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Koepf, Wolfram (D-KSSL)

Bieberbach's conjecture, the de Branges and Weinstein functions and the Askey-Gasper inequality. (English summary)

Ramanujan J. **13** (2007), *no. 1-3,* 103–129.

The first part of this richly referenced survey article provides an historical account of the Bieberbach, Robertson and Milin Conjectures.

Next, the author recounts the proof of the Milin Conjecture (which implies the Bieberbach Conjecture) given by L. de Branges [Acta Math. **154** (1985), no. 1-2, 137–152; MR0772434 (86h:30026)]. The proof involves a system of auxiliary hypergeometric functions $\tau_k^n(t)$ with special properties that can be established via the Askey-Gasper Identity. The author indicates how this crucial identity can be proven by using a short Maple package (included in the paper).

The author also recounts the proof of the Milin Conjecture given by L. Weinstein [Internat. Math. Res. Notices **1991**, no. 5, 61–64; MR1131432 (92m:30033)], involving a different system of functions $\Lambda_k^n(t)$, but without using the Askey-Gasper result. He also discusses how the final part of Weinstein's proof can be computerized.

It was shown independently by H. S. Wilf [Bull. London Math. Soc. **26** (1994), no. 1, 61–63; MR1246472 (95a:30019)] and P. G. Todorov [Acad. Roy. Belg. Bull. Cl. Sci. (6) **3** (1992), no. 12, 335–346; MR1266020 (95c:30024)] that the auxiliary functions of de Branges and Weinstein are related by the formula

$$\frac{d}{dt}\,\tau_k^n(t) = -k\,\Lambda_k^n(t),$$

and the author gives an elementary proof of this fact. The article concludes with a discussion of the generating functions for both sets of auxiliary functions.

For another easily accessible account of the proof, see the paper by A. Z. Grinshpan [Amer. Math. Monthly **106** (1999), no. 3, 203–214; MR1682341 (2000b:30027)].

Reviewed by *Stephen M. Zemyan*

References

1. Aharonov, D.: On Bieberbach Eilenberg functions. Bull. Amer. Math. Soc. **76**, 101–104 (1970) MR0257343 (41 #1994)
2. Askey, R., Gasper, G.: Positive Jacobi polynomial sums II. Amer. J. Math. **98**, 709–737 (1976) MR0430358 (55 #3363)
3. Baernstein, A., Drasin, D., Duren, P., Marden, A. (eds.): The Bieberbach conjecture. Proceedings of the Symposium on the Occasion of the Proof. Mathematical surveys and monographs, vol. 21. American Mathematical Society, Providence, R. I. (1986) MR0875226 (87k:30002)
4. Bieberbach, L.: Über die Koeffizienten derjenigen Potenzreihen, welche eine schlichte Abbildung des Einheitskreises vermitteln. S.-B. Preuss. Akad. Wiss. **38**, 940–955 (1916)

5. de Branges, L.: A proof of the Bieberbach conjecture. Acta Math. **154**, 137–152 (1985) MR0772434 (86h:30026)
6. Brickman, L.: Extreme points of the set of univalent functions. Bull. Amer. Math. Soc. **76**, 372–374 (1970) MR0255788 (41 #448)
7. Brickman, L., MacGregor, T. H., Wilken, D. R.: Convex hulls of some classical families of univalent functions. Trans. Amer. Math. Soc. **156**, 91–107 (1971) MR0274734 (43 #494)
8. Carathéodory, C.: Über den Variabilitätsbereich der Koeffizienten von Potenzreihen, die gegebene Werte nicht annehmen. Math. Ann. **64**, 95–115 (1907) MR1511425
9. Carathéodory, C.: Über den Variabilitätsbereich der Fourier'schen Konstanten von positiven harmonischen Funktionen. Rend. Circ. Mat. Palermo **32**, 193–217 (1911)
10. Carathéodory, C.: Untersuchungen über die konformen Abbildungen von festen und veränderlichen Gebieten. Math. Ann. **72**, 107–144 (1912) MR1511688
11. Charzyński, Z., Schiffer, M.: A new proof of the Bieberbach conjecture for the fourth coefficient. Arch. Rational Mech. Anal. **5**, 187–193 (1960) MR0114930 (22 #5746)
12. Collins, G.E., Krandick, W.: An efficient algorithm for infallible polynomial complex roots isolation. In: Wang, Paul S. (ed.) Proceedings of ISSAC'92, pp. 189–194 (1992)
13. Dieudonné, J.: Sur les fonctions univalentes. C. R. Acad. Sci. Paris **192**, 1148–1150 (1931)
14. Duren. P.L.: Coefficients of univalent functions. Bull. Amer. Math. Soc. **83**, 891–911 (1977) MR0470182 (57 #9943)
15. Duren, P.L.: Univalent functions. Grundlehren der mathematischen Wissenschaften, vol. 259. Springer-Verlag, New York-Berlin-Heidelberg-Tokyo (1983) MR0708494 (85j:30034)
16. Ekhad, S.B., Zeilberger, D.: A high-school algebra, "Formal calculus", proof of the Bieberbach conjecture [after L. Weinstein]. In: Barcelo et al. (eds.) Jerusalem combinatorics '93: an International Conference in Combinatorics, May 9–17, 1993, Jerusalem, Israel. Providence, RI: American Mathematical Society. Contemp. Math. **178**, 113–115 (1994) MR1310578 (95m:30027)
17. Fekete, M., Szegö, G.: Eine Bemerkung über ungerade schlichte Funktionen. J. London Math. Soc. **8**, 85–89 (1933)
18. FilzGerald, C.H., Quadratic inequalities and coefficient estimates for schlicht functions. Arch. Rational Mech. Anal. **46**, 356–368 (1972) MR0335777 (49 #557)
19. FitzGerald, C.H., Pommerenke, Ch.: The de Branges Theorem on univalent functions. Trans. Amer. Math. Soc. **290**, 683–690 (1985) MR0792819 (87b:30023)
20. Friedland, S.: On a conjecture of Robertson. Arch. Rational Mech. Anal. **37**, 255–261 (1970) MR0255796 (41 #456)
21. Garabedian, P. R., Schiffer, M.: A proof of the Bieberbach conjecture for the fourth coefficient. J. Rational Mech. Anal. **4**, 427–465 (1955) MR0070712 (17,24e)
22. Gautschi, W.: Reminiscences of my involvement in de Branges's proof of the Bieberbach conjecture. In: Baernstein, Drasin, Duren, Marden (Eds): The Bieberbach conjecture (West Lafayette, Ind., 1985), Math. Surveys Monogr. 21, Amer. Math. Soc., Providence, RI, pp. 205–211 (1986) MR0875243 (88f:30001b)
23. Gong, S.: The Bieberbach conjecture. Studies in Advanced Mathematics, vol. 12. American Mathematical Society, Providence, R.I. (1999) MR1699322 (2000j:30029)
24. Gronwall, T.H.: Some remarks on conformal representation. Ann. of Math. **16**, 72–76 (1914–

1915) MR1502490
25. Grunsky, H.: Koeffizientenbedingungen für schlicht abbildende meromorphe Funktionen. Math. Z. **45**, 29–61 (1939) MR1545803
26. Hamilton, D.H.: Extremal boundary problems. Proc. London Math. Soc. (3) **56**, 101–113 (1988) MR0915532 (89e:30042)
27. Hayman, W.K.: The asymptotic behaviour of p-valent functions. Proc. London Math. Soc. (3) **5**, 257–284 (1955) MR0071536 (17,142j)
28. Hayman, W.K., Hummel, J.A.: Coefficients of powers of univalent functions. Complex Variables **7**, 51–70 (1986) MR0877651 (88d:30020)
29. Heine, E.: Handbuch der Kugelfunctionen. Theorie und Anwendungen. Reimer, Berlin (1878)
30. Henrici. P.: Applied and Computational Complex Analysis, vol. 3: Discrete Fourier Analysis—Cauchy Integrals—Construction of Conformal maps—Univalent Functions. John Wiley & Sons, New York (1986) MR0822470 (87h:30002)
31. Horowitz, D.: A further refinement for coefficient estimates of univalent functions. Proc. Amer. Math. Soc. **71**, 217–221 (1978) MR0480979 (58 #1126)
32. Hummel, J. A.: A variational method for Gelfer functions. J. Analyse Math. **30**, 271–280 (1976) MR0440025 (55 #12906)
33. Koebe, P.: Über die Uniformisierung beliebiger analytischer Kurven. Nachr. Kgl. Ges. Wiss. Göttingen, Math-Phys. Kl, pp. 191–210 (1907)
34. Koebe, P.: Über die Unifomisierung der algebraischen Kurven durch automorphe Funktionen mit imaginärer Substitutionsgruppe. Nachr. Kgl. Ges. Wiss. Göttingen, Math-Phys. Kl, pp. 68–76 (1909)
35. Koebe, P.: Über eine neue Methode der konformen Abbildung und Uniformisierung. Nachr. Kgl. Ges. Wiss. Göttingen, Math-Phys. Kl, pp. 844–848 (1912)
36. Koepf, W.: On nonvanishing univalent functions with real coefficients. Math. Z. **192**, 575–579 (1986) MR0847006 (87m:30020)
37. Koepf, W.: Extrempunkte und Stützpunkte in Familien nichtverschwindender schlichter Funktionen. Complex Variables **8**, 153–171 (1987) MR0891758 (88j:30024)
38. Koepf, W.: Power series in Computer Algebra. J. Symb. Comp. **13**, 581–603 (1992) MR1177710 (93j:68087)
39. Koepf, W.: Hypergeometric Summation. Vieweg, Braunschweig/Wiesbaden (1998) MR1644447 (2000c:33002)
40. Koepf, W.: Power series, Bieberbach conjecture and the de Branges and Weinstein functions. In: Sendra, J.R. (ed.) Proceedings of ISSAC 2003, pp. 169–175. Philadelphia, ACM, New York (2003). MR2035209 (2004m:68293)
41. Koepf, W., Schmersau, D.: On the de Branges theorem. Complex Variables **31**, 213–230 (1996) MR1423252 (97m:30017)
42. Laplace, P.-S.: Théorie des attractions des sphéroïdes et de la figure des planètes. Mémoires de l'Academie Royale des Sciences de Paris 113–196 (1782).
43. Lebedev, N. A., Milin, I. M.: An inequality. Vestnik Leningrad Univ. **20**, 157–158 (1965) (Russian) MR0186793 (32 #4248)
44. Leeman, G. B.: The seventh coefficient of odd symmetric univalent functions. Duke Math. J.

43, 301–307 (1976) MR0409793 (53 #13545)
45. Legendre, A.-M.: Suite des recherches sur la figure des planètes. Mémoires de l'Academie Royale des Sciences de Paris, pp. 372–454 (1789).
46. Leung, Y.: Successive coefficients of starlike functions. Bull. London Math. Soc. **10**, 193–196 (1978) MR0480980 (58 #1127)
47. Leung, Y.: Robertson's conjecture on the coefficients of close-to-convex functions. Proc. Amer. Math. Soc. **76**, 89–94 (1979) MR0534395 (80i:30019)
48. Littlewood, J.E.: On inequalities in the theory of functions. Proc. London Math. Soc. (2) **23**, 481–519 (1925)
49. Littlewood, J.E. and Paley, R.E.A.C.: A proof that an odd schlicht function has bounded coefficients. J. London Math. Soc. **7**, 167–169 (1932)
50. Löwner, K.: Untersuchungen über die Verzerrung bei konformen Abbildungen des Einheitskreises $|z|$¡1, die durch Funktionen mit nichtverschwindender Ableitung geliefert werden. S.-B. Verh. Sächs. Ges. Wiss. Leipzig **69**, 89–106 (1917)
51. Löwner, K.: Untersuchungen über schlichte konforme Abbildungen des Einheitskreises I. Math. Ann. **89**, 103–121 (1923) MR1512136
52. Milin, I.M.: Estimation of coefficients of univalent functions. Dokl. Akad. Nauk SSSR **160** (1965), 769–771 (Russian) = Soviet Math. Dokl. **6**, 196–198 (1965)
53. Milin, I.M.: On the coefficients of univalent functions. Dokl. Akad. Nauk SSSR **176**, 1015–1018 (1967) (Russian) = Soviet Math. Dokl. **8**, 1255–1258 (1967) MR0222276 (36 #5328)
54. Milin, I.M.: Univalent functions and orthonormal systems. Izdat. "Nauka", Moskau, 1971 (Russian). English Translation: Amer. Math. Soc., Providence, R.I. (1977) MR0427620 (55 #651)
55. Milin, I.M.: De Branges' proof of the Bieberbach conjecture (Russian), Preprint (1984)
56. Nehari, Z.: On the coefficients of Bieberbach-Eilenberg functions. J. Analyse Math. **23**, 297–303 (1970). MR0273003 (42 #7884)
57. Nehari, Z.: A proof of $|a_4| \leq 4$ by Loewner's method. In: Clunie, J., Hayman, W.K. (ed.) Proceedings of the Symposium on Complex Analysis, Canterbury, 1973. London Math. Soc. Lecture Note Series, vol. 12, pp. 107–110. Cambridge University Press (1974). MR0402023 (53 #5846)
58. Nevanlinna, R.: Über die konforme Abbildung von Sterngebieten. Översikt av Finska Vetenskaps-Soc. Förh. **63**(A), Nr. 6, 1–21 (1920–1921)
59. Ozawa, M.: On the Bieberbach conjecture for the sixth coefficient. Kōdai Math. Sem. Rep. **21**, 97–128 (1969) MR0239073 (39 #432)
60. Ozawa, M.: An elementary proof of the Bieberbach conjecture for the sixth coefficient. Kōdai Math. Sem. Rep. **21**, 129–132 (1969) MR0247064 (40 #333)
61. Pederson, R.N.: A proof of the Bieberbach conjecture for the sixth coefficient. Arch. Rational. Mech. Anal. **31**, 331–351 (1968) MR0239072 (39 #431)
62. Pederson, R.N., Schiffer, M.: A proof of the Bieberbach conjecture for the fifth coefficient. Arch. Rational. Mech. Anal. **45**, 161–193 (1972) MR0311891 (47 #453)
63. Pommerenke, Ch.: Univalent functions. Vandenhoeck und Ruprecht. Göttingen-Zürich (1975) MR0507768 (58 #22526)

64. Pommerenke, Ch.: The Bieberbach Conjecture. Mathematical Intelligencer **7**(2), 23–25 (1985) MR0784940 (86c:30005)
65. Reade, M.O.: On close-to-convex univalent functions. Mich. Math. J. **3**, 59–62 (1955) MR0070715 (17,25c)
66. Robertson, M.S.: On the theory of univalent functions. Ann. of Math. **37**, 374–408 (1936) MR1503286
67. Rogosinski, W.: Über positive harmonische Entwicklungen und typisch-reelle Potenzreihen. Math. Z. **35**, 93–121 (1932) MR1545292
68. Salvy, B., Zimmermann, P.: GFUN: A Maple package for the manipulation of generating and holonomic functions in one variable. ACM Transactions on Mathematical Software **20**, 163–177 (1994)
69. Schiffer, M.: A method of variation within the family of simple functions. Proc. London Math. Soc. **44**, 432–449 (1938)
70. Study, E.: Vorlesungen über ausgewählte Gegenstände der Geometrie, 2. Heft: Konforme Abbildung einfach-zusammenhängender Bereiche. Teubner-Verlag, Leipzig-Berlin (1913)
71. Todorov, P.G.: A simple proof of the Bieberbach conjecture. Bull. Cl. Sci., VI. Sér, Acad. R. Belg. (3) **12**, 335–356 (1992) MR1266020 (95c:30024)
72. Weinstein, L.: The Bieberbach conjecture. Internat. Math. Res. Notices **5**, 61–64 (1991) MR1131432 (92m:30033)
73. Wilf, H.: A footnote on two proofs of the Bieberbach-de Branges Theorem. Bull. London Math. Soc. **26**, 61–63 (1994) MR1246472 (95a:30019)
74. Wilf, H., Zeilberger, D.: An algorithmic proof theory for hypergeometric (ordinary and ”q”) multisum/integral identities. Invent. Math. **103**, 575–634 (1992) MR1163239 (93k:33010)
75. Zeilberger, D.: A fast algorithm for proving terminating hypergeometric identities. Discrete Math. **80**, 207–211 (1990) MR1048463 (91d:33006)

Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.