A NOTE ON INTEGRABILITY OF ALMOST PRODUCT RIEMANNIAN STRUCTURES *

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الخلاصة:

سوف نثبت - في هذا البحث - باستخدام مؤثر Yano-Ako شرطاً كافياً جديداً لقابلية تكامل ما يقارب مضروب تركيبات Riemann المعرفة على المتنوعات القابلة للتفاضل، كما سنطبق هذا الشرط في حالة تقارب مضروب تركيبات Riemann المعرفة على حزمة الموتّر.

ABSTRACT

Using the Yano-Ako operator, we give a new sufficient condition of integrability for almost product Riemannian structures on the differentiable manifold and apply this condition to the case of almost product Riemannian structures defined on the tensor bundle.

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1. INTRODUCTION

Let M_n be an almost product manifold with almost product structure F. In order that the almost product structure F be integrable, it is necessary and sufficient that it is possible to introduce a torsion-free affine connection ∇ with respect to which the structure tensor F is covariantly constant (see [3], [8]), *i.e.* $\nabla F = 0$. Also, we know that the integrability of F is equivalent to the vanishing of the Nijenhuis tensor N_F [9, p. 240]. In this paper we consider another possible sufficient condition of the integrability of almost product structures on Riemannian manifolds.

If M_n admits a Riemannian metric g such that

$$g(FX, Y) = g(X, FY) \tag{1}$$

for $X, Y \in \Im_0^1(M_n)$, then M_n is called an almost product Riemannian manifold [5], [11]. Sometimes, such metrics are said to be pure with respect to F [6], [7].

We define the operator $\Phi_F : \Im_2^0(M_n) \to \Im_3^0(M_n)$ (see [10]) associated with F and applied to the pure metric g:

$$(\Phi_F g)(X, Y_1, Y_2) = (FX)(g(Y_1, Y_2)) - X(g(FY_1, Y_2)) + g((L_{Y_1} F)X, Y_2) + g(Y_1, (L_{Y_2} F)X), \tag{2}$$

where L_{Y_i}, $i = 1, 2$ denotes the Lie differentiation with respect to Y_i. By setting $X = \partial_k$, $Y_1 = \partial_i$, $Y_2 = \partial_j$ in Equation (2), we see that the components $(\Phi_F g)_{kij}$ of $\Phi_F g$ with respect to a local coordinate system $x^1, \ldots, x^n$ may be expressed as follows:

$$(\Phi_F g)_{kij} = F_k^m \partial_m g_{ij} - F_i^m \partial_k g_{mj} + g_{mj}(\partial_i F_k^m - \partial_k F_i^m) + g_{im} \partial_j F_k^m .$$

2. SUFFICIENT CONDITION OF THE INTEGRABILITY

Let there be given an almost product Riemannian manifold (M_n, F) with pure metric g. By virtue of (1) and $\nabla g = 0$ we have

$$g(Z, (\nabla_Y F)X) = g((\nabla_Y F)Z, X), \tag{3}$$

Using (3) and $[X, Y] = \nabla_X Y - \nabla_Y X$, we have transform (2) as follows:

$$(\Phi_F g)(X, Z_1, Z_2) = -g((\nabla_X F)Z_1, Z_2) + g((\nabla_{Z_1} F)X, Z_2) + g(Z_1, (\nabla_{Z_2} F)X) \tag{4}$$

From this we have

$$(\Phi_F g)(Z_2, Z_1, X) = -g((\nabla_{Z_2} F)Z_1, X) + g((\nabla_{Z_1} F)Z_2, X) + g(Z_1, (\nabla_X F)Z_2) \tag{5}$$

If we add (4) and (5), we find

$$(\Phi_F g)(X, Z_1, Z_2) + (\Phi_F g)(Z_2, Z_1, X) = 2g(X, (\nabla_{Z_1} F)Z_2) \tag{6}$$

Putting $\Phi_F g = 0$ in (6), we find $\nabla F = 0$

On the other hand, we know that the integrability of the almost product structure F is equivalent to the existence of a torsion-free affine connection with respect to which the equation $\nabla F = 0$ holds. Since the Levi–Civita connection ∇ of g is a torsion-free affine connection, we have

Theorem 1. *Let (M_n, F) be an almost product Riemannian manifold with pure metric g. Then F is integrable if $\Phi_F g = 0$.*

We note that, if $\nabla F = 0$, then the condition $\Phi_F g = 0$ follows from (4) or (5). Thus we have

Corollary. For an almost product Riemannian manifold with pure metric g, the condition $\Phi_F g = 0$ is equivalent to $\nabla F = 0$, where ∇ is the Levi-Civita connection of g.

An integrable almost product Riemannian manifold with structure tensor F is usually called a locally product Riemannian manifold. If the metric of a locally product Riemannian manifold M_n has the form

$$ds^2 = g_{ab}(x^c)dx^a dx^b + g_{\bar{a}\bar{b}}(x^{\bar{c}})dx^{\bar{a}}dx^{\bar{b}}, \qquad a,b,c,... = 1,...,m, \quad \bar{a},\bar{b},\bar{c},... = m+1,...,n$$

that is g_{ab} are functions of x^c only, $g_{a\bar{b}} = 0$ and $g_{\bar{a}\bar{b}}$ are functions of $x^{\bar{c}}$ only, then we call the manifold M_n a locally decomposable Riemannian manifold. On the other hand, we know that the locally product Riemannian manifold with structure tensor F is locally decomposable if and only if F is covariantly constant with respect to the Levi–Civita connection ∇ [11, p. 420].

Thus, by Theorem 1 and its corollary, we have

Theorem 2. *Let (M_n, F) be an almost product Riemannian manifold with pure metric g. A necessary and sufficient condition for (M_n, F) to be a locally decomposable Riemannian manifold is that $\Phi_F g = 0$.*

3. DECOMPOSABILITY OF AN ALMOST PRODUCT RIEMANNIAN STRUCTURES ON TENSOR BUNDLE

Let $T_q^p(M_n) = \bigcup_{P\in M_n} T_q^p(P)$ be a tensor bundle of type (p,q) over M_n with the natural projection $\pi : T_q^p(M_n) \to M_n$. If x^j are local coordinates on a neighborhood U of $P \in M_n$, then a tensor t at P, which is an element of $T_q^p(M_n)$, is expressible in the form $(x^j, t_{j_1...j_q}^{i_1...i_p})$, where $t_{j_1...j_q}^{i_1...i_p}$ are components of t with respect to the natural base. We may consider $(x^j, t_{j_1...j_q}^{i_1...i_p}) = (x^j, x^{\bar{j}}) = x^J$, $j = 1,...,n$, $\bar{j} = n+1,...,n+n^{p+q}$, $J = 1,...,n+n^{p+q}$ as local coordinates on a neighborhood $\pi^{-1}(U)$.

If $\alpha \in \Im_p^q(M_n)$, it is regarded, in a natural way, by contraction, as a function on $T_q^p(M_n)$, which we denote by $\iota\alpha$. If α has local expression

$$\alpha = \alpha_{i_1...i_p}^{j_1...j_q} \partial_{j_1} \otimes ... \otimes \partial_{j_q} \otimes dx^{i_1} \otimes ... \otimes dx^{i_p}$$

on a coordinate neighborhood $U(x^j) \subset M_n$, then $\iota\alpha = \alpha(t)$ has the local expression

$$\iota\alpha = \alpha_{i_1...i_p}^{j_1...j_q} t_{j_1...j_q}^{i_1...i_p}$$

with respect to the coordinates $(x^j, x^{\bar{j}})$ on $\pi^{-1}(U)$.

Let $A \in \Im_q^p(M_n)$. Then there is a unique vector field ${}^V A \in \Im_0^1(T_q^p(M_n))$ such that for $\alpha \in \Im_p^q(M_n)$

$${}^V A(\iota\alpha) = \alpha(A) \circ \pi = {}^V(\alpha(A))$$

where ${}^V(\alpha(A))$ is the vertical lift of the function $\alpha(A) \in \Im_0^0(M_n)$. We call ${}^V A$ the vertical lift of $A \in \Im_q^p(M_n)$ to $T_q^p(M_n)$ [1]. The vertical lift ${}^V A$ has components of the form

$${}^V A = \left({}^V A^j, {}^V A^{\bar{j}}\right) = \left(0, A_{j_1...j_q}^{i_1...i_p}\right)$$

with respect to the coordinates $(x^j, x^{\bar{j}})$ on $T_q^p(M_n)$.

We define the horizontal lift ${}^H X$ of X to $T_q^p(M_n)$ (see [4]) by ${}^H X(\iota\alpha) = \iota(\nabla_X \alpha)$, for all $\alpha \in \Im_p^q(M_n)$. The horizontal lift ${}^H X$ of $X \in \Im_0^1(M_n)$ to $T_q^p(M_n)$ has components

$${}^H X = \left(X^j, X^s \left(\sum_{\mu=1}^{q} \Gamma_{sj_\mu}^m t_{j_1...m...j_q}^{i_1...i_p} - \sum_{\lambda=1}^{p} \Gamma_{sm}^{i_\lambda} t_{j_1...j_q}^{i_1...m...i_p} \right)\right)$$

with respect to the coordinates $(x^j, x^{\bar{j}})$ on $T_q^p(M_n)$, where Γ_{ij}^k are local components of ∇ on M_n.

For each $P \in M_n$, the extension of scalar product g (denoted also by g) is defined on the tensor space $\pi^{-1}(P) = T_q^p(P)$ by

$$g(A,B) = g_{i_1 t_1} ... g_{i_p t_p} g^{j_1 l_1} ... g^{j_q l_q} A_{j_1 ... j_q}^{i_1 ... i_p} B_{l_1 ... l_q}^{t_1 ... t_p}$$

for all $A, B \in T_q^p(P)$. A Sasakian metric ${}^s g$ (or a diagonal lift of g) is defined on $T_q^p(M_n)$ by the three equations

$$^s g({}^V A, {}^V B) = {}^V(g(A,B)),\ A, B \in \Im_q^p(M_n)$$

$$^s g({}^V A, {}^H Y) = 0,$$

$$^s g({}^H X, {}^H Y) = {}^V(g(X,Y)),\ X, Y \in \Im_0^1(M_n).$$

These equations are easily seen to determine ${}^s g$ on $T_q^p(M_n)$ with respect to which the horizontal and vertical distributions are complementary and orthogonal. On the other hand, let J be the almost product structure on $T_q^p(M_n)$ determining the vertical distribution consisting of distribution of the vertical vectors and the horizontal distribution determined by the Riemannian connection ∇, *i.e.*

$$J\,{}^H X = {}^H X,\ J\,{}^V A = -{}^V A$$

for any $X \in \Im_0^1(M_n)$ and $A \in \Im_q^p(M_n)$. Then we easily obtain

$$^s g(J\,{}^V A, {}^V B) = {}^s g({}^V A, J\,{}^V B),$$

$$^s g(J\,{}^V A, {}^H Y) = {}^s g({}^V A, J\,{}^H Y),$$

$$^s g(J\,{}^H X, {}^V B) = {}^s g({}^H X, J\,{}^V B),$$

$$^s g(J\,{}^H X, {}^H Y) = {}^s g({}^H X, J\,{}^H Y)$$

for any $X, Y \in \Im_0^1(M_n)$ and $A, B \in \Im_q^p(M_n)$. Thus $(T_q^p(M_n), {}^s g, J)$ is an almost product Riemannian manifold.

The following equations are known $[2]$

$$\left[{}^V A, {}^V B\right] = 0\ , \tag{7}$$

$$\left[{}^H X, {}^V B\right] = {}^V\left(\nabla_X B\right), \tag{8}$$

$$\left[{}^H X, {}^H Y\right] = \bar{D}_{R(X,Y)} + {}^H\left[X, Y\right] \tag{9}$$

for all $A, B \in \Im_q^p(M_n)$ and $X, Y \in \Im_0^1(M_n)$. Here $\bar{D}_{R(X,Y)}$ means that a lift of derivation $D_{R(X,Y)}$ defined by an endomorphism $R(X, Y) \in \Im_1^1(M_n)$ (R - curvature tensor field of the connection ∇).

From (7) we see that the vertical distribution V is tangent to the fibres $\pi^{-1}(x)$ of $T_q^p(M_n)$ and is integrable, *i.e.* $\left[{}^V A, {}^V B\right] \in V$. Moreover, this distribution is also totally geodesic, *i.e.* $\nabla_{{}^V A}{}^V B \in V$. In fact, using (8) we have

$$\begin{aligned} 2\,{}^s g(\nabla_{{}^V A}{}^V B, {}^H X) &= {}^H X\ {}^V(g(A,B)) - {}^s g\left([{}^V A, {}^H X], {}^V B\right) - {}^s g\left([{}^V B, {}^H X], {}^H A\right) \\ &= {}^V(Xg(A,B) - g(\nabla_X A, B) - g(A, \nabla_X B)) \\ &= 0 \end{aligned}$$

From Equation (9) we see that the vertical part of $\left[{}^{H}X , {}^{H}Y \right]$ vanishes if and only if $\bar{D}_{R(X,Y)} = 0$ for all $X, Y \in \Im_0^1(M_n)$. It is easy to prove that this is equivalent to $R = 0$ if $p + q > 0$.

Thus, taking account of Theorem 2, we have

Theorem 3. *The almost product Riemannian manifold* $(T_q^p(M_n), {}^{s}g, J)$ *is locally decomposable if and only if* M_n *is locally flat* .

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