Gorenstein Homological Dimensions of Commutative Rings

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Abstract. The classical global and weak dimensions of rings play an important role in the theory of rings and have a great impact on homological and commutative algebra. In this paper, we define and study the Gorenstein homological dimensions of commutative rings (Gorenstein projective, injective, and flat dimensions of rings) which introduce a new theory similar to the one of the classical homological dimensions of rings.

Key Words. Classical homological dimensions of modules; global and weak dimensions of rings; Gorenstein homological dimensions of modules and of rings; strongly Gorenstein projective, injective, and flat modules; (n-)Gorenstein rings; quasi-Frobenius rings; n-FC rings; IF-rings (weakly quasi-Frobenius rings).

Contents

1 Introduction

Throughout this paper all rings are commutative with identity element and all modules are unital.

Setup and Notation: Let R be a ring, and let M be an R-module.
As usual we use $\mathrm{pd}_R(M)$, $\mathrm{id}_R(M)$, and $\mathrm{fd}_R(M)$ to denote, respectively, the classical projective, injective and flat dimensions of M. And we use $\mathrm{Gpd}_R(M)$, $\mathrm{Gid}_R(M)$, and $\mathrm{Gfd}_R(M)$ to denote, respectively, the Gorenstein projective, injective and flat dimensions of M. $\mathrm{gldim}(R)$ and $\mathrm{wdim}(R)$ are, respectively, the classical global and weak dimensions of R.

The history of Gorenstein homological dimensions starts in the sixties, when Auslander and Bridger introduced the Gorenstein dimension (G-dimension for short), for finitely generated modules over commutative Noetherian rings [2, 3]. From The reason behind this name (i.e., Gorenstein dimension) is that a commutative local Noetherian ring is Gorenstein (i.e., with finite self-injective dimension) if, and only if, every finitely generated module has finite Gorenstein dimension.

Several decades later, Enochs et al. [15, 16, 19] defined the Gorenstein projective dimension as an extension of the G-dimension to modules that are not necessarily finitely generated, and the Gorenstein injective dimension as a dual notion of the Gorenstein projective dimension. To complete the analogy with the classical homological dimension, Enochs et al. [18] introduced the Gorenstein flat dimension.

The principal importance of this three dimensions is that they are refinements of the classical homological dimensions (see [23, Proposition 2.27] and [6, page 13]), in the sense that, for a ring R and every R-module M:

1. $\mathrm{Gpd}_R(M) \leq \mathrm{pd}_R(M)$ with equality when $\mathrm{pd}_R(M)$ is finite.
2. $\mathrm{Gid}_R(M) \leq \mathrm{id}_R(M)$ with equality when $\mathrm{id}_R(M)$ is finite.
3. If R is coherent, then $\mathrm{Gfd}_R(M) \leq \mathrm{fd}_R(M)$ with equality when $\mathrm{fd}_R(M)$ is finite.

In the last years, the Gorenstein homological dimensions have become a vigorously active area of research (see [10]* for more details). In 2004, Holm [23] generalized several results which already obtained over Noetherian rings (see also [11]). Then, the notion of the Gorenstein dimensions witnessing a new phase, and becomes a recent active area of research.

In this paper, we study the Gorenstein dimensions of arbitrary rings, which are canonically defined, for any ring R, as follows:

1. The Gorenstein projective dimension of R is:
$$\mathbf{GPD}(\boldsymbol{R}) = \sup\{\mathbf{Gpd}_{\boldsymbol{R}}(\boldsymbol{M}) \mid \boldsymbol{M}\ \boldsymbol{R}\!-\!\boldsymbol{module}\}$$
2. The Gorenstein injective dimension of R is:
$$\mathbf{GID}(\boldsymbol{R}) = \sup\{\mathbf{Gid}_{\boldsymbol{R}}(\boldsymbol{M}) \mid \boldsymbol{M}\ \boldsymbol{R}\!-\!\boldsymbol{module}\}$$
3. The Gorenstein flat dimension of R is:
$$\mathbf{GFD}(\boldsymbol{R}) = \sup\{\mathbf{Gfd}_{\boldsymbol{R}}(\boldsymbol{M}) \mid \boldsymbol{M}\ \boldsymbol{R}\!-\!\boldsymbol{module}\}$$

*In [10] Christensen forgot some details in few results. For the correction, see errata on the Christensen's homepage:

http://www.math.unl.edu/∼1christensen3/publications.html

The majority of researches done in Gorenstein homological dimensions are influenced and based on ones existed in the classical homological dimensions, and proved that there is a strong similarity between them. In this direction it is natural to ask to what extent the results existed in the theory of classical homological dimensions of rings can hold in the Gorenstein homological dimensions. Then we ask the first question:

Question A: For any ring R, does the Gorenstein projective dimension of R agrees with its Gorenstein injective dimension ?

Over Noetherian rings, we have already an affirmative answer to this question (see Corollary 2.3). In Section 3, we generalize this result, such that we obtain an affirmative answer to Question A, that is, the Gorenstein projective and injective dimensions of any ring R coincide, $\mathbf{GPD(R) = GID(R)}$ (Theorem 3.2). So, according to the terminology of the classical theory of homological dimensions of rings, this common value of $\mathrm{GPD}(R)$ and $\mathrm{GID}(R)$ will be called *Gorenstein global dimension* of R, and denoted by $\mathbf{G-gldim(R)}$.

In Section 4, we investigate the Gorenstein flat dimension of rings. Following the same pattern as in the Gorenstein projective and injective dimensions of rings, we ask two questions:

Question B: For any ring R, do we have inequality $\mathrm{GFD}(R) \leq \mathrm{G-gldim}(R)$?

Over a Noetherian ring R, we have already equality $\mathrm{GFD}(R) = \mathrm{G-gldim}(R)$ (see Corollary 2.3). Over a general ring R, there is inequality $\mathrm{GFD}(R) \leq \mathrm{G-gldim}(R)$ (Theorem 4.2), this gives an affirmative answer to Question B. Then, the Gorenstein flat dimension of a ring R will be called *Gorenstein weak dimension* of R, and denoted by $\mathbf{G-wdim(R)}$.

Question C: If R is a perfect ring, do we have equality $\mathrm{G-wdim}(R) = \mathrm{G-gldim}(R)$?

We shall see this holds over rings of finite Gorenstein global dimension (Proposition 4.9).

Also, in the same section, we give a characterization of the Gorenstein weak dimension of coherent rings (Theorem 4.11). We will see that this deals with the notion of n-FC rings (see Definition 2.4), which is introduced in [9] to generalize the characterization of Gorenstein flat modules over Gorenstein rings (see Definition 2.4 and Theorem 2.5). This characterization gives a partial affirmative answer to each of the two following questions:

1. Foxby's Conjecture: For any ring R, if $\mathrm{Gfd}_R(M)$ is finite, then $\mathrm{Gfd}_R(M) = \mathrm{Rfd}_R(M)$.
   Recall, the *large restricted flat dimension* [12], $\mathrm{Rfd}_R(M)$, of an R-module M, is defined by:

   $$\mathrm{Rfd}_R(M) = \sup\{i \geq 0 \,|\, \exists L \in \overline{\mathcal{F}}(R) : \; Tor_i(L, M) \neq 0\}$$

   such that $\overline{\mathcal{F}}(R)$ is the class of all R-modules of finite flat dimension.
2. As in the classical case, have we, for any ring R, the equivalence: an R-module M is Gorenstein flat if, and only if, M_P is Gorenstein flat R_P-module for any prime ideal P of R ?

Finally, we conclude this paper with a brief discussion of the scopes and limits of our results (see Section 5).

Before starting, we give, in the next section, some definitions and results to complete the overview of our topic and to use later.

2 Preliminaries

The Gorenstein projective, injective, and flat dimensions of modules are defined in terms of resolutions by Gorenstein projective, injective, and flat modules, respectively, which are defined as follows:

Definition 2.1 ([23])

1. *An R-module M is said to be Gorenstein projective (G-projective for short), if there exists an exact sequence of projective modules*
$$\mathbf{P} = \ \cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$$
such that $M \cong \mathrm{Im}(P_0 \to P^0)$ and such that $\mathrm{Hom}_R(-, Q)$ leaves the sequence $\mathbf{P}$ exact whenever Q is a projective module.
The exact sequence $\mathbf{P}$ is called a complete projective resolution.
2. *The Gorenstein injective (G-injective for short) modules are defined dually.*
3. *An R-module M is said to be Gorenstein flat (G-flat for short), if there exists an exact sequence of flat modules*
$$\mathbf{F} = \ \cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots$$
such that $M \cong \mathrm{Im}(F_0 \to F^0)$ and such that $- \otimes I$ leaves the sequence $\mathbf{F}$ exact whenever I is an injective module.
The exact sequence $\mathbf{F}$ is called a complete flat resolution.

Recall that a ring R is said to be n-Gorenstein, for a positive integer n, if it is Noetherian with self-injective dimension less or equal than n (i.e., $\mathrm{id}_R(R) \leq n$) [24]. And R is said to be Gorenstein, if it is n-Gorenstein for some positive integer n. The Gorenstein dimensions were extensively studied over Gorenstein rings (see for example [17, Theorems 11.2.5, 11.5.7, and 11.7.5]). They are specially used to characterize the n-Gorenstein rings, as follows:

Theorem 2.2 ([17], Theorem 12.3.1) *If R is a Noetherian ring, then, for a positive integer n, the following are equivalent:*

1. *R is n-Gorenstein;*
2. *$\mathrm{Gpd}_R(M) \leq n$ for all R-modules M;*
3. *$\mathrm{Gid}_R(M) \leq n$ for all R-modules M;*
4. *$\mathrm{Gfd}_R(M) \leq n$ for all R-modules M.*

This may be rewrited using the Gorenstein dimensions of rings (see page 2) as follows:

Corollary 2.3 *For any Noetherian ring R, there is equalities:*
$$\mathrm{GID}(R) = \mathrm{GPD}(R) = \mathrm{GFD}(R).$$
Such that, for a positive integer n, this common value is less or equal than n if, and only if, R is n-Gorenstein.

The n-FC rings (Definition 2.4 (2) below) are introduced in [9] to generalize the characterization of Gorenstein flat modules over Gorenstein rings.

Definitions 2.4 ([9], [22] and [28]) *Let R be a ring, and let M be an R-module.*

1. *We say that M has FP-injective (or pure) dimension at most n (for some $n \geq 0$), denoted by FP-$\mathrm{id}_R(M) \leq n$, if and only if, $\mathrm{Ext}_R^{n+1}(P, M) = 0$ for all finitely presented R-modules P. The modules of FP-injective dimension 0 are called FP-injective (or absolutely pure) modules.*
2. *A ring R is said to be n-FC (for some $n \geq 0$), if it is coherent and it has self-FP-injective dimension at most n (i.e., FP-$\mathrm{id}_R(R) \leq n$).*

Obviously, over Noetherian rings, the FP-injective dimension coincide with the usual injective dimension. Then, the Noetherian n-FC rings are same the n-Gorenstein rings. From this fact, the following theorem is a generalization of Theorem 2.2.

Theorem 2.5 ([9], Theorem 7) *A coherent ring R is n-FC if, and only if, $\mathrm{Gfd}_R(M) \leq n$ for all R-modules M (i.e., $\mathrm{GFD}(R) \leq n$).*

3 Gorenstein global dimension of rings

We give in this section a detailed treatment of the Gorenstein projective and injective dimensions of rings, which are canonically defined as follows:

Definition 3.1 *For any ring R,*

1. *The Gorenstein projective dimension of R is:* $\mathrm{GPD}(R) = \sup\{\mathrm{Gpd}_R(M) \mid M\ R\text{−}module\}$.
2. *The Gorenstein injective dimension of R is:* $\mathrm{GID}(R) = \sup\{\mathrm{Gid}_R(M) \mid M\ R\text{−}module\}$.

As shown in Corollary 2.3, we have $\mathrm{GPD}(R) = \mathrm{GID}(R)$ for any Noetherian ring R. Our main result in this section is to prove this equality holds over arbitrary rings. Namely, this is an extension of the well-known result [27, Theorem 9.10], that is, the global dimension of a ring R is precisely the common value of the two equal quantities: $\mathrm{PD}(R) = \sup\{\mathrm{pd}_R(M) \mid M\ R\text{−}module\}$ and $\mathrm{ID}(R) = \sup\{\mathrm{id}_R(M) \mid M\ R\text{−}module\}$.

Theorem 3.2 *For any ring R,* $\mathrm{GPD}(R) = \mathrm{GID}(R)$.

The proof would use the following results:

We need the following (functorial) characterization of the Gorenstein projective dimension of rings.

Lemma 3.3 *Let R be a ring. If R has finite Gorenstein projective dimension, then, for a positive integer n, the following are equivalent:*

1. $\mathrm{GPD}(R) \leq n$;
2. $\mathrm{Gpd}_R(M) \leq n$ *for all finitely generated R-modules M;*
3. $\mathrm{Gpd}_R(R/I) \leq n$ *for all ideals I of R;*
4. $\mathrm{Ext}_R^i(M, P') = 0$ *for all $i > n$, all (finitely generated) R-modules M, and all R-modules P' with finite $\mathrm{pd}_R(P')$ (i.e., $\mathbf{id_R(P') \leq n}$ **for all R-modules $\mathbf{P'}$ with finite $\mathbf{pd_R(P')}$**);*
5. $\mathrm{Ext}_R^i(M, P) = 0$ *for all $i > n$, all (finitely generated) R-modules M, and all projective R-modules P (i.e., $\mathbf{id_R(P) \leq n}$ **for all projective R-modules P**).*

Proof. All no obvious implications follow immediately from [23, Theorem 2.20], [27, Theorem 9.8], and [27, Lemma 9.11]. ■

A dual argument of the proof of Lemma 3.3 gives the following (functorial) characterization of the Gorenstein injective dimension of rings.

Lemma 3.4 *Let R be a ring. If R has finite Gorenstein injective dimension, then, for a positive integer n, the following are equivalent:*

1. $\mathrm{GID}(R) \leq n$;
2. $\mathrm{Ext}_R^i(I', M) = 0$ *for all* $i > n$, *all* R*-modules* M, *and all* R*-modules* I' *with finite* $\mathrm{id}_R(I')$ *(i.e.,* $\mathbf{pd_R(I') \leq n}$ ***for all*** $\mathbf{R}$***-modules*** $\mathbf{I'}$ ***with finite*** $\mathbf{id_R(I')}$*).*
3. $\mathrm{Ext}_R^i(I, M) = 0$ *for all* $i > n$, *all* R*-modules* M, *and all injective* R*-modules* I *(i.e.,* $\mathbf{pd_R(I) \leq n}$ ***for all injective*** $\mathbf{R}$***-modules*** $\mathbf{I}$*).*

Lemma 3.5 *Let R be a ring.*

1. *If* $\mathrm{GPD}(R) \leq n$, *then* $\mathrm{pd}_R(E) \leq n$ *for every* R*-module* E *with finite* $\mathrm{id}_R(E)$.
2. *If* $\mathrm{GID}(R) \leq n$, *then* $\mathrm{id}_R(P) \leq n$ *for every* R*-module* P *with finite* $\mathrm{pd}_R(E)$.

Proof. It suffices to prove the first assertion. The second has a dual argument.

We prove by induction on $\mathrm{id}_R(E)$.

First, assume E to be an injective R-module. We have $\mathrm{Gpd}(E) \leq n$ (since $\mathrm{GPD}(R) \leq n$). The case where $\mathrm{Gpd}(E) = 0$ is easy. Then, we may assume that $0 < \mathrm{Gpd}(E) \leq n$. Thus, from [23, Theorem 2.10], there exists a short exact sequence

$$(F_1) \qquad 0 \longrightarrow P \longrightarrow G \longrightarrow E \longrightarrow 0$$

where G is Gorenstein projective and $\mathrm{pd}_R(P) \leq n-1$.
Applying the functor $\mathrm{Hom}_R(G_0, -)$ to (F_1), where G_0 is a Gorenstein projective R-module, we get the long exact sequence:

$$\cdots \to \mathrm{Ext}(G_0, P) \to \mathrm{Ext}(G_0, G) \to \mathrm{Ext}(G_0, E) \to \cdots$$

Since $\mathrm{Ext}(G_0, E) = 0$ (since E is injective), and $\mathrm{Ext}(G_0, P) = 0$ (from [23, Theorem 2.20] and since P is projective), $\mathrm{Ext}(G_0, G) = 0 \quad (\bigstar)$.
On the other hand, since G is Gorenstein projective, there exists (by Definition 2.1 (1)) a short exact sequence

$$(F_2) \qquad 0 \longrightarrow G \longrightarrow Q \longrightarrow G' \longrightarrow 0$$

where Q is projective and G' is Gorenstein projective.
By the equality $(\bigstar)$ above, $\mathrm{Ext}(G', G) = 0$, hence the sequence (F_2) is split. Then, G is projective as a direct summand of the projective module Q.
Finally, the standard inequality [8, §8, N$^{\mathrm{o}}$1, Corollary 2 (b)] applied to (F_1) gives $\mathrm{pd}_R(E) = \mathrm{pd}_R(P) + 1 \leq n$, as desired.

Now, assume that $\mathrm{id}_R(E) = m > 0$.
Pick a short exact sequence:

$$0 \longrightarrow E \longrightarrow I \longrightarrow L \longrightarrow 0$$

where I is an injective R-module, and $\mathrm{id}(L) = m-1$.
By induction, $\mathrm{pd}(I) \leq n$ and $\mathrm{pd}(L) \leq n$. Then, from also [8, §8, N$^{\mathrm{o}}$1, Corollary 2 (c)], we have $\mathrm{pd}(E) \leq \sup\{\mathrm{pd}(I), \mathrm{pd}(L) - 1\} \leq n$. Thus, the proof is complete. ■

Lemma 3.6 *For any short exact sequence of modules* $0 \to A \to B \to C \to 0$*, we have the inequality:*

$$\mathrm{Gid}(C) \leq \sup\{\mathrm{Gid}(B), \mathrm{Gid}(A) - 1\}$$

Proof. Using the dual version of Horseshoe's lemma (see the remark below [27, Lemma 6.20]), the argument is essentially dual to the proof of [10, Corollary 1.2.9 (a)]. ■

Also, we need the notion of strongly Gorenstein projective and injective modules, which are defined as follows:

Definition 3.7 ([6], Definition 2.1) *If we consider all projective (resp., injective) modules and homomorphisms of the complete projective (resp., injective) resolution of Definition 2.1 are the same,* M *will be called strongly Gorenstein projective (resp., injective) module.*

The principal role of these modules is to characterize the Gorenstein projective and injective modules, as follows:

Theorem 3.8 ([6], Theorems 2.7) *A module is Gorenstein projective if, and only if, it is a direct summand of a strongly Gorenstein projective module.*

Each of the strongly Gorenstein projective and injective module has simple characterizations [6, Proposition 2.9]. These characterizations may be modified as follows (we set only the characterization of a strongly Gorenstein projective module, and, from [6, remarks 2.10 (2)], a strongly Gorenstein injective module has a dual characterization):

Proposition 3.9 *A module* M *is strongly Gorenstein projective if, and only if, there exists a short exact sequence* $0 \to M \to P \to M \to 0$ *where* P *is a projective module, and* $\mathrm{Ext}^i(M, Q) = 0$ *for some integer* $i > 0$ *and for any module* Q *with finite projective dimension (or for any projective module* Q*).*

Proof. We prove only the projective case, and the flat case is analogous.
Note that if we have a short exact sequence $0 \to M \to P \to M \to 0$ where P is a projective module, then, for all modules L and all $i > 0$, we have the long exact sequence:

$$0 = \mathrm{Ext}^i(P, L) \to \mathrm{Ext}^i(M, L) \to \mathrm{Ext}^{i+1}(M, L) \to \mathrm{Ext}^{i+1}(P, L) = 0$$

This means that $\mathrm{Ext}^i(M, L) \cong \mathrm{Ext}^{i+1}(M, L)$, for all integer $i > 0$ and all R-modules L.
Therefore, the desired equivalence is easily obtained from the first characterization of strongly Gorenstein projective modules [6, Propositions 2.9]. ■

Proof of Theorem 3.2. To prove this theorem, it is sufficient, from Lemmas 3.3 and 3.4, using Lemma 3.5, to prove the following equivalence:

$$\mathrm{GPD}(R) < \infty \iff \mathrm{GID}(R) < \infty.$$

We claim only the implication $\mathrm{GPD}(R) < \infty \Rightarrow \mathrm{GID}(R) < \infty$, and the converse implication is proved dually.

Then, let $\mathrm{GPD}(R) \leq n$ (for some positive integer n).

First, we argue that for any strongly Gorenstein projective R-module L, $\mathrm{Gid}(L) \leq n$.

From Proposition 3.9, there exists a short exact sequence $0 \to L \to P \to L \to 0$ where P is projective.

From the dual version of Horseshoe's lemma (see the remark below [27, Lemma 6.20]), with an injective resolution of L, we have the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & L & \to & P & \to & L & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & I_0 & \to & I_0 \oplus I_0 & \to & I_0 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & I_n & \to & E_n & \to & I_n & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where I_i is injective for $i = 0, ..., n-1$.
We claim that I_n is strongly Gorenstein injective. So, $\mathrm{Gid}(L) \leq n$.
Since P is projective, $\mathrm{id}(P) \leq n$ (by Lemma 3.3), hence E_n is injective. Then, from the injective version of Proposition 3.9 (1), it remains to prove that $\mathrm{Ext}^i(E, I_n) = 0$ for all injective R-modules E, and some positive integer i.
Indeed, we have $\mathrm{pd}(E) \leq n$ for such injective R-module E (from Lemma 3.5), which implies that $\mathrm{Ext}^i(E, M) = 0$ for all $i \geq n+1$ and all R-modules M, as desired.
This implies, from [23, Proposition 2.19], that $\mathrm{Gid}(Q) \leq n$ for all Gorenstein projective R-modules Q, since every Gorenstein projective module is a direct summand of a strongly Gorenstein projective module (Theorem 3.8).
Now, consider any R-module M. We may assume that $0 < \mathrm{Gpd}(M) \leq n$. Then, there exists a short exact sequence

$$0 \to K \to N \to M \to 0$$

such that N is Gorenstein projective and $\mathrm{Gpd}(K) \leq n-1$ [23, Proposition 2.18].
Then, by induction, $\mathrm{Gid}(K) \leq n$ and $\mathrm{Gid}(N) \leq n$. Finally, from Lemma 3.6, $\mathrm{Gid}(M) \leq n$. Thus, the proof is complete. ■

From this, we denote by $\mathbf{G-gldim(R)}$ the common value of $\mathrm{GPD}(R)$ and $\mathrm{GID}(R)$, and we call it *Gorenstein global dimension* of R.

Remarks 3.10 *1. From Lemmas 3.3 and 3.4, we deduce that if, for a ring R, $\mathrm{G-gldim}(R)$ is finite, then $\mathrm{G-gldim}(R)$ is also determined by the formulas:*

$$\begin{aligned} \mathrm{G-gldim}(R) &= \sup\{\mathrm{Gpd}_R(R/I) \mid I \textit{ ideal of } R\} \\ &= \sup\{\mathrm{Gpd}_R(M) \mid M \textit{ finitely generated } R-\textit{module}\}. \end{aligned}$$

This may be shown as a generalization of [17, Corollary 12.3.2].

2. Recall the finitistic projective dimension of a ring R, denoted by $\mathrm{FPD}(R)$, is defined by the formula: $\mathrm{FPD}(R) = \sup\{\mathrm{pd}_R(M) | M\ R-module \text{ with } \mathrm{pd}_R(M) < \infty\}$.
From [23, Theorem 2.28], if $\mathrm{G-gldim}(R)$ is finite, then $\mathrm{G-gldim}(R) = \mathrm{FPD}(R)$.
However, the fact that $\mathrm{FPD}(R)$ is finite does not assure that the Gorenstein global dimension

of R is also finite. For example, consider the ring $R = k[[X, Y]]/(X^2, XY)$ where k is a field; we have $\mathrm{G-gldim}(R) = \infty$, but $\mathrm{FPD}(R) = 1$.
Indeed, R is Noetherian with Krull dimension 1, $\dim(R) = 1$, but it is not Gorenstein ring (see [10, page 15]). So, by Corollary 2.3, $\mathrm{G-gldim}(R) = \infty$, and, from [29, Theorem 0.16], $\mathrm{FPD}(R) = \dim(R) = 1$.

3. *From Corollary 2.3, we deduce that the Gorenstein global dimension of any Noetherian ring R may be determined from its self-injective dimension, such that we have $\mathrm{G-gldim}(R) = \mathrm{id}_R(R)$. In general, if $\mathrm{G-gldim}(R) = n < \infty$, then $\mathrm{id}_R(R) \leq n$ (from Lemma 3.3). Nevertheless, the self-injective dimension of a non-Noetherian ring does not assure any information about its Gorenstein global dimension. Indeed, we can see this, using Theorem 5.2.4, if we consider a non-Noetherian self-injective ring (for example an infinite product of fields).*

The Gorenstein projective and injective dimensions of modules are refinements of the usual projective and injective dimensions, respectively [23, Proposition 2.27]. Now, it is natural to investigate how much the usual global and weak dimensions of rings differ from the Gorenstein global dimension.

Proposition 3.11 *For any ring R, $\mathrm{G-gldim}(R) \leq \mathrm{gldim}(R)$.*
The equality holds if $\mathrm{wdim}(R) < \infty$.

To prove this, we need the following lemma, which is a strong version of [6, Corollary 3.11]:

Lemma 3.12 *A strongly Gorenstein projective module is projective if, and only if, it has finite flat dimension.*
Consequently, if R is a ring with finite weak dimension, then the class of all projective R-modules and the class of all Gorenstein projective R-modules are the same class.

Proof. Use [6, Corollary 3.11] and [7, Theorem 2.5]. ■

Proof of Proposition 3.11. The inequality holds since every projective module is Gorenstein projective, and the equality holds from Lemma 3.12. ■

Remark 3.13 *Recall a Noetherian ring is called regular, if it has finite global (=weak) dimension. Proposition 3.11 is a generalization of the obvious fact that a Gorenstein ring is regular if, and only if, it has finite weak dimension. Thus, we have already examples of rings showing that the inequality of Proposition 3.11 may be strict and so the condition $\mathrm{wdim}(R) < \infty$ cannot be removed. In fact, it suffices to consider a non-regular Gorenstein ring (see for example [10, page 15]).*

Now, we study the behavior of modules over rings of finite Gorenstein global dimension.

The next result is a generalization of a result due to Iwanaga (see [17, Proposition 9.1.10]). It implies that, for a ring of finite Gorenstein global dimension, the class of all modules which have finite projective dimension, the class of all modules which have finite injective dimension, and the class of all modules which have finite flat dimension are all the same class. We use $\mathbf{\Omega}$ to denote this class.

Proposition 3.14 *Let R be a ring, and let M be an R-module. If $\mathrm{G-gldim}(R) \leq n$ for some positive integer n, then the following are equivalent:*

1. $\mathrm{pd}_R(M) < \infty$;
2. $\mathrm{pd}_R(M) \leq n$;

3. $\mathrm{id}_R(M) < \infty$;
4. $\mathrm{id}_R(M) \leq n$;
5. $\mathrm{fd}_R(M) < \infty$;
6. $\mathrm{fd}_R(M) \leq n$.

Proof. From Lemmas 3.3 and 3.4, and Lemma 3.5, only the implication (5) $\Rightarrow$ (1) merits a proof. So, this implication is proved immediately using the fact that if $\mathrm{FPD}(R)$ is finite, then every flat R-module has finite projective dimension [29, Theorem 0.13]. ■

The following is an extension of [17, Proposition 11.5.7].

Proposition 3.15 *Let R be a ring, and let M be an R-module. If $\mathrm{G-gldim}(R) < \infty$, then, for a positive integer m, the following are equivalent:*

1. $\mathrm{Gpd}_R(M) \leq m$;
2. $\mathrm{Ext}_R^{m+1}(M, X) = 0$ *for all* $X \in \mathbf{\Omega}$;
3. $\mathrm{Ext}_R^{i}(M, X) = 0$ *for all* $i > m$ *and all* $X \in \mathbf{\Omega}$.

Proof. Easy from Lemma 3.3. ■

For the strongly Gorenstein projective modules we have:

Proposition 3.16 *Let R be a ring. If $\mathrm{G-gldim}(R) < \infty$, then an R-module M is strongly Gorenstein projective if, and only if, there exists a short exact sequence $0 \to M \to P \to M \to 0$ where P is a projective R-module.*

Proof. This is obtained immediately from Proposition 3.14 and Proposition 3.9 (1). ■

Remark 3.17 *Each of Propositions 3.16 and 3.15 has a Gorenstein injective version.*

4 Gorenstein weak dimension of rings

In this section we investigate the Gorenstein flat dimension of commutative rings, which is canonically defined as follows:

Definition 4.1 *For any ring R, the Gorenstein flat dimension of R, denoted by $\mathrm{GFD}(R)$, is defined by the formula:* $\mathrm{GFD}(R) = \sup\{\mathrm{Gfd}_R(M) \mid M\ R\text{-module}\}$.

From the reason below (Theorem 4.2) and according to the classical terminology of homological dimensions of rings, the Gorenstein flat dimension of a ring R will be called *Gorenstein weak dimension* of R, and denoted by $\mathbf{G-wdim(R)}$.

We begin with the relation that exists between the Gorenstein weak dimension and the Gorenstein global dimension of rings.

Theorem 4.2 *For any ring R,* $\mathrm{G-wdim}(R) \leq \mathrm{G-gldim}(R)$.

Proof. Follows immediately from Lemma 4.3 bellow. ■

Lemma 4.3 *Let R be a ring. If* $\mathrm{G-gldim}(R)$ *is finite, then every Gorenstein projective R-module is also Gorenstein flat.*

Proof. Let $\mathrm{G-gldim}(R) < \infty$ for some positive integer n.
From the proof of [23, Proposition 3.4], it suffices to prove that the character of I (i.e., the R-module $I^* = \mathrm{Hom}_{\mathbb{Z}}(I, \mathbb{Q}/\mathbb{Z})$) has finite projective dimension for every injective R-module I.
For such module I, $\mathrm{fd}_R(I) < \infty$ (from Proposition 3.14). Then, $\mathrm{id}_R(I^*) < \infty$ (from [27, Theorem 3.52]). Therefore, from also Proposition 3.14, $\mathrm{pd}_R(I^*) < \infty$. ■

Note that this result gives a new condition so that the Gorenstein projective module be Gorenstein flat (see [23, Propostion 3.4] and [6, Proposition 1.3]).

Remark 4.4 *From Corollary 2.3, we have* $\mathrm{G-wdim}(R) = \mathrm{G-gldim}(R)$ *for any Noetherian ring R. This is an extension of the well-known equality* $\mathrm{wdim}(R) = \mathrm{gldim}(R)$ *when R is a Noetherian ring.*

Next result shows that the Gorenstein weak dimension is a refinement of the weak dimension of arbitrary rings.

Proposition 4.5 *For any ring R,* $\mathrm{G-wdim}(R) \leq \mathrm{wdim}(R)$.
The equality holds if $\mathrm{wdim}(R) < \infty$.

Proof. The inequality holds since every flat module is Gorenstein flat, and the equality follows immediately from [6, Corollary 3.8]. ■

Similarly to Proposition 3.16, we give a characterization of strongly Gorenstein flat modules over rings which have finite Gorenstein weak dimension. Recall if we consider all flat modules and homomorphisms of the complete flat resolution of Definition 2.1 (3) are the same, M will be called strongly Gorenstein flat module [6, Definition 3.1]. Similarly to Proposition 3.9, we give the following simple modification of [6, Proposition 3.6]

Proposition 4.6 *A module M is strongly Gorenstein flat if, and only if, there exists a short exact sequence $0 \to M \to F \to M \to 0$ where F is a flat module, and $\mathrm{Tor}_i(M, I) = 0$ for some integer $i > 0$ and for any module I with finite injective dimension (or for any injective module I).*

Proposition 4.7 *Let R be a ring. If* $\mathrm{G-wdim}(R) < \infty$*, then an R-module M is strongly Gorenstein flat if, and only if, there exists a short exact sequence $0 \to M \to F \to M \to 0$ where F is a flat R-module.*

Proof. The result follows immediately from Proposition 4.6 and the following lemma. ■

Lemma 4.8 *Let R be a ring. If* $\mathrm{G-wdim}(R) = n < \infty$ *for some positive integer n, then* $\mathrm{fd}_R(I) \leq n$ *for all injective R-modules I (or all R-modules I with finite* $\mathrm{id}_R(I)$*).*

Proof. Assume that $\mathrm{G-wdim}(R) = n < \infty$. Then, for all R-modules M, $\mathrm{Tor}_i^R(M, I) = 0$ for all $i > n$ and all injective R-modules I (or all R-modules I with finite $\mathrm{id}_R(I)$). This implies the desired result. ■

Now we give a partial affirmative answer to Question C (page 3) under the condition $\mathrm{G-gldim}(R) < \infty$.
Recall a ring is called perfect, if every flat module is projective [4]; a ring is called quasi-Frobenius, if it is Noetherian and self-injective (see [1]); and a ring is called weakly quasi-Frobenius, if it is coherent and self-FP-injective [21]. From [28, Theorem 4.4], we have:

A weakly quasi-Frobenius ring is quasi-Frobenius if, and only if, it is perfect.

As shown later (Theorems 5.2.2 and 5.2.4), the weakly quasi-Frobenius rings are those of Gorenstein weak dimension 0, and the quasi-Frobenius rings are those of Gorenstein global dimension 0. Now, we set the following extension:

Proposition 4.9 *If R is a perfect ring with finite Gorenstein global dimension, then:* $\mathrm{G-wdim}(R) = \mathrm{G-gldim}(R)$.

Proof. Follows immediately from the following lemma. ■

The following result gives a characterization of perfect rings which have finite Gorenstein global dimension.

Lemma 4.10 *If R is a ring with finite Gorenstein global dimension, then the following are equivalent:*

1. *R is perfect (i.e., every flat module is projective);*
2. *Every strongly Gorenstein flat R-module is strongly Gorenstein projective;*
3. *Every Gorenstein flat R-module is Gorenstein projective;*

Proof. (1) $\Rightarrow$ (2). Follows from Propositions 3.16 and 4.7.

(2) $\Rightarrow$ (3). This is simple consequence of Theorem 3.8 and [6, Theorem 3.5].

(3) $\Rightarrow$ (1). Let F be a flat R-module (then Gorenstein flat), hence it is Gorenstein projective. From Proposition 3.14, $\mathrm{pd}(F) < \infty$. Therefore, from [23, Proposition 2.27], F is projective. ■

Note, from Lemma 3.12, we deduce easily the implication $2 \Rightarrow 1$ of Lemma 4.10 above without assuming that $\mathrm{G-gldim}(R) < \infty$.

Now, we give a (functorial) characterization of the Gorenstein weak dimension of coherent rings. We shall see this deals with the n-FC rings, then of the FP-injective dimension of modules (see Definition 2.4). In fact, using Gorenstein (flat) weak dimension approach, this characterization gives and extends the list of properties characterizing the n-FC rings [9, Theorem 7].

Theorem 4.11 *If R is a coherent ring, then, for a positive integer n, the following are equivalent:*

1. $\mathrm{G-wdim}(R) \leq n$*;*
2. $\mathrm{Gfd}_R(G) \leq n$ *for all Gorenstein injective R-modules G;*
3. $\mathrm{Gfd}_R(M) \leq n$ *for all finitely presented R-modules M;*
4. $\mathrm{Gfd}_R(R/I) \leq n$ *for all finitely generated ideals I of R;*
5. $\mathrm{Tor}_i^R(M, E) = 0$ *for all $i > n$, all (finitely presented) R-modules M, and all FP-injective R-modules E (i.e.,* $\mathbf{fd(E) \leq n}$ ***for all FP-injective* $\mathbf{R}$*-modules* $\mathbf{E}$*);*

6. $\mathrm{Tor}^i_R(M, E') = 0$ *for all* $i > n$, *all (finitely presented)* R*-modules* M, *and all* R*-modules* E' *with finite* FP-$\mathrm{id}_R(E')$ *(i.e.,* $\mathbf{fd(E')} \leq \mathbf{n}$ ***for all*** $\mathbf{R}$***-modules*** $\mathbf{E'}$ ***with finite*** $\boldsymbol{FP}$-$\mathbf{id_R(E')}$*);*
7. $\mathrm{Tor}_i^R(M, I) = 0$ *for all* $i > n$, *all (finitely presented)* R*-modules* M, *and all injective* R*-modules* I *(i.e.,* $\mathbf{fd(I)} \leq \mathbf{n}$ ***for all injective*** $\mathbf{R}$***-modules*** $\mathbf{I}$*);*
8. $\mathrm{Tor}^i_R(M, I') = 0$ *for all* $i > n$, *all (finitely presented)* R*-modules* M, *and all* R*-modules* I' *with finite* $\mathrm{id}_R(I')$ *(i.e.,* $\mathbf{fd(I')} \leq \mathbf{n}$ ***for all*** $\mathbf{R}$***-modules*** $\mathbf{I'}$ ***with finite*** $\mathbf{id_R(I')}$*);*
9. $\mathrm{Ext}^i_R(M, F') = 0$ *for all* $i > n$, *all finitely presented* R*-modules* M, *and all* R*-modules* F' *with finite* $\mathrm{fd}_R(F')$ *(i.e.,* $\boldsymbol{FP}$-$\mathbf{id_R(F')} \leq \mathbf{n}$ ***for all*** $\mathbf{R}$***-modules*** $\mathbf{F'}$ ***with finite*** $\mathbf{fd_R(F')}$*);*
10. $\mathrm{Ext}^i_R(M, F) = 0$ *for all* $i > n$, *all finitely presented* R*-modules* M, *and all flat* R*-modules* F *(i.e.,* $\boldsymbol{FP}$-$\mathbf{id_R(F)} \leq \mathbf{n}$ ***for all flat*** $\mathbf{R}$***-modules*** $\mathbf{F}$*);*
11. $\mathrm{Ext}^i_R(M, R) = 0$ *for all* $i > n$, *all finitely presented* R*-modules* M *(i.e.,* $\mathbf{R}$ ***is*** $\mathbf{n}$-$\boldsymbol{FC}$*).*

Consequently, the Gorenstein weak dimension of R is also determined by the formulas:

$$\begin{aligned} \mathrm{G-wdim}(R) &= \sup\{\mathrm{Gfd}_R(R/I) \,|\, I \;\; \textit{finitely generated ideal of } R\} \\ &= \sup\{\mathrm{Gfd}_R(M) \,|\, M \; \textit{finitely presented } R-\textit{module}\} \\ &= \sup\{\mathrm{Gpd}_R(R/I) \,|\, I \;\; \textit{finitely generated ideal of } R\} \\ &= \sup\{\mathrm{Gpd}_R(M) \,|\, M \; \textit{finitely presented } R-\textit{module}\}. \end{aligned}$$

To prove this theorem we need the following characterization of the FP-injective dimension of modules over coherent rings.

Lemma 4.12 *Let* R *be a coherent ring, and let* M *be an* R*-module. Then, for some positive integer* n*, the following are equivalent:*

1. FP-$\mathrm{id}_R(M) \leq n$*;*
2. $\mathrm{Ext}^i_R(P, M) = 0$ *for all* $i > n$ *and all finitely presented* R*-modules* P*;*
3. $\mathrm{fd}_R(M^*) \leq n$*, where* $M^* = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ *the character module of* M*.*

Proof. From Definition 2.4 (1), the implication (2) ⇒ (1) is obvious.

W claim (1) ⇒ (2). Since R is coherent, for every finitely presented R-module P, we have a short exact sequence: $0 \to K \to L \to P \to 0$ where L is a finitely generated free R-module and K is a finitely presented R-module. Thus, considering the long exact sequence:

$$0 = \mathrm{Ext}^i(L, M) \to \mathrm{Ext}^i(K, M) \to \mathrm{Ext}^{i+1}(P, M) \to \mathrm{Ext}^{i+1}(L, M) = 0$$

and by induction we obtain the desired result.

The equivalence (2) ⇔ (3) is simply deduced from the isomorphism [22, Theorem 2.6.6]:

$$(\mathrm{Ext}^i_R(P, M))^* \cong \mathrm{Tor}_i^R(M^*, P)$$

for all finitely presented R-modules P. ■

Proof of Theorem 4.11. First, note that in the last equalities we also have the Gorenstein projective dimension of modules, this follows from [6, Proposition 1.3].

The implications (1) ⇒ (3) ⇒ (4) are obvious.

(4) $\Rightarrow$ (7). Follows from [23, Theorem 3.14] and [22, Theorem 1.3.8].

(7) $\Rightarrow$ (8). Easy, using an induction on $\mathrm{id}_R(I')$.

(8) $\Rightarrow$ (9). Let M be finitely presented R-module. From [22, Theorem 2.6.6], we have the following isomorphism:

$$(\mathrm{Ext}_R^i(M,F))^* \cong \mathrm{Tor}_i^R(M,F^*)$$

for any $i \geq 0$, and any R-module F.
If such R-module F is flat, then F^* is an injective R-module (from [22, Theorem 1.2.1]). Then, by hypothesis, $\mathrm{Tor}_i^R(M,F^*) = 0$, hence $(\mathrm{Ext}_R^i(M,F))^* = 0$. Therefore, $\mathrm{Ext}_R^i(M,F) = 0$ (from [27, Lemma 3.51]).

(9) $\Rightarrow$ (10) $\Rightarrow$ (11) Obvious.

(11) $\Rightarrow$ (1). This is Theorem 2.5.

(1) $\Leftrightarrow$ (2). The direct implication is straightforward.
For the converse implication, it suffices, from the above implications to prove the implication (2) $\Rightarrow$ (7). So, let I be an injective R-module. We claim $\mathrm{fd}_R(I) \leq n$. Since I is injective, it is Gorenstein injective. Then, $\mathrm{Gfd}_R(I) \leq n$ (by (2)). The case $\mathrm{Gfd}_R(I) = 0$ (i.e., I is Gorenstein flat) is evident. Then, we may assume that $0 < \mathrm{Gfd}_R(I) \leq n$. Thus, from [23, Theorem 3.23], there exists a short exact sequence $0 \to K \to G \to I \to 0$ where G is Gorenstein flat and $\mathrm{fd}_R(K) \leq n-1$. From Definition 2.1 (2), there exists a short exact sequence $0 \to G \to F \to G' \to 0$ where F is flat and G' is Gorenstein flat. Consider the following pushout diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & K & \to & G & \to & I & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & K & \to & F & \to & D & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & G' & = & G' & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

From the middle horizontal sequence $\mathrm{fd}_R(D) \leq n$. On the other hand, since I is injective, the right vertical sequence is split. Therefore, $\mathrm{fd}_R(I) \leq n$.

(5) $\Leftrightarrow$ (7). The direct implication is easy since every injective module is FP-injective.
For the converse implication, it suffices, from the above implications to prove the implication (10) $\Rightarrow$ (5). So, this implication is simply deduced from Lemma 4.12 and the following isomorphism [22, Theorem 1.1.8] (1): $(\mathrm{Tor}_R^i(M,E))^* \cong \mathrm{Ext}_i^R(M,E^*)$.

Finally, similarly to the equivalence (5) $\Leftrightarrow$ (7) above we prove the equivalence (6) $\Leftrightarrow$ (8). ■

Since, over Noetherian rings the class of all FP-injective modules and the class of all injective modules are the same class, and since the n-Gorenstein rings are n-FC, the following result is an extension of [17, Proposition 9.1.10] to n-FC rings.

Corollary 4.13 *Assume R be an n-FC ring. Let M be an R-module. Then, for a positive integer n, then the following are equivalent:*

1. *FP-*$\mathrm{id}_R(M) < \infty$*;*

2. $FP\text{-id}_R(M) \leq n$;
3. $\text{fd}_R(M) < \infty$;
4. $\text{fd}_R(M) \leq n$.

Proof. Follows immediately from Theorem 4.11. ■

Also, as consequence of Theorem 4.11, we give the following partial affirmative answers to Questions 1 and 2 page 3. We begin by the relation that exists between the large restricted flat dimension (see page 3) and the Gorenstein flat dimension of modules.

In [23], Holm gave a partial affirmative answer to the Foxby's Conjecture (see Question 1, page 3) concerning the relation between the large restricted flat dimension and the Gorenstein flat dimension of modules, that is: Let R be a ring. For any R-module M, we have inequality: $\text{Rfd}_R(M) \leq \text{Gfd}_R(M)$. The equality holds if R is Noetherian.
Next, we give an other partial affirmative answer to this question.

Proposition 4.14 *If R is a coherent ring with finite Gorenstein weak dimension, then* $\text{Rfd}_R(M) = \text{Gfd}_R(M)$ *for any R-module M.*

Proof. From [23, Theorem 3.19], it remains to prove the inequality $\text{Rfd}_R(M) \geq \text{Gfd}_R(M)$.
We may assume that $\text{Rfd}_R(M) = n < \infty$. Let I be an injective R-module, hence $\text{fd}_R(I) < \infty$ (by Theorem 4.11). Then $\text{Tor}_i^R(M, I) = 0$ for all $i > n$ (by the definition of $\text{Rfd}_R(M)$ (see page 3)). Therefore, from [23, Theorem 3.14], $\text{Gfd}_R(M) \leq n$, as desired. ■

Under the conditions R to be coherent and $\text{G}-\text{wdim}(R) < \infty$ (i.e., R to be n-FC for some positive integer n), the Gorenstein flat modules behave much more like the usual flat modules, as shown by [9, Corollary 8] and the following result:

Proposition 4.15 *If R is a coherent ring with finite Gorenstein weak dimension, then an R-module is Gorenstein flat if, and only if, M_P is a Gorenstein flat R_P-module for any prime ideal P of R.*

Proof. The direct implication is a particular case of [23, Proposition 3.10].

Inversely, assume that M_P is Gorenstein flat R_P-module for any prime ideal P of R. We claim that M is Gorenstein flat. Let, I be an injective R-module (hence $\text{fd}_R(I) < \infty$ by Theorem 4.11). From [27, Theorem 9.49], we have:

$$(\text{Tor}_i^R(M, I))_P \cong \text{Tor}_i^{R_P}(M_P, I_P)$$

for all positive integer i and every prime ideal P of R.
On the other hand, $\text{Rfd}_{R_P}(M_P) = 0$ (from Proposition 4.14). Then, $\text{Tor}_i^{R_P}(M_P, I_P) = 0$ for all $i > 0$ (since $\text{fd}_{R_P}(I_P) < \infty$). Thus, $(\text{Tor}_i^R(M, I))_P = 0$ for all $i > 0$ and every prime ideal P of R. This implies from [27, Theorem 3.80] that $\text{Tor}_i^R(M, I) = 0$ for all $i > 0$. Therefore, M is Gorenstein flat (by [23, Theorem 3.14]). ■

5 Closing remarks

5.1 How to remove the condition "G−gldim(R) < ∞" in Lemmas 3.3 and 3.4 ?

An attempt to arrive at a deeper understanding of the Gorenstein global dimension of rings leads us to set the following definition of two homological invariants of rings:

Definition 5.1.1 1. ***IP-dimension:*** *we say that a ring R has an IP-dimension, denoted by* $\mathrm{IP-dim}(R)$*, at most n (for some integer $n \geq 0$), if every injective R-module has projective dimension at most n.*

2. ***PI-dimension:*** *we say that a ring R has an PI-dimension, denoted by* $\mathrm{PI-dim}(R)$*, at most n (for some integer $n \geq 0$), if every projective R-module has injective dimension at most n.*

From Theorem 3.2, [17, Proposition 9.1.10], and [17, Theorem 9.1.11], we deduce that, if a ring R is Noetherian or has finite Gorenstein global dimension, then:

$$\mathrm{G-gldim}(R) = \mathrm{IP-dim}(R) = \mathrm{PI-dim}(R).$$

This seems interesting and raises many questions. Obviously, one would like to have these last equalities for arbitrary rings.
Later, we characterize rings of Gorenstein global dimension 0 (see Theorem 5.2.4); this points out that the desired equality holds if $\mathrm{IP-dim}(R)$ or $\mathrm{PI-dim}(R)$ is equal to 0.
Also, a careful reading of the proof of [20, Theorem 2.1, (1) ⇒ (4)] gives us the following:

Proposition 5.1.2 *If R is a coherent ring with finite both* $\mathrm{FPD}(R)$ *and* $\mathrm{IP-dim}(R)$*, then:*

$$\mathrm{G-gldim}(R) = \mathrm{IP-dim}(R) = \mathrm{PI-dim}(R).$$

Remark 5.1.3 *In [14], Enochs and Jenda introduced the copure injective dimension of a module M, denoted by $cid(M)$, as the largest positive integer n such that* $\mathrm{Ext}^n(E, M) \neq 0$ *for some injective module E. Obviously, one can see that* $\mathrm{IP-dim}(R) = \sup\{cid(M) | M\ R\text{−}module\}$.

5.2 Rings of small Gorenstein homological dimensions

Following the context of this paper and motivating by the important role of the rings of the classical global and weak dimensions smaller or equal to one in several areas of algebra, it seems appropriate, according to the terminology of the classical ring theory, to set the following definitions:

Definition 5.2.1 *For a ring R:*

1. *R is called G-semisimple, if* $\mathrm{G-gldim}(R) = 0$.
2. *R is called G-hereditary, if* $\mathrm{G-gldim}(R) \leq 1$*, and R is called G-Dedekind, if it is G-hereditary domain.*
3. *R is called G-von Neumann regular, if* $\mathrm{G-wdim}(R) = 0$.
4. *R is called G-semihereditary, if it is coherent and* $\mathrm{G-wdim}(R) \leq 1$*, and R is called G-Prüfer, if it is G-semihereditary domain.*

Then, every semisimple ring is G-semisimple; every hereditary (Dedekind) ring is G-hereditary (G-Dedekind); every von Neumann regular ring is G-von Neumann regular; and every semihereditary (Prüfer) ring is G-semihereditary (G-Prüfer). The converse of each implication holds, if the ring has finite weak dimension (Propositions 3.11 and 4.5).

These rings are related, from Theorem 4.2, as follows:

$$\begin{array}{ccc} G-semisimple & \Longrightarrow & G-von\ Neumann\ regular \\ \Downarrow & & \Downarrow \\ G-hereditary & \Longrightarrow & G-semihereditary \end{array}$$

From Corollary 2.3, the converse implications of the horizontal implications hold if the rings are Noetherian, and so they become the well-known 0-Gorenstein (i.e., quasi-Frobenius) and 1-Gorenstein rings, respectively.

The following result shows that a ring R such that $\mathrm{G-wdim}(R) = 0$ is coherent, then it is 0-FC ring. The 0-FC rings are extensively studied in non-commutative case, they are also called weakly quasi-Frobenius rings (see for example [21, 25, 28]). In commutative case, we have, from [28, Proposition 4.2], that these kind of rings and the IF-rings (i.e., rings which satisfied every injective module is flat [13, 25]) are the same rings (see [13, 21]).
Next, we set and extend the list of some known properties characterizing these kind of rings (in commutative case) using the Gorenstein projectivity, injectivity, and flatness approach.

Theorem 5.2.2 *For any ring R, the following are equivalent:*

1. *R is G-von Neumann regular;*
2. *Every Gorenstein injective R-module is Gorenstein flat;*
3. *Every strongly Gorenstein injective R-module is strongly Gorenstein flat;*
4. *R is coherent and self-FP-injective (i.e., R is* 0*-FC ring);*
5. *Every injective R-module is flat (i.e., R is IF-ring);*
6. *Every FP-injective R-module is flat;*
7. *R is coherent and every flat R-module is FP-injective;*

Proof. The equivalences of all statements, except 3, follow from Theorem 4.11.

We claim 3 ⇒ 5. Let I be an injective R-module, then it is strongly Gorenstein injective, then strongly Gorenstein flat (by (3)). Hence, there exists a short exact sequence $0 \to I \to F \to I \to 0$ where F is a flat R-module. This sequence splits (since I is injective); so, I is flat as a direct summand of F, as desired.

5 ⇒ 3. Let M be a strongly Gorenstein injective R-module. Then, there exists a short exact sequence $0 \to M \to I \to M \to 0$ where I is an injective R-module. From (5), I is flat, and $Tor(M, E) = 0$ for every injective R-module E. This means from Proposition 4.6, that M is strongly Gorenstein flat. ■

From [13, Proposition 5], the IF-rings have either the weak dimension to be equal 0 (i.e., von Neumann regular rings) or ∞. So, this can be shown, from Theorem 5.2.2, as a particular case of Proposition 4.5.

Next we characterize G-semisimple rings; we will see that they are the same the well-known quasi-Frobenius rings. Recall that a ring R is called quasi-Frobenius (QF-ring for short), if it is Noetherian and self-injective (i.e., 0-Gorenstein). These kind of rings have the following characterization:

Theorem 5.2.3 ([1], Theorem 31.9) *For a ring R, the following are equivalent:*

1. *R is quasi-Frobenius;*
2. *Every projective R-module is injective;*
3. *Every injective R-module is projective.*

Next result extends the list of properties characterizing the quasi-Frobenius rings using the Gorenstein dimensions.

Theorem 5.2.4 *For any ring R, the following are equivalent:*

1. *R is G-semisimple;*
2. *Every Gorenstein injective R-module is Gorenstein projective;*
3. *Every strongly Gorenstein injective R-module is strongly Gorenstein projective;*
4. *Every Gorenstein projective R-module is Gorenstein injective;*
5. *Every strongly Gorenstein injective R-module is strongly Gorenstein projective;*
6. *R is quasi-Frobenius.*

Proof. $(1) \Rightarrow (6)$. Use Lemma 3.3 and Theorem 5.2.3.

$(6) \Rightarrow (1)$. This is a simple case of Corollary 2.3.

The implications $(1) \Rightarrow (2)$ and $(1) \Rightarrow (4)$ are obvious.

The argument of $(2) \Rightarrow (6)$, $(4) \Rightarrow (6)$, $(3) \Leftrightarrow (6)$, and $(5) \Leftrightarrow (6)$ is similar to the proof of the equivalence $3 \Leftrightarrow 5$ of Theorem 5.2.2. ∎

Thus, Theorems 5.2.4 and 5.2.2 show that the G-semisimple and G-von Neumann regular rings are well-known, and so we have many examples of G-von Neumann regular rings which are neither G-semisimple nor von Neumann regular (see [13, 21]).

From [26, Theorems 69 and 206], we can see that, for a positive integer n, a ring R is n-Gorenstein if, and only if, the polynomial ring $R[X]$ is $(n+1)$-Gorenstein. Then, using Corollary 2.3, and consider, for example, the non-semisimple quasi-Frobenius ring $R = k[X]/(X^2)$ where k is a field (see [27, Exercice 9.24]), the polynomial ring $R[X]$ is an example of a Noetherian G-hereditary (=G-semihereditary) ring which is neither G-semisimple (=G-von Neumann regular) nor hereditary (=semihereditary).

Obviously, one would like to have examples out of the class of Noetherian rings.

Also one would like to have examples of G-Dedekind domains which are not Dedekind; and G-Prüfer domains which are neither G-Dedekind nor Prüfer.
In the literature, the Noetherian G-Dedekind domains (i.e., 1-Gorenstein domains) are extensively studied, and so we have examples of Noetherian G-Dedekind rings which are not Dedekind (see for instance [5]). From Remarks 3.10 (1) and [22, Theorems 2.5.13 and 2.5.14], the coherent G-Dedekind domains are Noetherian. Then, naturally we ask whether G-Dedekind domains are Noetherian ? and generally, whether G-hereditary rings are coherent ?

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