Advanced Gain-Scheduling Techniques For Uncertain Systems

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Abstract

This paper is concerned with the design of gain-scheduled controllers for Linear Parameter-Varying systems. Two alternative LMI characterizations are investigated. Both characterizations are amenable to a finite number of LMI conditions either via a gridding of the parameter range or via grid-free techniques which rely on multi-convexity concepts. Practicality and implementation issues are discussed and examples are provided.

1 Introduction

The gain-scheduling problem has been the subject of a great deal of research over recent years, both from theoretical and practical viewpoints. This renewed interest probably stems from the development of new techniques and software which allow for a more rigorous and systematic treatment of the gain-scheduling problem. The classical approach to this problem essentially consists of repeated design syntheses associated with some scheduling strategy connecting locally designed controllers. Such schemes, however, lack supporting theories that guarantee the behavior of the scheduled controller. A significant contribution toward the elimination of such weaknesses is the formulation of the gain-scheduling problem in the context of convex semi-definite programming [1], an elegant and solidly based branch of optimization theory [2, 3, 4]. Expressed in terms of Linear Matrix Inequalities (LMIs), the gain-scheduling problem is readily and globally solved using currently available efficient optimization software [5]. LMI techniques now appear as very natural mechanisms for the formulation of gain-scheduling problems as well as for a vast array of other problems in the control field. Reference [6] gives an overview of the scope of application of such techniques.

As emphasized in H_∞ control theory, a key stage in the characterization of gain-scheduled controllers is the search for adequate Lyapunov functions that establish stability and a performance bound for the closed-loop system. The LFT gain-scheduling techniques in [7, 8, 9, 10] or the so-called quadratic gain-scheduled techniques in [11, 12] make use of a fixed Lyapunov function, as opposed to one which depends on the scheduled variables, to characterize stability and performance. According to [13], such approaches are potentially very conservative because they allow for arbitrary rates of variation in the scheduled variables. More dramatically, it has been shown in [13] that some systems are not even quadratically stabilizable, that is, are not stabilizable on the basis of a single Lyapunov function. A significant improvement over such techniques can be obtained by exploiting the concept of parameter-dependent Lyapunov functions. This is discussed in the context of robustness analysis and synthesis in [14, 15, 16] and for the gain-scheduling problem in [13, 16]. Parameter-dependent Lyapunov functions allow the incorporation of knowledge on the rate of variation in the analysis or synthesis technique, and therefore lead to much less conservative answers. The reader is referred to [17, 13] for earlier work related to the approaches considered here. The discretization of continuous-time gain-scheduled controllers is considered in [18].
In this paper we investigate two different techniques: [19, 20] and an extension of [21, 22] to the gain-scheduling problem. These techniques provide a simple and streamlined treatment of the gain-scheduling problem. The focus is on practical issues of real-time controller calculation and implementation. Refinements directed at overpassing the gridding phase and reducing conservatism are also explored. The reader is referred to [23] for a full version of the paper. For notational simplicity, in long symmetric expressions, terms denoted $\star$ are deduced by symmetry.

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2 Output-Feedback Synthesis

In this section we recap some known results on the gain-scheduling technique with bounded parameter variation rates and point out connections between different approaches. We first give a general characterization of gain-scheduled controllers, the solution to which involves both intermediate controller matrices and Lyapunov variables X and Y. This formulation will be referred as the *basic* characterization, emphasizing the fact that it can be easily extended to multiple objective problems, pole clustering problems, etc... [19, 20]. Next, a second formulation of gain-scheduled controllers is presented. It will be referred as the *projected* characterization, as the intermediate controller matrices have been eliminated through projections [21]. Reconstructing the controller state-space data from the projected conditions has been addressed in [21, 22] for the customary H_∞ control problem. The reconstruction procedure is again described here, in the case of the gain-scheduling problem, for completeness of the discussion. The reader is referred to [17, 24, 13] for details, insights and applications of analogous gain-scheduling techniques.
The problem addressed throughout the paper is the following. Suppose we are given a Linear Parameter-Varying (LPV) plant $G(\theta)$ with state-space realization

$$\begin{array}{rcl} \dot{x} &=& A(\theta)x + B_1(\theta)w + B_2(\theta)u \\ z &=& C_1(\theta)x + D_{11}(\theta)w + D_{12}(\theta)u \\ y &=& C_2(\theta)x + D_{21}(\theta)w \ , \end{array} \quad (1)$$

where

$$A \in \mathbf{R}^{n\times n}, \quad D_{12} \in \mathbf{R}^{p_1 \times m_2}, \text{and} \quad D_{21} \in \mathbf{R}^{p_2\times m_1}$$

define the problem dimension. The time-varying parameter $\theta := (\theta_1, \cdots, \theta_L)^T$ as well as its rates of variation $\dot\theta$ are assumed bounded as follows,

(a) each parameter θ_i ranges between known extremal values $\underline{\theta}_i$ and $\bar\theta_i$:

$$\theta_i(t) \in [\underline{\theta}_i, \bar\theta_i], \quad \forall t \geq 0 \quad (2)$$

(b) the rate of variation $\dot\theta_i$ is assumed well-defined at all times and satisfies

$$\dot\theta_i(t) \in [\underline{\nu}_i, \bar\nu_i], \quad \forall t \geq 0 \quad (3)$$

where $\underline{\nu}_i \leq \bar\nu_i$ are known lower and upper bounds on $\dot\theta_i$.

The first assumption means that the parameter vector θ is valued in a hypercube Θ with vertex set

$$\mathcal{V} := \left\{ (\theta_1, \ldots, \theta_L)^T \ : \ \theta_i \in \{\underline{\theta}_i, \bar\theta_i\} \right\} . \quad (4)$$

Similarly, (3) defines a hypercube Θ_d of $\mathbf{R}^L$ with vertices in

$$\mathcal{T} := \left\{ (\tau_1, \ldots, \tau_L)^T \ : \ \tau_i \in \{\underline{\nu}_i, \bar\nu_i\} \right\} . \quad (5)$$

The gain-scheduled output-feedback control problem consists of finding a dynamic LPV controller, $K(\theta)$, with state-space equations

$$\begin{array}{rcl} \dot{x}_K & = & A_K(\theta,\dot{\theta})x_K + B_K(\theta,\dot{\theta})y \\ u & = & C_K(\theta,\dot{\theta})x_K + D_K(\theta,\dot{\theta})y \ , \end{array} \quad (6)$$

which ensures internal stability and a guaranteed L_2-gain bound γ for the closed-loop operator (1)-(6) from the disturbance signal w to the error signal z, that is

$$\int_0^T z^T z\, d\tau \le \gamma^2 \int_0^T w^T w\, d\tau, \quad \forall T \ge 0$$

and all admissible trajectories $(\theta, \dot{\theta})$ and zero state initial conditions. Note that A and A_K have the same dimensions, since we restrict the discussion to the full-order case. The formulation of such controllers can be handled via an extension of the Bounded Real Lemma with quadratic parameter-dependent Lyapunov functions $V(x_{c\ell}, \theta) = x_{c\ell}^T P(\theta) x_{c\ell}$ where $x_{c\ell}$ stands for the state vector of the closed-loop system. See [13, 14, 15, 19] for details. Note that the controller state-space matrices are allowed to depend explicitly on the derivative of the time-varying parameter θ. Different techniques to remove the dependence on $\dot{\theta}$ will be extensively discussed in Section 3, see also [24].

Except the usual smoothness assumptions on the dependence on θ, the problem data and variables will be unrestricted in the subsequent derivations. The basic characterization of gain-scheduled controllers with guaranteed L_2-gain performance is presented in the next theorem where the dependence of data and variables on θ and $\dot{\theta}$ has been dropped for simplicity.

Theorem 2.1 (Basic Characterization) *Consider the LPV plant governed by (1), with parameter trajectories constrained by (2), (3). There exists a gain-scheduled output-feedback controller (6) enforcing internal stability and a bound γ on the L_2 gain of the closed-loop system (1) and (6), whenever there exist parameter-dependent symmetric matrices Y and X and a parameter-dependent quadruple of state-space data $(\widehat{A}_K, \widehat{B}_K, \widehat{C}_K, D_K)$ such that for all pairs $(\theta, \dot{\theta})$ in $\Theta \times \Theta_d$ the following infinite-dimensional LMI problem holds,*

$$\begin{bmatrix} \dot{X} + XA + \widehat{B}_K C_2 + (\star) & \star & \star & \star \\ \widehat{A}_K^T + A + B_2 D_K C_2 & -\dot{Y} + AY + B_2 \widehat{C}_K + (\star) & \star & \star \\ (XB_1 + \widehat{B}_K D_{21})^T & (B_1 + B_2 D_K D_{21})^T & -\gamma I & \star \\ C_1 + D_{12} D_K C_2 & C_1 Y + D_{12} \widehat{C}_K & D_{11} + D_{12} D_K D_{21} & -\gamma I \end{bmatrix} < 0 \quad (7)$$

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} > 0. \quad (8)$$

In such case, a gain-scheduled controller of the form (6) is readily obtained with the following two-step scheme:

- *solve for N, M, the factorization problem*

$$I - XY = NM^T. \quad (9)$$

- *compute A_K, B_K, C_K with*

$$\begin{array}{rcl} A_K & = & N^{-1}(X\dot{Y} + N\dot{M}^T + \widehat{A}_K - X(A - B_2 D_K C_2)Y \\ & & -\widehat{B}_K C_2 Y - X B_2 \widehat{C}_K) M^{-T} \quad (10) \\ B_K & = & N^{-1}(\widehat{B}_K - X B_2 D_K) \quad (11) \\ C_K & = & (\widehat{C}_K - D_K C_2 Y) M^{-T}. \quad (12) \end{array}$$

Proof
See [19, 20]. ■

Note that since all variables are involved linearly, the constraints (7) and (8) constitute an LMI system. This system is, however, infinite due to its dependence on $(\theta, \dot{\theta})$ ranging over $\Theta \times \Theta_d$. Using the Projection Lemma, detailed in [21], the controller variables can be eliminated, leading to a characterization involving the variables X and Y, only. This is presented in the next theorem.

Theorem 2.2 (Projected Solvability Conditions) *Consider the LPV plant governed by (1), with parameter trajectories constrained by (2) and (3). There exists a gain-scheduled output-feedback controller (6) enforcing internal stability and a bound γ on the L_2 gain of the closed-loop system (1) and (6), whenever there exist parameter-dependent symmetric matrices $Y(\theta)$ and $X(\theta)$ such that for all pairs $(\theta, \dot{\theta})$ in $\Theta \times \Theta_d$ the following infinite-dimensional LMI problem holds,*

$$\left[\begin{array}{c|c} \mathcal{N}_X & 0 \\ \hline 0 & I \end{array}\right]^T \left[\begin{array}{cc|c} \dot{X} + XA + A^T X & XB_1 & C_1^T \\ B_1^T X & -\gamma I & D_{11}^T \\ \hline C_1 & D_{11} & -\gamma I \end{array}\right] [\star] < 0 \quad (13)$$

$$\left[\begin{array}{c|c} \mathcal{N}_Y & 0 \\ \hline 0 & I \end{array}\right]^T \left[\begin{array}{cc|c} -\dot{Y} + YA^T + AY & YC_1^T & B_1 \\ C_1 Y & -\gamma I & D_{11} \\ \hline B_1^T & D_{11}^T & -\gamma I \end{array}\right] [\star] < 0 \quad (14)$$

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} > 0. \quad (15)$$

where $\mathcal{N}_X$ and $\mathcal{N}_Y$ designate any bases of the null spaces of $[\,C_2 \quad D_{21}\,]$ and $[\,B_2^T \quad D_{12}^T\,]$, respectively.

Proof
[21]. ■

Theorem 2.2 only provides existence conditions for controllers of the form (6). These conditions become necessary and sufficient if we confine the involved Lyapunov functions to the set of quadratic forms

$$V(x_{c\ell}, \theta) := x_{c\ell}^T P(\theta) x_{c\ell}, \qquad \text{with} \qquad x_{c\ell} := \begin{bmatrix} x \\ x_K \end{bmatrix}.$$

The controller state space data are easily constructed from X and Y at any value of θ using the techniques in [21, 22, 13].

It should be noted that in spite of their different structures, the characterizations given in Theorems 2.1 and 2.2 are equivalent and can virtually be used interchangeably for controller synthesis. In contrast, when the focus is on computational complexity or practical implementation, these techniques exhibit significant differences. This is discussed in Section 4. Finally, the case where only some parameters θ_i are subject to constraints on their derivatives is easily handled by removing the unconstrained parameters from the matrix functions $X(.)$ and $Y(.)$. The reader may also refer to [25, 23, 26] for a thorough discussion of the practical advantages of the basic technique in dealing with multi-objective problems.

3 Practical Validity of Gain-Scheduling

It must be stressed out that an LPV controller derived from Theorem 2.1 or Theorems 2.2 is not gain-scheduled in the usual sense of the term. Its implementation requires not only the real-time measurement of the parameter θ, but also of its time-derivative $\dot{\theta}$. This is generally prohibitive, since parameter derivatives either are not available or are difficult to estimate during system operation. Gain-scheduled controllers that do not require a measurement of $\dot{\theta}$ will be called *practically valid* hereafter. As discussed in [17], there is no systematic and tractable approach for removing the dependence on $\dot{\theta}$ while maintaining the generality of Theorems 2.1 or 2.1 and 2.2. As suggested by the controller formula (10), a simple but conservative approach has been proposed in [24]. It consists of restricting the variable $Y(\theta)$ to $\dot{Y} = 0$, that is, Y not depending on θ. This operation amounts to using a fixed Lyapunov function for the parameter-dependent control problem described in (14). It thereby sacrifices some performance, resulting in a higher γ.

Keeping in mind that the dependence of the controller data on $\dot{\theta}$ stems from the term $X\dot{Y} + N\dot{M}^T$, (10), the general characterization of Theorem 2.1 or 2.2 offers additional freedom that is worth pointing out. The discussion is summarized in the next table.

	Variables X, Y	Variables N, M
$\frac{d\theta}{dt} = 0$	$X := X(\theta)$,$Y := Y(\theta)$	$NM^T = I - X(\theta)Y(\theta)$
$\frac{d\theta}{dt} \in \Theta_d$	$X := X(\theta)$,$Y := Y(\theta)$	$NM^T = I - X(\theta)Y(\theta)$
$\frac{d\theta}{dt} \in \Theta_d$	$X := X(\theta)$,$Y := Y_0$	$N := I - X(\theta)Y_0$, $M := I$
$\frac{d\theta}{dt} \in \Theta_d$	$X := X_0$,$Y := Y(\theta)$	$N := I$, $M := I - Y(\theta)X_0$
$\frac{d\theta}{dt}$ unbounded	$X := X_0$,$Y := Y_0$	$NM^T = I - X_0Y_0$

Table 1: Selection of variables in the gain-scheduled control problem

In Row #1 of the table, the scheduled variable is assumed constant in time, a practically valid gain-scheduled controller can theoretically be constructed using Theorem 2.1 or alternatively Theorem 2.2, for any matrix functions $X(.)$ and $Y(.)$ of θ. Such an approach ignores possible time variations of θ and provides neither performance nor stability guarantees for the closed-loop system in the face of time-variations. With the same choice of matrix functions $X(.)$ and $Y(.)$, but the rate of variations of θ being confined to a compact Θ_d, row #2, there is no known techniques to compute a practically valid gain-scheduled controller. In rows #3 and #4, we have assumed the conservative choices that X or Y are constant matrix variables. In both cases, the gain-scheduling problem with bounded variation rates admits practically valid controller solutions, provided the variables N and M are adequately selected in Theorems 2.1 and 2.2. With further conservatism, that is, $\dot{\theta}$ is unbounded, row #5, the problem is again tractable and solvable using the same techniques. The case of time-varying parameters with bounds on the rate of variation can be constructively handled by the choices of rows #3 and #4. However, due to the loss of duality in the variables X and Y, such choices are not equivalent. As a consequence, there are some problems for which it is better to take a parameter-dependent X and a constant Y while others will require the converse. Hence, both alternatives must be tried to get a less conservative design . In the controller construction scheme, the variables N and M are subject to the algebraic constraint $I - XY = NM^T$ from which one easily infers the identity

$$\dot{X}Y + \dot{N}M^T = -(X\dot{Y} + N\dot{M}^T).$$

In light of this identity, a practically valid gain-scheduled controller in the cases of rows #3 and #4 can be derived using the same formulas (11) and (12), but with A_K suitably updated to

$$A_K = N^{-1}(\widehat{A}_K - X(A - B_2D_KC_2)Y - \widehat{B}_KC_2Y - XB_2\widehat{C}_K)M^{-T}. \quad (16)$$

The same formulas are still valid for the case of frozen-in-time parameters, row #1, and for arbitrarily varying parameters, row #5, the variables X and Y being replaced by their constant values X_0 and Y_0, in the latter case. Summing up, Table 1 displays all options to handle any situations from the frozen-in-time parameters to arbitrarily time-varying parameters. However, the case in which both X and Y depend on θ with a bounded $\dot{\theta}$ still resists a convex formulation for a practically valid controller.

4 Towards Finite-Dimensional Problems

Even with the simplifications of Table 1 in place, the characterizations of Theorems 2.1 or 2.2 involve the solution of a convex but infinite-dimensional and infinitely constrained problem. This is the price to pay for allowing a general parameter dependence in the plant (1). Generally speaking, there is no systematic rule for selecting the functional dependence of the matrix functions X and Y on θ. We are therefore led to some simple heuristics in order to simplify the computation of solutions to the LMI problems (7)-(8) or (13)-(15). A simple but practical technique has been proposed in [13]. The key idea is to "mimic" the parameter dependence of the plant in the Lyapunov function variables X and Y. Interestingly, the same idea can be used in the more general context of the basic characterization of Theorem 2.1. In return, this offers new potential approaches for the synthesis of gain-scheduled controllers with multiple objective constraints (mixed $H_2 - H_\infty$, pole clustering, and others still to find). When the plant state-space data (1) have an affine expansion in nonlinear and differentiable functions ρ_i ($i = 1, \ldots, N$) of the scheduled variable θ a practically useful approach is to select the quadruple $(\widehat{A}_K(.), \widehat{B}_K(.), \widehat{C}_K(.), D_K(.))$ and the pair $(X(.), Y(.))$ with the very same affine expansion

$$\widehat{A}_K(\theta) := \widehat{A}_{K,0} + \sum_{i=1}^{N} \rho_i(\theta)\widehat{A}_{K,i},\; X(\theta) := X_0 + \sum_{i=1}^{N} \rho_i(\theta)X_i, \ldots \quad (17)$$

The functional dependence of X and Y being fixed, the matrices $\widehat{A}_{K,0}$, $\widehat{A}_{K,i}$,..., play the role of decision variables in the infinitely constrained LMI problems (7)-(8) or (13)-(15). A simple remedy for turning such problems into a finite set of LMIs is to grid the value set of θ [13]. Since the derivative $\dot{\theta}$ appears linearly in the LMIs (7) and (13)-(14), there is only need to check the extreme points of the set Θ_d for all admissible values of θ. The reader can consult references [13, 23] for details.

When restricted to the parameterization (17), the basic and projected characterizations are no longer equivalent. In the first one, we have further restrictions on the structure of the quadruple $(\widehat{A}_K(.), \widehat{B}_K(.), \widehat{C}_K(.), D_K(.))$. As a result, the first approach is generally more conservative, although we have observed very little difference in practice. See the application Section 7 for comparisons. From a complexity viewpoint, the first technique requires a larger number of scalar variables to be optimized; the number of additional variables being approximately $n(n + m_2 + p_2)N$, where N is number of nonlinear functions ρ_i. Its scope of application is therefore more restricted. In contrast, the controller equations resulting from the basic characterization are significantly less complex than those resulting from the projected characterization. See [23] for details.

Since they offer complementary advantages, the techniques described above can be used together to yield a more effective methodology. Confirmed by practical experience, the following rules have proven useful. All necessary tunings, requiring repeated computations should be based on the less costly projected technique. The procedure is completed by running the basic technique, for controller implementation purposes. Though the last phase may be very slow, it is run only once in the whole design process.

5 Overpassing the Gridding Phase

As discussed earlier, for LPV systems having a general nonlinear θ -dependence, there is no systematic technique to overpass the gridding phase, hence making the design more direct. For polynomial LPV systems, however, it is possible to take advantage of some geometric properties of the functions involved to convert infinitely LMI-constrained problems into a finite number of LMI constraints. In view of the preliminary results in [14], this can be done with little induced extra conservatism. The proposed technique relies on the following preparatory lemmas.

Lemma 5.1 *Consider a scalar quadratic function of $\theta \in \mathbf{R}^L$:*

$$f(\theta_1, \ldots, \theta_L) = \alpha_0 + \sum_i \alpha_i\theta_i + \sum_{i<j} \beta_{ij}\theta_i\theta_j + \sum_i \gamma_i\theta_i^2 \quad (18)$$

Then $f(\cdot)$ is negative (resp. positive) in the hypercube (2) whenever

$$f(\theta) < 0, \quad (resp. > 0) \quad \forall\theta \in \mathcal{V}, \quad (19)$$

and

$$\gamma_i = \frac{1}{2}\frac{\partial^2 f}{\partial {\theta_i}^2}(\theta) \geq 0 \quad (resp. \leq 0), \qquad i = 1, \ldots, L\,. \quad (20)$$

Proof

A proof of this result can be found in [14]. ■

Property (20) is referred to as the multi-convexity (resp. multi-concavity) property since it merely amounts to expressing that the function is convex (resp. concave) with respect to each variable θ_i separately. With this lemma, a sufficient condition for checking the sign invariance of f is reduced to a finite number of linear algebraic conditions in the polynomial coefficients. A similar result is now presented for higher-order polynomial functions.

Lemma 5.2 *Consider a general polynomial function $f(\theta_1, \ldots, \theta_L)$ of arbitrary order. Denote d_k the partial degree with respect to the variable θ_k, $k = 1, \ldots, L$ and d the total degree of the polynomial* *function. Then $f(\cdot)$ is negative (resp. positive) in the hypercube (2) whenever*

$$f(\theta) < 0, \qquad (resp. > 0) \qquad \forall \theta \in \mathcal{V}\,, \tag{21}$$

and

$$(-1)^m \frac{\partial^{2m}}{\partial\theta_{l_1}^2 \dots \partial\theta_{l_m}^2} f(\theta) \le 0, \qquad (resp. \ge 0), \quad \forall \theta \in \mathcal{V}\,, \tag{22}$$

where

$$1 \le l_1 \le l_2 \le \dots \le l_m \le L, \quad 1 \le m \le \frac{d}{2}$$

$$2\#\{l_j = k : \ j \in \{1,\dots,m\}\} \le d_k, \quad k = 1,2,\dots,L.$$

Proof

The proof is obtained by a recursive use of Lemma 5.1 and using the fact that partial derivative orders exceeding the partial degree of a variable are immaterial. ■

Here again, checking the sign invariance of a general polynomial has been reduced, potentially conservatively, to a finite number of linear algebraic constraints on its coefficients. An application of these results to checking the feasibility of a class of parameterized LMIs is presented. We are considering polynomially θ-dependent LMIs of the form

$$\Gamma(\theta, z) := \sum_{\nu \in J} \theta^{[\nu]} M_\nu(z) < 0\,, \tag{23}$$

where $M_\nu(z)$ stand for symmetric matrix-valued functions of the decision variable z that are linear in z. The notation $[\nu]$ is the vector of partial degrees $[\nu] = [\nu_1, \dots, \nu_L]$ associated with the lexicographically ordered term

$$\theta^{[\nu]} = \theta_1^{\nu_1}\theta_2^{\nu_2}\dots\theta_L^{\nu_L}, \qquad \theta \in \Theta = \mathrm{Co}\mathcal{V}$$

and the convention $\theta^{[0]} = 1$. J is a set of L-tuples of partial degrees describing the polynomial expansion (23). Checking or invalidating the feasibilty of (23) in z is not tractable in general since it involves infinitely many LMI constraints. In virtue of Lemma 5.2, it is possible to conservatively reduce this problem to a finite number of LMI conditions in z. As before, d_k and d designate the partial and total degrees in the matrix polynomial expansion.

Lemma 5.3 *Consider the parameterized LMI (23), where θ ranges over a hypercube (2). Then the (uncountable infinite number) LMI conditions*

$$\Gamma(\theta, z) < 0, \qquad \forall \theta \in \Theta \tag{24}$$

hold for some z, whenever the finite family of LMI conditions:

$$\Gamma(\theta, z) < 0, \qquad \forall \theta \in \mathcal{V} \tag{25}$$

$$(-1)^m \frac{\partial^{2m}}{\partial\theta_{l_1}^2 \dots \partial\theta_{l_m}^2} \Gamma(\theta, z) \le 0, \quad \forall \theta \in \mathcal{V}\,, \tag{26}$$

where

$$1 \le l_1 \le l_2 \le \dots \le l_m \le L, \quad 1 \le m \le \frac{d}{2}$$

$$2\#\{l_j = k : \ j \in \{1,\dots,m\}\} \le d_k, \quad k = 1,2,\dots,L.$$

are feasible for some z.

Proof

Note that condition (24) is equivalent to

$$x^T\Gamma(\theta, z)x < 0, \quad \forall \theta \in \Theta, \quad \forall x \ne 0\,. \tag{27}$$

With fixed x and z, Lemma 5.2 is applicable to the function $f(\theta) = x^T\Gamma(\theta, z)x$, and the algebraic conditions (21) and (22) enforce the negativity of f over Θ. Repeating the argument for all $x \ne 0$ yields the conditions (25) and (26) that consequently enforce (24), as required. ■

Lemma 5.3 can be exploited in the context of LPV synthesis, Section 2, to formulate sufficient solvability conditions in terms of a finite number of LMI constraints. This can be done for the projected and basic synthesis techniques with an L_2-gain performance, and more generally for multi-channel and multi-objective control problems when both the plant data and the variables X, Y, $\widehat{A}_K$,... assume a polynomial dependence in θ. As an illustration, we provide the charaterization for an affine problem, keeping in mind that any polynomial dependence can be handled via Lemma 5.3.

Theorem 5.4 *Consider the LPV plant governed by (1), with parameter trajectories constrained by (2) and (3) and assume the plant data (A, B_1, C_1, D_{11}) are affine in θ. That is,*

$$\begin{bmatrix} A(\theta) & B_1(\theta) \\ C_1(\theta) & D_{11}(\theta) \end{bmatrix} = \begin{bmatrix} A_0 & B_{1,0} \\ C_{1,0} & D_{11,0} \end{bmatrix} + \sum_{i=1}^{L} \theta_i \begin{bmatrix} A_i & B_{1,i} \\ C_{1,i} & D_{11,i} \end{bmatrix}.$$

Then, there exists a gain-scheduled output-feedback controller (6) enforcing internal stability and a bound γ on the L_2 gain of the closed-loop system (1) and (6), whenever there exist affinely parameter-dependent matrices X, Y, $\widehat{B}_K$, $\widehat{C}_K$ and D_K

$$Y(\theta) = Y_0 + \sum_{i=1}^{L} \theta_i Y_i, \; X(\theta) = X_0 + \sum_{i=1}^{L} \theta_i X_i \dots$$

such that the following finite family of LMIs are feasible.

$$\begin{bmatrix} \dot{X} + XA + \widehat{B}_K C_2 + (\star) & \star & \star \\ (XB_1 + \widehat{B}_K D_{21})^T & -\gamma I & \star \\ C_1 + D_{12} D_K C_2 & (D_{11} + D_{12} D_K D_{21}) & -\gamma I \end{bmatrix} < 0 \tag{28}$$

$$\begin{bmatrix} -\dot{Y} + AY + B_2 \widehat{C}_K + (\star) & \star & \star \\ (B_1 + B_2 D_K D_{21})^T & -\gamma I & \star \\ C_1 Y + D_{12} \widehat{C}_K & (D_{11} + D_{12} D_K D_{21}) & -\gamma I \end{bmatrix} < 0 \tag{29}$$

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} > 0, \tag{30}$$

$$\begin{bmatrix} X_i A_i + (\star) & \star \\ B_{1,i}^T X_i & 0 \end{bmatrix} \ge 0, \qquad \begin{bmatrix} A_i Y_i + (\star) & \star \\ C_{1,i} Y_i & 0 \end{bmatrix} \ge 0 \tag{31}$$

for $(\theta, \dot{\theta}) \in \mathcal{V} \times \mathcal{T}$ and $i = 1, 2, \dots, L$.

5.1 Illustration

We now give a simple illustration of the technique and provide comparison results with existing gain-scheduling control approaches. The LPV plant under consideration is taken from [16] and has the state-space descripton in LFT format:

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 \\ -3.25 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ -2.75 \end{bmatrix} w_\theta + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v \\ z_\theta &= \begin{bmatrix} 1 & 0 \end{bmatrix} x \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} x \\ w_\theta &= \theta z_\theta\,, \end{aligned}$$

where x, v, y and θ denote the state vector, the control input, the measurement output and the scheduled parameter, respectively. Actuator dynamics $v = \frac{1}{s+1} u$ are also incorporated to reflect a more realistic situation. The synthesis interconnection used in this problem is depicted in Figure 1, with the following weighting functions

$$W_u = 0.32, \quad W_d = 1.33, \quad W_p = 0.5, \quad W_n = 10\frac{s+1}{s+200}.$$

We are seeking an LPV controller providing internal stability and minimal L_2-gain performance between exogeneous and error signals, with parameter trajectories $\theta(t) \in [-1, 1]$, $t \ge 0$. The syntheses are conducted with 3 different techniques:

- the LFT gain-scheduled control technique in [27], suitably modified to incorporate skew-symmetric scalings. See also [16]. Such techniques disregard parameter rate of variation constraints.
- the gain-scheduling control technique in Section 2, combined with a gridding of the parameter range.
- the gain-scheduling control technique in Section 5 with no gridding.

Recall that the last two approaches exploit informations on the parameter rate of variation. The achieved performance levels for rate of variation bounds from 0 to 10 are shown in Figure 2. We can see that the last two techniques behave as theoretically expected. They provide far better answers for low rates of variation than the LFT control approach. The observed gap is likely to be even larger for problems with multiple scheduled variables. Also interesting is the fact that the technique based on multi-convexity concepts, hence, with the gridding ruled out, gives performance levels close to those of the synthesis with gridding. It therefore provides a potential alternative for problems of reasonable size.

6 Reducing Conservatism by Scaling

The techniques in Sections 2 and 5 are readily modified to handle problems with structured uncertainties. That is, w and z are related through a structured operator Δ. The associated theoretical characterizations involve possibly parameter-dependent scaling matrices but are not reducible to LMIs. Simple heuristics such as $D-K$-like iterative schemes can however be utilized to improve the design. See [23].

7 Two-Link Flexible Manipulator

The gain-scheduled control of a two-link flexible manipulator is a non trivial problem. The dynamics of such a system include both rigid body and lightly damped structural modes. The problem is complicated by uncertainty in the high frequency dynamics of the system and by the variation of dynamics with manipulator geometry. The first of these complications drives the requirement for closed-loop robust stability while the second drives the requirement for gain-scheduling. In addition, a rapid closed-loop response to position commands is desired. The ability of a control synthesis approach to handle the tradeoffs between robustness, performance, and gain scheduling with the least possible conservatism is thus critical for such a system. Due to lack of space, the reader is encouraged to consult reference [23] to see illustrations of concepts and techniques previously introduced.

8 Conclusions

Advanced gain-scheduling design approaches for LPV systems have been presented with emphasis on the practical goals of reduced computational burden and ease of implementation. Two complementary LMI characterizations for the calculation of such controllers have been investigated which, when used together, achieve these two objectives. The methodology is completed with a grid-free approach and a scaling technique directed at facilitating the design task and reducing conservatism, respectively. The challenging problem of the control of a two-link flexible manipulator is introduced in this context and used to demonstrate the validity of the theoretical solutions.

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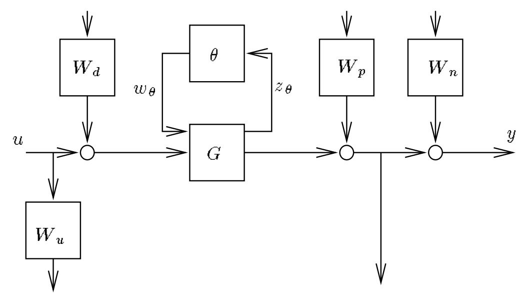

Figure 1: Synthesis interconnection

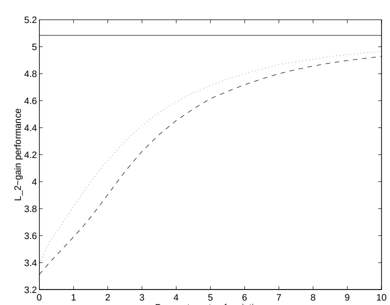

Figure 2: Guaranteed L_2-performance vs. rate of variation LFT control (solid); synthesis with gridding (dashed); no gridding (dotted)