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STRONG AND WEAK F-REGULARITY ARE EQUIVALENT FOR GRADED RINGS

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Abstract. It is shown that the tight closure of a submodule in a Artinian module is the same as its finitistic tight closure, when the modules are graded over a finitely generated $\mathbb{N}$-graded ring over a perfect field. As a corollary, it is deduced that for such a graded ring, strong and weak F-regularity are equivalent. As another application, the following conjecture of Hochster and Huneke is proved: Let (R, m) be a finitely generated $\mathbb{N}$-graded ring over a field with unique homogeneous maximal ideal m, then R is (weakly) F-regular if and only if R_m is (weakly) F-regular.

1. Introduction. Tight closure is a closure operation performed on ideals in rings of prime characteristic, and by standard reduction to characteristic p techniques, also for ideals in any ring containing a field. Introduced by Hochster and Huneke in 1987, it has found powerful applications in commutative algebra.

Rings in which all ideals are tightly closed play a special role, and are called *weakly F-regular rings.* The "F" here stands for the Frobenius (or p^{th}-power) map central to the definition of tight closure. The term "F-regular" is supposed to hint at a class of rings similar to regular rings with respect to Frobenius. F-regular rings enjoy nice properties such as normality, Cohen-Macaulayness, and pseudorationality. The class of F-regular rings includes, for example, rings of invariants for linearly reductive groups acting on regular rings.

Despite the importance of weakly F-regular rings, it is not known whether the localization of a weakly F-regular ring is weakly F-regular. This is reminiscent of an analogous question about regular rings finally settled independently by Serre and by Auslander and Buchsbaum in the fifties: is the localization of a regular ring always regular? In this paper, we prove that if R is a weakly F-regular ring that is $\mathbb{N}$-graded, then all localizations of R remain weakly F-regular; see Corollary 4.4.

Our main theorem, Theorem 3.3, shows that the tight closure of any graded submodule in an Artinian graded module is equal to its finitistic tight closure for any $\mathbb{N}$-graded ring over a perfect field (see Section 2 for definitions). This allows us to deduce in Corollary 4.3 that strong and weak F-regularity are equivalent for such graded rings. Strong F-regularity is a condition introduced by Hochster and Huneke, who, motivated by a suspicion that it is equivalent to weak F-regularity,

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observed that strong F-regularity localizes [HH1]. Hochster and Huneke also showed that weak and strong F-regularity are equivalent for Gorenstein rings, a result since strengthened to include rings with $\mathbb{Q}$-Gorenstein singularities except at isolated points [W], [Mac].

As a byproduct of our work, we also settle a conjecture of Hochster and Huneke regarding F-regular graded rings. They predicted that R is weakly F-regular if and only if R_m is weakly F-regular, where R_m is the localization of R at its unique homogeneous maximal ideal [HH4, p. 609]. We will show in Corollary 4.6 that this is true.

Our main theorem flows from several sources. The method of using the finitistic tight closure to establish the equivalence of strong and weak F-regularity was described in [S1, 7.1.2]. While there is no direct connection with [L], the ideas of [L], especially those involved in the proof of [L, 3.4], were helpful in proving the main theorem. The grading of $R^{(e)}$ as an R-module is analyzed also in [SV, 3.1.6], to produce related finiteness results for graded rings.

2. Preliminaries. Throughout this paper, all rings are commutative, Noetherian, and contain a field of prime characteristic $p > 0$.

We quickly recall the definitions of tight closure and related concepts. For detailed discussions, we refer to the original sources [HH1] and [HH2]. Huneke's exposition [Hu] also gives a good, student friendly, account.

For any ring of prime characteristic p, the Frobenius map (or its e^{th}-iterate) is the ring map $R \xrightarrow{F^e} R$ raising elements to their p^eth powers. The phrase R *is F-finite* means that the Frobenius map $R \xrightarrow{F} R$ is finite. One big class of F-finite rings are the algebras essentially of finite type over a perfect field, or even over an F-finite field. Every F-finite ring is excellent [Ku]. In particular, the nonregular locus of the spectrum of an F-finite ring is a Zariski closed set.

2.1. The notation $R^{(e)}$ denotes the R-bimodule which is R as an abelian group, has left R-structure as usual, and right R-structure via F^e. For any R-module M, the notation $F^e(M)$ denotes the R-bimodule $R^{(e)} \otimes_R M$. In particular, $F^e(R)$ is naturally isomorphic to $R^{(e)}$ as an R-bimodule. Note that F^e is a right exact functor from the category of R-modules to the category of R-bimodules.

2.2. Let $N \subset M$ be an inclusion of R-modules. An element $\eta \in M$ is in the *tight closure* N^*_M of N in M if there exists $c \in R$ not in any minimal prime such that $c \otimes \eta$ is in the image of $F^e(N)$ in $F^e(M)$ for all $e \gg 0$. The *finitistic tight closure* of N in M, denoted N^{*fg}_M, is the union over all finitely generated submodules $M' \subset M$ of the tight closures of $N \cap M'$ in M'. It is easy to check that $N^{*fg}_M \subset N^*_M$, but the reverse inclusion fails in general.

2.3. A ring is said to be *weakly F-regular* if all ideals are tightly closed. It is not hard to see that R is weakly F-regular if and only if the local rings R_m are weakly F-regular for all maximal ideals m [HH2, 4.15]. However, it is not known whether localization at an arbitrary prime ideal preserves weak F-regularity. The term *F-regular* refers to weakly F-regular rings all of whose localizations are weakly F-regular.

One property of weakly F-regular rings we will use is that they are reduced. In fact, it is not hard to see that a ring in which all principal ideals are tight closed is normal [HH2, 5.9]. Thus a weakly F-regular ring is a product of normal domains.

2.4. A ring R is *strongly F-regular* if it is a reduced, F-finite ring with the following property: For every c not in any minimal prime of R, there exists a $q = p^e$ such that the inclusion $Rc^{1/q} \hookrightarrow R^{1/q}$ splits as a map of R-modules. It is not hard to see that strongly F-regular rings are F-regular. Furthermore, the locus of primes $P \in \operatorname{Spec} R$ such that R_P is not strongly F-regular is a Zariski closed set, so that strong F-regularity is preserved by localization [HH1]. Hochster and Huneke conjectured that strong and weak F-regularity are equivalent (in F-finite rings). In addition to eliminating the need for awkward adjectives, this would imply that the property of all ideals being tightly closed passes to localizations.

The following characterization of strong F-regularity will be useful.

PROPOSITION 2.4.1. [S1, 7.1.2] *An F-finite reduced local ring is strongly F-regular if and only if the zero module is tightly closed in the injective hull of its residue field.*

Proof. Recall that if $M \xrightarrow{\phi} N$ is an injective map of finitely generated modules over Noetherian local ring R, then ϕ splits if and only if the map $M \otimes E \xrightarrow{\phi\otimes id} N \otimes E$ is injective, where E is the injective hull of the residue field of R [HH5, 2.1e]. Let η be an element in E, the injective hull of the residue field of R. The element η is in the tight closure of zero if and only if there exists an element c not in any minimal prime of R such that the element $c \otimes \eta$ is zero in the module $R^{(e)} \otimes E$ for all $e \gg 0$, or equivalently, if $c^{1/q} \otimes \eta$ is zero in $R^{1/q} \otimes E$ for all $q = p^e \gg 0$. Equivalently, η is in the tight closure of zero if and only if $c^{1/q} \otimes \eta$ is in the kernel of the map $R \otimes E \to R^{1/q} \otimes E$ induced by sending 1 to $c^{1/q}$. Thus the map $R \to R^{1/q}$ sending 1 to $c^{1/q}$ splits for all c and all $q \gg 0$ if and only if zero is tightly closed in E. ☐

To show that strong and weak F-regularity are equivalent for a local ring, it is sufficient to show that the tight closure of zero in the injective hull of the residue field is the same as its finitistic tight closure. This follows from the proposition above, since it is also known that a local ring is weakly F-regular if and only if the finitistic tight closure of zero in E [HH2, 8.23].

We will show that weak and strong F-regularity are equivalent by showing the finitistic tight closure is equal to the tight closure for any submodule of the injective hull of the residue field of the localization of a graded ring at its unique maximal homogeneous ideal.

3. The main theorem. For the remainder of the paper, we will fix the following notation. By (R, m) we denote a finitely generated $\mathbb{N}$-graded ring over a field k of prime characteristic p, with $R_0 = k$ and m the unique homogeneous maximal ideal $\oplus_{i>0} R_i$. Equivalently, R is a Noetherian $\mathbb{N}$-graded ring with $R_0 = k$. We assume in this section also that R is F-finite, which is equivalent to the assumption that the ground field k is F-finite (e.g. perfect).

Note that if R is graded, then we may regard the underlying abelian group of $R^{(e)}$ (defined in §2.1) as graded in the same way, since R and $R^{(e)}$ are, after all, the same abelian groups. As a left R module, $R^{(e)}$ is a graded R-module, but as a right R-module it is not graded. Indeed, for $r \in R$ and $x \in R^{(e)}$, the element $x \cdot r$ has degree $\deg x + (p^e \deg r)$, not $\deg x + \deg r$ as would be required for a graded right R-module. Likewise, if M is a graded R-module, this induces a natural grading on $F^e(M)$: if $r \in R$ has degree d and $m \in M$ has degree d', then $r \otimes m \in R^{(e)} \otimes M$ has degree $d + p^e d'$. Again, this grading is compatible with the left (but not the right) graded R-module structure.

The main point is the following lemma.

MAIN LEMMA 3.1. *Let R be an F-finite $\mathbb{N}$-graded ring. There exists an integer t, depending only on R, such that, whenever*

$$M \xrightarrow{\phi} N$$

is a degree preserving map of graded R-modules that is bijective in degrees greater than s, the induced map

$$F^e(M) \xrightarrow{F^e(\phi)} F^e(N)$$

is bijective in all degrees greater than $p^e(s + t)$.

The Main Lemma follows from the following technical lemma.

LEMMA 3.2. *Let R be an $\mathbb{N}$-graded ring, and let L be a finitely generated R-module that is graded as an abelian group (but not necessarily as an R-module). Assume that there exists a positive integer d such that for all $i \geq 0$ and all j, $R_i L_j \subset L_{di+j}$. Then there exists a t such that whenever*

$$M \xrightarrow{\phi} N$$

is a degree preserving map of graded R-modules that is bijective in degrees greater than s, the induced map

$$L \otimes_R M \to L \otimes_R N$$

is bijective in all degrees greater than $ds + t$*. Here, the R-module* $L \otimes_R M$ *is graded as an abelian group with* $\deg(x \otimes m) = (\deg x) + d \deg m$.

To see that the Main Lemma follows from Lemma 3.2, we take L to be $R^{(1)}$. Note that the right R-module structure of $R^{(1)}$ satisfies

$$[R^{(1)}]_j R_i \subset [R^{(1)}]_{pi+j},$$

so the integer d can be taken to be p. Also, because R is F-finite, the right R-module $R^{(1)}$ is finitely generated.

Thus Lemma 3.2 implies that $F(M) \xrightarrow{F(\phi)} F(N)$ is bijective in degrees greater than $ps + t$, where t is independent of the map $M \to N$. Applying Lemma 3.2 again, we see that

$$F^2(M) = F(F(M)) \to F(F(N)) = F^{(2)}(N)$$

is bijective in degrees larger than $p(ps + t) + t$. Iterating, we find that the map

$$F^{(e)}(M) \xrightarrow{F^e(\phi)} F^{(e)}(N)$$

is bijective in degrees exceeding $p^e s + p^{e-1}t + p^{e-2}t + \cdots + pt + t$. Because $(p^{e-1} + p^{e-2} + \cdots + p + 1) < p^e$, we conclude that

$$R^{(e)} \otimes M \xrightarrow{F^e(\phi)} R^{(e)} \otimes N$$

is bijective in degrees greater than $p^e(s + t)$. This will complete the proof of the main Lemma, Lemma 3.1, as soon as we have established Lemma 3.2.

Proof of Lemma 3.2. Although L is not graded as an R module (unless $d = 1$), we can interpret L as a graded R module by suitably regrading R. Namely, we let R' be the ring R graded as follows:

$$R'_n = 0 \text{ if } d \text{ does not divide } n$$
$$R'_n = R_{n/d} \text{ if } d \text{ divides } n.$$

Then L is a finitely generated graded R'-module. This means that L has a finite homogeneous presentation: an exact sequence

$$(*) \qquad \oplus R'(-c_j) \xrightarrow{\psi'} \oplus R'(-b_i) \xrightarrow{\psi} L \to 0$$

of finitely generated graded R'-modules where the R-module maps ψ' and ψ are degree preserving maps of graded R'-modules and $R'(-a)$ is defined as the R'-

module R' which has been graded so that $[R'(-a)]_n = [R']_{n-a}$ which is $[R]_{\frac{n-a}{d}}$ if $\frac{n-a}{d}$ is an integer and zero otherwise. Let t be the maximum shift that occurs, $t = \max_{i,j}\{b_i, c_j\}$.

Let M be a graded R-module. If H is a graded R'-module, remembering that $R' = R$, we can form $H \otimes_R M$ which is graded as an abelian group with $\deg h \otimes m = \deg h + d \deg m$. With this grading, if $H \to H'$ is a degree-preserving map of graded R'-modules, the induced map $H \otimes_R M \to H' \otimes_R M$ is a degree-preserving map of graded abelian groups.

Now note that for any a, the induced map $R'(-a) \otimes M \to R'(-a) \otimes N$ is bijective in degrees greater than $ds + a$. Indeed, since $R'(-a)$ is flat as an R-module, the sequence

$$0 \to R'(-a)\otimes(\ker\phi) \to R'(-a)\otimes M \xrightarrow{1\otimes\phi} R'(-a)\otimes N \to R'(-a)\otimes(\operatorname{coker}\phi) \to 0$$

is exact. Thus it is sufficient to check that if K (for K the kernel or cokernel of ϕ) vanishes in degrees greater than s, then $R'(-a) \otimes K$ vanishes in degrees greater than $ds + a$. But if $er \otimes x = e \otimes rx \in R'(-a) \otimes K$ has degree greater than $ds + a$, where e is a degree a generator of the free cyclic R'-module $R'(-a)$, then

$$\deg(e \otimes rx) = \deg e + d(\deg(rx)) > a + ds.$$

This would force $rx \in K$ to have degree exceeding s, contrary to our assumption.

We can now see that $L \otimes M \to L \otimes N$ is bijective in degrees greater than $ds + t$. This is just a matter of chasing the commutative diagram with exact rows and degree preserving maps

$$\begin{array}{ccccccc}
\oplus R'(-c_j) \otimes M & \longrightarrow & \oplus R'(-b_i) \otimes M & \longrightarrow & L \otimes M & \longrightarrow & 0 \\
{\scriptstyle f_1}\downarrow & & {\scriptstyle f_2}\downarrow & & \downarrow & & \\
\oplus R'(-c_j) \otimes N & \longrightarrow & \oplus R'(-b_i) \otimes N & \longrightarrow & L \otimes N & \longrightarrow & 0,
\end{array}$$

keeping in mind that f_1 and f_2 are bijective in degrees greater than $ds + t$. □

It is now easy to prove that tight closure and finitistic tight closure are equivalent for graded modules over graded F-finite rings.

THEOREM 3.3. *Let R be an $\mathbb{N}$-graded F-finite ring, and let $N \subset M$ be any inclusion of $\mathbb{Z}$-graded R-modules. Then*

$$N_M^* = \bigcup_{d\in\mathbb{Z}} (N_{\geq d})^*_{M_{\geq d}},$$

where $M_{\geq d}$ denotes the R-submodule of M of elements of degree d and higher. In particular, if M is Artinian, then

$$N_M^* = N_M^{*fg},$$

where N_M^{*fg} has been defined in §2.2.

Proof. First note that N_M^* is a graded R-module. To check this, we may assume without loss of generality that R is a domain: an element $z \in M$ is in N_M^* if and only if its image in M/PM is in the tight closure of N/PN computed over R/P for every minimal prime P of R ([HH2, 6.25a]; the proof given there for ideals works for arbitrary modules, as is written down in [AHH, 2.10 c1]). Now suppose that R is a domain, and $z \in N_M^*$, so for some nonzero c, the element $c \otimes z$ is in the image of the natural map $R^{(e)} \otimes N \to R^{(e)} \otimes M$ for all large e. Since this image module under this map is graded, then $c' \otimes z'$ also belongs to the image, where c' and z' denote the lowest degree nonzero homogeneous components of c and z respectively. This implies that z' is in N_M^*, and by induction, that each homogeneous component of z is in N_M^*.

Let $z \in N_M^*$ be homogeneous of degree, say, d. Let $M' \subset M$ be the submodule of M consisting of elements of degree at least $d - t$ (where t is as in Lemma 3.1). Let $N' = N \cap M'$. We claim that $z \in {N'}^*_{M'}$.

To see this, note that the inclusion map $M' \hookrightarrow M$ is bijective in degrees at least $d - t$, so the induced horizontal maps

$$\begin{array}{ccc} F^e(M') & \longrightarrow & F^e(M) \\ \uparrow & & \uparrow \\ F^e(N') & \longrightarrow & F^e(N) \end{array}$$

are bijective in degrees at least $p^e d$. In particular, for any $c \in R$, the element $c \otimes z \in F^e(M')$ has degree at least dp^e, so it lies in the image of $F^e(N')$ if and only if its image $c \otimes z \in F^e(M)$ lies in the image of $F^e(N)$. This says that $z \in N_M^*$ if and only if $z \in {N'}^*_{M'}$. We conclude that $N_M^* = \cup_d (N_{\geq d})^*_{M_{\geq d}}$, whenever M and N are graded R-modules.

Now, if M is Artinian, then because R is $\mathbb{N}$-graded, each module $M_{\geq d}$ is finitely generated. This implies that $N_M^* = N_M^{*fg}$. □

Remark 3.3.1. We conjecture that $N_M^* = N_M^{*fg}$ in general for any Artinian module M over a Noetherian ring.

The most important example of the Theorem occurs for the case of the injective hull of the residue field. For an $\mathbb{N}$-graded ring (R, m), let E denote an injective hull of its residue field R/m. Recall that E is an Artinian, $\mathbb{Z}$-graded R module. For basic facts on injective modules, see [Eis], including [Eis, Exercise A3.5c].

COROLLARY 3.4. *If (R, m) is an F-finite $\mathbb{N}$-graded ring, then $0^*_E = 0^{*fg}_E$, where E is an injective hull of R/m.*

This result is analogous to the fact that $0^*_{H^d_m(R)} = 0^{*fg}_{H^d_m(R)}$ where $H^d_m(R)$ is the highest local cohomology module of R with supports in the maximal ideal [S2, 3.3].

4. Applications. We now prove that strong and weak F-regularity are equivalent in F-finite graded rings, and deduce that weak F-regularity localizes for arbitrary graded rings. We also settle the conjecture of Hochster and Huneke that a graded ring is F-regular if and only if its localization at the unique homogeneous maximal ideal is F-regular.

First, we need two easy lemmas. The first is proved in considerably more generality than we need, but is included at the suggestion of Mel Hochster to fill a gap in the literature.

LEMMA 4.1. *Let $R \to S$ be a map of F-finite rings. If $R \to S$ is flat with geometrically regular fibers, and R is strongly F-regular, then S is strongly F-regular. If $R \to S$ is pure, and S is strongly F-regular, then R is strongly F-regular.*

In particular, if R is an algebra essentially of finite type over a perfect field k, and $k \to L$ is any algebraic extension, then R is strongly F-regular if and only if $R \otimes_k L$ is strongly F-regular.

Proof. Assume that $R \to S$ is flat with geometrically regular fibers (some authors call this a *regular* map, the term *smooth* map being reserved for the case where S is finitely presented over R). Let R be strongly F-regular.

Choose c not in any minimal prime of R (or S) such that R_c is regular. Then S_c is also regular. It suffices to show that there exists $q = p^e$ such that the inclusion $Sc^{1/q} \hookrightarrow S^{1/q}$ splits over S [HH1, 3.3]. First we consider the case where $R \to S$ is étale. In this case, there is a natural isomorphism $R^{1/q} \otimes_R S \cong S^{1/q}$ [HH2, 6.3]. So the R-module splitting of $Rc^{1/q} \hookrightarrow R^{1/q}$ can be tensored with S to get an S-module splitting of $Sc^{1/q} \hookrightarrow S \otimes_R R^{1/q} = S^{1/q}$. This shows that strong F-regularity is preserved by étale maps. The case of a general regular map $R \to S$ follows by writing S as a filtered inductive limit of smooth (finite type) R-algebras. (This is Popescu's "General Néron Desingularization" [P1, P2]. The controversy surrounding the correctness of Popescu's arguments seems to have been settled; see the careful exposition [Sw].) This allows us to reduce to the case where $R \to S$ is smooth (finite type). A smooth extension is locally a polynomial extension followed by an étale extension, so each can be treated separately. We have already considered the étale case. The reader will have no trouble verifying the easy polynomial extension case, thus completing the proof that S is strongly F-regular.

Now assume $R \to S$ is pure and S is strongly F-regular. Without loss of generality, both R and S may be assumed local, so that S is a domain, and hence its pure subring R is a domain. Given a nonzero $c \in R$, we need to show that the map $Rc^{1/q} \subset R^{1/q}$ splits over R. Because $R^{1/q}$ is a finitely generated R-module, the splitting of this map is equivalent to the purity of the map. By assumption, the map $Sc^{1/q} \to S^{1/q}$ splits over S. Because the composition $Rc^{1/q} \subset Sc^{1/q} \subset S^{1/q}$ is a pure map of R-modules, it restricts to a pure map of R modules $Rc^{1/q} \hookrightarrow R^{1/q}$. The proof that R is strongly F-regular is complete.

The final statement follows immediately. The map $k \to L$ is separable field extension, and so $R \to R \otimes_k L$ is a regular, pure map. □

LEMMA 4.2. *Let R be an F-finite graded ring. The radical ideal defining the locus of points that are not strongly F-regular is homogeneous.*

Proof. First assume the ground field $k = R_0$ is infinite. In this case, an ideal $I \subset R$ is homogeneous if and only if it is preserved by the natural k^*-action given by the grading (that is, $\lambda \in k^* = k - \{0\}$ acts on a degree e element by $\lambda \cdot r = \lambda^e r$). Let I be the defining ideal for the nonstrongly F-regular locus. An element $c \in R$ is in I if and only if R_c is strongly F-regular. Since each $\lambda \in k^*$ acts by an automorphism of R, the ring $R_{\lambda \cdot c}$ is isomorphic to R_c. Thus I is preserved by the k^* action, and so I is homogeneous.

Now assume $R_0 = k$ is finite, and let L be an algebraic closure. If R_c is strongly F-regular, then Lemma 4.1 implies that $R_c \otimes_k L \cong (R \otimes_k L)_c$ is strongly F-regular (where c now denotes the image of c under the inclusion $R \hookrightarrow R \otimes_k L$). This means that c is in the defining ideal of the nonstrongly F-regular locus of $R \otimes_k L$, which is homogeneous because L is infinite. So each of the homogeneous components c_i of c are in this ideal. Thus $(R \otimes_k L)_{c_i} = R_{c_i} \otimes_k L$ is strongly F-regular, and hence R_{c_i} is strongly F-regular again by Lemma 4.1. We conclude that if $c \in R$ is in the radical ideal defining the nonstrongly F-regular locus of R, then so are each of its homogeneous components, and the ideal is homogeneous. □

COROLLARY 4.3. *An $\mathbb{N}$-graded F-finite ring is weakly F-regular if and only if it is strongly F-regular.*

Proof. The injective hull of the residue field of R_m is the same as the injective hull of R/m over R or over $\hat{R}_m$. Since elements of $R_m - R$ act as units on E, the reader will easily verify that the tight closure (and the finitistic tight closure) of the zero module in E is the same whether computed over R or R_m. By Theorem 3.3, we know then that $0_E^{*fg} = 0_E^*$ for R or R_m.

The vanishing of 0_E^{*fg} is equivalent to weak F-regularity of R_m [HH2, 8.23], whereas the vanishing of module 0_E^* is equivalent to strong F-regularity of R_m, as is shown in [S1, Ch 7] and also in [LS, 2.9]. By Theorem 3.3, we know that $0_E^{*fg} = 0_E^*$ and we conclude that R_m is strongly F-regular if and only if R_m is weakly F-regular.

Finally, say that R is weakly F-regular. Then R_m is weakly F-regular [HH2, 4.15]. From above, we know R_m is strongly F-regular. Because the nonstrongly F-regular locus of R is defined by a *homogeneous ideal*, this locus must contain m if it is nonempty. It follows that R is strongly F-regular. □

COROLLARY 4.4. *A weakly F-regular $\mathbb{N}$-graded ring is F-regular.*

Proof. Of course, if R is F-finite, this follows immediately from Corollary 4.3. We must reduce to the F-finite case. Suppose that R is weakly F-regular. Then R_m is weakly F-regular. Let $L = k^{1/p^\infty}$ be the perfect closure of the base field $k = R_0$. The map $R_m \to R_m \otimes_k L$ is a flat, purely inseparable map such that the maximal ideal of R expands to the maximal ideal m' of $R_m \otimes_k L$. By [HH3, 6.17b], we conclude that $R_m \otimes_k L$ is weakly F-regular.

The $\mathbb{N}$-graded ring $R \otimes_k L$ is finitely generated over the perfect field L and hence F-finite. Thus by Corollary 4.3, because $R_m \otimes_k L \cong (R \otimes_k L)_{m'}$ is weakly F-regular, it is also strongly F-regular. Now Lemma 4.2 implies that $R \otimes_k L$ is strongly F-regular. Finally, this implies that $R \otimes_k L$ is F-regular, which is to say, it is weakly F-regular after localization at *any* multiplicative system. Because the ring R is a direct summand of $R \otimes_k L$ (as an R-module), it follows finally that R is F-regular. □

There are several easy ways to deduce the next corollary, whose proof we include for completeness.

COROLLARY 4.5. *Let S be a local ring that is either the localization or the completion of an $\mathbb{N}$-graded ring at its unique homogeneous maximal ideal.*

(1) *If the ground field k is F-finite, then S is weakly F-regular if and only if S is strongly F-regular.*

(2) *The ring S is weakly F-regular if and only if it is F-regular.*

Proof. (1) The case where S is the localization of a graded ring R was proved in the proof of Corollary 4.3. The statement about the completion of R at the maximal ideal follows as well, since both 0^*_E and 0^{*fg}_E are the same subsets of E whether computed over R or over $\hat{R}$ (as easily follows from considering the grading).

Alternatively, one can check the complete case by considering the completion map $R \to S$ (which is pure, and flat with geometrically regular fibers). If S is weakly F-regular, then so is its pure subring R. In this case, Corollary 4.3 implies that R is strongly F-regular, whence so is its completion S, by Lemma 4.1.

(2) If S is the localization a graded ring R, and S is weakly F-regular, then so is R (using Corollary 4.6 below). It follows from Corollary 4.4 that R is F-regular, whence so is its localization S.

If S is the completion of a graded ring R, then if S is weakly F-regular, so is R, as the completion map $R \to S$ is pure. Then R is F-regular by Corollary 4.4

and since the completion map $R \to S$ has regular fibers, S also is F-regular by [HH3, 7.3c]. □

Finally, we settle the conjecture of Hochster and Huneke [HH4, p. 609].

COROLLARY 4.6. *An* $\mathbb{N}$*-graded ring* (R, m) *is weakly* F*-regular if and only if* R_m *is weakly* F*-regular.*

Proof. We already know that if R is weakly F-regular, then so is R_m [HH2, 4.15]. Now assume R_m is weakly F-regular. As in the proof of Corollary 4.4, we conclude also that $R_m \otimes_k L$ is weakly F-regular, where $L = k^{1/p^\infty}$. Corollary 4.3 implies that $R_m \otimes_k L$ is strongly F-regular, whence Lemma 4.2 implies that $R \otimes_k L$ is strongly F-regular, and so of course also F-regular. Therefore, its pure subring R is F-regular, and so weakly F-regular, and the proof is complete. □

We can also prove that the identity $0^*_E = 0^{*fg}_E$ has many interesting consequences in addition to the equivalence of strong and weak F-regularity. For example, if it holds in general, then the test ideal commutes with localization. It also implies that the test ideal commutes with completion, as conjectured in [HH2]. In [LS], some of these consequences are described, together with a further study of when $0^*_E = 0^{*fg}_E$.

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