RING HOMOMORPHISMS AND FINITE GORENSTEIN DIMENSION

Luchezar L. Avramov and Hans–Bjørn Foxby

Contents

Introduction
1. Homological algebra
2. Dualizing complexes
3. Dualizing equivalences
4. Gorenstein dimension
5. Relative dualizing complexes: Properties
6. Relative dualizing complexes: Proofs
7. Bass numbers of local homomorphisms
8. Quasi-Gorenstein homomorphisms
References

Introduction

This paper, in which all rings are commutative and noetherian, is devoted to the study of the local structure of ring homomorphisms.

Given a ring homomorphism $\varphi\colon R \to S$, various numerical invariants have been attached in [8], [5], [6], to its *localizations* $\varphi_{\mathfrak{q}}\colon R_{\mathfrak{q}\cap R} \to S_{\mathfrak{q}}$ at prime ideals $\mathfrak{q}$ of S. Some of these numbers, like dimension, depth, or type, express quantitative characteristics of φ; others, like Cohen–Macaulay defect, or complete intersection defect, capture its qualitative aspects. An upshot of the work in [4–8] is the realization that when $\operatorname{fd}\varphi_{\mathfrak{q}}$ is finite – that is, the R–module $S_{\mathfrak{q}}$ has *finite flat dimension* – at all $\mathfrak{q}$, then the properties of the homomorphism φ control the transfer of local properties between the rings R and S.

This point of view has put on common ground many phenomena perceived earlier as different, and treated accordingly. The new perspective has also led to the determination of large classes of ring homomorphisms, like the *Gorenstein* or *Cohen–Macaulay* ones, with

L.L.A. was partially supported by National Science Foundation Grant No. DMS-9102951.

H.B.F. was partially supported by the Danish Natural Science Research Council and the Research Academy of Denmark.

remarkable stability properties. Even in the classical framework of flat homomorphisms, our approach has produced new results, like the solutions in [5] and [6] of a series of open "localization" problems, going back to Grothendieck, for Cohen–Macaulay properties.

In the present paper we expand the scope of our study of homomorphisms by weakening the homological assumption on the maps. Namely, we replace the condition on the flat dimension of the R–modules $S_{\mathfrak{q}}$ by one on the finiteness of their *Gorenstein dimension*, or G-dimension. This concept was initially defined for finite (that is, finitely generated) modules by Auslander [1]. It is a finer invariant than the classical projective dimension: they are equal when the latter is finite, but the Gorenstein dimension may be finite without the projective one being so. In fact, over a Gorenstein ring of finite Krull dimension all finite modules have finite G-dimension, a result which is doubtless responsible for the name of this homological dimension.

As we want to impose – at least locally – a finiteness condition on the G-dimension of the R–module S, it is crucial to have an operational notion for not necessarily finite R–modules. While a concept of G-dimension is introduced in that generality in [2], its properties are not sufficiently developed for the purpose at hand. Therefore, we take a different approach, based on a relative version of the Cohen Structure Theorem for complete local rings.

It is proved in [8] that for each local homomorphism $R \to S$, the induced map from R to the completion $\widehat{S}$ of S has a *Cohen factorization* $R \to R' \to \widehat{S}$ into a local flat extension with regular closed fiber, followed by a surjective homomorphism. We say φ has *finite G-dimension*, and write $\operatorname{G-dim}\varphi < \infty$, if the R'–module $\widehat{S}$ has finite G-dimension. This calls for a proof that the concept is independent of the choice of the factorization. The argument is given in Section 4, where some of the more immediate consequences of the finiteness of $\operatorname{G-dim}\varphi$ are also established.

In Section 5 we describe our main tool for the study of local homomorphisms of finite G-dimension: the *dualizing complex* for a local homomorphism φ. When $\operatorname{G-dim}\varphi$ is finite we present a roster of properties of that complex, closely paralleling those of dualizing complexes for local rings. Thus, φ has a dualizing complex when S is complete, or when both R and S have such complexes; dualizing complexes are unique up to isomorphism and translation in the derived category $\mathbf{D}(S)$ of S–modules; they localize when R has Gorenstein formal fibers; etc. The proofs of those results are presented in Section 6.

In [4] we defined a local homomorphism $\varphi\colon (R,\mathfrak{m}) \to (S,\mathfrak{n})$ to be *Gorenstein at* $\mathfrak{n}$ if $\operatorname{fd}\varphi$ is finite and the *Bass numbers* of both rings essentially coincide: $\mu_R^{i+\operatorname{depth} R} = \mu_S^{i+\operatorname{depth} S}$ for *all* $i \in \mathbb{Z}$. Weakening the homological condition to finite G-dimension, we obtain a larger class of local homomorphisms, which we call *quasi-Gorenstein* at $\mathfrak{n}$. By using a normalized dualizing complex for such a φ, we define a formal Laurent series with non-negative integer coefficients $\mathrm{I}_\varphi(t)$, the *Bass series* of φ, and establish a product formula

$$\mathrm{I}_S(t) = \mathrm{I}_R(t)\,\mathrm{I}_\varphi(t)$$

where $\mathrm{I}_R(t) = \sum_{i\in\mathbb{Z}} \mu_R^i t^i$ is the usual Bass series of R. In the case of finite flat dimension, such a formula is established in [9], with a very different definition of $\mathrm{I}_\varphi(t)$. Our present approach provides a better hold on the Bass series of a homomorphism, and allows us to

prove it has no "gaps," solving a problem raised in [4]. This result has the practical consequence, that now (quasi-)Gorenstein homomorphisms can be defined by the corresponding homological condition, plus an equality $\mu_R^{i+\operatorname{depth} R} = \mu_S^{i+\operatorname{depth} S}$ for a *single* integer $i > 0$.

The preceding results are proved in Section 7, where we also characterize quasi-Gorenstein homomorphisms without reference to G-dimension. They are shown to be precisely the local homomorphisms which base change dualizing complexes properly: if D is dualizing for R, then the derived tensor product $D \otimes^{\mathbf{L}}_R S$ is dualizing for S. In Section 8 we study homomorphisms of noetherian rings all of whose localizations are quasi-Gorenstein, and prove that this class enjoys essentially all the stability properties of Gorenstein homomorphisms, established in [4] and [6].

On a couple of occasions in this Introduction we have mentioned concepts defined in terms of derived categories of modules. In the body of the paper derived categories provide both the language and the proof techniques for most results, so in Section 1 we fix some terminology and notation. In Section 2, for the reader's and our own convenience, we briefly recall a few standard properties of dualizing complexes for local rings.

In Section 3 we prove that when R is local with dualizing complex D, the endofunctors $D \otimes^{\mathbf{L}}_R -$ and $\mathbf{R}\operatorname{Hom}_R(D, -)$ of $\mathbf{D}(R)$ establish an equivalence of the full subcategories $\mathbf{F}(R)$ and $\mathbf{I}(R)$ of $\mathbf{D}(R)$, consisting of complexes isomorphic to bounded complexes of flat and injective modules, respectively. This is a vast generalization of a theorem of Sharp [19] on the equivalence of the categories of finite modules of finite projective and finite injective dimension over a Cohen–Macaulay local ring with dualizing module. For our purposes, even the generalized form of such an equivalence is insufficient, so we extend it further to full subcategories $\mathbf{A}(R)$ and $\mathbf{B}(R)$ of $\mathbf{D}(R)$. The category $\mathbf{A}(R)$, which contains both $\mathbf{F}(R)$ and the full subcategory of finite R–modules of finite G-dimension, provides a particularly flexible environment for the study of homomorphisms of finite G-dimension.

1. Homological algebra

(1.1) *Complexes.* A *complex* M of R–modules, or *R–complex*, is a sequence of R–linear homomorphisms $\{\partial_n \colon M_n \to M_{n-1}\}_{n\in\mathbb{Z}}$ such that $\partial_n\partial_{n+1} = 0$ for all n. (We only use subscripts and all differentials have degree -1.) The *infimum*, *supremum*, and *amplitude* of M are defined by $\inf M = \inf\{n \in \mathbb{Z} \mid \mathrm{H}_n(M) \neq 0\}$, $\sup M = \sup\{n \in \mathbb{Z} \mid \mathrm{H}_n(M) \neq 0\}$, and $\operatorname{amp} M = \sup M - \inf M$. For $s \in \mathbb{Z}$ we denote by $\Sigma^s M$ the complex with $(\Sigma^s M)_n = M_{n-s}$ and $\partial^{\Sigma^s M}_n = (-1)^s \partial^M_{n-s}$. If N is an R–module then we also denote by N the complex of R–modules with $N_n = 0$ for $n \neq 0$ and $N_0 = N$.

If N is an R–complex, then a *morphism* $\alpha \colon M \to N$ is a sequence of R–linear homomorphisms $\alpha_n \colon M_n \to N_n$, such that $\partial^N_n \alpha_n = \alpha_{n-1}\partial^M_n$ for $n \in \mathbb{Z}$. A *quasi-isomorphism* is a morphism α such that $\mathrm{H}_n(\alpha)$ is an isomorphism for all n; we indicate quasi-isomorphisms by $\simeq$, while $\cong$ is our notation for isomorphisms of complexes (and thereby of modules).

(1.2) *Derived functors.* The *derived category* of the category of R–modules, cf. [22] or [16], is denoted by $\mathbf{D}(R)$. Isomorphisms in $\mathbf{D}(R)$ are labeled with $\simeq$ (since a morphism of complexes is a quasi-isomorphism if and only if its image in $\mathbf{D}(R)$ is an isomorphism, no notational confusion arises). We use $\sim$ to denote isomorphisms up to translation; in particular, $\mathrm{H}_i(M) = 0$ for $i \neq n$, precisely when $M \sim \mathrm{H}_n(M)$.

We write $\mathbf{D}^{\mathrm{f}}(R)$ for the full subcategory of $\mathbf{D}(R)$ consisting of complexes M with $\mathrm{H}_n(M)$ a finite R–module for each $n \in \mathbb{Z}$. Also, $\mathbf{D}_+(R)$, $\mathbf{D}_-(R)$, $\mathbf{D}_{\mathrm{b}}(R)$, $\mathbf{D}_0(R)$, denote the full subcategories defined by $\mathrm{H}_n(M) = 0$ for, respectively, $n \ll 0$, $n \gg 0$, $|n| \gg 0$, $n \neq 0$. For a subcategory $\mathbf{S}(R) \subseteq \mathbf{D}(R)$ we set $\mathbf{S}^{\mathrm{f}}(R) = \mathbf{S} \cap \mathbf{D}^{\mathrm{f}}(R)$, etc. Obvious equivalences identify $\mathbf{D}_0(R)$ with the category of R–modules, and $\mathbf{D}_0^{\mathrm{f}}(R)$ with that of finite R–modules.

The left derived functor of the tensor product functor of R–complexes is denoted by $- \otimes^{\mathbf{L}}_R -$, and the right derived functor of the homomorphism functor of R–complexes is denoted by $\mathbf{R}\mathrm{Hom}_R(-,-)$ (no boundedness conditions are imposed on the arguments, due to the existence of appropriate resolutions, cf. [10], [21]). Thus, for arbitrary $M, N \in \mathbf{D}(R)$ there are complexes $M \otimes^{\mathbf{L}}_R N$ and $\mathbf{R}\mathrm{Hom}_R(M,N)$ which are defined uniquely up to isomorphism in $\mathbf{D}(R)$, and possess the expected functorial properties. As usual, we set

$$\mathrm{Tor}^R_n(M,N) = \mathrm{H}_n(M \otimes^{\mathbf{L}}_R N) \qquad \text{and} \qquad \mathrm{Ext}^n_R(M,N) = \mathrm{H}_{-n}(\mathbf{R}\mathrm{Hom}_R(M,N))$$

for $n \in \mathbb{Z}$. These are the classical notions when M and N are modules.

When $\varphi\colon R \to S$ is a ring homomorphism, M is an R–complex, and N is an S–complex, standard spectral sequence arguments, cf. e.g. [3; (4.7.I.Proof)], yield:

(1.2.1) If $M \in \mathbf{D}^{\mathrm{f}}_+(R)$ and $N \in \mathbf{D}^{\mathrm{f}}_+(S)$, then $M \otimes^{\mathbf{L}}_R N \in \mathbf{D}^{\mathrm{f}}_+(S)$.

(1.2.2) If $M \in \mathbf{D}^{\mathrm{f}}_+(R)$ and $N \in \mathbf{D}^{\mathrm{f}}_-(S)$, then $\mathbf{R}\mathrm{Hom}_R(M,N) \in \mathbf{D}^{\mathrm{f}}_-(S)$.

The following result is useful in locating quasi-isomorphisms.

(1.2.3) **Lemma.** *Let $D \in \mathbf{D}^{\mathrm{f}}_{\mathrm{b}}(R)$ have $\mathrm{H}(D_{\mathfrak{p}}) \neq 0$ for all $\mathfrak{p} \in \operatorname{Spec} R$, and let $M \in \mathbf{D}_{\mathrm{b}}(R)$.*

(a) *If $D \in \mathbf{D}^{\mathrm{f}}_0(R)$, then $\inf(D \otimes^{\mathbf{L}}_R M) = \inf M$ and $\sup \mathbf{R}\mathrm{Hom}_R(D,M) = \sup M$.*

(b) *If $\alpha\colon M \to N$ is a morphism in $\mathbf{D}_{\mathrm{b}}(R)$, such that $D \otimes^{\mathbf{L}}_R \alpha$ or $\mathbf{R}\mathrm{Hom}_R(D,\alpha)$ is an isomorphism, then α is an isomorphism.*

Proof. By [14; (2.2), (2.1.1)] there are inequalities

$$\sup M - \sup D \leq \sup \mathbf{R}\mathrm{Hom}_R(D,M) \leq \sup M - \inf D\,.$$

Let $\mathrm{E}_R(A)$ be the injective envelope of an R–module A, and set $I = \prod_{\mathfrak{m} \in \operatorname{Max}(R)} \mathrm{E}_R(R/\mathfrak{m})$. The module I is injective, with $\mathrm{Hom}_R(B,I) \neq 0$ for each non-zero R–module B, so we have an isomorphism $\mathrm{H}(\mathrm{Hom}_R(M,I)) \cong \mathrm{Hom}_R(\mathrm{H}(M),I)$, with equalities $\sup M = -\inf \mathrm{Hom}_R(M,I)$ and $\inf M = -\sup \mathrm{Hom}_R(M,I)$. The isomorphism $\mathrm{Hom}_R(D \otimes^{\mathbf{L}}_R M, I) \cong \mathbf{R}\mathrm{Hom}_R(D, \mathrm{Hom}_R(M,I))$ now translates the inequalities above into

$$\inf M + \inf D \leq \inf(D \otimes^{\mathbf{L}}_R M) \leq \inf M + \sup D\,.$$

The two sets of inequalities immediately imply (a).

A standard mapping cone argument reduces (b) to the claim that M is exact if $D \otimes^{\mathbf{L}}_R M$ or $\mathbf{R}\mathrm{Hom}_R(D,M)$ is. In the first case, $\infty \leq \inf M$ by the last inequality. In the second case, $\sup M \leq -\infty$ by the first inequality. Either result means that M is exact. $\square$

(1.3) *Homological dimensions.* In $\mathbf{D}(R)$ we consider the full subcategories $\mathbf{F}(R)$, $\mathbf{I}(R)$, and $\mathbf{P}(R)$, consisting of complexes isomorphic to bounded complexes of – respectively – flat, injective, or projective modules. We note that $\mathbf{P}^{\mathrm{f}}(R) = \mathbf{F}^{\mathrm{f}}(R)$, cf. [3; (2.10.F)], and that $\mathbf{P}(R) = \mathbf{F}(R)$ and $\mathbf{P}_0(R) = \mathbf{F}_0(R)$ when $\dim R < \infty$, cf. [15; (21.17)]. Furthermore, $\mathbf{F}_0(R)$, $\mathbf{I}_0(R)$, and $\mathbf{P}_0(R)$ are equivalent – respectively – to the (full) subcategories of modules of finite flat, injective, or projective dimension.

If $M \in \mathbf{D}_{\mathrm{b}}(R)$, $F \in \mathbf{F}(R)$, $I, I' \in \mathbf{I}(R)$, and $P \in \mathbf{P}(R)$, then

(a) $M \otimes^{\mathbf{L}}_R F \in \mathbf{D}_{\mathrm{b}}(R)$; (b) $\mathbf{R}\mathrm{Hom}_R(M, I) \in \mathbf{D}_{\mathrm{b}}(R)$; (c) $\mathbf{R}\mathrm{Hom}_R(P, M) \in \mathbf{D}_{\mathrm{b}}(R)$;

(d) $I \otimes^{\mathbf{L}}_R F \in \mathbf{I}(R)$; (e) $\mathbf{R}\mathrm{Hom}_R(I, I') \in \mathbf{F}(R)$.

(1.4) *Canonical morphisms.* Several canonical morphisms in $\mathbf{D}(R')$ are associated with a homomorphism of rings $R \to R'$, complexes $K, M \in \mathbf{D}(R)$, and $K', M', N' \in \mathbf{D}(R')$.

Without comment we use the *associativity* and *adjointness* isomorphisms

$$(K \otimes^{\mathbf{L}}_R M') \otimes^{\mathbf{L}}_{R'} N' \simeq K \otimes^{\mathbf{L}}_R (M' \otimes^{\mathbf{L}}_{R'} N');$$
$$\mathbf{R}\mathrm{Hom}_{R'}(K \otimes^{\mathbf{L}}_R M', N') \simeq \mathbf{R}\mathrm{Hom}_R(K, \mathbf{R}\mathrm{Hom}_{R'}(M', N')),$$

and their special cases

$$(K \otimes^{\mathbf{L}}_R R') \otimes^{\mathbf{L}}_{R'} N' \simeq K \otimes^{\mathbf{L}}_R N';$$
$$\mathbf{R}\mathrm{Hom}_{R'}(K \otimes^{\mathbf{L}}_R R', N') \simeq \mathbf{R}\mathrm{Hom}_R(K, N').$$

They easily yield an isomorphism

$$\mathbf{R}\mathrm{Hom}_{R'}(K' \otimes^{\mathbf{L}}_R M, N') \simeq \mathbf{R}\mathrm{Hom}_{R'}(K', \mathbf{R}\mathrm{Hom}_R(M, N')). \tag{1.4.1}$$

Although the *evaluation* morphisms

$$\omega_{KM'N'} : \mathbf{R}\mathrm{Hom}_R(K, M') \otimes^{\mathbf{L}}_{R'} N' \to \mathbf{R}\mathrm{Hom}_R(K, M' \otimes^{\mathbf{L}}_{R'} N');$$
$$\theta_{KM'N'} : K \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_{R'}(M', N') \to \mathbf{R}\mathrm{Hom}_{R'}(\mathbf{R}\mathrm{Hom}_R(K, M'), N'),$$

are not always isomorphisms, by [3; (4.4)] the following hold for $K \in \mathbf{D}^{\mathrm{f}}_+(R)$:

(1.4.2) $\omega_{KM'N'}$ is an isomorphism when $M' \in \mathbf{D}_-(R')$, and $N' \in \mathbf{F}(R')$ or $K \in \mathbf{P}(R)$.
(1.4.3) $\theta_{KM'N'}$ is an isomorphism when $M' \in \mathbf{D}_{\mathrm{b}}(R')$, and $N' \in \mathbf{I}(R')$ or $K \in \mathbf{P}(R)$.

We also systematically use the *biduality* morphism

$$\delta_{MK} : M \to \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(M, K), K)$$

and the *homothety* morphism

$$\chi_M : R \to \mathbf{R}\mathrm{Hom}_R(M, M).$$

(1.5) *Numerical and formal invariants.* Let R be a local ring with residue field k.

If $M \in \mathbf{D}^{\mathrm{f}}_{+}(R)$, then the *Betti number* $\beta^R_i(M) = \operatorname{rank}_k \operatorname{Tor}^R_i(M,k)$ is finite for all i and vanishes for $i \ll 0$ by (1.2.1). Thus, $\mathrm{P}^R_M(t) = \sum_{i\in\mathbb{Z}} \beta^R_i(M)t^i$ is a formal Laurent series, known as the *Poincaré series* of M. By Nakayama, the order of $\mathrm{P}^R_M(t)$ is equal to $\inf M$.

Similarly, when $M \in \mathbf{D}^{\mathrm{f}}_{-}(R)$ the *Bass number* $\mu^i_R(M) = \operatorname{rank}_k \operatorname{Ext}^i_R(k,M)$ is finite for all i and vanishes for $i \ll 0$ by (1.2.2). The formal Laurent series $\mathrm{I}^M_R(t) = \sum_{i\in\mathbb{Z}} \mu^i_R(M)t^i$ is known as the *Bass series* of M. We set $\mu^i_R = \mu^i_R(R)$ and $\mathrm{I}_R(t) = \mathrm{I}^R_R(t)$.

The formal invariants of M determine membership in $\mathbf{P}(M)$ or $\mathbf{I}(M)$, cf. e.g. [3; (5.5)]:

(1.5.1) $M \in \mathbf{D}^{\mathrm{f}}_{+}(R)$ is in $\mathbf{P}(M)$ if and only if $\mathrm{P}^R_M(t)$ is a Laurent polynomial. Furthermore, $M \sim R$ if and only if $\mathrm{P}^R_M(t) = t^d$ for some $d \in \mathbb{Z}$.

(1.5.2) $M \in \mathbf{D}^{\mathrm{f}}_{-}(R)$ is in $\mathbf{I}(M)$ if and only if $\mathrm{I}^M_R(t)$ is a Laurent polynomial.

A homomorphism $\varphi\colon R \to S$ is said to be *local* if the ring S is local, and the *closed fiber* $k \otimes_R S$ of φ is nontrivial. The next result generalizes [15; (11.21), (13.19)].

(1.5.3) **Lemma.** *Let $\varphi\colon R \to S$ be a local homomorphism $\varphi\colon R \to S$.*

(a) *If $M \in \mathbf{D}^{\mathrm{f}}_{+}(R)$, $N \in \mathbf{D}^{\mathrm{f}}_{+}(S)$ and $L = M \otimes^{\mathbf{L}}_R N$, then*

$$\mathrm{P}^S_L(t) = \mathrm{P}^R_M(t)\,\mathrm{P}^S_N(t)\ .$$

(b) *If $M \in \mathbf{D}^{\mathrm{f}}_{+}(R)$, $N \in \mathbf{D}^{\mathrm{f}}_{-}(S)$ and $L = \mathbf{R}\operatorname{Hom}_R(M,N)$, then*

$$\mathrm{I}^L_S(t) = \mathrm{P}^R_M(t)\,\mathrm{I}^N_S(t)\ .$$

Proof. Let ℓ denote the residue field of S. A first sequence of isomorphisms

$$(M \otimes^{\mathbf{L}}_R N) \otimes^{\mathbf{L}}_S \ell \simeq M \otimes^{\mathbf{L}}_R (N \otimes^{\mathbf{L}}_S \ell) \simeq (M \otimes^{\mathbf{L}}_R k) \otimes_k (N \otimes^{\mathbf{L}}_S \ell)$$

and the Künneth formula yield $\operatorname{Tor}^S_*(M \otimes^{\mathbf{L}}_R N, \ell) \cong \operatorname{Tor}^R_*(M,k) \otimes_k \operatorname{Tor}^S_*(N,\ell)$, which gives (a). Formula (1.4.1) starts a second sequence of isomorphisms

$$\begin{aligned}
\mathbf{R}\operatorname{Hom}_S(\ell, \mathbf{R}\operatorname{Hom}_R(M,N)) &\simeq \mathbf{R}\operatorname{Hom}_S(\ell \otimes^{\mathbf{L}}_R M, N)\\
&\simeq \mathbf{R}\operatorname{Hom}_S((M \otimes^{\mathbf{L}}_R k) \otimes_k \ell, N)\\
&\simeq \operatorname{Hom}_k(M \otimes^{\mathbf{L}}_R k, \mathbf{R}\operatorname{Hom}_S(\ell,N))\\
&\simeq \operatorname{Hom}_k(M \otimes^{\mathbf{L}}_R k, k) \otimes_k \mathbf{R}\operatorname{Hom}_S(\ell,N)\,,
\end{aligned}$$

hence $\operatorname{Ext}^*_S(\ell, \mathbf{R}\operatorname{Hom}_R(M,N)) \cong \operatorname{Hom}_k(\operatorname{Tor}^R_*(M,k),k) \otimes_k \operatorname{Ext}^*_S(\ell,N)$, producing (b). □

2. Dualizing complexes

In this section $(R, \mathfrak{m}, k)$ is a local ring with maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$.

The *Cohen–Macaulay defect* of R is the positive integer $\operatorname{cmd} R = \dim R - \operatorname{depth} R$. The first positive Bass number μ^i_R appears at $i = \operatorname{depth} R$; known as the *type* of R, it is denoted $\operatorname{type} R$. We write $\nu_R M$ for the minimal number of generators of a finite R–module M.

Definition. An R–complex D is said to be *dualizing for* R if the homothety morphism $\chi_D : R \to \mathbf{R}\operatorname{Hom}_R(D, D)$ is an isomorphism, and $D \in \mathbf{I}^{\mathrm{f}}(R)$.

We record some important properties of dualizing complexes [16], cf. also [17] and [15], with a dual purpose in mind. On the one hand, they appear in several proofs. On the other, they presage properties of the relative dualizing complexes of Section 5.

(2.1) *Example.* A ring R is Gorenstein if and only if the R–module R is a dualizing complex, cf. [16; (V.10.1)], [15; (15.5)].

(2.2) *Completion.* A complex $D \in \mathbf{D}(R)$ is dualizing for R if and only if $D \otimes_R \widehat{R}$ is dualizing for $\widehat{R}$, cf. [16; (V.3.5)], [15; (22.28)].

(2.3) *Existence.* A homomorphic image of a Gorenstein ring has a dualizing complex.
In particular, each complete ring has a dualizing complex, cf. [16; (V.10.4)], [15; (17.16)].

(2.4) *Uniqueness.* If $D, D' \in \mathbf{D}(R)$ are dualizing complexes for R, then $D \sim D'$, cf. [16; (V.3.1)], [15; (15.14)].

(2.5) *Size.* If D is a dualizing complex for R, then $\operatorname{amp} D = \operatorname{cmd} R$ and $\nu_R \operatorname{H}_i(D) = \operatorname{type} R$ for $i = \inf D$, cf. [15; (15.18), (15.23.a)].

We say a dualizing complex D for R is *normalized* if $\inf D = \operatorname{depth} R$ (this convention differs from the one in [4; p. 1031]); by (2.4) such a complex is unique up to isomorphism.

(2.6) *Formal invariants.* A complex $D \in \mathbf{D}^{\mathrm{f}}_{\mathrm{b}}(R)$ is dualizing for R if and only if $\operatorname{I}^D_R(t) = t^d$ for some $d \in \mathbb{Z}$, cf. [16; (V.3.4)], [15; (15.14)].
When D is normalized, $\operatorname{I}^D_R(t) = 1$ and $\operatorname{P}^R_D(t) = \operatorname{I}_R(t)$, cf. [15; (15.18.b), (15.23.a)].

(2.7) *Biduality.* If $D \in \mathbf{D}(R)$ is dualizing for R, then the biduality morphism

$$\delta_{MD} : M \to \mathbf{R}\operatorname{Hom}_R(\mathbf{R}\operatorname{Hom}_R(M, D), D)$$

is an isomorphism for each $M \in \mathbf{D}^{\mathrm{f}}_{+}(R)$, cf. [16; (V.2.1)], [15; (15.10)].

(2.8) *Localization.* If D is a dualizing complex for R, then for each $\mathfrak{p} \in \operatorname{Spec} R$ the complex $D_{\mathfrak{p}} \in \mathbf{D}(R_{\mathfrak{p}})$ is dualizing for $R_{\mathfrak{p}}$, cf. [16; (V.8.1)], [15; (15.17)].

Recall that the *fiber* of a homomorphism of rings $\varphi : R \to S$ at a prime ideal $\mathfrak{p}$ in R is the ring $k(\mathfrak{p}) \otimes_R S$, where $k(\mathfrak{p}) = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ is the residue field of the local ring $R_{\mathfrak{p}}$. The fibers of the $\mathfrak{m}$-adic completion map $R \to \widehat{R}$ are called the *formal fibers* of R.

(2.9) *Formal fibers.* A ring with a dualizing complex has Gorenstein formal fibers, cf. [16; (V.10.1)], [15; (22.26)].

The next property appears undocumented; in view of (5.1.a), it is a special case of (5.10).

(2.10) *Closed fiber.* If $\varphi\colon R \to S$ is a flat local homomorphism, R is Gorenstein, and E is a dualizing complex for S, then the complex $k \otimes^{\mathbf{L}}_R E$ is dualizing for $k \otimes_R S$.

A homomorphism $\varphi\colon R \to S$ has *finite flat dimension*, denoted $\operatorname{fd}\varphi < \infty$, if S admits a finite resolution by flat R–modules. A local homomorphism φ is *Gorenstein*, cf. [4], or more precisely, *Gorenstein at* $\mathfrak{n}$, if $\operatorname{fd}\varphi < \infty$ and $\mu_R^{i+\operatorname{depth} R} = \mu_S^{i+\operatorname{depth} S}$ for $i \in \mathbb{Z}$. By [4; (4.2)], cf. also (8.3) below, a flat homomorphism is Gorenstein if and only if the ring $S/\mathfrak{m}S$ is Gorenstein. The relevance of this notion in the present context comes from the next result, proved in [4; (5.1)].

(2.11) *Base change.* Let $\varphi\colon R \to S$ be a local homomorphism. The following conditions on $D \in \mathbf{D}^{\mathrm{f}}_{-}(R)$ are equivalent.

(i) D is dualizing for R, and φ is Gorenstein at $\mathfrak{n}$.
(ii) $D \otimes^{\mathbf{L}}_R S$ is dualizing for S, and φ has finite flat dimension.

(2.12) *Finite ascent.* Let $\varphi\colon R \to S$ be a finite homomorphism of local rings. If D is a dualizing complex for R then $\mathbf{R}\operatorname{Hom}_R(S, D)$ is one for S, cf. [16; (V.10.2)], [15; (15.31)].

3. Dualizing equivalences

In this section R denotes a local ring with dualizing complex D.

We exhibit an equivalence of the categories $\mathbf{F}(R)$ and $\mathbf{I}(R)$, which is provided by the restriction of endofunctors of the entire derived category $\mathbf{D}(R)$. The existence of such an equivalence leads to natural extensions of both subcategories considered above.

(3.1) *Auslander categories.* Let $\mathbf{A}(R)$ denote the full subcategory of $\mathbf{D}_{\mathrm{b}}(R)$, consisting of those complexes M for which $D \otimes^{\mathbf{L}}_R M \in \mathbf{D}_{\mathrm{b}}(R)$ and the canonical morphism

$$\gamma_M\colon M \to \mathbf{R}\operatorname{Hom}_R(D, D \otimes^{\mathbf{L}}_R M)\,,$$

induced by $m \mapsto (d \mapsto (-1)^{|m||d|} d \otimes m)$, is an isomorphism.

Similarly, let $\mathbf{B}(R)$ denote the full subcategory of $\mathbf{D}_{\mathrm{b}}(R)$, consisting of those complexes M for which $\mathbf{R}\operatorname{Hom}_R(D, M) \in \mathbf{D}_{\mathrm{b}}(R)$ and the canonical morphism

$$\iota_M\colon D \otimes^{\mathbf{L}}_R \mathbf{R}\operatorname{Hom}_R(D, M) \to M\,,$$

induced by $d \otimes \alpha \mapsto (-1)^{|d||\alpha|}\alpha(d)$, is an isomorphism.

(3.2) **Theorem.** *If D is a dualizing complex for R, then there is a commutative diagram*

$$\begin{array}{ccc}
\mathbf{D}(R) & \underset{\mathbf{R}\operatorname{Hom}_R(D,-)}{\overset{D\otimes^{\mathbf{L}}_R -}{\rightleftarrows}} & \mathbf{D}(R) \\
\bigcup\vert & & \bigcup\vert \\
\mathbf{A}(R) & \rightleftarrows & \mathbf{B}(R) \\
\bigcup\vert & & \bigcup\vert \\
\mathbf{F}(R) & \rightleftarrows & \mathbf{I}(R)
\end{array}$$

in which the vertical inclusions are full embeddings, and the unlabeled horizontal arrows are quasi-inverse equivalences of categories.

Furthermore, for $C, E \in \mathbf{D}_{\mathrm{b}}(R)$ the following hold:

(a) $D \otimes^{\mathbf{L}}_R C \in \mathbf{B}(R)$ *implies* $C \in \mathbf{A}(R)$; (b) $\mathbf{R}\mathrm{Hom}_R(D, E) \in \mathbf{A}(R)$ *implies* $E \in \mathbf{B}(R)$;

(f) $D \otimes^{\mathbf{L}}_R C \in \mathbf{I}(R)$ *implies* $C \in \mathbf{F}(R)$; (i) $\mathbf{R}\mathrm{Hom}_R(D, E) \in \mathbf{F}(R)$ *implies* $E \in \mathbf{I}(R)$.

Remark. If R is Gorenstein, then it is clear that $\mathbf{A}(R) = \mathbf{B}(R) = \mathbf{D}_{\mathrm{b}}(R)$, and (3.2) contains the well known fact that then also $\mathbf{F}(R) = \mathbf{I}(R)$.

Proof. If $C \in \mathbf{F}(R)$, then $D \otimes^{\mathbf{L}}_R C \in \mathbf{D}_{\mathrm{b}}(R)$ by (1.3.a), and ω_{DDC} is a isomorphism by (1.4.2). The commutative diagram

$$\begin{array}{ccc} C & \xrightarrow{\gamma_C} & \mathbf{R}\mathrm{Hom}_R(D, D \otimes^{\mathbf{L}}_R C) \\ \downarrow\simeq & & \simeq\uparrow{\scriptstyle \omega_{DDC}} \\ R \otimes^{\mathbf{L}}_R C & \xrightarrow[\chi_D \otimes^{\mathbf{L}}_R C]{\simeq} & \mathbf{R}\mathrm{Hom}_R(D, D) \otimes^{\mathbf{L}}_R C \end{array}$$

shows that γ_C is an isomorphism. Similarly, if $E \in \mathbf{I}(R)$, then $\mathbf{R}\mathrm{Hom}_R(D, E) \in \mathbf{D}_{\mathrm{b}}(R)$ by (1.3.b), and θ_{DDE} is an isomorphism by (1.4.3), and from the commutative diagram

$$\begin{array}{ccc} D \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(D, E) & \xrightarrow{\iota_E} & E \\ {\scriptstyle \theta_{DDE}}\downarrow\simeq & & \simeq\uparrow \\ \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(D, D), E) & \xrightarrow[\mathbf{R}\mathrm{Hom}_R(\chi_D, E)]{\simeq} & \mathbf{R}\mathrm{Hom}_R(R, E) \end{array}$$

we see that ι_E is an isomorphism. We have established the embeddings of categories.

For the remainder of the proof, C and E denote complexes in $\mathbf{D}_{\mathrm{b}}(R)$.

Set $F = D \otimes^{\mathbf{L}}_R C$, and consider the morphisms $\iota_F \colon D \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(D, F) \to F$ and $\gamma_C : C \to \mathbf{R}\mathrm{Hom}_R(D, F)$. The induced morphism

$$F = D \otimes^{\mathbf{L}}_R C \xrightarrow{D \otimes^{\mathbf{L}}_R \gamma_C} D \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(D, F)$$

satisfies $\iota_F(D \otimes^{\mathbf{L}}_R \gamma_C) = 1_F$, hence we see that

(∗) ι_F is an isomorphism if and only if $D \otimes^{\mathbf{L}}_R \gamma_C$ is one.

Starting with $C \in \mathbf{A}(R)$, the definition of $\mathbf{A}(R)$ shows that C, F, and $\mathbf{R}\mathrm{Hom}_R(D, F)$ are in $\mathbf{D}_{\mathrm{b}}(R)$, and that $D \otimes^{\mathbf{L}}_R \gamma_C$ is an isomorphism. Thus, ι_F is an isomorphism, hence $F \in \mathbf{B}(R)$. We have shown that $D \otimes^{\mathbf{L}}_R -$ restricts to a functor $\mathbf{A}(R) \to \mathbf{B}(R)$.

Set $B = \mathbf{R}\mathrm{Hom}_R(D, E)$, and consider the morphisms $\gamma_B : B \to \mathbf{R}\mathrm{Hom}_R(D, D \otimes^{\mathbf{L}}_R B)$ and $\iota_E \colon D \otimes^{\mathbf{L}}_R B \to E$. The induced morphism

$$\mathbf{R}\mathrm{Hom}_R(D, D \otimes^{\mathbf{L}}_R B) \xrightarrow{\mathbf{R}\mathrm{Hom}_R(D, \iota_E)} \mathbf{R}\mathrm{Hom}_R(D, E) = B$$

satisfies $\mathbf{R}\mathrm{Hom}_R(D,\iota_E)\,\gamma_B = 1_B$, hence

$(**)$ γ_B is an isomorphism if and only if $\mathbf{R}\mathrm{Hom}_R(D,\iota_E)$ is one.

Arguing as above, we see that $E \in \mathbf{B}(R)$ implies $B \in \mathbf{A}(R)$, so that $\mathbf{R}\mathrm{Hom}_R(D,-)$ restrict to a functor $\mathbf{B}(R) \to \mathbf{A}(R)$. The fact that those functors between $\mathbf{A}(R)$ and $\mathbf{B}(R)$ are quasi-inverse equivalences is built in the definitions of these categories.

If $C \in \mathbf{F}(R)$, then $D \otimes^{\mathbf{L}}_R C \in \mathbf{I}(R)$ by (1.3.d); if $E \in \mathbf{I}(R)$, then $\mathbf{R}\mathrm{Hom}_R(D,E) \in \mathbf{F}(R)$ by (1.3.e). Thus, $D \otimes^{\mathbf{L}}_R -$ and $\mathbf{R}\mathrm{Hom}_R(D,-)$ restrict to functors between $\mathbf{F}(R)$ and $\mathbf{I}(R)$. They are quasi-inverse because this is true for their extensions to $\mathbf{A}(R)$ and $\mathbf{B}(R)$.

It remains to establish the last four assertions.

If $F = D \otimes^{\mathbf{L}}_R C$ is in $\mathbf{B}(R)$, then the complexes F and $\mathbf{R}\mathrm{Hom}_R(D,F)$ are homologically bounded, and ι_F is an isomorphism, hence by $(*)$ so is $D \otimes^{\mathbf{L}}_R \gamma_C$. As D satisfies the assumption of (1.2.3.b), cf. (2.8), we conclude that γ_C is an isomorphism, hence C is in $\mathbf{A}(R)$. This proves (a). The argument for (b) is similar, using $(**)$.

If $D \otimes^{\mathbf{L}}_R C$ is in $\mathbf{I}(R)$, then $C \in \mathbf{A}(R)$ in view of the commutative diagram and (a). This yields $C \simeq \mathbf{R}\mathrm{Hom}_R(D, D \otimes^{\mathbf{L}}_R C) \in \mathbf{F}(R)$, as desired. The proof of (i) is similar. □

(3.3) **Lemma.** *The following conditions are equivalent.*

(i) *The ring R is Cohen–Macaulay.*
(ii) $\operatorname{amp}(D \otimes^{\mathbf{L}}_R M) = \operatorname{amp} M$ *for all* $M \in \mathbf{A}(R)$.
(iii) $\operatorname{amp} \mathbf{R}\mathrm{Hom}_R(D,M) = \operatorname{amp} M$ *for all* $M \in \mathbf{B}(R)$.
(iv) $\operatorname{amp} D = 0$.

Proof. (i) $\iff$ (iv) is clear (and contained in (2.5)).
(ii) $\implies$ (iv): set $M = R$.
(iii) $\implies$ (iv): set $M = D$.
(iv) $\implies$ (ii). For $M \in \mathbf{A}(R)$ we have

$$\sup M = \sup \mathbf{R}\mathrm{Hom}_R(D, D \otimes^{\mathbf{L}}_R M) = \sup(D \otimes^{\mathbf{L}}_R M)$$

with second equality coming from (1.2.3.a), which also provides $\inf M = \inf(D \otimes^{\mathbf{L}}_R M)$.
(iv) $\implies$ (iii) is similar. □

When R is Cohen–Macaulay, the single non-zero homology module of a dualizing complex D, defined uniquely up to isomorphism, is known as the *canonical module* of R.

(3.4) **Proposition.** *Let R be a Cohen–Macaulay ring with canonical module D.*

An R–module M is in $\mathbf{A}_0(R)$ if and only if $\operatorname{Tor}^R_i(D,M) = 0 = \operatorname{Ext}^i_R(D, D \otimes_R M)$ for $i > 0$, and the canonical map $M \to \operatorname{Hom}(D, D \otimes_R M)$ is bijective.

An R–module M is in $\mathbf{B}_0(R)$ if and only if $\operatorname{Ext}^i_R(D,M) = 0 = \operatorname{Tor}^R_i(D, \operatorname{Hom}_R(D,M))$ for $i > 0$, and the canonical map $D \otimes_R \operatorname{Hom}_R(D,M) \to M$ is bijective.

Proof. We prove the first assertion, and leave the second one to the reader.

Note that the morphism $\alpha\colon \mathbf{R}\mathrm{Hom}_R(D, D \otimes_R M) \to \operatorname{Hom}_R(D, D \otimes_R M)$ is an isomorphism if and only if $\operatorname{Ext}^i_R(D, D \otimes_R M) = 0$ for $i > 0$ and that the morphism

$\beta\colon D\otimes^{\mathbf{L}}_R M\to D\otimes_R M$ is an isomorphism if and only if $\operatorname{Tor}^R_i(D,M)=0$ for $i>0$. Furthermore, $\mathrm{H}(\alpha)\,\mathrm{H}(\mathbf{R}\operatorname{Hom}_R(D,\beta))\,\mathrm{H}(\gamma_M)$ is the canonical map $\gamma'\colon M\to\operatorname{Hom}(D,D\otimes_R M)$.

Assume $M\in\mathbf{A}_0(R)$, so that γ_M is an isomorphism. By (3.3) the Tor's vanish, hence β is an isomorphism. Thus, $\mathbf{R}\operatorname{Hom}_R(D,\beta)\gamma_M\colon M\to\mathbf{R}\operatorname{Hom}_R(D,D\otimes_R M)$ is an isomorphism. It follows that the Ext's vanish, implying α is an isomorphism, hence γ' is bijective.

Conversely, the vanishing of Ext's implies that α is an isomorphism. The vanishing of Tor's implies that $D\otimes^{\mathbf{L}}_R M$ is bounded, and that β is an isomorphism. Now from the bijectivity of γ' we get that of $\mathrm{H}(\gamma_M)$ is an isomorphism, hence so is γ_M. □

(3.5) *Remark.* It follows from the first part of the proposition and [11; (2.4), (3.3), (3.4)] that a module M over a Cohen–Macaulay ring R is in $\mathbf{A}_0(R)$ if and only if there exists an exact sequence $0\to G_n\to\cdots\to G_0\to M\to 0$, such that each module G_i is *Gorenstein flat* in the sense of Enochs et al. Similarly, the second part of the proposition and [11; (2.5)] imply that M is in $\mathbf{B}(R)$ if and only if it there exists an exact sequence $0\to M\to J_0\to\cdots\to J_n\to 0$, such that each J_i is *Gorenstein injective*.

Combining (3.2) and (3.4), we obtain the following

(3.6) **Corollary.** *When R is Cohen–Macaulay with canonical module D there are commutative diagrams of categories of R–modules*

$$\begin{array}{ccc}
\mathbf{D}_0(R) & \underset{\operatorname{Hom}_R(D,-)}{\overset{D\otimes_R-}{\rightleftarrows}} & \mathbf{D}_0(R)\\
\bigcup| & & \bigcup|\\
\mathbf{A}_0(R) & \rightleftarrows & \mathbf{B}_0(R)\\
\bigcup| & & \bigcup|\\
\mathbf{F}_0(R) & \rightleftarrows & \mathbf{I}_0(R)
\end{array}
\qquad\text{and}\qquad
\begin{array}{ccc}
\mathbf{D}^{\mathrm{f}}_0(R) & \underset{\operatorname{Hom}_R(D,-)}{\overset{D\otimes_R-}{\rightleftarrows}} & \mathbf{D}^{\mathrm{f}}_0(R)\\
\bigcup| & & \bigcup|\\
\mathbf{A}^{\mathrm{f}}_0(R) & \rightleftarrows & \mathbf{B}^{\mathrm{f}}_0(R)\\
\bigcup| & & \bigcup|\\
\mathbf{F}^{\mathrm{f}}_0(R) & \rightleftarrows & \mathbf{I}^{\mathrm{f}}_0(R)
\end{array}$$

in which the vertical inclusions are full embeddings, and the unlabeled horizontal arrows are quasi-inverse equivalences of categories. □

Remark. In view of (3.4), the corollary can be obtained as a direct consequence of [13; (1.4), (2.1)]. Furthermore, *loc. cit.* establishes an analog of the last part of (3.2).

The equivalence above between $\mathbf{F}^{\mathrm{f}}_0(R)$ and $\mathbf{I}^{\mathrm{f}}_0(R)$ is due to Sharp, [19; (2.9)].

Finally, we record some stability properties of $\mathbf{A}(R)$.

(3.7) **Proposition.** *Let $\varphi\colon R\to S$ be a local homomorphism and let $N\in\mathbf{D}(S)$.*

(a) *$N\in\mathbf{A}(R)$ if and only if $N\otimes_S\widehat{S}\in\mathbf{A}(\widehat{R})$.*

(b) *If φ is Gorenstein, then $N\in\mathbf{A}(R)$ if and only $N\in\mathbf{A}(S)$.*

(c) *If $S\to S'$ is a local flat extension, then $N\in\mathbf{A}(R)$ if and only if $N\otimes_S S'\in\mathbf{A}(R)$.*

(d) *If $\mathfrak{q}\in\operatorname{Spec}S$ and $\mathfrak{p}=\mathfrak{q}\cap R$, then $N\in\mathbf{A}(R)$ implies $N_{\mathfrak{q}}\in\mathbf{A}(R_{\mathfrak{p}})$.*

Proof. When φ is Gorenstein, $E=D\otimes^{\mathbf{L}}_R S$ is a dualizing complex for S by (2.11), hence (b) results from the canonical isomorphisms

$$D\otimes^{\mathbf{L}}_R N\simeq E\otimes^{\mathbf{L}}_S N\,,\qquad \mathbf{R}\operatorname{Hom}_R(D,D\otimes^{\mathbf{L}}_R N)\simeq\mathbf{R}\operatorname{Hom}_S(E,E\otimes^{\mathbf{L}}_S N)\,.$$

(c) follows by faithful flatness from the canonical isomorphisms

$$\mathrm{H}(D \otimes^{\mathbf{L}}_R N) \otimes_S S' \cong \mathrm{H}(D \otimes^{\mathbf{L}}_R N \otimes_S S')\,,$$
$$\mathbf{R}\mathrm{Hom}_R(D, D \otimes^{\mathbf{L}}_R N) \otimes_S S' \simeq \mathbf{R}\mathrm{Hom}_R(D, D \otimes^{\mathbf{L}}_R N \otimes_S S')\,.$$

By (c), N is in $\mathbf{A}(R)$ if and only if $N \otimes_S \widehat{S}$ is in $\mathbf{A}(R)$, and by (b) the last condition is equivalent to $N \otimes_S \widehat{S}$ being in $\mathbf{A}(\widehat{R})$. This establishes (a).

As $D_{\mathfrak{p}}$ is a dualizing complex for $R_{\mathfrak{p}}$ by (2.8), the canonical isomorphisms

$$D_{\mathfrak{p}} \otimes^{\mathbf{L}}_{R_{\mathfrak{p}}} N_{\mathfrak{q}} \simeq (D \otimes^{\mathbf{L}}_R N)_{\mathfrak{q}}\,,$$
$$\mathbf{R}\mathrm{Hom}_{R_{\mathfrak{p}}}(D_{\mathfrak{p}}, D_{\mathfrak{p}} \otimes^{\mathbf{L}}_{R_{\mathfrak{p}}} N_{\mathfrak{q}}) \simeq \mathbf{R}\mathrm{Hom}_R(D, D \otimes^{\mathbf{L}}_R N)_{\mathfrak{q}}$$

yield (d). □

4. Gorenstein dimension

In this section R denotes a local ring.

The dimension discussed below is introduced for modules by Auslander [1], [2].

(4.1) *Gorenstein dimension.* A finite R–module M has G-dimension 0 if $\mathrm{Ext}^i_R(M,R) = 0 = \mathrm{Ext}^i_R(\mathrm{Hom}_R(M,R),R)$ for $i > 0$, and the canonical map $M \to \mathrm{Hom}_R(\mathrm{Hom}_R(M,R),R)$ is bijective. It is of *G-dimension at most* n if there exists an exact sequence

$$0 \to G_n \to G_{n-1} \to \ldots \to G_1 \to G_0 \to M \to 0$$

in which G_j has G-dimension 0 for $0 \le j \le n$; in this case we write $\text{G-dim}_R M \le n$.

For each finite R–module M the following hold:

(4.1.1) $\text{G-dim}_R M \le \mathrm{pd}_R M$, with equality if $\mathrm{pd}_R M < \infty$, cf. [2; (3.14)].

(4.1.2) If $\text{G-dim}_R M < \infty$, then $\text{G-dim}_R M = \operatorname{depth} R - \operatorname{depth}_R M$, cf. [2; (4.13.b)].

(4.1.3) If $\text{G-dim}_R M < \infty$, then $\text{G-dim}_R M = -\inf \mathbf{R}\mathrm{Hom}_R(M,R)$, cf. [2; (4.13.a)].

(4.1.4) If $R \to S$ is a flat local homomorphism, then $\text{G-dim}_S(M \otimes_R S) = \text{G-dim}_R M$.

(4.1.5) $\text{G-dim}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \le \text{G-dim}_R M$ for each $\mathfrak{p} \in \operatorname{Spec} R$, cf. [2; (4.15)].

(4.1.6) R is Gorenstein if and only if $\text{G-dim}_R M < \infty$ for all finite M, cf. [2; (4.20)].

A result of Foxby, cf. [24; (2.7)], provides a connection with Auslander categories:

(4.1.7) If R has a dualizing complex, then $\text{G-dim}_R M < \infty$ if and only if $M \in \mathbf{A}(R)$.

(4.2) *Factorizations.* A *factorization* of a local homomorphism $\varphi\colon (R,\mathfrak{m}) \to (S,\mathfrak{n})$ is a commutative triangle of local homomorphisms

$$\begin{array}{ccc} & R' & \\ {\scriptstyle\dot\varphi}\nearrow & & \searrow{\scriptstyle\varphi'} \\ R & \xrightarrow[\varphi]{} & S\,. \end{array}$$

with $\dot\varphi$ flat and φ' surjective. A factorization is said to be *regular*, respectively, *Gorenstein*, if the local ring $R'/\mathfrak{m}R'$ has the corresponding property. It is clear that

(4.2.1) φ has a regular factorization if is essentially of finite type.

A *Cohen factorization* is a regular factorization with a complete local ring R'. It often exists, by a relative form [8; (1.1)] of Cohen's Structure Theorem for complete local rings:

(4.2.2) For each local homomorphism φ, the composition $\dot\varphi\colon R \to \widehat{S}$ of φ and the $\mathfrak{n}$-adic completion map $S \to \widehat{S}$ has a Cohen factorization.

By [7; (2.7)] or [8; (3.3)], Cohen and regular factorizations reflect the finiteness of $\operatorname{fd}\varphi$:

(4.2.3) If $\operatorname{fd}\varphi$ is finite, then $\operatorname{pd}_{R'}\widehat{S}$ is finite in each regular factorization $R \to R' \to \widehat{S}$ of $\dot\varphi$; conversely, if $\operatorname{pd}_{R'}\widehat{S}$ is finite in some Cohen factorization of $\dot\varphi$, then $\operatorname{fd}\varphi$ is finite.

The next theorem establishes a corresponding result for G-dimension.

(4.3) **Theorem.** *For a local homomorphism $\varphi\colon R \to S$ the following are equivalent.*

(i) $\operatorname{G-dim}_{R'}\widehat{S}$ *is finite for some Cohen factorization $R \to R' \to \widehat{S}$ of $\dot\varphi$.*
(ii) $\operatorname{G-dim}_{R'}\widehat{S}$ *is finite for some Gorenstein factorization $R \to R' \to \widehat{S}$ of $\dot\varphi$.*
(iii) $\operatorname{G-dim}_{R'}\widehat{S}$ *is finite for each Gorenstein factorization $R \to R' \to \widehat{S}$ of $\dot\varphi$.*
(iv) $\widehat{S}$ *belongs to* $\mathbf{A}(\widehat{R})$.

When φ has a Gorenstein factorization $R \to R'' \to S$, they are also equivalent to

(ii') $\operatorname{G-dim}_{R''}S$ *is finite.*

When R has a dualizing complex, they are also equivalent to

(iv') S *belongs to* $\mathbf{A}(R)$.

Proof. The implications (iii) $\implies$ (i) $\implies$ (ii) are clear.

By (4.1.4), conditions (ii) and (iii) do not change if we replace $R \to R' \to \widehat{S}$ with the Gorenstein factorization $R \to \widehat{R'} \to \widehat{S}$. Once this is done, we see from (3.7.b) that (iv) is equivalent to the condition $\widehat{S} \in \mathbf{A}(\widehat{R'})$. By (4.1.7) this inclusion is equivalent to the finiteness of $\operatorname{G-dim}_{R'}\widehat{S}$. It follows that (ii) $\implies$ (iv) $\implies$ (iii).

If $R \to R'' \to S$ is a Gorenstein factorization of φ, then $R \to \widehat{R''} \to \widehat{S}$ is one of $\dot\varphi$, hence (ii) $\iff$ (ii') by (4.1.4).

If R has a dualizing complex, then (iv) $\iff$ (iv') by (3.7.a). □

Definition. A local homomorphism $\varphi\colon R \to S$ which satisfies the equivalent conditions of the theorem is said to have *finite Gorenstein dimension*, denoted $\operatorname{G-dim}\varphi < \infty$.

Some basic properties of such homomorphisms follow easily from (4.1) and (4.3).

(4.4.1) *Gorenstein source.* If R is Gorenstein, then so is the ring R' in any Cohen factorization of $\dot\varphi$, hence by (4.1.6) each local homomorphism $\varphi\colon R \to S$ has $\operatorname{G-dim}\varphi < \infty$.

(4.4.2) *Finite flat dimension.* If $\operatorname{fd}\varphi < \infty$, then $\operatorname{G-dim}\varphi < \infty$ by (4.2.3) and (4.1.1).

(4.4.3) *Completion.* $\operatorname{G-dim}\varphi < \infty$ if and only if $\operatorname{G-dim}\dot\varphi < \infty$, if and only if $\operatorname{G-dim}\widehat{\varphi} < \infty$.

(4.4.4) *Finite homomorphism.* When φ is finite, $\operatorname{G-dim}\varphi<\infty$ if and only if $\operatorname{G-dim}_R S<\infty$. Indeed, by (4.1.7) the condition $\widehat{S}\in\mathbf{A}(\widehat{R})$ is equivalent to the finiteness of $\operatorname{G-dim}_{\widehat{R}}\widehat{S}$. By (4.1.4) the latter is tantamount to the finiteness of $\operatorname{G-dim}_R S$.

The property of φ to have finite flat dimension localizes: If $\operatorname{fd}\varphi<\infty$, then for each prime ideal $\mathfrak{q}$ in S the induced local homomorphism $\varphi_{\mathfrak{q}}\colon R_{\mathfrak{q}\cap R}\to S_{\mathfrak{q}}$ satisfies $\operatorname{fd}\varphi_{\mathfrak{q}}<\infty$. For the Gorenstein dimension we have the following partial results.

(4.5) **Proposition.** *If $\varphi\colon R\to S$ is a local homomorphism with $\operatorname{G-dim}\varphi<\infty$, then $\operatorname{G-dim}\varphi_{\mathfrak{q}}<\infty$ for each $\mathfrak{q}\in\operatorname{Spec}S$ under each one of the following conditions:*

(1) *φ is essentially of finite type; or*
(2) *R has Gorenstein formal fibers.*

Proof. Fix $\mathfrak{q}$, and set $\mathfrak{p}=\mathfrak{q}\cap R$.

Under condition (1), φ has a regular factorization $R\to R'\to S$ in which R' is the localization of a ring of polynomials over R, at a prime ideal lying above $\mathfrak{m}$. If $\mathfrak{p}'=\mathfrak{q}\cap R'$, then the sequence $R_{\mathfrak{p}}\to R'_{\mathfrak{p}'}\to S_{\mathfrak{q}}$ is a regular factorization of $\varphi_{\mathfrak{q}}$, and we have $\operatorname{G-dim}_{R'_{\mathfrak{p}'}}S_{\mathfrak{q}}\le\operatorname{G-dim}_{R'}S$ by (4.1.5). Thus, $\operatorname{G-dim}\varphi_{\mathfrak{q}}$ is finite by (4.3).

By the same result, we see that under condition (2) it suffices to show that $\widehat{S}\in\mathbf{A}(\widehat{R})$ implies $(S_{\mathfrak{q}})\widehat{\ }\in\mathbf{A}((R_{\mathfrak{p}})\widehat{\ })$. Choose a prime ideal $\mathfrak{q}^*$ lying over $\mathfrak{q}$ and set $\mathfrak{p}^*=\mathfrak{q}^*\cap\widehat{R}$. By (3.7.d) and (3.7.a) we have $(\widehat{S}_{\mathfrak{q}^*})\widehat{\ }\in\mathbf{A}((\widehat{R}_{\mathfrak{p}^*})\widehat{\ })$. By hypothesis, the flat local homomorphism $R_{\mathfrak{p}}\to\widehat{R}_{\mathfrak{p}^*}$ has a Gorenstein closed fiber, and hence so does its completion $(R_{\mathfrak{p}})\widehat{\ }\to(\widehat{R}_{\mathfrak{p}^*})\widehat{\ }$. Thus, from (3.7.b) we see that $(\widehat{S}_{\mathfrak{q}^*})\widehat{\ }\in\mathbf{A}((R_{\mathfrak{p}})\widehat{\ })$. The desired assertion now follows from (3.7.c), due to the identification of $(\widehat{S}_{\mathfrak{q}^*})\widehat{\ }$ with $(S_{\mathfrak{q}})\widehat{\ }\otimes_{(S_{\mathfrak{q}})\widehat{\ }}(\widehat{S}_{\mathfrak{q}^*})\widehat{\ }$. □

Next we look at the behavior of Auslander categories under descent.

(4.6) **Proposition.** *Let R be a local ring with a dualizing complex, and let $\varphi\colon R\to S$ be a local homomorphism.*

(a) *If $\operatorname{G-dim}\varphi$ is finite then $\mathbf{F}(S)\subseteq\mathbf{A}(R)$ and $\mathbf{I}(S)\subseteq\mathbf{B}(R)$.*
(b) *If $\operatorname{fd}\varphi$ is finite then $\mathbf{F}(S)\subseteq\mathbf{F}(R)$ and $\mathbf{I}(S)\subseteq\mathbf{I}(R)$.*

Proof. Let $D\in\mathbf{I}^{\mathrm{f}}(R)$ be dualizing for R.

Consider first a complex $M\in\mathbf{F}(S)$. When $\operatorname{fd}\varphi$ is finite, the canonical isomorphism $-\otimes^{\mathbf{L}}_R M\simeq(-\otimes^{\mathbf{L}}_R S)\otimes^{\mathbf{L}}_S M$ and (1.3.a) yield $M\in\mathbf{F}(R)$.

When $\operatorname{G-dim}\varphi$ is finite, by (4.3) we have $S\in\mathbf{A}(R)$. On the one hand, this implies that $D\otimes^{\mathbf{L}}_R S\in\mathbf{D}_{\mathrm{b}}(S)$, hence $\omega_{D(D\otimes^{\mathbf{L}}_R S)M}$ is an isomorphism by (1.4.2), and that $\operatorname{amp}(D\otimes^{\mathbf{L}}_R M)=\operatorname{amp}((D\otimes^{\mathbf{L}}_R S)\otimes^{\mathbf{L}}_S M)$ is finite. On the other hand, it implies $\gamma_S\colon S\to\mathbf{R}\operatorname{Hom}_R(D,D\otimes^{\mathbf{L}}_R S)$ is an isomorphism, hence $\gamma_S\otimes^{\mathbf{L}}_R M$ is one. Now the commutative diagram

$$\begin{array}{ccc} S\otimes^{\mathbf{L}}_S M=M & \xrightarrow{\gamma_M} & \mathbf{R}\operatorname{Hom}_R(D,D\otimes^{\mathbf{L}}_R M) \\ {\scriptstyle\gamma_S\otimes^{\mathbf{L}}_R M}\downarrow\simeq & & \simeq\downarrow \\ \mathbf{R}\operatorname{Hom}_R(D,D\otimes^{\mathbf{L}}_R S)\otimes^{\mathbf{L}}_S M & \xrightarrow[\omega_{D(D\otimes^{\mathbf{L}}_R S)M}]{\simeq} & \mathbf{R}\operatorname{Hom}_R(D,(D\otimes^{\mathbf{L}}_R S)\otimes^{\mathbf{L}}_S M) \end{array}$$

shows that γ_M is an isomorphism. We have proved that M is in $\mathbf{A}(R)$.

The proofs for a complex $M \in \mathbf{I}(S)$ are similar, using (1.3.b) and (1.4.3). □

(4.7) **Proposition.** *Let $\psi\colon Q \to R$ and $\varphi\colon R \to S$ be local homomorphisms, with $\operatorname{fd}\varphi$ finite. If $\text{G-dim}\,\psi < \infty$, then $\text{G-dim}\,\varphi\psi < \infty$. The converse holds when φ is flat.*

Proof. In view of (4.4.3) the assumption on $\text{G-dim}\,\psi$ is invariant under completion; that on $\operatorname{fd}\varphi$ has the same property, cf. e.g. [8; (3.3)]. Thus, we may assume Q has a dualizing complex. The previous proposition then shows that $S \in \mathbf{F}(R) \subseteq \mathbf{A}(Q)$, and hence $\text{G-dim}\,\varphi\psi$ is finite by (4.3). The converse follows from (3.7.c), in view of of (4.3). □

(4.8) *Remark.* We do not know whether the composition of local homomorphisms of finite G-dimension has the same property. The construction of [6; (4.4)] shows that this is equivalent to the validity of the following property of G-dimension: If $Q \to R \to S$ are surjective homomorphisms of local rings, such that $\text{G-dim}_Q R$ and $\text{G-dim}_R S$ are both finite, then $\text{G-dim}_Q S$ is finite.

Extending this to finite modules, we raise the

Question. Let $Q \to R$ be a finite homomorphism of local rings, and let M be a finite R–module. If $\text{G-dim}_Q R$ and $\text{G-dim}_R M$, are finite, is then $\text{G-dim}_Q M$ finite?

The argument for (4.7) easily generalizes to yield a positive answer when $\operatorname{pd}_R N$ is finite, and another case is established in (7.11), but the general case appears to be open.

5. Relative dualizing complexes: Properties

In this section $\varphi\colon (R,\mathfrak{m}) \to (S,\mathfrak{n})$ denotes a local homomorphism.

When M is a finite R–module, we write $\operatorname{codim}_R M$ for the height of its annihilator ideal, and $\operatorname{grade}_R M$ for the maximal length of an R–regular sequence contained in this ideal. The *dimension*, *depth*, *Cohen–Macaulay defect*, and *type* of φ are defined from a Cohen factorization $R \to R' \to \widehat{S}$ of $\grave{\varphi}$, cf. (4.2), as follows:

$$\begin{aligned}
\dim\varphi &= \dim R' - \dim R - \operatorname{codim}_{R'}\widehat{S}\,;\\
\operatorname{depth}\varphi &= \operatorname{depth} S - \operatorname{depth} R\,;\\
\operatorname{cmd}\varphi &= \dim\varphi - \operatorname{depth}\varphi\,;\\
\operatorname{type}\varphi &= \nu_R\widehat{S}\,|\operatorname{Ext}^d_{R'}(\widehat{S},R')\big) \quad \text{for} \quad d = \operatorname{depth} R' - \operatorname{depth} S\,.
\end{aligned}$$

It is proved in [8; (2.1)] that the right hand side of the first (and hence, the third) formula is independent of the factorization; the corresponding result for the last formula is established in [6; (7.1)]. Unlike the prototype invariants for rings, the dimension, depth, or Cohen–Macaulay defect of a homomorphism may be negative, and its type may be zero.

Definition. An S–complex C is *dualizing for* φ if the homothety $\chi_C\colon S \to \mathbf{R}\operatorname{Hom}_S(C,C)$ is an isomorphism, $C \in \mathbf{D}^{\mathrm{f}}_{\mathrm{b}}(S)$, and $D' \otimes^{\mathbf{L}}_{\widehat{R}} (C \otimes_S \widehat{S}) \in \mathbf{I}(\widehat{S})$ for a dualizing complex D' for $\widehat{R}$.

By applying (a) below to the local structure homomorphism $\mathbb{Z}_{(p)} \to S$, where $p = \operatorname{char} S/\mathfrak{n}$, one sees that the properties which follow specialize to the correspondingly numbered in Section 2 properties of absolute dualizing complexes:

(5.1) **Examples.** (a) If R is Gorenstein, then $C \in \mathbf{D}(S)$ is dualizing for φ if and only if it is dualizing for S: this follows from (2.1).

(b) The identity homomorphism 1_R has a dualizing complex, namely R.

(c) If $\rho\colon R \to R^*$ is the completion map for the adic topology defined by a proper ideal $\mathfrak{a}$ in R, then R^* is a dualizing complex for ρ: as $\widehat{\rho} = 1_{\widehat{R}}$, this follows from (b) and (5.2).

(5.2) **Completion.** *Let C be an S–complex.*

(a) *C is dualizing for φ if and only if $C \otimes_S \widehat{S} \in \mathbf{D}(\widehat{S})$ is dualizing for $\grave{\varphi}$ and/or $\widehat{\varphi}$.*

(b) *If R has a dualizing complex D, then C is dualizing for φ if and only if χ_C is an isomorphism, $C \in \mathbf{D}^{\mathrm{f}}_{\mathrm{b}}(S)$, and $D \otimes^{\mathbf{L}}_R C \in \mathbf{I}(S)$.*

Proof. (a) By faithful flatness, χ_C is an isomorphism if and only if $\chi_{C\otimes_S \widehat{S}}$ is one, and $C \in \mathbf{D}^{\mathrm{f}}_{\mathrm{b}}(S)$ if and only if $C \otimes_S \widehat{S} \in \mathbf{D}^{\mathrm{f}}_{\mathrm{b}}(\widehat{S})$.

(b) By (2.2) $D' = D \otimes_R \widehat{R}$ is dualizing or $\widehat{R}$. Since $D' \otimes^{\mathbf{L}}_{\widehat{R}} (C \otimes_S \widehat{S}) \simeq (D \otimes^{\mathbf{L}}_R C) \otimes_S \widehat{S}$ as $\widehat{S}$–complexes, $D' \otimes^{\mathbf{L}}_{\widehat{R}} (C \otimes_S \widehat{S}) \in \mathbf{I}(\widehat{S})$ if and only if $D \otimes^{\mathbf{L}}_R C \in \mathbf{I}(S)$ by [3; (5.5.I)]. □

The proofs of the following theorems are collected at the end of Section 6.

Hypothesis for Theorems (5.3) **through** (5.9): $\operatorname{G-dim}\varphi$ *is finite.*

(5.3) **Existence.** *If both R and S have dualizing complexes, or if φ has a Gorenstein factorization, then φ has a dualizing complex.*

In particular, each homomorphism to a complete ring has a dualizing complex.

Dualizing complexes are actually constructed in (6.1.b), (6.5), and (6.7) below.

(5.4) **Uniqueness.** *If $C, C' \in \mathbf{D}(S)$ are dualizing for φ, then $C \sim C'$.*

(5.5) **Size.** *Let C be a dualizing complex for φ.*

If $R \to R' \to \widehat{S}$ is a Gorenstein factorization for $\grave{\varphi}$, then

$$\operatorname{amp} C = \operatorname{cmd}\varphi + \operatorname{codim}_{R'} \widehat{S} - \operatorname{grade}_{R'} \widehat{S} \geq \operatorname{cmd}\varphi$$

with equality if $\operatorname{fd}\varphi < \infty$ or if R is Cohen–Macaulay; in the latter case, $\operatorname{amp} C = \operatorname{cmd} S$.

For $i = \inf C$ there are equalities $\nu_S \operatorname{H}_i(C) = \operatorname{type}\varphi = \dfrac{\operatorname{type} S}{\operatorname{type} R}$.

We do not know if $\operatorname{cmd}\varphi \geq 0$ when $\operatorname{G-dim}\varphi$ is finite, unless $\operatorname{fd}\varphi < \infty$, cf. [8; (3.6)].

We say a dualizing complex C for φ is *normalized* if $\inf C = \operatorname{depth} S - \operatorname{depth} R$; by (5.4) such a complex is unique up to isomorphism. (When R is Gorenstein, C is also dualizing for S by (5.1.a), but as such is not normalized unless R is artinian.)

(5.6) **Formal invariants.** *If C is a normalized dualizing complex for φ, then*

$$\operatorname{P}^S_C(t) = \frac{\operatorname{I}_S(t)}{\operatorname{I}_R(t)} \qquad \textit{and} \qquad \operatorname{I}^C_S(t) = \operatorname{I}_R(t)\ .$$

(5.7) **Biduality.** *If $C \in \mathbf{D}(S)$ is dualizing for φ, then the biduality morphism $\delta_{NC}\colon N \to \mathbf{R}\mathrm{Hom}_S(\mathbf{R}\mathrm{Hom}_S(N,C),C)$ is an isomorphism for each $N \in \mathbf{D}^{\mathrm{f}}_{\mathrm{b}}(S)$ with $N \otimes_S \widehat{S} \in \mathbf{A}(\widehat{R})$.*

If R has a dualizing complex, then δ_{NC} is an isomorphism for $N \in \mathbf{D}^{\mathrm{f}}_{\mathrm{b}}(S) \cap \mathbf{A}(R)$.

Relative dualizing complexes do not always localize nicely. Indeed, Ferrand and Raynaud [12; (4.2.i)] have constructed a one-dimensional local domain R, such that $\widehat{R}$ has a minimal prime $\mathfrak{q}$ with $\widehat{R}_{\mathfrak{q}}$ non-Gorenstein. From (5.1.c) we see that $C = \widehat{R}$ is a dualizing complex for $\rho\colon R \to \widehat{R}$. On the other hand, as $R_{(0)}$ is a field and the ring $R_{\mathfrak{q}}$ is not Gorenstein, (5.1.a) shows that $C_{\mathfrak{q}} = \widehat{R}_{\mathfrak{q}}$ is not a dualizing complex for $\rho_{\mathfrak{q}}\colon R_{(0)} \to \widehat{R}_{\mathfrak{q}}$.

Thus, the failure of the formal fibers of R to be Gorenstein is an obstruction for the dualizing complex of φ to localize properly. The following result proves it is the only one.

(5.8) **Localization.** *If $C \in \mathbf{D}(S)$ is a dualizing complex for φ, then for each $\mathfrak{q} \in \operatorname{Spec} S$ the complex $C_{\mathfrak{q}} \in \mathbf{D}(S_{\mathfrak{q}})$ is dualizing for $\varphi_{\mathfrak{q}}$, under any one of the following conditions:*

(1) *φ is essentially of finite type; or*
(2) *R has Gorenstein formal fibers.*

(5.9) **Formal fibers.** *If R has Gorenstein formal fibers and φ has a dualizing complex, then S has Gorenstein formal fibers.*

(5.10) **Closed fiber.** *If φ is flat and $C \in \mathbf{D}(S)$ is a (normalized) dualizing complex for φ, then the complex $k \otimes^{\mathbf{L}}_R C$ is (normalized) dualizing for $k \otimes_R S$.*

Theorems (5.11) through (5.13) involve a second local homomorphism, $\psi\colon Q \to R$.

(5.11) **Base change.** *When $\operatorname{G-dim} \psi < \infty$ and $A \in \mathbf{D}^{\mathrm{f}}_{\mathrm{b}}(R)$ the following are equivalent.*

(i) *A is dualizing for ψ, and φ is Gorenstein at $\mathfrak{n}$.*
(ii) *$A \otimes^{\mathbf{L}}_R S$ is dualizing for $\varphi\psi$, and φ has finite flat dimension.*

(5.12) **Finite ascent.** *If $\operatorname{G-dim} \psi < \infty$, $\operatorname{G-dim} \varphi\psi < \infty$, the homomorphism φ is finite, and the complex $A \in \mathbf{D}(R)$ is dualizing for ψ, then $\mathbf{R}\mathrm{Hom}_R(S,A)$ is dualizing for $\varphi\psi$.*

(5.13) **Composition.** *If $\operatorname{G-dim} \psi < \infty$, $\operatorname{fd} \varphi < \infty$, $A \in \mathbf{D}(R)$ is dualizing for ψ, and $C \in \mathbf{D}(S)$ is dualizing for φ, then $A \otimes^{\mathbf{L}}_R C \in \mathbf{D}(S)$ is dualizing for $\varphi\psi$.*

6. Relative dualizing complexes: Proofs

In this section $\varphi\colon (R,\mathfrak{m}) \to (S,\mathfrak{n})$ denotes a local homomorphism.

The following result is pivotal for most of the arguments to follow.

(6.1) **Theorem.** *Assume $\operatorname{G-dim} \varphi < \infty$ and D is a (normalized) dualizing complex for R.*

(a) *$C \in \mathbf{D}(S)$ is (normalized) dualizing for φ if and only if $D \otimes^{\mathbf{L}}_R C$ is (normalized) dualizing for S.*

(b) *$E \in \mathbf{D}(S)$ is dualizing for S if and only if $\mathbf{R}\mathrm{Hom}_R(D,E)$ is dualizing for φ.*

We precede its proof by

(6.2) **Lemma.** *Assume that R has a dualizing complex, and let C and E be S–complexes. When* $\operatorname{G-dim}\varphi<\infty$*, respectively,* $\operatorname{fd}\varphi<\infty$*, the following hold.*

(a) *If C is dualizing for φ, then $C\in\mathbf{A}(R)$, respectively, $C\in\mathbf{F}(R)$.*

(b) *If E is dualizing for S, then $E\in\mathbf{B}(R)$, respectively, $E\in\mathbf{I}(R)$.*

Proof. Let D be a dualizing complex for R.

(a) By definition, $D\otimes^{\mathbf{L}}_R C$ is in $\mathbf{I}(S)$, hence is in $\mathbf{B}(R)$ by (4.6.a), respectively, in $\mathbf{I}(R)$ by (4.6.b). Now $C\in\mathbf{A}(R)$, respectively, $C\in\mathbf{F}(R)$, from (3.2.g), respectively, (3.2.f).

(b) follows directly from (4.6.a), respectively, from (4.6.b). □

Proof of Theorem (6.1). (a') Assume that C is dualizing for φ. The complex $E=D\otimes^{\mathbf{L}}_R C$ is in $\mathbf{I}(S)$ by (5.2.b), and in $\mathbf{D}^{\mathrm{f}}(S)$ by (1.2.1). On the other hand, $C\in\mathbf{A}(R)$ by (6.2.a), hence $\gamma_C\colon C\to\mathbf{R}\mathrm{Hom}_R(D,E)$ is an isomorphism. Thus, the commutative diagram

$$\begin{array}{ccc} S & \xrightarrow{\chi_E} & \mathbf{R}\mathrm{Hom}_S(E,E) \\ {\scriptstyle\chi_C}\big\downarrow{\scriptstyle\simeq} & & {\scriptstyle\simeq}\big\downarrow \\ \mathbf{R}\mathrm{Hom}_S(C,C) & \xrightarrow[\mathbf{R}\mathrm{Hom}_S(C,\gamma_C)]{\simeq} & \mathbf{R}\mathrm{Hom}_S(C,\mathbf{R}\mathrm{Hom}_R(D,E))\,, \end{array}$$

in which the unnamed isomorphism is as in (1.4.1), implies that χ_E is an isomorphism, so that E is dualizing for S.

(b') Assume that E is dualizing for S, set $C=\mathbf{R}\mathrm{Hom}_R(D,E)$, and note that $C\in\mathbf{D}^{\mathrm{f}}_{\mathrm{b}}(S)$ by (1.2.2) and (1.3.b). On the other hand, (6.2.b) yields $E\in\mathbf{B}(S)$, hence $\iota_E\colon D\otimes^{\mathbf{L}}_R C\to E$ is an isomorphism, and thus $D\otimes^{\mathbf{L}}_R C\in\mathbf{I}(S)$. Finally, the commutative diagram

$$\begin{array}{ccc} S & \xrightarrow{\chi_C} & \mathbf{R}\mathrm{Hom}_S(C,C) \\ {\scriptstyle\chi_E}\big\downarrow{\scriptstyle\simeq} & & {\scriptstyle\simeq}\big\downarrow \\ \mathbf{R}\mathrm{Hom}_R(E,E) & \xrightarrow[\mathbf{R}\mathrm{Hom}_C(\iota_E,E)]{\simeq} & \mathbf{R}\mathrm{Hom}_S(D\otimes^{\mathbf{L}}_R C,E) \end{array}$$

shows that χ_C is an isomorphism, hence C is dualizing for φ.

(a") Assume that $E=D\otimes^{\mathbf{L}}_R C$ is a dualizing complex for S. By applying consecutively (6.2.b) and (3.2.g), we see that C belongs to $\mathbf{A}(R)$. It follows that C is isomorphic to $\mathbf{R}\mathrm{Hom}_R(D,E)$, which is a dualizing complex for φ by the part of (b) established in (b').

(b") If $C=\mathbf{R}\mathrm{Hom}_R(D,E)$ is dualizing for φ, then (6.2.a) and (3.2.j) yield $E\in\mathbf{B}(R)$, hence $D\otimes^{\mathbf{L}}_R C\simeq E$. By the established part of (a), E is dualizing for S.

Finally, to get (a) in the form which involves normalizations, it suffices to remark that $\inf(D\otimes^{\mathbf{L}}_R C)=\inf D+\inf C=\operatorname{depth}R+\inf C$. □

(6.3) **Lemma.** *If* $\operatorname{G-dim}\varphi<\infty$*, the R–complex D is dualizing for R, the S–complex E is dualizing for S, and the S–complex C is dualizing for φ, then $C\sim\mathbf{R}\mathrm{Hom}_R(D,E)$.*

Proof: It suffices to prove the relations $C\simeq\mathbf{R}\mathrm{Hom}_R(D,D\otimes^{\mathbf{L}}_R C)\sim\mathbf{R}\mathrm{Hom}_R(D,E)$. The first one is due to the inclusion $C\in\mathbf{A}(R)$, established in (6.2.a). The second one is induced by the relation $D\otimes^{\mathbf{L}}_R C\sim E$, which expresses the uniqueness (2.4) of dualizing complexes for S, since $D\otimes^{\mathbf{L}}_R C$ is one by (6.1.a). □

(6.4) **Lemma.** *A complex $C \in \mathbf{D}(S)$ is dualizing for φ if and only if the complex $C \otimes_S \widehat{S} \in \mathbf{D}(\widehat{S})$ is dualizing for $\grave{\varphi}$ and/or for $\widehat{\varphi}$.*

Proof. By the faithful flatness of $\widehat{S}$ over S, the isomorphism $\mathrm{H}(C \otimes_S \widehat{S}) \cong \mathrm{H}(C) \otimes_S \widehat{S}$ shows $C \otimes_S \widehat{S} \in \mathbf{D}^{\mathrm{f}}_{\mathrm{b}}(\widehat{S})$ if and only if $C \in \mathbf{D}^{\mathrm{f}}_{\mathrm{b}}(S)$. When this holds, $\chi_{C \otimes_S} \widehat{S}$ is an isomorphism together with $\chi_C \otimes_S \widehat{S}$ due to the commutative square

$$\begin{array}{ccc} \widehat{S} & \xrightarrow{\chi_C \otimes_S \widehat{S}} & \mathbf{R}\mathrm{Hom}_S(C,C) \otimes_S \widehat{S} \\ \| & & \simeq \downarrow \\ \widehat{S} & \xrightarrow[\chi_{C \otimes_S \widehat{S}}]{} & \mathbf{R}\mathrm{Hom}_{\widehat{S}}(C \otimes_S \widehat{S}, C \otimes_S \widehat{S}) \,. \end{array}$$

By faithful flatness, the latter morphism is an isomorphism together with χ_C. □

(6.5) **Lemma.** *If φ is finite and $\operatorname{G-dim} \varphi < \infty$, then $\mathbf{R}\mathrm{Hom}_R(S,R)$ is dualizing for φ.*

Proof. By the preceding lemma, we may assume that R and S are complete, and then choose a dualizing complex D for R. Note the canonical isomorphisms

$$\mathbf{R}\mathrm{Hom}_R(S,R) \simeq \mathbf{R}\mathrm{Hom}_R(S, \mathbf{R}\mathrm{Hom}_R(D,D)) \simeq \mathbf{R}\mathrm{Hom}_R(D, \mathbf{R}\mathrm{Hom}_R(S,D)) \,.$$

As $\mathbf{R}\mathrm{Hom}_R(S,D)$ is a dualizing complex for S by (2.12), conclude by (6.1.b). □

(6.6) **Lemma.** *If $\psi\colon Q \to R$ is a local homomorphism which is Gorenstein at $\mathfrak{m}$, and $\operatorname{G-dim} \varphi < \infty$, then $C \in \mathbf{D}^{\mathrm{f}}_{\mathrm{b}}(S)$ is dualizing for φ if and only if it is dualizing for $\varphi\psi$.*

Proof. As above, we may assume that Q has a dualizing complex, B. By (2.11) the complex $D = B \otimes^{\mathbf{L}}_Q R$ is dualizing for R. Using (3.7.b) and (4.3), we see that $\operatorname{G-dim} \varphi\psi$ is finite. In view of the isomorphism $D \otimes^{\mathbf{L}}_R C \simeq B \otimes^{\mathbf{L}}_Q C$, the assertion follows from (6.1.a). □

Combining the last two lemmas, we get:

(6.7) **Lemma.** *If $R \to R' \to S$ is a Gorenstein factorization of φ and $\operatorname{G-dim} \varphi < \infty$, then $\mathbf{R}\mathrm{Hom}_{R'}(S,R')$ is a dualizing complex for φ.* □

The proofs of the results of Section 5 follow a logical – not numerical – order.

Proof of Theorem (5.3): apply (6.1.b) and (6.7). □

Proof of Theorem (5.7). By (6.4) the complex $C \otimes_S \widehat{S}$ is dualizing for $\widehat{\varphi}$.

The commutative triangle

$$\begin{array}{ccc} & \mathbf{R}\mathrm{Hom}_S(\mathbf{R}\mathrm{Hom}_S(N,C),C) \otimes_S \widehat{S} & \\ {\scriptstyle \delta_{NC} \otimes_S \widehat{S}} \nearrow & & \searrow \simeq \\ N \otimes_S \widehat{S} & \xrightarrow[\delta_{(N \otimes_S \widehat{S})\,(C \otimes_S \widehat{S})}]{} & \mathbf{R}\mathrm{Hom}_{\widehat{S}}(\mathbf{R}\mathrm{Hom}_{\widehat{S}}(N \otimes_S \widehat{S}, C \otimes_S \widehat{S}), C \otimes_S \widehat{S}) \end{array}$$

and faithful flatness show that we may assume R and S are complete.

So let D and E be dualizing complexes for R and S, respectively. By (6.3) we may further assume that $C = \mathbf{R}\mathrm{Hom}_R(D, E)$. Set $K = D \otimes^{\mathbf{L}}_R S$ and $L = D \otimes^{\mathbf{L}}_R N$. The canonical isomorphisms $\mathbf{R}\mathrm{Hom}_R(D, L) \xrightarrow{\simeq} \mathbf{R}\mathrm{Hom}_S(K, L)$, $\mathbf{R}\mathrm{Hom}_S(L, E) \xrightarrow{\simeq} \mathbf{R}\mathrm{Hom}_S(N, C)$, and $C \xrightarrow{\simeq} \mathbf{R}\mathrm{Hom}_S(K, E)$ induce a commutative diagram

$$\begin{array}{ccc}
N & \xrightarrow{\delta_{NC}} & \mathbf{R}\mathrm{Hom}_S(\mathbf{R}\mathrm{Hom}_S(N, C), C) \\
{\scriptstyle\gamma_N}\downarrow{\scriptstyle\simeq} & & {\scriptstyle\simeq}\downarrow \\
\mathbf{R}\mathrm{Hom}_R(D, L) & & \mathbf{R}\mathrm{Hom}_S(\mathbf{R}\mathrm{Hom}_S(L, E), C) \\
\downarrow{\scriptstyle\simeq} & & {\scriptstyle\simeq}\downarrow \\
\mathbf{R}\mathrm{Hom}_S(K, L) & & \mathbf{R}\mathrm{Hom}_S(\mathbf{R}\mathrm{Hom}_S(L, E), \mathbf{R}\mathrm{Hom}_S(K, E)) \\
{\scriptstyle\mathbf{R}\mathrm{Hom}_S(K,\delta_{LE})}\downarrow{\scriptstyle\simeq} & & {\scriptstyle\simeq}\downarrow \\
\mathbf{R}\mathrm{Hom}_S(K, \mathbf{R}\mathrm{Hom}_S(\mathbf{R}\mathrm{Hom}_S(L, E), E)) & \xrightarrow{\simeq} & \mathbf{R}\mathrm{Hom}_S(K \otimes^{\mathbf{L}}_S \mathbf{R}\mathrm{Hom}_S(L, E), E)\,.
\end{array}$$

Thus, δ_{NC} is an isomorphism.

The last assertion of the theorem follows from (3.7.a). □

Proof of Theorem (5.4). Since $C' \otimes_S \widehat{S}$ and $C \otimes_S \widehat{S}$ are both dualizing for $\widehat{\varphi}\colon \widehat{R} \to \widehat{S}$, we get $C' \otimes_S \widehat{S} \sim C \otimes_S \widehat{S}$ by (6.3). It follows that

$$\mathbf{R}\mathrm{Hom}_S(C', C) \otimes_S \widehat{S} \simeq \mathbf{R}\mathrm{Hom}_{\widehat{S}}(C' \otimes_S \widehat{S}, C \otimes_S \widehat{S}) \sim \mathbf{R}\mathrm{Hom}_{\widehat{S}}(C \otimes_S \widehat{S}, C \otimes_S \widehat{S}) \simeq \widehat{S}\,.$$

Faithful flatness shows that $\mathrm{H}(\mathbf{R}\mathrm{Hom}_S(C', C))$ is concentrated in a single degree, where it is isomorphic to S. Thus, $\mathbf{R}\mathrm{Hom}_S(C', C) \sim S$. It remains to note that in the sequence

$$C' \xrightarrow{\delta_{C'C}} \mathbf{R}\mathrm{Hom}_S(\mathbf{R}\mathrm{Hom}_S(C', C), C) \sim \mathbf{R}\mathrm{Hom}_S(S, C) \simeq C\,.$$

the morphism $\delta_{C'C}$ is an isomorphism by (5.7). □

Proof of Theorem (5.8.1). We have $\operatorname{G-dim} \varphi_{\mathfrak{q}} < \infty$ by (4.5.1). Let $R \to R' \to S$ be a regular factorization of φ as in the proof of that proposition. By (6.7) the complex $\mathbf{R}\mathrm{Hom}_{R'}(S, R')$ is dualizing for φ, hence $C \sim \mathbf{R}\mathrm{Hom}_{R'}(S, R')$ by (5.4). The last relation localizes to $C_{\mathfrak{q}} \sim \mathbf{R}\mathrm{Hom}_{R'_{\mathfrak{p}'}}(S_{\mathfrak{q}}, R'_{\mathfrak{p}'})$, where $\mathfrak{p}' = \mathfrak{q} \cap R'$. Since $R_{\mathfrak{p}} \to R'_{\mathfrak{p}'} \to S_{\mathfrak{q}}$ is a regular factorization of $\varphi_{\mathfrak{q}}$, by (6.7) again we conclude that $C_{\mathfrak{q}}$ is dualizing for $\varphi_{\mathfrak{q}}$. □

Proof of Theorem (5.13). By (4.4.2) and (4.7), all three homomorphisms ψ, φ, and $\varphi\psi$ have finite G-dimension. We may assume that Q has a dualizing complex B. By (6.2.a) we have $C \in \mathbf{F}(R)$, hence $A \otimes^{\mathbf{L}}_R C \in \mathbf{D}^{\mathrm{f}}_{\mathrm{b}}(S)$ by (1.2.1) and (1.3.a). On the other hand, by (6.1.a) $B \otimes^{\mathbf{L}}_Q A$ is dualizing for R. The isomorphism $(B \otimes^{\mathbf{L}}_Q A) \otimes^{\mathbf{L}}_R C \simeq B \otimes^{\mathbf{L}}_Q (A \otimes^{\mathbf{L}}_R C)$ and another application of (6.1.a) show that $A \otimes^{\mathbf{L}}_R C$ is dualizing for $\varphi\psi$. □

Proof of Theorem (5.11). We may assume all rings are complete, and choose dualizing complexes B for Q and D for R.

(i) $\implies$ (ii). The complex $D \otimes^{\mathbf{L}}_R S$ is dualizing for S by (2.11). Furthermore, as $\operatorname{fd} \varphi$ is finite, we have $S \simeq \mathbf{R}\operatorname{Hom}_R(D, D \otimes^{\mathbf{L}}_R S)$ by (3.2), hence S is a dualizing complex for φ by (6.1.b). Now (5.13), shows that $A \otimes^{\mathbf{L}}_R S$ is dualizing for $\varphi\psi$.

(ii) $\implies$ (i). By (6.1.a) the complex $B \otimes^{\mathbf{L}}_Q (A \otimes^{\mathbf{L}}_R S)$ is dualizing for S. Since it is isomorphic to $(B \otimes^{\mathbf{L}}_Q A) \otimes^{\mathbf{L}}_R S$, we see from (2.11) that φ is Gorenstein at $\mathfrak{n}$ and $B \otimes^{\mathbf{L}}_Q A$ is dualizing for R. Applying once more (6.1.a), we conclude that A is dualizing for ψ. □

Proof of Theorem (5.8.2) *and Theorem* (5.9). Fix $\mathfrak{q} \in \operatorname{Spec} S$, set $\mathfrak{p} = \mathfrak{q} \cap R$, and note that the induced homomorphism $\varphi_\mathfrak{q} \colon R_\mathfrak{p} \to S_\mathfrak{q}$ has finite G-dimension by (4.5.2).

In the special case when R has a dualizing complex D, the S–complex $E = D \otimes^{\mathbf{L}}_R C$ is dualizing for S by (6.1.a), and $C \sim \mathbf{R}\operatorname{Hom}_R(D, E)$ by (6.3). It follows that $C_\mathfrak{q} \sim \mathbf{R}\operatorname{Hom}_{R_\mathfrak{p}}(D_\mathfrak{p}, E_\mathfrak{q})$. By (2.8), $D_\mathfrak{p}$ and $E_\mathfrak{q}$ are dualizing for $R_\mathfrak{p}$ and $S_\mathfrak{q}$, respectively. As $\text{G-dim}\, \varphi_\mathfrak{q} < \infty$, we conclude by (6.1.b) that $C_\mathfrak{q}$ is dualizing for $\varphi_\mathfrak{q}$.

In general, we remark that $C \otimes_S \widehat{S}$ is a dualizing complex for $\widehat{\varphi}$ by (6.4), choose a prime ideal $\mathfrak{q}^* \in \operatorname{Spec} \widehat{S}$ lying over $\mathfrak{q}$, and set $\mathfrak{p}^* = \mathfrak{q}^* \cap \widehat{R}$. By the special case, the complex $(C \otimes_S \widehat{S})_{\mathfrak{q}^*}$ is dualizing for $\widehat{\varphi}_{\mathfrak{q}^*} \colon \widehat{R}_{\mathfrak{p}^*} \to \widehat{S}_{\mathfrak{q}^*}$. At this point, we have a commutative square

$$\begin{array}{ccc} R_\mathfrak{p} & \xrightarrow{\varphi_\mathfrak{q}} & S_\mathfrak{q} \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle\sigma} \\ \widehat{R}_{\mathfrak{p}^*} & \xrightarrow[\widehat{\varphi}_{\mathfrak{q}^*}]{} & \widehat{S}_{\mathfrak{q}^*} \end{array}$$

of local homomorphisms, in which σ is flat, ρ is flat with Gorenstein closed fiber, $\text{G-dim}\, \varphi_\mathfrak{q}$ is finite (observed above), and $\text{G-dim}\, \widehat{\varphi}_{\mathfrak{q}^*}$ is finite (from (4.4.3) and (4.5.2)).

The $\widehat{S}_{\mathfrak{q}^*}$–complex $(C \otimes_S \widehat{S})_{\mathfrak{q}^*}$ is dualizing for $\widehat{\varphi}_{\mathfrak{q}^*}\rho$ by (6.6). As it is isomorphic to $C_\mathfrak{p} \otimes_{S_\mathfrak{q}} \widehat{S}_{\mathfrak{q}^*}$, and $\widehat{\varphi}_{\mathfrak{q}^*}\rho = \sigma\varphi_\mathfrak{q}$, we see that the last complex is dualizing for $\sigma\varphi_\mathfrak{q}$. Since σ is flat, (5.11) applies and shows that $C_\mathfrak{q}$ is dualizing for $\varphi_\mathfrak{q}$, and the ring $\widehat{S}_{\mathfrak{q}^*}/\mathfrak{q}\widehat{S}_{\mathfrak{q}^*}$ is Gorenstein. It is a localization of the formal fiber $k(\mathfrak{q}) \otimes_S \widehat{S}$. In choosing $\mathfrak{q}^*$ we had the freedom to pick it as the contraction of any prime ideal of $k(\mathfrak{q}) \otimes_S \widehat{S}$, so we conclude that this formal fiber is Gorenstein. □

Proof of Theorem (5.6). We may assume that $\varphi = \widehat{\varphi}$, and choose a normalized dualizing complex D for R. By (6.1.a) the complex $E = D \otimes^{\mathbf{L}}_R C$ is normalized dualizing for S. By (1.5.3.a) we have $\mathrm{P}^S_E(t) = \mathrm{P}^R_D(t)\,\mathrm{P}^S_C(t)$, and (2.6) translates the preceding equality into $\mathrm{I}_S(t) = \mathrm{I}_R(t)\,\mathrm{P}^S_C(t)$, as desired. Furthermore, as $C \in \mathbf{A}(R)$ by (6.2.a), we have $C \simeq \mathbf{R}\operatorname{Hom}_R(D, E)$. Thus, (1.5.3.b) and (2.6) yield $\mathrm{I}^C_S(t) = \mathrm{P}^R_D(t)\,\mathrm{I}^E_S(t) = \mathrm{I}_R(t)$. □

Proof of Theorem (5.5). We may assume that S and R are complete, and let D be a dualizing complex for R.

We start with a Gorenstein factorization $R \to R' \to S$ of φ. By (6.7), the complex $\mathbf{R}\operatorname{Hom}_{R'}(S, R')$ is dualizing for φ. By the uniqueness (5.4) of such complexes, we may assume $C = \mathbf{R}\operatorname{Hom}_{R'}(S, R')$. Note that $\inf C = \operatorname{depth} S - \operatorname{depth} R'$ by (4.1.3) and (4.1.2).

As $\sup C = -\operatorname{grade}_{R'} S$ by the homological characterization of grade, we get

$$\operatorname{amp} C = \operatorname{depth} R' - \operatorname{depth} S - \operatorname{grade}_{R'} S\,.$$

On the other hand, since $R \to R'$ is flat with Gorenstein closed fiber, we have $\dim R' - \dim R + \operatorname{depth} R = \operatorname{depth} R'$, hence

$$\begin{aligned}\operatorname{cmd}\varphi &= (\dim R' - \dim R - \operatorname{codim}_{R'} \widehat{S}) - (\operatorname{depth} S - \operatorname{depth} R)\\ &= \operatorname{depth} R' - \operatorname{depth} S - \operatorname{codim}_{R'} \widehat{S}\,.\end{aligned}$$

Thus, we obtain $\operatorname{amp} C - \operatorname{cmd}\varphi = \operatorname{codim}_{R'} S - \operatorname{grade}_{R'} S \geq 0$, as desired.

If R is Cohen–Macaulay, then so is the ring R', hence $\operatorname{codim}_{R'} S = \operatorname{grade}_{R'} S$, so $\operatorname{amp} C = \operatorname{cmd}\varphi$. By (6.1.a) the complex $D \otimes^{\mathbf{L}}_R C$ is dualizing for S, hence $\operatorname{amp} C = \operatorname{amp}(D \otimes^{\mathbf{L}}_R C) = \operatorname{cmd} S$ by (3.3) and (2.5).

From now on we assume that $R \to R' \to S$ is a Cohen factorization, cf. (4.2.2). If $\operatorname{fd}\varphi$ is finite, then by (4.2.3) the projective dimension of the R'–module S is finite, hence $\operatorname{codim}_{R'} S = \operatorname{grade}_{R'} S$ by [6; (2.5)], and so $\operatorname{amp} C = \operatorname{cmd}\varphi$.

As noted above, $i = \inf C$ is equal to $\operatorname{depth} S - \operatorname{depth} R'$, so that $\operatorname{H}_i(C) = \operatorname{Ext}^{-i}_{R'}(S, R')$. By definition, $\operatorname{type}\varphi$ is equal to the minimal number of generators of the last S–module, hence $\nu_S \operatorname{H}_i(C) = \operatorname{type}\varphi$. On the other hand, the equality $\nu_S \operatorname{H}_i(C) = \operatorname{type} S / \operatorname{type} R$ follows immediately from the expression for $\operatorname{P}^S_C(t)$ obtained in (5.6). □

Proof of Theorem (5.10). We assume that the rings are complete, cf. (6.4), and choose dualizing complexes D and E for R and S. As $\mathbf{R}\operatorname{Hom}_R(k, D) \sim k$ by (2.6), we have

$$\begin{aligned}k \otimes^{\mathbf{L}}_R \mathbf{R}\operatorname{Hom}_R(D, E) &\simeq \mathbf{R}\operatorname{Hom}_R(\mathbf{R}\operatorname{Hom}_R(k, D), E)\\ &\sim \mathbf{R}\operatorname{Hom}_R(k, E)\\ &\simeq \mathbf{R}\operatorname{Hom}_S(k \otimes_R S, E)\,,\end{aligned}$$

where the first isomorphism is due to (1.4.3), which applies because E is in $\mathbf{I}(R)$ by (4.6.b). As $k \otimes^{\mathbf{L}}_R \mathbf{R}\operatorname{Hom}_R(D, E) \sim k \otimes^{\mathbf{L}}_R C$ by (6.3), and the complex $\mathbf{R}\operatorname{Hom}_S(k \otimes_R S, E)$ is dualizing for $k \otimes_R S$ by (2.12). If furthermore C is normalized, then $\inf(k \otimes^{\mathbf{L}}_R C) = \inf C = \operatorname{depth} S - \operatorname{depth} R = \operatorname{depth} k \otimes_R S$, hence $k \otimes^{\mathbf{L}}_R C$ is normalized as well. □

Proof of Theorem (5.12). Again, we may assume all rings complete, and take dualizing complexes B and D for Q and R, respectively. Consider now the canonical isomorphisms

$$\mathbf{R}\operatorname{Hom}_R(S, \mathbf{R}\operatorname{Hom}_Q(B, D)) \simeq \mathbf{R}\operatorname{Hom}_R(S \otimes^{\mathbf{L}}_Q B, D) \simeq \mathbf{R}\operatorname{Hom}_Q(B, \mathbf{R}\operatorname{Hom}_R(S, D))\,,$$

where the first one is given by (1.4.1). As $\mathbf{R}\operatorname{Hom}_R(S, D)$ is a dualizing complex for S by (2.12), and $\operatorname{G-dim}\varphi\psi$ is finite by assumption, we see that the isomorphic complexes above are dualizing for $\varphi\psi$. It remains to note that $A \sim \mathbf{R}\operatorname{Hom}_Q(B, D)$ by (6.3). □

7. Bass numbers of local homomorphisms

In this section $\varphi\colon (R,\mathfrak{m},k)\to(S,\mathfrak{n},\ell)$ denotes a local homomorphism.

If $\operatorname{G-dim}\varphi$ is finite, then so is $\operatorname{G-dim}\widehat{\varphi}$ by (4.4.3), and the latter homomorphism has a normalized dualizing complex C' by (5.3). Its i'th Betti number $\beta_i^{\widehat{S}}(C')$ is defined in (1.5).

Definition. The number $\mu^i_\varphi=\beta_i^{\widehat{S}}(C')$ is called the i'th *Bass number* of φ; the formal Laurent series $\mathrm{I}_\varphi(t)=\sum\mu^i_\varphi t^i$ is called the *Bass series* of φ.

Clearly, $\mu^i_\varphi=\mu^i_{\dot\varphi}=\mu^i_{\widehat\varphi}$.

(7.1) **Theorem.** *When* $\operatorname{G-dim}\varphi$ *is finite there is an equality of formal Laurent series*

$$\mathrm{I}_S(t)\ =\mathrm{I}_R(t)\,\mathrm{I}_\varphi(t)\ .$$

In particular, $\mu_R^{i+\operatorname{depth}R}\le\mu_S^{i+\operatorname{depth}S}$ *for each* $i\in\mathbb{Z}$.

Proof. The equality comes from (5.6). The inequalities follow, since for $d=\operatorname{depth}S-\operatorname{depth}R$ we have $\mu^i_\varphi=0$ when $i<d$, $\mu^d_\varphi\ne0$, and $\mu^i_\varphi\ge0$ when $i>d$. □

Now we compare the present Bass invariants with those introduced in [9].

(7.2) *Homotopy fiber.* The *homotopy fiber* $F(\varphi)$ of φ is $k\otimes^{\mathbf{L}}_R S$, equipped with its natural structure of differential graded algebra (which is unique up to isomorphism in the subcategory of $\mathbf{D}(R)$ generated by DG R–algebras and their morphisms). The "Bass series" of φ is defined in [9; p. 512] to be the "series" $\mathrm{I}_{F(\varphi)}(t)=\sum_{i\in\mathbb{Z}}\operatorname{rank}_\ell\operatorname{Ext}^i_{F(\varphi)}(\ell,F(\varphi))t^i$, where quotation marks indicate that the ranks involved need not be finite.

Assume that $\operatorname{fd}\varphi<\infty$. It follows from [9; (5.1)] that then $\mathrm{I}_{F(\varphi)}(t)$ is actually a formal Laurent series, and satisfies $\mathrm{I}_S(t)\ =\mathrm{I}_R(t)\,\mathrm{I}_{F(\varphi)}(t)$. As in this case $\operatorname{G-dim}\varphi<\infty$ by (4.4.2), comparison of this equation with that of the theorem yields $\mathrm{I}_{F(\varphi)}(t)\ =\mathrm{I}_\varphi(t)$.

The following example shows that (7.1) extends part of [9; (5.1)] in an essential way.

(7.3) **Example.** Let $(Q,\mathfrak{q})$ be a Gorenstein local ring, let $\mathbf{x}$ and $\mathbf{y}$ be Q–regular sequences such that $0\ne(\mathbf{x})\subseteq\mathfrak{q}(\mathbf{y})$, and let φ be the canonical map $R=Q/(\mathbf{x})\to Q/(\mathbf{y})=S$. As $\operatorname{G-dim}\varphi<\infty$ by (4.4.1), we have $\mathrm{I}_S(t)\ =\mathrm{I}_R(t)\,\mathrm{I}_\varphi(t)$ by (7.1). On the other hand, it is shown in [9; (5.5), (5.7)] that $\mathrm{I}_{F(\varphi)}(t)$ is a formal Laurent series, but $\mathrm{I}_S(t)\ne\mathrm{I}_R(t)\,\mathrm{I}_{F(\varphi)}(t)$.

Next we show that the Bass series of homomorphisms exhibits the same rigidity as that of local rings. When $\operatorname{fd}\varphi$ is finite, it is proved in [4; (4.4)] that conditions (i) and (ii) below are equivalent, and the question is raised in [4; (3.11)] of their equivalence with (iii). Shida [20] has obtained an affirmative answer when R is Cohen–Macaulay, or when $\operatorname{cmd}\varphi\le1$. We get a positive answer in the wider framework of finite G-dimension.

(7.4) **Theorem.** *If* $\operatorname{G-dim}\varphi$ *is finite, then the following conditions are equivalent.*

(i) $\mathrm{I}_\varphi(t)\ =t^{\operatorname{depth}S-\operatorname{depth}R}$.

(ii) $\mathrm{I}_\varphi(t)$ *is a Laurent polynomial.*

(iii) $\mu^i_\varphi=0$ *for some* $i>\operatorname{depth}S-\operatorname{depth}R$.

Proof. We may assume φ has a a normalized dualizing complex C, and set $d = \inf C$.

(iii) $\implies$ (ii). Let $i > d$ be such that $\mu^i_\varphi = 0$. By [17; (II.2.4.1)] or [15; (11.30)], $C \in \mathbf{D}^{\mathrm{f}}_+(S)$ is isomorphic to a complex F of finite free S–modules with $F_n = 0$ for $n < d$, and $\partial(F_n) \subseteq \mathfrak{n}F_{n-1}$ for each $n \in \mathbb{Z}$. We then have $\operatorname{rank}_k F_n = \beta^S_n(F) = \beta^S_n(C) = \mu^n_\varphi$, and hence $F_i = 0$. Thus, $C \simeq F' \oplus F''$, where $F' = F_{<i} \neq 0$ and $F'' = F_{>i}$. Consequently:

$$S \cong \operatorname{Ext}_S(C,C) \cong \operatorname{Ext}_S(F',F') \oplus \operatorname{Ext}_S(F',F'') \oplus \operatorname{Ext}_S(F'',F') \oplus \operatorname{Ext}_S(F'',F'')\,.$$

The S–module S is indecomposable and $\operatorname{Ext}_S(F',F') \neq 0$, hence $\operatorname{Ext}_S(F'',F'') = 0$. This means that $F'' = 0$ or, in other words, that $\mu^n_\varphi = 0$ for $n > i$.

(ii) $\implies$ (i). Our assumption means that C is in $\mathbf{P}(S)$, hence by (1.4.2) we have

$$S \simeq \mathbf{R}\operatorname{Hom}_S(C,C) \simeq C^* \otimes^{\mathbf{L}}_S C\,,$$

where $C^* = \mathbf{R}\operatorname{Hom}_S(C,S)$. The latter complex is in $\mathbf{D}^{\mathrm{f}}_{\mathrm{b}}(S)$ by (1.3.c) and (1.2.2), hence (1.5.3.a) gives $1 = \mathrm{P}^S_S(t) = \mathrm{P}^S_{C^*}(t)\,\mathrm{P}^S_C(t)$. This implies $\mathrm{P}^S_C(t) = t^m$ for some $m \in \mathbb{Z}$. As the order of $\mathrm{P}^S_C(t)$ is equal to $d = \operatorname{depth} S - \operatorname{depth} R$, it follows that $m = d$. □

Definition. A local homomorphism φ which has finite G-dimension and satisfies the equivalent conditions of (7.4) is said to be *quasi-Gorenstein at* $\mathfrak{n}$.

As an immediate consequence of (7.1) and (7.4) we have

(7.5) **Theorem.** *The following condition are equivalent when φ has finite G-dimension.*

(i) *φ is quasi-Gorenstein at $\mathfrak{n}$.*
(ii) *$\mu^{i+\operatorname{depth} R}_R = \mu^{i+\operatorname{depth} S}_S$ for each $i \in \mathbb{Z}$.*
(iii) *$\mu^{i+\operatorname{depth} R}_R = \mu^{i+\operatorname{depth} S}_S$ for some $i > 0$.* □

Condition (iii) cannot be weakened any further: equality in (ii) holds trivially for $i < 0$, and the next example shows that equality for $i = 0$ needs *not* imply equalities for $i > 0$.

(7.6) **Example.** The composition $R \to R' = R[\![X,Y]\!] \to R[\![X,Y]\!]/X(X,Y) = S$ is a flat local homomorphism with non-Gorenstein closed fiber $F = k[\![X,Y]\!]/X(X,Y)$, hence $\mu^i_F > 0$ for $i \geq 0$. Since $\mu^0_F = 1$, and $\mathrm{I}_S(t) = \mathrm{I}_R(t)\,\mathrm{I}_F(t)$ by (7.1) and (5.10), we see that $\mu^{\operatorname{depth} R}_R = \mu^{\operatorname{depth} S}_S$ and $\mu^{i+\operatorname{depth} R}_R < \mu^{i+\operatorname{depth} S}_S$ for each $i > 0$. More precisely, we have

$$\mathrm{I}_S(t) = \mathrm{I}_R(t) \cdot t^{-2}\frac{1+t-t^2}{1-t-t^2}$$

either by expressing $\mathrm{I}_F(t)$ from [23; Satz 8], or by noting that $\varphi\colon R' \to S$ is Golod by [18; Theorem 3] and of finite flat dimension, and applying [9; (5.7.b)].

In view of (4.4.2) and (4.4.1), the following remarks result by comparison of definitions.

(7.7.1) φ is Gorenstein at $\mathfrak{n}$ if and only if it is quasi-Gorenstein at $\mathfrak{n}$ and $\operatorname{fd}\varphi$ is finite.

(7.7.2) The following conditions are equivalent:

(i) R and S are Gorenstein.
(ii) R is Gorenstein and φ is quasi-Gorenstein at $\mathfrak{n}$.
(iii) S is Gorenstein and $\operatorname{G-dim}\varphi$ is finite.

(7.7.3) φ is quasi-Gorenstein at $\mathfrak{n}$ if and only if $\grave{\varphi}$ and/or $\widehat{\varphi}$ is quasi-Gorenstein at $\mathfrak{n}\widehat{S}$.

Next we describe the homomorphisms which properly base change dualizing complexs.

(7.8) **Theorem.** *The following conditions are equivalent.*

(i) *φ is quasi-Gorenstein at $\mathfrak{n}$.*

(ii) *S is a dualizing complex for φ, and φ has finite G-dimension.*

(iii) *$D' \otimes^{\mathbf{L}}_{\widehat{R}} \widehat{S}$ is a dualizing complex for $\widehat{S}$ if D' is one for $\widehat{R}$.*

If R has a dualizing complex D, they are also equivalent to

(iii') *$D \otimes^{\mathbf{L}}_{R} S$ is a dualizing complex for S.*

Proof. (iii) $\iff$ (iii') by (2.2) and (5.2.a).

By (7.7.3), (5.2.a) and (4.4.3), conditions (i) and (ii) are invariant under passage from φ to $\widehat{\varphi}$, so for the rest of the proof we may assume that R has a dualizing complex D, and φ has a normalized dualizing complex C. Now (ii) $\implies$ (iii') by (6.1.a).

(iii') $\implies$ (ii). As $D \otimes^{\mathbf{L}}_{R} S$ is a dualizing complex for S, it has bounded homology along with S and, and produces a commutative diagram

$$\begin{array}{ccc} S & \xrightarrow[\simeq]{\chi_{D\otimes^{\mathbf{L}}_R S}} & \mathbf{R}\mathrm{Hom}_S(D \otimes^{\mathbf{L}}_{R} S, D \otimes^{\mathbf{L}}_{R} S) \\ \| & & \simeq\uparrow \\ S & \xrightarrow[\gamma_S]{} & \mathbf{R}\mathrm{Hom}_R(D, D \otimes^{\mathbf{L}}_{R} S) \end{array}$$

which shows that γ_S is an isomorphism. This implies $S \in \mathbf{A}(R)$, thus G-dim φ is finite by (4.3). Now (6.1.b) applies and shows that $\mathbf{R}\mathrm{Hom}_R(D, D \otimes^{\mathbf{L}}_{R} S)$ is a dualizing complex for φ. Using once more the isomorphism γ_S, we see that S has the same property.

(i) amounts by (7.1) to $\mathrm{I}^C_S(t) = 1$, which by (1.5.1) means $C \sim S$, which is (ii). □

(7.9) **Corollary.** *Assume that φ is quasi-Gorenstein at $\mathfrak{n}$ and that R has a dualizing complex. An S–complex M is in $\mathbf{A}(S)$, respectively, $\mathbf{B}(S)$, if and only if it is in $\mathbf{A}(R)$, respectively, $\mathbf{B}(R)$.*

Proof. By (7.8) the complex $E = D \otimes^{\mathbf{L}}_{R} S$ is dualizing for S, hence the functors $D \otimes^{\mathbf{L}}_{R} -$ and $E \otimes^{\mathbf{L}}_{S} -$, respectively, $\mathbf{R}\mathrm{Hom}_R(D, -)$ and $\mathbf{R}\mathrm{Hom}_S(E, -)$, are naturally equivalent. □

(7.10) **Corollary.** *When a local homomorphism $\psi\colon Q \to R$ is quasi-Gorenstein at $\mathfrak{m}$, then* G-dim $\varphi < \infty$ *if and only if* G-dim $\varphi\psi < \infty$.

Proof. By (7.7.3) and (4.4.3) we may assume all rings are complete, so by (4.3) we have to prove that $R \in \mathbf{A}(Q)$ is equivalent to $S \in \mathbf{A}(R)$. This follows from (7.9). □

We now recall the useful result [2; (4.32)] of Auslander and Bridger: if $S = R/(x)$ for an R–regular x, and N is a finite S–module with G-dim$_S N < \infty$, then G-dim$_R N =$ G-dim$_S N + 1 < \infty$. Since the map $R \to R/(x)$ is Gorenstein, and thus quasi-Gorenstein by (7.7.1), the next theorem provides a broad generalization, as well as a converse. That it holds is a remarkable property of G-dimension with (almost) no counterpart for the familiar projective dimension: if $x \in \mathfrak{m}\,\mathrm{Ann}_R N$, then $\mathrm{pd}_S N = \infty$, cf. [18; §3].

(7.11) **Theorem.** *Let $\varphi\colon (R,\mathfrak{m}) \to (S,\mathfrak{n})$ be a finite local homomorphism, which is quasi-Gorenstein at $\mathfrak{n}$. If N is a finite S–module, then*

$$\operatorname{G-dim}_R N = \operatorname{G-dim}_R S + \operatorname{G-dim}_S N\,.$$

In particular, $\operatorname{G-dim}_R N < \infty$ *if and only if* $\operatorname{G-dim}_S N < \infty$.

Proof. By the Auslander–Bridger equality (4.1.2), it suffices to prove the last assertion. By (4.1.4) and (4.2.3) we may assume that both R and S are complete. From (4.1.7) we see that the finiteness of $\operatorname{G-dim}_R N$ is equivalent to the inclusion $N \in \mathbf{A}(R)$, and that of $\operatorname{G-dim}_S N$ – to the inclusion $N \in \mathbf{A}(S)$. As φ is quasi-Gorenstein, (7.9) shows that these inclusions hold simultaneously. □

8. Quasi-Gorenstein homomorphisms

In this section $\varphi\colon R \to S$ denotes a homomorphism of noetherian rings.

Definition. The homomorphism φ is *quasi-Gorenstein at* a prime $\mathfrak{q}$ of S if the induced local homomorphism $\varphi_{\mathfrak{q}}\colon R_{\mathfrak{q}\cap R} \to S_{\mathfrak{q}}$ is quasi-Gorenstein at $\mathfrak{q}S_{\mathfrak{q}}$; it is *quasi-Gorenstein* if it has this property at all $\mathfrak{q} \in \operatorname{Spec} S$.

We list a series of properties of quasi-Gorenstein homomorphisms, with indications of the results from which they are obtained by localization.

(8.1) **Gorenstein homomorphisms.** *φ is Gorenstein (called "locally Gorenstein" in* [4]*) if and only if it is quasi-Gorenstein and locally of finite flat dimension.* Cf. (7.7.1).

(8.2) **Ascent and Descent.** *When R is Gorenstein, φ is quasi-Gorenstein if and only if S is Gorenstein. If S is Gorenstein and φ is locally of finite G-dimension, then φ is quasi-Gorenstein and R is Gorenstein at the prime ideals contracted from S.* Cf. (7.7.2).

(8.3) **Flat homomorphisms.** *A flat homomorphism φ is quasi-Gorenstein if and only if all the non-trivial fibers of φ are Gorenstein.* Cf. (5.10) and (2.1).

(8.4) **Essentially finite type.** *If φ is essentially of finite type and is quasi-Gorenstein at all maximal ideals of S, then it is quasi-Gorenstein.* Cf. (4.5.1) and (5.8.1).

(8.5) **Gorenstein formal fibers.** *If the formal fibers of R are Gorenstein and φ is quasi-Gorenstein at all maximal ideals of S, then φ is quasi-Gorenstein and all formal fibers of S are Gorenstein.* Cf. (4.5.2), (5.8.2), and (5.9).

The proofs of the next two theorems follow those of [6; (6.10), (6.11)], and are omitted.

(8.6) **Flat base change.** *Let φ be essentially of finite type, and let $\tau\colon R \to T$ be a flat homomorphism. If φ is quasi-Gorenstein, then so is the induced homomorphism $\varphi \otimes_R T\colon T \to S \otimes_R T$; when τ is faithfully flat the converse holds as well.* □

(8.7) **Completion.** *Let $\mathfrak{a} \subset R$ and $\mathfrak{b} \subset S$ be ideals such that $\varphi(\mathfrak{a}) \subseteq \mathfrak{b} \neq S$, and let $\varphi^*\colon R^* \to S^*$ be the induced homomorphism of the corresponding ideal-adic completions. If R has Gorenstein formal fibers and φ is quasi-Gorenstein, then so is φ^*.* □

The final results, involving a second homomorphism $\psi\colon Q \to R$, have no analog for rings.

(8.8) **Flat descent.** *If φ is faithfully flat and $\varphi\psi$ is quasi-Gorenstein, then ψ is quasi-Gorenstein and φ is Gorenstein.*

(8.9) **Composition.** *If ψ and φ are quasi-Gorenstein, then so is $\varphi\psi$.*

(8.10) **Decomposition.** *Assume that $\varphi\psi$ is quasi-Gorenstein.*

(a) *If ψ is quasi-Gorenstein, then so is φ.*
(b) *If ψ an φ are locally of finite G-dimension, then φ is quasi-Gorenstein, and ψ is quasi-Gorenstein at the prime ideals of R contracted from S.*

Proof of Theorems (8.8) *through* (8.10). We may assume that ψ, φ, and hence $\varphi\psi$, are local. All three homomorphisms have finite G-dimension: this is the hypothesis in (8.10.b), follows from (4.7) in (8.8), and from (7.10) in the remaining two cases. From (7.1) we obtain the double inequalities

$$\mu_Q^{i+\operatorname{depth} Q} \le \mu_R^{i+\operatorname{depth} R} \le \mu_S^{i+\operatorname{depth} S} \quad \text{for} \quad i \in \mathbb{Z}\,,$$

which become equalities if and only if $\mu_Q^{i+\operatorname{depth} Q} = \mu_S^{i+\operatorname{depth} S}$. The assertions follow. □

Precisely as the concept introduced in this section extends that of Gorenstein homomorphism [4], one can generalize the Cohen–Macaulay homomorphisms of [6] as follows:

(8.11) *Quasi-Cohen–Macaulay homomorphisms.* A local homomorphism $\varphi\colon R \to (S, \mathfrak{n})$ is *quasi-Cohen–Macaulay at* $\mathfrak{n}$ if $\operatorname{G-dim}\varphi < \infty$, and a dualizing complex C' for $\widehat{\varphi}$ has $\operatorname{amp} C' = 0$. It is easily seen that φ is quasi-Gorenstein at $\mathfrak{n}$ if and only it is quasi-Cohen–Macaulay at $\mathfrak{n}$ and has type 1.

Several properties of quasi-Gorenstein homomorphisms hold in the more general context of quasi-Cohen–Macaulay homomorphisms. However, if S is Cohen–Macaulay and $\operatorname{G-dim}\varphi < \infty$, then φ is quasi-Cohen–Macaulay, but we do not know whether R is Cohen–Macaulay. This incomplete descent result is one reason not to pursue the venue. Another is that – at this time – we do not know whether the larger class is closed under composition, partly due to the unknown transitivity of the finiteness of G-dimension, cf. (4.8).

References

1. M. Auslander, *Anneaux de Gorenstein et torsion en algèbre commutative*, Exposés rédigés par M. Mangeney, C. Peskine et L. Szpiro, Séminaire d'algèbre commutative dirigé par P. Samuel, Secrétariat math., Paris, 1967.
2. _____ and M. Bridger, *Stable module theory*, Mem. Amer. Math. Soc., vol. 94, Amer. Math. Soc., Providence, R.I., 1969.
3. L. L. Avramov and H.–B. Foxby, *Homological dimensions of unbounded complexes*, J. Pure Appl. Algebra **71** (1991), 129–155.
4. _____ and _____, *Locally Gorenstein homomorphisms*, Amer. J. Math. **114** (1992), 1007–1047.
5. _____ and _____, *Grothendieck's localization problem*, Syzygies, multiplicities, and birational algebra (W. J. Heinzer, C. L. Huneke, and J. D. Sally, eds.), Contemp. Math., vol. 159, Amer. Math. Soc., Providence, R. I., 1994, pp. 1–13.
6. _____ and _____, *Cohen–Macaulay properties of ring homomorphisms*, Adv. in Math. **133** (1998), 54–92.

7. ———, ———, and S. Halperin, *Descent and ascent of local properties along homomorphisms of finite flat dimension*, J. Pure Appl. Algebra **38** (1986), 167–185.
8. ———, ———, and B. Herzog, *Structure of local homomorphisms*, J. Algebra **164** (1994), 124–145.
9. ———, ———, and J. Lescot, *Bass series of local ring homomorphisms of finite flat dimension*, Trans. Amer. Math. Soc. **335** (1993), 497–523.
10. ———and S. Halperin, *Through the looking glass: A dictionary between rational homotopy theory and local algebra*, Algebra, algebraic topology, and their interactions (J.-E. Roos, ed.), Lecture Notes in Math., vol. 1183, Springer, Berlin, 1986, pp. 1–27.
11. E. E. Enochs, O. M. G. Jenda, and J. Xu, *Foxby duality and Gorenstein injective and projective modules*, Trans. Amer. Math. Soc. **348** (1996), 3223–3234.
12. D. Ferrand and M. Raynaud, *Fibres formelles d'un anneau local noethérien*, Ann. Sci. Éc. Norm. Sup. (4) **3** (1970), 293–311.
13. H.–B. Foxby, *Gorenstein modules and related modules*, Math. Scand. **31** (1972), 267–284.
14. ———, *Isomorphisms between complexes with applications to the homological theory of modules*, Math. Scand. **40** (1977), 5–19.
15. ———, *A homological theory of complexes of modules*, Københavns Univ. Mat. Inst. Preprint Ser. No. 19a&b, 1981.
16. R. Hartshorne, *Residues and duality*, Lecture Notes in Math., vol. 20, Springer, Berlin, 1971.
17. P. Roberts, *Homological invariants of modules over commutative rings*, Sém. Math. Sup., vol. 72, Presses Univ. Montréal, Montréal, 1980.
18. J. Shamash, *The Poincaré series of a local ring.*, J. Algebra **12** (1969), 453–470.
19. R. Sharp, *Finitely generated modules of finite injective dimension over certain Cohen–Macaulay rings*, Proc. London Math. Soc. (3) **25** (1972), 303–328.
20. A. Shida, *Nonvanishing of Bass numbers for DG fiber*, Master Thesis, Nagoya Univ., Nagoya, 1994.
21. N. Spaltenstein, *Resolutions of unbounded complexes*, Compositio Math. **65** (1988), 121–154.
22. J.–L. Verdier, *Catégories dérivées. Quelques résultats (État 0)*, SGA $4\frac{1}{2}$, Lecture Notes in Math., vol. 569, Springer, Berlin, 1977, pp. 262–311.
23. H. Wiebe, *Über homologische Invarianten lokaler Ringe*, Math. Ann. **179** (1969), 257–274.
24. S. Yassemi, *G-dimension*, Math. Scand. **77** (1995), 161–174.

Department of Mathematics, Purdue University, West Lafayette, Indiana 47907, U. S. A.
E-mail address: avramov@math.purdue.edu

Matematisk Institut, Universitetsparken 5, DK–2100 København Ø, Denmark
E-mail address: foxby@math.ku.dk