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CHILDREN'S SOLUTIONS TO MULTIPLICATION AND DIVISION WORD PROBLEMS: A LONGITUDINAL STUDY

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Children's solution strategies to a variety of multiplication and division word problems were analysed at four interview stages in a 2-year longitudinal study. The study followed 70 children from Year 2 into Year 3, from the time where they had received no formal instruction in multiplication and division to the stage where they were being taught basic multiplication facts. Ten problem structures, five for multiplication and five for division, were classified on the basis of differences in semantic structure. The relationship between problem condition (i.e. small or large number combinations and use of physical objects or pictures), on performance and strategy use was also examined.

The results indicated that 75% of the children were able to solve the problems using a wide variety of strategies even though they had not received formal instruction in multiplication or division for most of the 2 year period. Performance level generally increased for each interview stage, but few differences were found between multiplication and division problems except for Cartesian and Factor problems.

Solution strategies were classified for both multiplication and division problems at three levels:

(i) direct modelling with counting;
(ii) no direct modelling, with counting, additive or subtractive strategies;
(iii) use of known or derived facts (addition, multiplication).

A wide range of counting strategies were classified as counting-all, skip counting and double counting. Analysis of intuitive models revealed preference for a repeated addition model for multiplication, and a 'building-up 'model for division.

In recent years there has been a steady growth in mathematics education research investigating how children develop mathematical concepts and processes (Bell, Costello & Kuchemann, 1981; Carpenter, Moser & Romberg, 1982; Ginsburg, 1983; Hiebert & Behr, 1989; Hart, 1981; Lesh & Landau, 1983; Steffe & Wood, 1990). While much of this research has focussed on secondary school children's understanding of mathematical ideas there has also been important research developments in young children's acquisition of specific concepts and skills. The development of early number concepts and counting procedures (Ginsburg, 1977; Hughes, 1986; Steffe, Von Glasersfeld, Richards & Cobb, 1983; Steffe, Cobb & Richards, 1988), addition and subtraction processes (Carpenter et al., 1982; Carpenter & Moser, 1984; De Corte & Verschaffel, 1987), and rational number concepts (Hunting, 1989) are examples of this research.

Research investigating early number concepts and processes has been largely influenced by the constructivist view of learning. While the constructivist movement has been based on the work of Ernst von Glasersfeld (1985) and related studies (Steffe et al., 1983; Steffe et al., 1988; Labinowicz, 1985), constructivists subscribe to a variety of different viewpoints about how knowledge is acquired (Kilpatrick, 1987). However diverse in their definitions or methodologies, constructivists ascribe to the belief that knowledge is actively constructed by the child, adapting to their environment. In common, constructivists focus on the observation of children's constructive processes first hand (Steffe et al., 1983) often through the use of clinical interviewing and teaching experiments. Moreover, the constructivist approach has provided new direction for mathematics education research by focussing on mathematical thinking processes from the child's viewpoint.

Informal and Formal Strategies

Another aspect of the research investigating children's development of mathematical concepts and processes has been the widespread evidence of children's informal or intuitive strategies. Related studies have indicated that children's informal strategies may be developed prior to instruction (Carpenter, Hiebert & Moser, 1981; Fuson, 1982; Gelman & Gallistel, 1978; Groen & Resnick, 1977; Hughes, 1986; Steffe et al., 1988), and that children may continue to use these despite formal instruction (Booth, 1981; Fischbein, Deri, Nello & Merino, 1985; Hart, 1981). When children experience formal instruction it cannot be assumed that their conceptualisations are linked with formal mathematical ideas, or that their own strategies match those encouraged by instruction.

It appears then, that the absence of these connections induces a shift from intuitive and meaningful problem-solving approaches, to mechanical and meaningless ones (Hiebert, 1984; 1990; Hughes, 1986). Similarly, Carpenter & Moser (1982) found that once children learned formal arithmetic procedures they stopped analysing the addition and subtraction problems they had previously been able to solve.

Addition and Subtraction Word Problems

Research investigating the development of addition and subtraction concepts and processes in the past decade has focussed on analysing children's solution strategies and the influence of different problem structures

(Carpenter & Moser, 1982, 1984; De Corte & Verschaffel, 1987; Riley, Greeno & Heller, 1983). In solving simple addition and subtraction problems, children used a variety of informal strategies such as modelling and counting, that reflected the semantic structure of the problem (Carpenter & Moser, 1984; De Corte &Verschaffel, 1987; Nesher, 1982). These studies have contributed to a more coherent picture of how children develop addition and subtraction processes, and in conjunction with further studies (Carpenter, Moser & Bebout, 1988; Cobb & Merkel, 1989), some new direction for instruction using a cognitive approach has been advanced.

Researchers have also been involved in building explicit models of the knowledge structures and solution processes underlying children's performance in addition and subtraction word problems (Briars & Larkin, 1984; De Corte & Verschaffel, 1989; Kintsch, 1986; Langford, 1988; Riley et al.,1983). The development of these models has indicated that there are complex variations in children's problem-solving processes, but these processes may not always be consistent with a particular model. However, solution strategies have been classified and described in terms of the modelling and abstractness of the mathematical processes involved, and this has given more insight into the development of addition and subtraction processes.

Multiplication and Division Word Problems

In the past decade, researchers have also investigated children's performance and solution processes to multiplication and division word problems, with most studies focussing on secondary students (Bell, Fischbein & Greer, 1984; Bell, Greer, Grimison & Mangan, 1989; De Corte, Verschaffel & Van Coillie, 1988). Earlier, the Concepts in Secondary Mathematics & Science (C.S.M.S.) studies (Brown and Kuchemann; 1976, 1977; Brown, 1981) revealed the use of additive-type strategies for solving multiplication and division problems and provided some direction for research comparing differences between problem structures.

Attempts to build classification schemes have been based on differences in semantic structure, mathematical structure, size of quantities used, and pupil's intuitive models (Bell et al., 1989; Fischbein et al., 1985; Nesher, 1988; Schwartz, 1988; Vergnaud, 1988). More recently, studies have analysed young children's solution strategies to multiplication and division problems (Anghileri, 1985; 1989; Boero, Ferrari & Ferrero,1989; Keranto, 1984; Kouba, 1989; Steffe, 1988). These studies have provided complementary evidence that the semantic structure of the problems, and the development of counting, grouping and addition strategies influence solution process.

The longitudinal study reported in this paper extends the research on multiplication and division word problems by analysing young children's solution strategies over a 2-year period, and addresses the question of how children develop informal and formal multiplication and division strategies and the relationship between these. More specifically, this research;

(i) develops a broader classification scheme for multiplication and division problem structures for young children,
(ii) classifies solution strategies into levels of modelling, counting and abstractness,
(iii) analyses the relationship between problem structure, problem condition and strategy use, and
(iv) provides evidence of children's intuitive models for multiplication and division.

Methodology

The methodology was based essentially on Carpenter and Moser's (1984) longitudinal study of children's solutions to addition and subtraction word problems, and was trialled in a cross-sectional pilot study of 35 children conducted prior to the longitudinal study (Mulligan, 1988). Intensive clinical interviewing was conducted at four stages over a 2-year period, and a classification scheme for problem structures and solution strategies was developed. Carpenter and Moser's research design was appropriate for this study because it allowed the researcher to directly examine solution strategies and how these changed over a 2-year period.

Sample

The interview sample controlled for sex differences and comprised Year 2 girls ranging from 7 to 8 years in age. These were randomly selected from 8 Catholic schools in the Sydney Metropolitan area, after a postal questionnaire was administered to 47 schools in the region. Each child in the sample (n=72), was interviewed and tested twice, for reading comprehension and oral comprehension. Two children having very inadequate reading comprehension ability, as indicated by the ACER Primary Reading Survey Test, were eliminated from the sample. Others moved from the region during the interview period, and the sample retained 60 girls at the final interview.

Procedures

The researcher conducted 261 individual interviews at four stages over the 2-year period. These took place during March/April, and November/December of the school year for 2 years. The first interview took place at a time when children had received no teacher instruction in multiplication and division concepts. At the time of the final interview all children had been instructed in basic multiplication facts but not in division facts.

Subjects were interviewed by the researcher in a room separate from the classroom and an audio-tape was made so that transcripts could be analysed. Each problem was presented for small and large number problem combinations, in written form on cards, with the availability of counters. The problems were re-read to the child if requested. The large number problems were asked only if the child was successful on the small number problems. Each interview lasted from 15 to 55 minutes.

If the child was unable to solve the Repeated Addition (a), Array, or Partition (b) problems (see Table l) with small numbers, a picture representing the problem was presented to the child. These problems were selected for trialling because they were the easiest examples from which a picture could provide a model to assist the child in finding a solution. Responses were recorded on an Interview Record Form and observations of solution strategies and the child's behaviour were also noted.

Table 1
Word Problems (Small Numbers)

Multiplication	**Division**
Repeated Addition (a) There are 2 tables in the classroom and 4 children are seated at each table. How many children are there altogether? (b) Peter had 2 drinks at lunchtime every day for 3 days. How many drinks did he have altogether? (c) I have three 5c pieces. How much money do I have?	*Partition (Sharing)* (a) There are 8 children and 2 tables in the classroom. How many children are seated at each table? (b) 6 drinks were shared equally between 3 children. How many drinks did they have each?
Rate If you need 5c to buy one sticker how much money do you need to buy two stickers?	*Rate* Peter bought 4 lollies with 20c. If each lolly cost the same price how much did one lolly cost? How much did 2 lollies cost?
Factor John has 3 books and Sue has 4 times as many. How many books does Sue have?	*Factor* Simone has 9 books and this is 3 times as many as Lisa. How many books does Lisa have?
Array There are 4 lines of children with 3 children in each line. How many children are there altogether?	*Quotition* (a) There are 16 children and 2 children are seated at each table. How many tables are there? (b) 12 toys are shared equally between the children. If they each had 3 toys, how many children were there?
Cartesian Product You can buy chicken chips or plain chips in small, medium or large packets. How many different choices can you make?	*Sub-division* I have 3 apples to be shared evenly between six people. How much apple will each person get?

Problem Structure

At each interview the child was asked to solve ten different problem types (Table l). These were developed from previous classification schemes (Anghileri, 1985; Bell et al., 1984; Brown, 1981; Kouba, 1986; Mulligan, 1988; Vergnaud, 1988), but extended the range of multiplication and division problem structures. Sub-categories representing a variation in linguistic terms or semantics were used in some problem types because differences were found in solution strategies in the pilot study. There were 14 small number problems and 11 large number problems asked in total.

Problem Condition

All the problems contained numbers representing discrete quantities only. The problems were presented using two different groups of number size, for example, those products and related division facts between 4 and 20 for small numbers (see Table 1) and between 20 and 40 for larger combinations. Number triples involving 0 and 1, squared numbers, or multiples of 10 were not included. For each problem structure, number triples were consistent across the four interview stages so that any differences in performance and strategy use could be attributed to other factors.

Results

Analysis of individual profiles across the four interview stages indicated that 75% of the children were able to solve most of the small number problems at some stage, even though they had not been instructed in multiplication or division for most of the 2-year period. Table 2 indicates that the performance level generally increased for each interview stage but varied according to the difficulty of the problem structure and size of number combinations used. However, there were few differences found in the performance level and solution strategies between multiplication and division problems, except that performance was much lower for Cartesian and Factor problems.

In comparing small and large number problems, a marked decrease in performance overall was found for large number problems with many children reverting to direct modelling and counting procedures. Further analysis of individual profiles revealed some contrasting evidence that 25% of the children were unable to solve two or more of the easiest 11 small number problems at any interview stage. Many of these children relied on immature strategies, such as using key words, looking at the size of the numbers to choose an operation, or applying number facts incorrectly. These strategies showed that the children were unable to analyse and apply meaning to the range of situations and number combinations required. This was consistent with research findings by Sowder (1988) and Carpenter & Moser (1984).

Primary Strategies Used Across Problem Structures and Interview Stages

There was a wide variety of strategies used to solve the ten different problem structures. These were largely based on grouping, counting and additive procedures, and the increased use of known addition and multiplication facts at Interview Stages 3 and 4. There were few differences found between the solution strategies for multiplication and division problems except for sharing, one-to-many correspondence, and trial-and-error used exclusively for division.

Primary strategies were those strategies used in 10% or more of the total correct responses at each level. For emphasis, the **bold type** represents more than 50% of the total correct strategies, and the strategies are placed in order of predominant use. When comparing one problem structure with another, however, it must be considered that variations in total correct responses were found, and Table 2 may need to be referred to here. Most compelling was the evidence that the solution strategy reflected the semantic structure of the problem and in general, the children tended to model the action or relationship described in the problem.

Table 2
Percentage of Correct Responses for Each Problem Structure and Number Size: Interviews 1 to 4

PROBLEM STRUCTURE		SMALL NO.				LARGE NO.				PICTURE			
		Interviews[a]				Interviews				Interviews			
		1	2	3	4	1	2	3	4	1	2	3	4
MULTIPLICATION													
Repeated Addition	(a)	50	77	79	92	27	45	54	80	50	20	20	8
	(b)	51	74	84	95	27	52	68	65				
	(c)	59	74	85	95								
Rate		72	82	89	98								
Factor		11	29	44	57	0	16	35	47				
Array		46	77	84	92	39	70	76	78	39	16	11	5
Cartesian		1	1	3	18	1	1	2	10				
DIVISION													
Partition (Sharing)	(a)	66	69	74	75	23	33	29	55				
	(b)	61	80	81	97	34	64	64	83	14	14	14	2
Rate		51	54	66	85								
Factor		4	3	6	17	0	0	0	10				
Quotition	(a)	34	58	55	85	26	36	44	72				
	(b)	47	64	69	93	34	45	50	73				
Subdivision		41	60	73	82	10	23	35	43				

a The numbers interviewed were 70, 69,62 and 60 respectively.

Table 3 indicates the broad differences in strategy use across four multiplication problem structures. The strategies used for the fifth problem structure, Cartesian Product, are not shown as performance was minimal. While some consistency in primary strategies is found across interview stages, the use of known facts (addition) was prevalent for the Repeated Addition (a) problem, and skip counting and multiplication facts for the Repeated Addition (b) and (c) problems. Counting-all with direct modelling was found for the Multiplying Factor and Array problems, but skip counting was widely used across problem structures. The Rate problem was easily solved with a simple addition fact.

Table 3
Primary Strategies Used at Each Interview Stage on Multiplication Small Number Problems

PROBLEM		Interview 1	Interview 2	Interview 3	Interview 4
Repeated Addition	(a)	**addition fact**[a] *counting-all*[b]	**addition fact** *counting-all*	**addition fact**	**multi. fact** addition fact
	(b)	skip counting rep. addition	**skip counting**	skip counting multi. fact	**multi. fact** skip counting
	(c)	skip counting rep. addition	**skip counting** rep. addition	skip counting multi. fact	**multi. fact** skip counting
Rate		**addition fact**	**addition fact** skip counting	**addition fact** skip counting	**addition fact** multi. fact
Factor		skip counting	*counting-all* skip counting	*counting-all* multi. fact	**multi. fact**
Array		*counting-all*	skip counting *skip counting* *counting-all*	*counting-all* multi. fact skip counting	**multi. fact** skip counting *counting-all*

a Bold type indicates strategy representing more than 50% of the total correct strategies
b Italics type indicates Direct Modelling

The absence of skip counting for the Repeated Addition (a) problem can be explained possibly by the nature of number combination used where it was more common to add "4 and 4 are 8". Skip counting was not appropriate here, hence the predominance of the addition fact. The strong emergence of multiplication facts at interview stage 4 was consistent with the solution strategies for other problem structures.

Table 4 indicates that common patterns in strategy use were found across the four interview stages for division problems, but differences in solution strategies were shown between division problem structures. Halving and addition was preferred for the Partition (a) problem, whereas one-to-many correspondence was preferred for the Partition (b) problem. Skip counting was used consistently for the Rate problem possibly because children liked to count in the 5's pattern. The Quotition problems revealed widespread preference for direct modelling with counting, and halving (using sub-division) was used exclusively for the Sub-division problem. The tendency to use known multiplication facts at interview stage 4 was similar to the pattern of strategy use found with multiplication problems.

Table 4
Primary Strategies used at Each Interview Stage on Division Small Number Problems

PROBLEM		Interview 1	Interview 2	Interview 3	Interview 4
Partition	(a)	halving addition fact	one-to-many addition fact	halving addition fact	**halving**[a] addition fact
	(b)	one-to-many *one-to-many*[b]	one-to-many skip counting	skip counting one-to-many	one-to-many multi. fact
Rate		skip counting	**skip counting**	**multi. fact** skip counting	**multi. fact** skip counting
Quotition	(a)	***counting-all***	***counting-all***	*counting-all* *double count*	*counting-all* *skip counting* skip counting
	(b)	***counting-all***	***counting-all*** *double count*	*counting-all* *double count*	*counting-all* multi. fact skip counting
Sub-division		**halving**	**halving**	**halving**	**halving**

a- Bold type indicates strategy representing more than 50% of the total correct strategies
b Italics indicates Direct Modelling

In summary, the analysis of primary strategies revealed marked differences in strategy use between solutions of the various multiplication and division problem structures. Less pronounced were the changes in strategy use across interview stages for each problem structure, but the use of known facts was found to be generally more common by the fourth interview.

Levels of Strategy Use For Multiplication and Division Problems

From the broad range of strategies used to solve the ten problem types across four interview stages, (small and large number combinations), three basic levels of strategy use were identified (See Appendix A for definitions of the strategies). The scheme for classifying levels of strategy use was devised by integrating two characteristics; the level of abstractness, and the level of modelling identified by the solution strategy. Three basic levels are described as:

(i) strategies based on direct modelling and counting, using counters or fingers;

(ii) strategies based on counting and addition and subtraction without direct modelling; and

(iii) strategies based on known or derived addition and multiplication facts.

Table 5 shows the three basic levels of strategy use reflecting increasing levels of modelling and abstractness across the four interviews. At Level 1 for multiplication solutions, grouping and counting strategies were combined where children formed equivalent sets representing the quantity given in the problem and then counted-all (one-by-one to gain total), skip counted ("3, 6, 9") or double counted (two counts made simultaneously for number of groups and number in the group). At Level 2 strategies were identical to those at Level 1 but were identified by the children verbalising the solution process and describing their visualisation of the model of the problem. This showed much more advanced mental processing.

For division, the grouping and counting strategies differed because the size of the group (Partition, Rate, Sub-Division) or the number of groups (Quotition) was unknown. Children estimated the number in the group and then formed groups of equal size. In most cases, counting-all (dividend), skip counting or one-to-one counting accompanied the grouping process, but sharing one-by-one was rarely used. If children were unsuccessful in their first attempt, trial-and- error grouping was used where new estimates were formed on the basis of each trial. This was prevalent for large number problems because it was more difficult for children to form estimates from larger dividends. Children were reluctant to estimate the number of groups for Quotition problems and relied on modelling equivalent groups and counting-all to check the dividend. At Level 3, use of known and derived addition and multiplication facts for multiplication and division emerged clearly at interview stage 4.

Although the solution strategies for multiplication and division problems were more complex and diverse, the levels of modelling, counting, and use of known facts, were found to be analogous to the addition and subtraction study (Carpenter and Moser, 1984). The use of additive and subtractive strategies, revealed in both multiplication and division problem solutions, showed common characteristics with the strategies found in the addition and subtraction study and were consistent with the strategies found in Kouba's (1989) study with multiplication and division problems.

Table 5
Levels of Strategy Use on Small Number Problems Across the Four Interviews

MULTIPLICATION		DIVISION	
Level (i) Direct Modelling			
(1)	Grouping, counting-all	(1)	Grouping, counting-all Sharing one-by-one One-to-many correspondence Trial-and-error grouping
(2)	Grouping, double counting	(2)	Grouping, double counting
(3)	Grouping, skip counting	(3)	Grouping, skip counting
(4)	Additive and subtractive: Repeated addition Doubling Halving Repeated subtraction	(4)	Additive and subtractive: Repeated addition Doubling Halving Repeated subtraction
Level (ii) No Direct Modelling			
(1)	Grouping, counting-all	(1)	Grouping, counting-all Sharing one-by-one One-to-many correspondence
(2)	Grouping, skip counting	(2)	Grouping, skip counting
(3)	Grouping, double counting	(3)	Grouping, double counting
(4)	Additive and subtractive: Repeated addition Doubling Halving Repeated subtraction	(4)	Additive and subtractive: Repeated addition Doubling Halving Repeated subtraction
Level (iii)Known Facts			
(1)	Known addition fact	(1)	Known addition fact
(2)	Known multiplication fact	(2)	Known multiplication fact
(3)	Derived multiplication fact	(3)	Derived multiplication fact
		(4)	Known division fact

Intuitive Models

Fishbein et al. (1985) identified a repeated addition model for multiplication and two models for division; partitive and quotative. In the process of analysing solution strategies, underlying intuitive models appeared to overarch the method of solution but it seemed to the author that these models were more complex than those previously described by Fischbein et al. (1985) with older pupils. Further analysis of the solution strategies revealed predominance of the repeated addition model for multiplication, although an 'operate on the set', an array, and cartesian models were also found.

Three underlying intuitive models for division appeared to the author to overarch the method of solution: sharing one-by-one, 'building-up' (additive) and 'building-down' (subtractive). These were consistent with recent findings by Kouba (1989). Analysis of Partition, Quotition and Rate problems across the four interview stages showed widespread preference for the 'building-up' model (Mulligan, 1991a) and this was based on counting or additive strategies where the child 'built-up' to the dividend. For example, in the Quotition (a) problem, the child 'built-up' groups of 2 until 16 was reached, and verbalised counting-all, skip counting or double counting "2, 4, 6, 8 ...". 'Building-down' was distinctive because the child always modelled or counted the dividend first such as " 16 take away 2, take away 2". A change in problem type may have affected the intuitive model used which supports the notion that children can develop more than one intuitive model. Some children who were consistent in their intuitive model across problems tended to be restricted to 'building-down'. Those children who were more successful across problems were more likely to change their model. The increase in 'building-up' strategies was shown as interview stages progressed. There was a clear indication, however, that the 'building-down' model was used initially and when children could relate additive strategies or multiplication facts to the problem , they 'built-up'.

These findings raise questions for teaching and learning methods that rely on the sharing and repeated subtraction models for division. It is proposed that the rigidity of traditional partitive and quotative models may restrict the solution process, rather than building on children's intuitive understandings. It can be questionned whether the use of additive and estimation strategies, especially efficient use of multiple and group counting might be a more effective way of teaching division. Further analysis of the data will reveal whether children preferred one underlying model consistently or whether specific problem types affected their underlying model. It will also reveal whether children's underlying models changed over the 2-year period.

Implications for Teaching

This study has provided evidence that young children can solve a variety of multiplication and division problems prior to instruction in these concepts. The importance of counting and additive procedures in the development of multiplication and division was shown with the use of efficient skip counting, and double counting as central to this development. In the classroom situation, the transition from counting, to mental strategies and then to known facts could be monitored across a variety of multiplication and division situations. Teaching programs could incorporate the development of informal strategies rather than focussing only on mastering number facts and computational skills that may not relate to the child's level of strategy development. Teachers could facilitate more meaningful learning by establishing links between children's intuitive strategies and the formal teaching of addition, subtraction, multiplication and division. Perhaps the teaching of these processes in an integrated fashion, and based on the child's experience of a range of related situations might best reflect the natural development of these processes. Children could also be assisted in solving multiplication and division problems by encouraging modelling with materials, or by presenting or drawing pictures. The relative difficulty of different problem structures and number combinations has been more clearly identified and thus, teachers could expose children to these with a bettter understanding of the relative ease or difficulty which children may encounter.

The analysis of intuitive models for multiplication and division indicated that children can develop different underlying models for these processes. The widespread preference for a repeated addition model for multiplication, and a 'building-up' model for division may influence more complex applications of these operations throughout primary and high school. The preference for additive strategies in the development of division warrants careful attention, and further research, as common teaching practice focusses on using sharing and repeated subtraction strategies. The data clearly indicated that children's informal use of multiple and additive procedures was both efficient and meaningful in solving division problems.

Teaching strategies that reflect the informal development and intuitive models of multiplication and division were successfully integrated into a constructivist teaching experiment conducted with 10 children in the later part of the longitudinal study (Mulligan, 1991b). The Teaching Experiment focussed on representing a range of multiplication and division situations through language, modelling, drawing, symbolising and writing. Children related their informal strategies to more formal symbolic representations by linking counting and additive recordings to multiplication and division. As well, children were able to find patterns and relationships between problems and devise their own problems showing understanding for the operation involved. Some direct teaching strategies were employed and these were influential in assisting children represent and solve the problems; relating skip counting to the problem situation, and using a hundred square to represent patterns. Further evidence of children's underlying intuitive models for multiplication and division in pictorial and symbolic form were consistent with the results of the longitudinal study.

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Appendix A

Direct Modelling: The physical use of unifix cubes or fingers to represent the action or relationship described in the problem.
Grouping: The formation of equivalent groups representing quantities in the problem. This occurred sometimes with modelling, and sometimes without modelling when students indicated that they were visualising a mental image.
Counting all: Counting each item by ones to sort or check groups or quantities represented by the problem. This occurred sometimes with modelling and grouping, and sometimes without modelling when students indicated that they were visualising a mental image.
Skip counting: Counting in a particular pattern or sequence, e.g. "two, four, six". This occurred with modelling and grouping and the number of groups may have been counted physically. Skip counting was also used in situations where child was visualising, e.g. the child verbalises that they can see counters grouped in twos.
Double Counting: Counting-all with a simultaneous count of the number of groups at the same time, e.g. "one, two, three (one); ... four, five six (two)".
Repeated Addition: Adding the number in a group n times, where use of

term "and" is verbalised as a distinguishing feature, e.g. "five and five are ten and five are fifteen".

Repeated Subtraction Subtracting the number in a group in times, where use of the term "take away", "subtract" or "minus" is verbalised as a distinguishing feature, e.g. "20 take away 5 is 15, take away".

Doubling: The use of adding in a doubling pattern where the strategy is verbalised by the term "double" as a distinguishing feature, e.g. "I knew 3 and 3 is 6 and double 6 makes 12".

Halving: Dividing the quantity into two equivalent groups through visualisation or modelling, e.g. "cut 8 into half to make 4". This procedure may have been repeated, e.g. "broke up 20 to 10 then to 5".

Sub-dividing: Dividing the quantity into more than two equivalent groups, e.g. "I divided one into four bits".

Sharing: Forming groups of objects one by one, e.g. "one for you, one for you, one for you" involving a mental count "one each... two each" or with counting all afterwards. Sharing could occur sometimes with modelling or sometimes without modelling, where children indicated that they were visualising a mental image.

One-to-Many Correspondence: Matching one item to equivalent groups already formed or vice versa by modelling, e.g. "four children to one table". This also occurred without modelling where children indicated that they were visualising a mental image.

Trial- and- Error Grouping: Estimating and forming equivalent groups and changing the size ofthe group according to previous error. This strategy occurred onlywith modelling.

Known Addition Fact: Retrieving an addition fact automatically with no apparentcounting, e.g. "four and four are eight".

Known Multiplication Fact: Retrieving a multiplication fact automatically with no apparent counting, e.g. "two fours are eight".

Known Division Fact: Retrieving a division fact automatically with no apparent counting, e.g. "18 divided by 3 is 6".

Derived Fact: Using a known fact to find another fact, e.g. "two threes are six and three more makes nine".

Number Fact strategies were also characterised by the retrieval of a fact through a taught procedure, e.g. saying the multiplication or division table in sequence from the beginning until the relevant fact is recalled.