac{1}{n} \sum_{i=1}^n [P_{1i} + P_{2i} + 2|\rho_i| \sqrt{P_{1i}P_{2i}} + 1] \right) \\ &\leq \frac{1}{2} \log(2\pi e) \left(P_1 + P_2 + \frac{1}{n} \sum_{i=1}^n 2\sqrt{\rho_i^2 P_{1i} P_{2i}} + 1 \right) \end{aligned}$$

From Cauchy-Schwarz inequality, we have

$$\begin{aligned} \frac{1}{n} \sum_{i=1}^n \sqrt{\rho_i^2 P_{1i} P_{2i}} &\leq \sqrt{\left(\frac{1}{n} \sum_{i=1}^n \rho_i^2 P_{1i} \right) \left(\frac{1}{n} \sum_{i=1}^n P_{2i} \right)} \\ &\leq \sqrt{\rho_a^2 P_1 P_2} \end{aligned}$$

Thus, we have

$$\frac{1}{n} h(Y^n) \leq \frac{1}{2} \log(2\pi e) \left(P_1 + P_2 + 2\rho_a \sqrt{P_1 P_2} + 1 \right) \quad (17)$$

Similarly, there exists a $\rho_b \in [0, 1]$ such that

$$\rho_b^2 P_2 = \frac{1}{n} \sum_{i=1}^n \rho_i^2 P_{2i}$$

and we have

$$R \leq \frac{1}{2} \log [(1 - \rho_b^2)P_2 + 1] + R_1 + 2\epsilon_n \quad (18)$$

$$\frac{1}{n} h(Y^n) \leq \frac{1}{2} \log(2\pi e) \left(P_1 + P_2 + 2\rho_b \sqrt{P_1 P_2} + 1 \right) \quad (19)$$

Let us define $\rho \in [0, 1]$, which is a function of $h(Y^n)$ as

follows: If

$$\frac{1}{n} h(Y^n) \leq \frac{1}{2} \log(2\pi e)(1 + P_1 + P_2) \quad (20)$$

then $\rho = 0$; otherwise, ρ is such that

$$\frac{1}{n} h(Y^n) = \frac{1}{2} \log(2\pi e)(1 + P_1 + P_2 + 2\rho \sqrt{P_1 P_2}) \quad (21)$$

For the case when $\rho = 0$, from (16), (18), (9), (20) and (3), and letting $n \rightarrow \infty$, we have

$$R \leq \frac{1}{2} \log [(1 - \rho^2)P_1 + 1] + R_2 \quad (22)$$

$$R \leq \frac{1}{2} \log [(1 - \rho^2)P_2 + 1] + R_1 \quad (23)$$

$$R \leq \frac{1}{2} \log(1 + P_1 + P_2 + 2\rho \sqrt{P_1 P_2}) \quad (24)$$

$$R \leq R_1 + R_2 - \frac{1}{2} \log \frac{1}{1 - \rho^2} \quad (25)$$

which means, for the case of $\rho = 0$, $R \leq T_1$.

For the case of $\rho > 0$, since $h(Y^n)$ must satisfy (17) and (19), we see that $\rho \leq \min(\rho_a, \rho_b)$. This means from (16), (18), (9) and (12), that we have

$$R \leq \frac{1}{2} \log (1 + (1 - \rho^2)P_1) + R_2 + 2\epsilon_n \quad (26)$$

$$R \leq \frac{1}{2} \log (1 + (1 - \rho^2)P_2) + R_1 + 2\epsilon_n \quad (27)$$

$$\begin{aligned} \frac{1}{2} \log (1 + P_1 + P_2 + 2\rho \sqrt{P_1 P_2}) &\leq R \\ &\leq \frac{1}{2} \log (1 + P_1 + P_2 + 2\rho \sqrt{P_1 P_2}) + 2\epsilon_n \end{aligned} \quad (28)$$

If ρ further satisfy $0 < \rho \leq \rho^*$, which is equivalent to $\sqrt{P_1 P_2} \left(\frac{1}{\rho} - \rho \right) - 1 \geq 0$, we define additional random variables

$$Z_i = Y_i + U'_i \quad i = 1, \dots, n$$

where U'^n is an i.i.d. Gaussian sequence with mean zero and variance

$$N = \sqrt{P_1 P_2} \left(\frac{1}{\rho} - \rho \right) - 1 \quad (29)$$

and is independent to everything else. We have

$$2nR \leq 2H(X_1^n, X_2^n) + 2n\epsilon_n \quad (30)$$

$$\begin{aligned} &= H(X_1^n, X_2^n) - I(X_1^n; X_2^n) + H(X_1^n) + H(X_2^n) + 2n\epsilon_n \\ &\leq H(X_1^n, X_2^n) - I(X_1^n; X_2^n) + nR_1 + nR_2 + 2n\epsilon_n \\ &\leq I(X_1^n, X_2^n; Y^n) - I(X_1^n; X_2^n) + nR_1 + nR_2 + 3n\epsilon_n \end{aligned} \quad (31)$$

where (30) follows because of (7), and (31) follows from (8).

Note that

$$\begin{aligned}
I(X_1^n; X_2^n) &= I(X_1^n; Z^n) - I(X_1^n; Z^n | X_2^n) + I(X_1^n; X_2^n | Z^n) \\
&\geq I(X_1^n; Z^n) - I(X_1^n; Z^n | X_2^n) \\
&= I(X_1^n, X_2^n; Z^n) - I(X_2^n; Z^n | X_1^n) - I(X_1^n; Z^n | X_2^n)
\end{aligned} \tag{32}$$

We further have

$$I(X_1^n; Z^n | X_2^n) \leq \sum_{i=1}^n \frac{1}{2} \log \frac{(1 - \rho_i^2)P_{1i} + 1 + N}{1 + N} \tag{33}$$

$$\leq \frac{n}{2} \log \frac{(1 - \rho^2)P_1 + 1 + N}{1 + N} \tag{34}$$

where (33) follows by similar arguments as (15) and (34) follows by similar arguments as (16) and (26). Similarly, we have

$$I(X_2^n; Z^n | X_1^n) \leq \frac{n}{2} \log \frac{(1 - \rho^2)P_2 + 1 + N}{1 + N} \tag{35}$$

We also have

$$\begin{aligned}
I(X_1^n, X_2^n; Z^n) &= h(Z^n) - h(Z^n | X_1^n, X_2^n) \\
&= h(Z^n) - \sum_{i=1}^n \frac{1}{2} \log(2\pi e)(1 + N)
\end{aligned}$$

From Entropy Power Inequality (EPI) [8, Lemma I], we have

$$h(Z^n) \geq \frac{n}{2} \log \left[2^{\left(\frac{2}{n} h(Y^n)\right)} + 2\pi e N \right]$$

Therefore,

$$\begin{aligned}
&h(Z^n) - h(Y^n) \\
&\geq \frac{n}{2} \log \left[1 + \frac{2\pi e N}{2^{\left(\frac{2}{n} h(Y^n)\right)}} \right] \\
&= \frac{n}{2} \log \left[1 + \frac{N}{P_1 + P_2 + 2\rho\sqrt{P_1 P_2} + 1} \right] \\
&= \frac{n}{2} \log \frac{P_1 + P_2 + 2\rho\sqrt{P_1 P_2} + 1 + N}{P_1 + P_2 + 2\rho\sqrt{P_1 P_2} + 1}
\end{aligned} \tag{36}$$

where (36) follows from (21). Thus,

$$\begin{aligned}
&I(X_1^n, X_2^n; Y^n) - I(X_1^n, X_2^n; Z^n) \\
&\leq \frac{n}{2} \log \frac{(N + 1)(P_1 + P_2 + 2\rho\sqrt{P_1 P_2} + 1)}{P_1 + P_2 + 2\rho\sqrt{P_1 P_2} + 1 + N}
\end{aligned} \tag{37}$$

Using (31), (32), (34), (35) and (37), we have

$$\begin{aligned}
&2nR \\
&\leq \frac{n}{2} \log(P_1 + P_2 + 2\rho\sqrt{P_1 P_2} + 1) \\
&\quad - \frac{n}{2} \log \frac{(P_1 + P_2 + 2\rho\sqrt{P_1 P_2} + 1 + N)(1 + N)}{((1 - \rho^2)P_1 + 1 + N)((1 - \rho^2)P_2 + 1 + N)} \\
&\quad + nR_1 + nR_2 + 3n\epsilon_n
\end{aligned} \tag{38}$$

plugging in N in (29), we have

$$\begin{aligned}
2R &\leq \frac{1}{2} \log(P_1 + P_2 + 2\rho\sqrt{P_1 P_2} + 1) - \frac{1}{2} \log \frac{1}{1 - \rho^2} \\
&\quad + R_1 + R_2 + 3\epsilon_n
\end{aligned} \tag{39}$$

Hence, for the case of $0 < \rho \leq \rho^*$, from (26), (27), (28) and (39), and letting $n \rightarrow \infty$, we have proved $R \leq T_1$.

Finally, for the case where $\rho > \rho^*$, though we do not have (39), (26), (27), (28) and (3) still hold, and by letting $n \rightarrow \infty$, we have proved $R \leq T_2$.

Hence, for all cases of $\rho \in [0, 1]$, we have proved that the achievable rate either satisfy $R \leq T_1$ or $R \leq T_2$, and thus, Theorem 1 is proved. \blacksquare

IV. A LOWER BOUND

Theorem 2 *The lower bound of the capacity of the above Gaussian multiple access diamond channel is*

$$\max_{0 \leq \rho \leq \rho^\circ} \min \left\{ \begin{array}{l} R_1 + \frac{1}{2} \log[1 + (1 - \rho^2)P_2] \\ R_2 + \frac{1}{2} \log[1 + (1 - \rho^2)P_1] \\ \frac{1}{2} \log[1 + P_1 + P_2 + 2\rho\sqrt{P_1 P_2}] \\ R_1 + R_2 - \frac{1}{2} \log \frac{1}{1 - \rho^2} \end{array} \right\} \tag{40}$$

where

$$\rho^\circ = \sqrt{1 - \exp(-2 \min(R_1, R_2))} \tag{41}$$

Proof: Consider a pair of zero-mean jointly Gaussian random variables (X_1, X_2) , such that the covariance of X_k is P_k , $k = 1, 2$ and the correlation coefficient between X_1 and X_2 is ρ .

The condition

$$0 \leq \rho \leq \rho^\circ$$

is equivalent to

$$\min(R_1, R_2) \geq \frac{1}{2} \log \frac{1}{1 - \rho^2}$$

Codebook generation: Randomly generate 2^{nR_1} independent codewords $x_1^n(i)$, $i = 1, \dots, 2^{nR_1}$ according to $p(x_1)$ and randomly generate 2^{nR_2} independent codewords $x_2^n(i)$, $i = 1, \dots, 2^{nR_2}$ according to $p(x_2)$. Then, with probability 1, for every codeword $x_1^n(i)$, $i = 1, \dots, 2^{nR_1}$, there are $2^{n(R_2 - \frac{1}{2} \log \frac{1}{1 - \rho^2})}$ x_2^n sequences joint typical with $x_1^n(i)$ according to the given Gaussian distribution. Similarly, with probability 1, for every codeword $x_2^n(i)$, $i = 1, \dots, 2^{nR_2}$, there are $2^{n(R_1 - \frac{1}{2} \log \frac{1}{1 - \rho^2})}$ x_1^n sequences joint typical with $x_2^n(i)$. We collect all the joint typical codeword pairs $(x_1^n(i), x_2^n(j))$ among all the possible (i, j) combinations and index them as $(x_1^n, x_2^n)(k)$, for $k = 1, \dots, 2^{nR}$, where

$$R = R_1 + R_2 - \frac{1}{2} \log \frac{1}{1 - \rho^2} \tag{42}$$

Encoding: When the message $W = w$, for $w = 1, \dots, 2^{nR}$, the source nodes finds the pair (i, j) that corresponds to $(x_1^n, x_2^n)(w)$. It sends index $i \in \{1, 2, \dots, 2^{nR_1}\}$ to Relay 1 and index $j \in \{1, 2, \dots, 2^{nR_2}\}$ to Relay 2. Relay 1 upon

receiving index i , sends $x_1^n(i)$ into the multiple access channel. Relay 2 upon receiving index j , sends $x_2^n(j)$ into the multiple access channel.

Decoding: Upon receiving Y^n , the receiver declares w is sent if $(x_1^n, x_2^n)(w)$ is jointly typical with the received codeword. If no such w exists, or if there is more than one such, an error is declared.

Probability of Error: By a similar argument as in [2, Sec. 14.3.1], we can show that the probability of error goes to zero if following conditions are satisfied

$$R_1 \leq I(X_1; Y, X_2)$$

$$R_2 \leq I(X_2; Y, X_1)$$

$$R \leq I(X_1, X_2; Y)$$

which means

$$R \leq R_2 + \frac{1}{2} \log[1 + (1 - \rho^2)P_1] \quad (43)$$

$$R \leq R_1 + \frac{1}{2} \log[1 + (1 - \rho^2)P_2] \quad (44)$$

$$R \leq \frac{1}{2} \log[1 + P_1 + P_2 + 2\rho\sqrt{P_1P_2}] \quad (45)$$

Thus, based on (42), (43), (44) and (45), Theorem 2 is proved. ■

V. SYMMETRIC CASE AND CAPACITY

Comparing the upper and lower bounds proposed in Theorem 1 and 2, we see that they take on similar forms, more specifically, the four functions after the minimum in (1) is exactly the same as that in (40). Thus, if the parameters of the Gaussian multiple access diamond channel, R_1, R_2, P_1 and P_2 , is such that $\rho^\circ \geq \rho^*$ and $T_1 \geq T_2$, the upper and lower bounds meet providing us with the exact capacity of the channel.

To show that there indeed exist channels such that the upper and lower bounds meet, in this section, we focus on the symmetric case, i.e., $P_1 = P_2 = P$ and $R_1 = R_2 = R_0$.

If the channel is such that

$$R_0 \geq \frac{1}{2} \log(1 + 4P)$$

it is clear that the multiple access channel in the second stage is the bottleneck of the whole network, and thus, the capacity is equal to $\frac{1}{2} \log(1 + 4P)$. On the other hand, if the channel is such that

$$\frac{1}{2} \log(1 + 2P) \geq 2R_0$$

it is clear that the two separate links in the first stage is the bottleneck of the whole network, and the capacity is equal to $2R_0$. Thus, we only need to focus on the nontrivial cases where

$$\frac{1}{4} \log(1 + 2P) < R_0 < \frac{1}{2} \log(1 + 4P) \quad (46)$$

To simplify presentation, let us define the following func-

tions of ρ :

$$f_1(\rho) \triangleq R_0 + \frac{1}{2} \log[1 + (1 - \rho^2)P]$$

$$f_2(\rho) \triangleq \frac{1}{2} \log[1 + 2(1 + \rho)P]$$

$$f_3(\rho) \triangleq 2R_0 - \frac{1}{2} \log \frac{1}{1 - \rho^2}$$

Then, for the symmetric case, Theorem 1 and Theorem 2 becomes

Corollary 1 *An upper bound on the capacity of the symmetric Gaussian multiple access diamond channel is*

$$\max(T_1, T_2) \quad (47)$$

where

$$T_1 = \max_{0 \leq \rho \leq \rho^*} \min \{f_1(\rho), f_2(\rho), f_3(\rho)\}$$

$$T_2 = \max_{\rho^* \leq \rho \leq 1} \min \{f_1(\rho), f_2(\rho), f_3(\rho)\}$$

and

$$\rho^* = \sqrt{1 + \frac{1}{4P^2} - \frac{1}{2P}}$$

Corollary 2 *A lower bound on the capacity of the symmetric Gaussian multiple access diamond channel is*

$$\max_{0 \leq \rho \leq \rho^\circ} \min \{f_1(\rho), f_2(\rho), f_3(\rho)\}$$

where

$$\rho^\circ = \sqrt{1 - 2^{(-2R_0)}}$$

Comparing Corollaries 1 and 2, we obtain the following theorem:

Theorem 3 *For the symmetric Gaussian multiple access diamond channel, the above upper and lower bound on the capacity meet iff the channel parameters are such that*

$$\rho^\circ \geq \bar{\rho}_2 \quad (48)$$

$$\rho^* \geq \bar{\rho}_1 \quad (49)$$

$$f_1(\rho^*) \leq f_3(\bar{\rho}_2) \quad (50)$$

where $\bar{\rho}_1$ and $\bar{\rho}_2$ are the positive roots of the second order equations $f_1(\rho) = f_2(\rho)$ and $f_3(\rho) = f_2(\rho)$, respectively. In this case, the capacity is $f_3(\bar{\rho}_2)$.

Proof: It is straightforward to check that for channels that satisfy (46), the second order equations $f_1(\rho) = f_2(\rho)$ and $f_3(\rho) = f_2(\rho)$ both have one and only one positive root. Furthermore, $\bar{\rho}_1, \bar{\rho}_2 \in (0, 1)$.

It can be seen that both $f_1(\rho)$ and $f_3(\rho)$ are strictly decreasing in ρ , while $f_2(\rho)$ is strictly increasing in ρ . If the channel is such that $1 - \frac{1}{2^{2R_0 - P}} < 0$, then $f_1(\rho) \geq f_3(\rho)$ for any $\rho \in [0, 1]$, such as Figure 2. If $1 - \frac{1}{2^{2R_0 - P}} \geq 0$, $f_1(\rho) < f_3(\rho)$ for $\rho < \sqrt{1 - \frac{1}{2^{2R_0 - P}}}$ and $f_1(\rho) \geq f_3(\rho)$

otherwise, such as Figure 3. In either case, it means $\bar{\rho}_1$ and $\bar{\rho}_2$ either satisfy

$$\bar{\rho}_1^2 < \bar{\rho}_2^2 < 1 - \frac{1}{2^{2R_0} - P} \quad (51)$$

which corresponds to the dotted line representing $f_2(\rho)$ in Figure 3 or

$$1 - \frac{1}{2^{2R_0} - P} \leq \bar{\rho}_2^2 \leq \bar{\rho}_1^2 \quad (52)$$

which corresponds to the solid line representing $f_2(\rho)$ in Figures 2 and 3.

Since for any $p(x_1, x_2)$, we always have

$$I(X_1; Y|X_2) + I(X_2; Y|X_1) \geq I(X_1, X_2; Y) - I(X_1; X_2)$$

Evaluating the above equation with jointly Gaussian zero-mean P -variance X_1, X_2 with correlation ρ , we obtain

$$2(f_1(\rho) - R_0) \geq f_2(\rho) + (f_3(\rho) - 2R_0), \quad \forall P, R_0 \geq 0, \rho \in [0, 1] \quad (53)$$

Based on the definition of $\bar{\rho}_1$ and $\bar{\rho}_2$, from (53), we have

$$\begin{aligned} f_2(\bar{\rho}_1) &\geq f_3(\bar{\rho}_1) \\ f_2(\bar{\rho}_2) &\leq f_1(\bar{\rho}_2) \end{aligned}$$

This means only (52), i.e., the solid line in Figures 2 and 3 is possible. This also means that the lower bound is no larger than $f_2(\bar{\rho}_2)$, in fact, when $\bar{\rho}_2$ satisfy $\bar{\rho}_2 \leq \rho^o$, the lower bound is $f_2(\bar{\rho}_2)$, otherwise, it is $f_2(\rho^o)$.

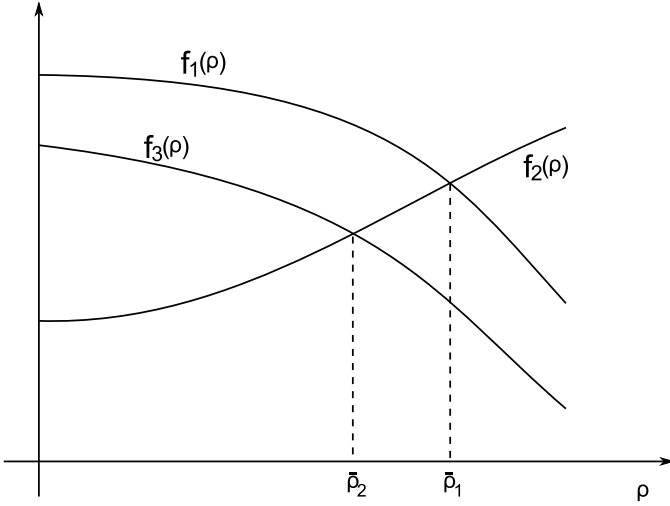


Fig. 2. $f_1(\rho) \geq f_3(\rho), \forall \rho \in [0, 1]$

First, let us consider the case where $\bar{\rho}_1 > \bar{\rho}_2$. Under this assumption, suppose $\bar{\rho}_2$ satisfies $\bar{\rho}_2 > \rho^*$, then in the upper bound, $T_1 = f_2(\rho^*)$ and $T_2 = \min(f_2(\bar{\rho}_1), 2R_0)$. Since $f_2(\rho^*) < f_2(\bar{\rho}_1)$ and $f_2(\rho^*) < f_2(\bar{\rho}_2) = f_3(\bar{\rho}_2) < 2R_0$, the upper bound is $\min(f_2(\bar{\rho}_1), 2R_0)$. The lower bound is no larger than $f_2(\bar{\rho}_2)$, which means the lower and upper bounds do not meet.

Suppose $\bar{\rho}_2$ and $\bar{\rho}_1$ satisfy $\bar{\rho}_2 \leq \rho^* < \bar{\rho}_1$, then in the

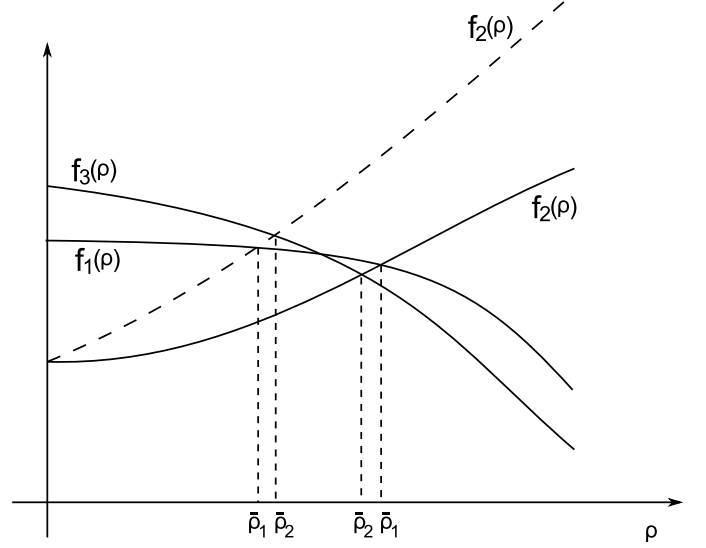


Fig. 3. $f_1(\rho)$ and $f_3(\rho)$ has a crossing point

upper bound, $T_1 = f_2(\bar{\rho}_2)$ and $T_2 = \min(f_2(\bar{\rho}_1), 2R_0)$. Since $f_2(\bar{\rho}_2) < f_2(\bar{\rho}_1)$ and $f_2(\bar{\rho}_2) = f_3(\bar{\rho}_2) < 2R_0$, the upper bound is $\min(f_2(\bar{\rho}_1), 2R_0)$. The lower bound is no larger than $f_2(\bar{\rho}_2)$, which means the lower and upper bounds do not meet.

Suppose $\bar{\rho}_1$ satisfies $\rho^* \geq \bar{\rho}_1$, then in the upper bound, $T_1 = f_2(\bar{\rho}_2)$ and $T_2 = \min(f_1(\rho^*), 2R_0)$. Note that $f_2(\bar{\rho}_2) = f_3(\bar{\rho}_2) \leq 2R_0$. In this case, the upper and lower bounds meet iff $T_2 \leq T_1$ and the lower bound is equal to $f_2(\bar{\rho}_2)$. The above two conditions are satisfied iff $\bar{\rho}_2 \leq \rho^o$, $2R_0 \geq f_1(\rho^*)$ and $f_1(\rho^*) \leq f_2(\bar{\rho}_2)$. Finally, noting that $f_1(\rho^*) \leq f_2(\bar{\rho}_2)$ implies $f_1(\rho^*) \leq f_3(\bar{\rho}_2) < 2R_0$, we have proved Theorem 3 when $\bar{\rho}_1 > \bar{\rho}_2$.

Next, let us consider the special case where $\bar{\rho}_1 = \bar{\rho}_2 = \sqrt{1 - \frac{1}{2^{2R_0} - P}} \triangleq \bar{\rho}$. This is equivalent to

$$P = \frac{2^{2R_0}(2^{2R_0} - 1)}{2^{2R_0+1} - 1} \quad (54)$$

In this case, since $\rho^o \geq \bar{\rho}$, the lower bound is $f_2(\bar{\rho})$. The upper bound is also $f_2(\bar{\rho})$. Thus, when the channel parameters satisfy (54), the upper and lower bounds meet. It is straightforward to check that when the channel parameters satisfy (54), $\rho^* = \bar{\rho}$ and conditions (48)-(50) satisfy also. Thus, we have proved Theorem 3 for all possible $(\bar{\rho}_1, \bar{\rho}_2)$. ■

To show that there indeed exist symmetric Gaussian multiple access diamond channels that satisfy (48)-(50), take for example, $P = 3$ and $R_0 = 1.2$, we then have

$$\begin{aligned} \rho^o &= 0.9003, \rho^* = 0.8471, \bar{\rho}_1 = 0.7734, \bar{\rho}_2 = 0.7643, \\ f_1(\rho^*) &= 1.6426, f_3(\bar{\rho}_2) = 1.7671 \end{aligned}$$

Thus, for $P = 3$ and $R_0 = 1.2$, (48)-(50) are satisfied and the achievability results and the converse results coincide, yielding the capacity, which is 1.7671.

To illustrate further, we plot the upper and lower bounds in Corollaries 1 and 2 and depict them in Fig. 4 and Fig. 5 for

the cases of $P = 3$ and $P = 30$, respectively. As can be seen, the gap between the lower and upper bounds is rather small, especially when R_0 is relatively small and/or P is relatively large.

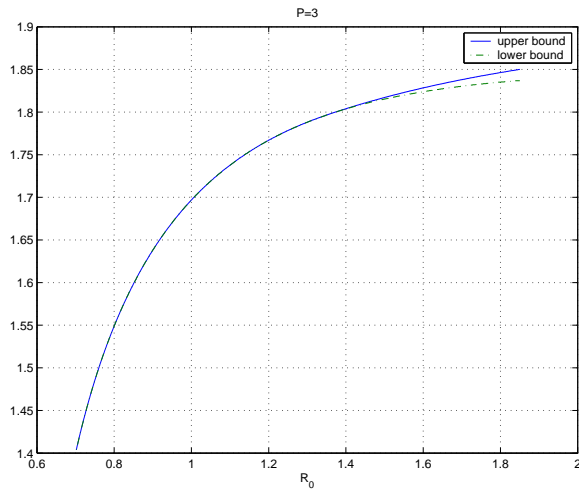


Fig. 4. Comparison of upper and lower bounds with $P = 3$

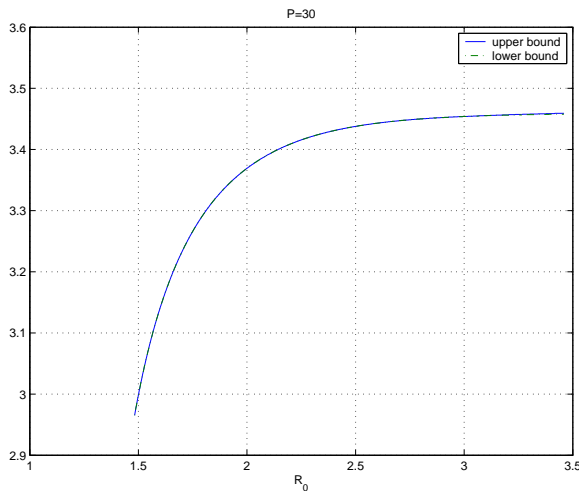


Fig. 5. Comparison of upper and lower bounds with $P = 30$

VI. CONCLUSIONS

We have studied the Gaussian multiple access diamond channel and provided upper and lower bounds on the capacity. Focusing on the symmetric case, we gave necessary and sufficient conditions that the upper and lower bounds meet. Thus, for a symmetric Gaussian multiple access diamond channels that satisfies these conditions, we have found its capacity.

REFERENCES

[1] B. E. Schein. *Distributed Coordination in Network Information Theory*. PhD thesis, Massachusetts Institute of Technology, 2001.

[2] T. M. Cover and J. A. Thomas. *Elements of Information Theory*. John Wiley and Sons, 1991.

[3] W. Kang and S. Ulukus. Capacity of a class of diamond channels. In *46th Annual Allerton Conference on Communications, Control and Computing*, pages 1426–1431, Monticello, IL, September 2008.

[4] R. Tandon and S. Ulukus. Diamond channel with partially separated relays. In *Proc. IEEE International Symp. on Information Theory (ISIT)*, pages 644–648, Austin, TX, June 2010.

[5] R. Ahlswede and T. S. Han. On source coding with side information via a multiple-access channel and related problems in multi-user information theory. *IEEE Trans. Inform. Theory*, 29(3):396–412, 1983.

[6] D. Traskov and G. Kramer. Reliable communication in networks with multi-access interference. In *Proc. Conf. IEEE Information Theory Workshop (ITW)*, Lake Tahoe, CA, Sep. 2007.

[7] L. Ozarow. On a source-coding problem with two channels and three receivers. *Bell Syst. Tech. J.*, 59:1909–1921, December 1980.

[8] P. Bergmans. A simple converse for broadcast channels with additive white Gaussian noise. *IEEE Trans. on Information Theory*, 20:279–280, March 1974.