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ON STRONG IFP NEAR-RINGS

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Abstract: In this paper, we introduce the notion of strong IFP and weak IFP near-rings. Weak IFP near-ring is a generalization of IFP near-ring. We study the basic properties of right weak IFP near-rings. We show that if N is a 2-primal near-ring and if N is strong IFP, then N is left weakly regular if and only if every prime ideal of N is maximal.

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1. Introduction

Throughout this paper, N denotes a zero-symmetric near-ring not necessarily with identity unless otherwise stated. Let $P(N)$ denote the prime radical and $N(N)$ the set of nilpotent elements of the near-ring N. For $X \subseteq N$, $l(X)$ (resp. $r(X)$) and $< x >$ denote the left (resp. right) annihilator of X and the ideal of N generated by x respectively.

For any subsets A, B of N, we denote $(A : B) = \{n \in N/nB \subseteq A\}$. It is trival to check that if A is left ideal of N and B is a N-subgroup of N, then $(A : B)$ is an ideal of N by [8, Corollary 1.43].

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A near-ring N is said to be reduced if $N(N) = 0$. A near-ring N is said to be regular if for any $a \in N$, there exists $x \in N$ such that $a = axa$.

Recall that a near-ring N is said to be 2-primal if $P(N) = N(N)$. A near-ring N is subdirectly irreducible if N has nonzero intersection of nonzero ideals. A near-ring N is said to be strong IFP if $xy \in P(N)$ implies $xNy = 0$ for $x, y \in N$. A near-ring N is said to be IFP if $ab = 0$ implies $anb = 0$ for all $n \in N$ and $a, b \in N$. Clearly every strong IFP near-ring is a IFP near-ring. If N is reduced, then the notions of IFP and strong IFP coincide

A near-ring N is said to be left weak IFP if $ab = 0$ for $a(\neq 0), b \in N$ implies $a^{'}Nb = 0$ for some $a^{'}(\neq 0) \in < a >$. The right weak IFP can be defined symmetrically. A near-ring N is said to be weak IFP if $ab = 0$ for any nonzero elements $a, b \in N$ implies $a^{'}Nb^{'} = 0$ for some $a^{'}(\neq 0) \in < a >$ and $b^{'}(\neq 0) \in < b >$.

Clearly IFP near-ring is a weak IFP near-ring, but the converse need not be true as the following example shows.

Example 1.1. Let $N = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$ where $F = \{0, 1\}$ is the field under addition and multiplication modulo 2. Then N is a weak IFP near-ring but not IFP near-ring, since if $x = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $y = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, then $xy = 0$ and $xNy \neq 0$. Here N is neither left weak IFP nor right weak IFP.

Clearly every strong IFP near-rings are IFP near-rings, however IFP near-ring need not be strong IFP as can be seen by the following example.

Example 1.2. Let $(N, +)$ (where $N = \{0, a, b, c\}$) be the klein's four group. Define multiplication in N as follows

.	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	0	0	0
c	0	a	b	a

Then $(N, +, .)$ is a near-ring (see Pilz[8], P-408, Scheme-11) which is a IFP near-ring but not a strong IFP near-ring, since $ab \in P(N)$, but $aNb \neq 0$.

Clearly every reduced near-ring is a 2-primal and strong IFP near-ring, but the converse need not be true as the following example shows.

Example 1.3. Let $(N, +)$ (where $N = \{0, a, b, c\}$) be the klein's four group. Define multiplication in N as follows

.	0	a	b	c
0	0	0	0	0
a	0	a	0	a
b	0	0	0	0
c	0	a	0	a

Then $(N, +, .)$ is a near-ring (see Pilz[8], P-408, Scheme-12) which is a 2-primal and strong IFP near-ring, but not reduced.

G.F.Birkenmeier, J.Y.Kim and J.K.Park [2] have shown that a reduced ring R is weakly regular if and only if every prime ideal of R is maximal. We extend this result to strong IFP near-rings which are 2-primal. For basic terminology in near-ring we refer to Pilz [8].

2. Main Results

Lemma 2.1. *Let N be a near-ring with identity. If N is left weak IFP, then for any $x, y \in N$ with $xy = 1$ implies $yx = 1$.*

Proof. Let N be a left weak IFP near-ring and $xy = 1$. Suppose $yx \neq 1$. Then $(1 - yx)yx = 0$. Since N is left weak IFP, we have $x^{'}Nyx = 0$ for some $x^{'}(\neq 0) \in < 1 - yx >$. Now, $x^{'}Ny = x^{'}Nyxy = 0$. Then $x^{'} = x^{'}xy \in x^{'}Ny = 0$, a contradiction. □

Proposition 2.2. *Let N be a regular near-ring. Then the following conditions are equivalent:*
i) N is a right weak IFP near-ring
ii) If $x(\neq 0) \in r(a)$, then $r(a)$ contains a non-zero ideal I with $I \subseteq < x >$
iii) If $x(\neq 0) \in r(a)$, $r(aN)$ contains a non-zero ideal I with $I \subseteq < x >$
iv) If $x(\neq 0) \in r(a)$, $i \in r(aN)$ for some $i(\neq 0) \in < x >$

Proof. $i) \Rightarrow ii)$ Let $x(\neq 0) \in r(a)$. Then $aNx^{'} = 0$ for some non-zero element $x^{'} \in < x >$. For any $n \in N$, $x^{'}na = (x^{'}na)t(x^{'}na) = x^{'}n(atx^{'})na = 0$ for some $t \in N$. Thus $x^{'}Na = 0$ and so $< x^{'} > Na = 0$. Let $y \in < x^{'} >$. Then by regularity of N, we have $ay = 0$. Thus $a < x^{'} >= 0$.

$ii) \Rightarrow iii)$ Let $x(\neq 0) \in r(a)$. Then $aI = 0$ and so $aNI = 0$.

$iii) \Rightarrow iv)$ It is obvious.

$iv) \Rightarrow i)$ Let $ab = 0$ for $a(\neq 0), b \in N$. Then $b^{'} \in r(aN)$ for some $b^{'}(\neq 0) \in < b >$. Thus $aNb^{'} = 0$. □

We now give an example to show that Proposition 2.2 is not true if N is not a regular near-ring.

Example 2.3. Consider the dihedral group $N=\{0, a, 2a, 3a, b, a+b, 2a+b, 3a+b\}$ with addition and multiplication defined as in Pilz ([9, P-339, Scheme-2]).

.	0	a	2a	3a	b	a+b	2a+b	3a+b
0	0	0	0	0	0	0	0	0
a	0	a	2a	3a	b	a+b	2a+b	3a+b
2a	0	2a	0	2a	0	0	0	0
3a	0	3a	2a	a	b	a+b	2a+b	3a+b
b	0	b	2a	b	b	0	2a+b	0
a+b	0	a+b	0	3a+b	0	a+b	0	3a+b
2a+b	0	2a+b	2a	2a+b	b	0	2a+b	0
3a+b	0	3a+b	0	a+b	0	a+b	0	3a+b

Then $(N, +, .)$ is a near-ring. Clearly $(N, +, .)$ is a right weak IFP and $a + b \in r(b)$, but $r(b)$ does not contains a non-zero ideal I with $I \subseteq < a + b >$.

Proposition 2.4. *Let N be a near-ring with identity. Then the following conditions are equivalent:*
i) N is a left weak IFP near-ring
ii) If $x(\neq 0) \in l(a)$, then $l(a)$ contains a non-zero ideal I with $I \subseteq < x >$
iii) If $x(\neq 0) \in l(a)$, then $l(Na)$ contains a non-zero ideal I with $I \subseteq < x >$
iv) If $x(\neq 0) \in l(a)$, then $i \in l(Na)$ for some $i(\neq 0) \in < x >$

Proof. Proof is as similar in Proposition 2.2. □

Lemma 2.5. *Let N be a regular near-ring. If N is subdirectly irreducible, then the following conditions are equivalent:*
i) N is a right weak IFP near-ring
ii) if $x(\neq 0) \in r(S)$, then $r(S)$ contains a non-zero ideal I of N with $I \subseteq < x >$ for any subset S of N.

Proof. $i) \Rightarrow ii)$ Let N be a right weak IFP near-ring and $x \in r(S)$ for any subset S of N. For any $s_i \in S$, by Proposition 2.2, we have $r(s_i)$ contains a non-zero ideal I_i of N with $I_i \subseteq < x >$ and so $0 \neq \cap I_i \subseteq < x >$ with $S(\cap I_i) = 0$.
$ii) \Rightarrow i)$ It is trivial. □

Proposition 2.6. *Let N be a regular near-ring. If N is subdirectly irreducible, then the following conditions are equivalent:*
i) N is a right weak IFP near-ring
ii) N is a reduced near-ring
iii) N is a strong IFP near-ring
iv) N is a IFP near-ring.

Proof. $i) \Rightarrow ii)$ Let $a(\neq 0) \in N$ such that $a^2 = 0$. Since N is regular, we have $a = axa$ for some $x \in N$. Set $e = ax$. Let $S = \{n - ne/n \in N\}$. Then $r(S)$ contains a non-zero ideal J with $J \subseteq < e >$ and so $nj = nej$ for all $n \in N$ and for all $j \in J$. Let $j(\neq 0) \in J$. Then there exists $y \in N$ such that $j = jyj = j(yj) = je(yj) = ja(xyj) = (ja)e(xyj) = ja^2x^2yj = 0$, a contradiction.

$ii) \Rightarrow iii)$ It follows from Proposition 2.94 of [8].

$iii) \Rightarrow iv)$ and $iv) \Rightarrow i)$ are trival. □

Hereafter N denote a zero-symmetrc near-ring with left identity. Following G. F. Birkenmeier and N. J. Groenewald [1], a near-ring N is said to be left (resp. right) weakly π-regular if $x^n \in < x^n > x^n$ (resp. $x^n \in x^n < x^n >$) for all $x \in N$ and for some natural number $n = n(x)$. A near-ring N is called weakly π-regular if N is both left and right weakly π-regular. A weakly π-regular near-ring is called weakly regular when $n = 1$.

A near-ring N is said to be left (resp. right) pseudo π-regular if $x^n \in < x > x^n$ (resp. $x^n \in x^n < x >$) for all $x \in N$ and for some natural number $n = n(x)$. A near-ring N is called pseudo π-regular if N is both left and right pseudo π-regular.

Proposition 2.7. *Let P be a completely prime ideal of N. If $N/P(N)$ is a left weakly π-regular near-ring, then P is a maximal ideal of N.*

Proof. Let P be a completely prime ideal of N and $N/P(N)$ be a left weakly π-regular near-ring. Suppose M is an ideal of N such that $P \subset M$. Let $a \in M \backslash P$. Then $P+ < a > \subseteq M$. Since $N/P(N) = \overline{N}$ is a left weakly π-regular, we have $\overline{N}\overline{a}^n = < \overline{a}^n > \overline{a}^n$ for some positive integer n. So $\overline{N}\overline{a}^n = \overline{M}\overline{a}^n$. Hence $\overline{a}^n = \overline{b}\overline{a}^n$ for some $\overline{b} \in \overline{M}$ and so $(1 - b)a^n \in P$ which implies $N = M$. □

Corollary 2.8. *(5 Theorem 2.3) Let P be a completely prime ideal of a ring R. If $P/P(R)$ is a left weakly π-regular near-ring, then P is a maximal ideal of R.*

G.F.Birkenmeier, J.Y.Kim and J.K.Park [2] have shown that a reduced ring R is weakly regular if and only if R is right weakly regular and if and only if every prime ideal of R is maximal. We shall prove this result under generalized conditions.

Proposition 2.9. *Let N be a 2-primal near-ring. If N is strong IFP, then the following conditions are equivalent:*

i) N is left weakly regular

ii) N is left weakly π-regular

iii) $N/P(N)$ is left weakly π-regular

iv) $N/P(N)$ is left pseudo π-regular
v) Every prime ideal of N is maximal.

Proof. $i) \Rightarrow ii)$, $ii) \Rightarrow iii)$ and $iii) \Rightarrow iv)$ Proofs are trival.
$iv) \Rightarrow v)$ It follows from Corollary 3.10 of [1].
$v) \Rightarrow i)$ Suppose N is not a left weakly regular. Then there exists an element $a \in N$ such that $a \notin < a > a$. Let T be a union of all prime ideals of N, such that each of them contain a. Let $S = N \backslash T$. Then S is a multiplicative closed subset of N by Theorem 5 of [4]. Let F be the multiplicative closed system generated by $\{a\} \cup S$. Suppose $0 \notin F$. Then there exists a proper prime ideal M of N with $M \cap F = \phi$ by Proposition 2.81 of [8]. Since $a \notin M$, we have $M + < a >= N$ and so there exists $b \in M$ and $c \in < a >$ such that $b + c = 1$. Clearly $b \notin T$, which implies $b \in F \cap M = \phi$, a contradiction. Thus $0 \in F$.

So $0 = a^{n_1} s_1 a^{n_2} ... a^{n_t} s_t$ where $s_i \in S$ and $n_1, n_2, ..., n_t$ are positive integers. For any prime ideal P, we have $a^{n_1} s_1 a^{n_2} ... a^{n_t} s_t \in P$. Since P is completely prime, we have $a \in P$ or $s_i \in P$ for some i. Let $s = s_1 s_2 ... s_t$. Then for any prime ideal P; either $a \in P$ or $s \in P$. Then $sa \in P(N)$. Since N is a strong IFP near-ring, we have $sNa = 0$. Then $< s > Na = 0$. Observe that a prime ideal can not contains both a and s; otherwise a prime ideal would contain both of them, which contradicts the definitions of S and T which implies $< s > + < a >= N$ and so N is a left weakly regular near-ring. □

Corollary 2.10 (2, Theorem 8)**.** *Let R be a reduced ring. Then the following conditions are equivalent:*
i) R is weakly regular
ii) R is right weakly π-regular
iii) Every prime ideal of R is maximal

Proof. The proof is an immediate consequence of Proposition 2.9 and Theorem 12 of [3].

Corollary 2.11 (2, Corollary 9)**.** *Let R be a 2-primal ring. Then the following conditions are equivalent:*
i) $R/P(R)$ is weakly regular
ii) $R/P(R)$ is right weakly regular
iii) Every prime ideal of R is maximal.

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