A NOTE ON UNENRICHED HOMOTOPY COENDS

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Abstract. Homotopy coends are an example of an indexed homotopy colimit. In this expository note, we give a definition of homotopy coend for arbitrary unenriched model categories and show how homotopy coends may be computed either using the cyclic bar resolution or with Grothendieck's derivator axioms.

1. Introduction

Homotopy colimits and their explicit realization with the bar resolution are two of the most useful tools in the topologist's kit: the decomposition of a space as a homotopy colimit of its constituent cells or skeleta is fundamental to many inductive arguments. Consequently the theory of homotopy colimits is (or ought to be) fundamental to any abstract axiomatization of homotopy theory. By now, there are many treatments of homotopy colimits and limits in the context of Quillen's model categories [21] and various generalizations [2, 3, 4, 9, 12, 13, 15, 18, 23]. This note is a brief description of the properties of homotopy coends, a special kind of homotopy colimit. It is written in the language of Quillen model categories, but most of the arguments rely only on the sorts of properties possessed by weak left Grothendieck derivators.

Homotopy coends appear frequently in the guise of a homotopy-invariant tensor product: e.g., the homotopy left Kan extension of a diagram $X : \mathscr{I} \to s\mathbf{Set}$ along $f : \mathscr{I} \to \mathscr{K}$ at $k \in \mathscr{K}$ ought to be the derived $\mathscr{I}$-tensor product of the contravariant $\mathscr{I}$-functor $\mathscr{K}(f-, k) : \mathscr{I}^{\mathrm{op}} \to \mathbf{Set}$ and the covariant $\mathscr{I}$-functor X. We expect to compute this by the realization of the two-sided bar resolution $B.(\mathscr{K}(f-, k), \mathscr{I}, X)$. However, in general there is no reason to restrict oneself to homotopy coends of $\mathscr{I}^{\mathrm{op}} \times \mathscr{I}$-diagrams that are obtained by products of $\mathscr{I}^{\mathrm{op}}$ and $\mathscr{I}$-diagrams. Indeed, the so-called cyclic bar resolution [20] (discussed in Section 5) is a workable generalization for homotopy coends in $s\mathbf{Set}$. However, when we replace $s\mathbf{Set}$ by an arbitrary model category $\mathscr{C}$, we may lose the simplicial enrichment; we can't use the bar resolution without the machinery of framings [9, 13, 15] or Dugger's structure theorems for model categories [7, 8, 6]. Moreover, although we can compute homotopy coends with the cyclic bar construction, we may wish to use another resolution. In fact this is possible. By regarding homotopy coends as homotopy colimits of certain diagrams over the twisted arrow category, we obtain a construction that works in any model category, agrees with the traditional definition of homotopy coend, and does not rely on an explicit resolution.

The material in this paper has appeared in some form or another in the literature of homotopy colimits: Dwyer and Kan introduced a homotopy end via the twisted arrow category in [10] and a comparison theorem for products of diagrams appears in [14].

2. Axiomatic derived left extensions and Grothendieck derivators

In this section, we'll give a brief recapitulation of homotopy left Kan extensions following [4]. Suppose $\mathscr{C}$ is a cofibrantly generated model category with all small limits and colimits. Recall that for any small category $\mathscr{I}$, the category $\mathscr{C}^{\mathscr{I}}$ of diagrams admits a model structure known as the *projective model structure* in which weak equivalences (respectively fibrations) are precisely those natural transformations $X \to Y$ for which $X_i \to Y_i$ is a weak equivalence (respectively fibration) for all $i \in \mathscr{I}$ [13, 15]. We can then construct the homotopy category $\operatorname{Ho}\mathscr{C}^{\mathscr{I}}$. Given any functor $f : \mathscr{I} \to \mathscr{K}$, the pullback $f^* : \mathscr{C}^{\mathscr{K}} \to \mathscr{C}^{\mathscr{I}}$ clearly preserves weak equivalences. In fact, something more is true: f^* is a right Quillen functor, since it preserves projective fibrations as well. Its left adjoint is left Kan extension $f_!$ [19]. This adjunction descends to an adjunction

$$\mathbf{L}f_! : \operatorname{Ho}\mathscr{C}^{\mathscr{I}} \rightleftarrows \operatorname{Ho}\mathscr{C}^{\mathscr{K}} : f^*$$

on the level of homotopy categories. When $\mathscr{K} = *$, the derived pushfoward $\mathbf{L}f_!$ is also called the *homotopy colimit functor* and denoted $\operatorname{hocolim}_{\mathscr{I}}$. Note that although the projective model structure ensures that all these derived functors exist, they are uniquely determined simply by the weak equivalences of $\mathscr{C}$.

Let's examine this construction from a global perspective. We make the following observations:

(1) The pullback functors fit together to give a strict 2-functor $\mathbf{Cat}^{\mathrm{op}} \to \mathbf{CAT}$ which assigns to $\mathscr{I}$ the category $\operatorname{Ho}\mathscr{C}^{\mathscr{I}}$. Note that this reverses both the direction of 1-cells and 2-cells. By the uniqueness of adjunctions (or, if the reader prefers, the fact that derived functors compose), $\mathbf{L}g_!\mathbf{L}f_! \simeq \mathbf{L}(gf)_!$ naturally if f and g are composable.

(2) Suppose $\mathscr{I}_s$, $s \in S$ is a (possibly empty) set of small categories and $\mathscr{I} = \coprod_{s\in S} \mathscr{I}_s$. The resulting product of pullback maps

$$\operatorname{Ho}\mathscr{C}^{\mathscr{I}} \to \prod_{s\in S} \operatorname{Ho}\mathscr{C}^{\mathscr{I}_s}$$

is an equivalence of categories.

(3) Suppose $\mathscr{I}$ is a small category; let $j : \mathscr{I}^\circ \to \mathscr{I}$ denote the inclusion of the lluf discrete subcategory of $\mathscr{I}$ (so $\operatorname{ob}\mathscr{I}^\circ = \operatorname{ob}\mathscr{I}$). Then the functor $j^* : \operatorname{Ho}\mathscr{C}^{\mathscr{I}} \to \operatorname{Ho}\mathscr{C}^{\mathscr{I}^{\mathrm{op}}}$ is conservative: if j^*F is a weak equivalence, $F \in \operatorname{ar}\mathscr{C}^{\mathscr{I}}$, then F is a weak equivalence.

(4) Suppose that $f : \mathscr{I} \to \mathscr{K}$ is a functor and $k \in \mathscr{K}$. Write $f \downarrow k$ for the comma category of pairs $i \in \mathscr{I}$, $fi \to k$. The 2-commutativity of the square

$$\begin{array}{ccc} f \downarrow k & \xrightarrow{\pi} & \mathscr{I} \\ {\scriptstyle p}\downarrow & \Leftarrow & \downarrow{\scriptstyle f} \\ * & \xrightarrow[k]{} & \mathscr{K} \end{array} \tag{2.1}$$

gives a change-of-base natural transformation

$$\mathbf{L}p_!\pi^* \to k^*\mathbf{L}f_!. \tag{2.2}$$

This is a weak equivalence: if X is a diagram $\mathscr{I} \to \mathscr{C}$, then

$$(\mathbf{L}f_!X)_k \simeq \operatorname*{hocolim}_{f\downarrow k}(X \circ \pi).$$

In the case that $\mathscr{C}$ is cofibrantly generated, one can prove this by noting that π^* and k^* are both left and right Quillen and then reducing (2.2) to the usual formula for left Kan extensions—see [5, Proposition 3.4.14] or [4]. A more geometric way to see this, when $\mathscr{C}$ is simplicially enriched, is using the bar resolution; we shall display the resolution in Section 5, but we'll be using (2.2) there, taking this item as a given.

A fancy way of saying this is that the functor $\mathrm{Ho}\,\mathscr{C}^{-} : \mathbf{Cat} \to \mathbf{CAT}$ is a weak left Grothendieck derivator. Our properties 2–4 are axioms Der1, Der2, and Der4g [4] (see also [12]).[1]

The properties we enumerated above have some far-reaching consequences. A simple observation, which we'll be making use of later, is the following:

Lemma 2.1. *Suppose $f : \mathscr{K} \to \mathscr{L}$ is a functor between small categories and $\mathscr{I}$ is a small category. The commutative square*

$$\begin{array}{ccc} \mathscr{I}\times\mathscr{K} & \xrightarrow{\mathrm{id}\times f} & \mathscr{I}\times\mathscr{L} \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ \mathscr{K} & \xrightarrow[f]{} & \mathscr{L} \end{array} \tag{2.3}$$

induces a natural isomorphism

$$\mathbf{L}p_!(\mathrm{id}\times f)^* \to f^*\mathbf{L}\pi_!. \tag{2.4}$$

Here p and π are the projection functors onto $\mathscr{K}$ and $\mathscr{L}$ respectively.

This ought to be true because f^* preserves colimits and weak equivalences; but it is not left Quillen in general. An example of a counterexample is the functor $f : G \to *$, G a nontrivial group: the cofibrant objects in $s\mathbf{Set}^G$ must be levelwise free G-sets, but the objects in the essential image of f^* have trivial G-action. Our proof is a consequence of the axioms 1–4 enumerated above; the reader is invited to translate it into the language of model categories.

Proof. If $\mathscr{K}$ is a point, the lemma says that homotopy colimits are computed pointwise in a functor category. We will first make a reduction to that case. Suppose $k \in \mathscr{K}$; we'll also write k for the functor $* \to \mathscr{K}$ mapping the unique object of $*$ to k. It is sufficient to check that

$$k^*\mathbf{L}p_!(\mathrm{id}\times f)^* \to k^*f^*\mathbf{L}\pi_!$$

is a weak equivalence, since all the functors k^* collectively are conservative. Now extend (2.3) to the left, observing that $p \downarrow k$ is isomorphic to $\mathscr{I} \times (\mathscr{K} \downarrow k)$:

$$\begin{array}{ccccccc} \mathscr{I} & \xrightarrow{\mathrm{id}\times g} & \mathscr{I}\times(\mathscr{K}\downarrow k) & \xrightarrow{\mathrm{id}\times \rho} & \mathscr{I}\times\mathscr{K} & \xrightarrow{\mathrm{id}\times f} & \mathscr{I}\times\mathscr{L} \\ {\scriptstyle r}\big\downarrow & & \big\downarrow{\scriptstyle q} & \Leftarrow & \big\downarrow{\scriptstyle p} & & \big\downarrow{\scriptstyle \pi} \\ * & = & * & \xrightarrow[k]{} & \mathscr{K} & \xrightarrow[f]{} & \mathscr{L} \end{array}$$

We use the base-change equivalence (2.2) and functoriality to reduce our problem to showing that

$$\mathbf{L}q_!(\mathrm{id}\times f\rho)^* \to (fk)^*\mathbf{L}\pi_!$$

[1]Axiom Der3g is the existence of the adjunctions $\mathbf{L}f_! \dashv f^*$. The "g" abbreviates *gauche;* the author is reluctant to translate this established notation).

is a weak equivalence.

Let $g : * \to \mathscr{K} \downarrow k$ be the inclusion of the terminal object. Then g is right adjoint to the unique functor $\ell : \mathscr{K} \downarrow k \to *$. This forces ℓ^* to be right adjoint to g^* by functoriality, so $g^* = \mathbf{L}\ell_!$. What's more, since $\ell g \to \mathrm{id}$ is an isomorphism, we see that $\mathrm{id} \to \ell^* \mathbf{L}\ell_!$ is an isomorphism. This is [4, Lemme 1.9]. Noting that $f\rho g = fk$, we obtain a further reduction: we need to show that

$$\mathbf{L}r_!(\mathrm{id} \times fk)^* \to (fk)^* \mathbf{L}\pi_!$$

is a weak equivalence. This follows from an analogous consideration of the diagram

$$\begin{array}{ccccc} \mathscr{I} & \longrightarrow & \mathscr{I} \times (\mathscr{L} \downarrow fk) & \longrightarrow & \mathscr{I} \times \mathscr{L} \\ {\scriptstyle r}\big\downarrow & & \big\downarrow & \Leftarrow & \big\downarrow{\scriptstyle \pi} \\ * & = & * & \xrightarrow{fk} & \mathscr{L}. \end{array} \qquad \square$$

We'll also need a much deeper property of homotopy colimits:

Definition 2.2. Suppose $\mathscr{I}$ is a small category. Write $\mathrm{N}\mathscr{I}$ for the nerve of $\mathscr{I}$, i.e, the simplicial set whose n-simplices are functors $[n] \to \mathscr{I}$. Now suppose $g : \mathscr{I} \to \mathscr{K}$ is a functor between small categories. We say f is *right homotopy cofinal* if $\mathrm{N}(k \downarrow f)$ is weakly contractible for all $k \in \mathscr{K}$.

Theorem 2.3 ([13, Theorem 19.6.7 (a)])**.** *With notation as above, suppose $X : \mathscr{K} \to \mathscr{C}$ is a diagram in an arbitrary small-cocomplete model category $\mathscr{C}$. The natural map*

$$\operatorname*{hocolim}_{\mathscr{I}} g^* X \to \operatorname*{hocolim}_{\mathscr{K}} X$$

is a weak equivalence.

We won't reprise a proof of this theorem. Note that when g is a right adjoint, it is right homotopy cofinal. The proof in this special case is straightforward (either using the cofibrant generation of $\mathscr{C}$, or the properties outlined above). One of the deep consequences of Cisinski's work [5] is that Theorem 2.3 holds in much greater generality, in essence for any Grothendieck derivator. Our paper will only discuss model categories, but the translation into the language of an arbitrary derivator should be clear.

We close this section with a remark. What if $\mathscr{C}$ is *not* cofibrantly generated? The diagram categories $\mathscr{C}^{\mathscr{I}}$ may not have a model structure with weak equivalences the objectwise weak equivalences. It is a remarkable fact that $\mathrm{Ho}\,\mathscr{C}^{\mathscr{I}}$ and all associated homotopy left and right Kan extensions may be constructed for an arbitrary small-complete and cocomplete model category $\mathscr{C}$. Conceptually, we can take $\mathrm{Ho}\,\mathscr{C}^{\mathscr{I}}$ to be the fundamental category of $(\mathrm{N}_{\mathrm{coh}}\mathscr{C})^{\mathscr{I}}$, where $\mathrm{N}_{\mathrm{coh}}$ is the coherent nerve. There are many approaches to this in the literature: in addition to [4], some references are [2, 3, 9, 13, 15, 18] and [23] for the enriched case. The key point is that the enumerated properties 1–4 still hold for an *arbitrary* model category $\mathscr{C}$. In particular, we can dualize to consider homotopy right Kan extensions in $\mathscr{C}$. In the remainder of this paper, we'll only use these properties of homotopy colimits and drop the cofibrant generation hypothesis, so our results are readily dualized to homotopy limits and ends.

3. Homotopy coends

Suppose $\mathscr{I}$ is a small category and $\mathscr{C}$ a category with all small colimits. Write $e\mathscr{I}$ for the category $\mathscr{I}^{\mathrm{op}} \times \mathscr{I}$. A *special cocone* on a diagram $F : e\mathscr{I} \to \mathscr{C}$ is a collection of arrows $\varphi_i : F_i^i \to X$ so that given any arrow $f : j \to i$ in $\mathscr{I}$, the square

$$\begin{array}{ccc} F_j^i & \xrightarrow{f^*} & F_j^j \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle \varphi_j} \\ F_i^i & \xrightarrow[\varphi_i]{} & X \end{array} \tag{3.1}$$

commutes. The initial special cocone on F is the *coend* of F and is denoted $\int^{i\in\mathscr{I}} F_i^i$ [19]. Coends can be manipulated in some ways similar to integrals; hence the suggestive notation. We can express $\int^{i\in\mathscr{I}} F_i^i$ as a colimit in the following way. Associated to $\mathscr{I}$ is a category we shall call the *lower twisted arrow category* $\mathrm{ar}_\tau\,\mathscr{I}$. The objects of $\mathrm{ar}_\tau\,\mathscr{I}$ are arrows in $\mathscr{I}$. A map $f \to g$ is a diagram

$$\begin{array}{ccc} \cdot & \longrightarrow & \cdot \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ \cdot & \longleftarrow & \cdot, \end{array} \tag{3.2}$$

i.e., a factorization of f through g. The lower twisted arrow category is the opposite of the Grothendieck construction on the Hom-functor $\mathscr{I}(-,-) : e\mathscr{I} \to \mathbf{Set}$ (for this definition, see, e.g., [24]). It is equipped with source and target functors $s : \mathrm{ar}_\tau\,\mathscr{I} \to \mathscr{I}$ and $t : \mathrm{ar}_\tau\,\mathscr{I} \to \mathscr{I}^{\mathrm{op}}$. The coend $\int^{i\in\mathscr{I}} F_i^i$ may be expressed as the colimit of $(t\times s)^*F$ over $\mathrm{ar}_\tau\,\mathscr{I}$ [19]. We also record another formulation of left Kan extensions in [19]: given $f : \mathscr{I} \to \mathscr{K}$, a diagram $F : \mathscr{I} \to \mathscr{C}$ and $D \in \mathscr{K}$, there is a natural isomorphism

$$\int^{i\in\mathscr{I}} \mathscr{K}(fi, D) \times F_i \cong (f_!F)_D. \tag{3.3}$$

Now suppose $\mathscr{C}$ is a small-cocomplete model category. The coend functor does not preserve weak equivalences of diagrams. But the above discussion suggests a homotopy-invariant replacement for $\int^{\mathscr{I}}$: given a diagram $F : e\mathscr{I} \to \mathscr{C}$, we set

$$\int_{\mathbf{L}}^{i\in\mathscr{I}} F_i^i = \operatorname*{hocolim}_{\mathrm{ar}_\tau\,\mathscr{I}} (t\times s)^*F. \tag{3.4}$$

Dwyer and Kan give a dual construction of homotopy ends in [10] in order to compute the derived mapping space between two diagrams of simplicial sets. We immediately have the following:

Proposition 3.1. *Homotopy coend preserves weak equivalences of diagrams.*

In this paper we will prove some other properties of the hoomotopy coend functor, the most important of which is the following:

Theorem 3.2. *Let $f : \mathscr{I} \to \mathscr{K}$ be a functor between small categories. Suppose $\mathscr{C}$ is a small-cocomplete model category and $F : \mathscr{I} \to \mathscr{C}$ is a diagram such that F_i is cofibrant for all $i \in \mathscr{I}$. Then there is a weak equivalence*

$$\int_{\mathbf{L}}^{i\in\mathscr{I}} \mathscr{K}(fi, D) \times F_i \to (\mathbf{L}f_!F)_D \tag{3.5}$$

natural in F and $D \in \mathscr{K}$.

Theorem 3.2 is a derived formulation of (3.3). Note that we need F_i to be cofibrant since the copower functor $\mathscr{K}(fi, D) \times -$ may not preserve weak equivalences. If coproducts in $\mathscr{C}$ preserve weak equivalences, then this assumption may be dropped.

We shall prove Theorem 3.2 in Section 4. Although the proof is involved, it is straightforward. Before proceeding, we'll list some other properties of the homotopy coend functor whose proofs are all straightforward.

Proposition 3.3. *Suppose $X : e\mathscr{I} \to \mathscr{C}$ is a diagram. Let $\tau : \mathscr{I}^{\mathrm{op}} \times \mathscr{I} \to \mathscr{I} \times \mathscr{I}^{\mathrm{op}}$ denote the interchange functor. Then there is a natural weak equivalence*

$$\int_{\mathbf{L}}^{i\in\mathscr{I}} X_i^i \simeq \int_{\mathbf{L}}^{i\in\mathscr{I}^{\mathrm{op}}} (\tau^* X)_i^i. \tag{3.6}$$

Proposition 3.4 (Fubini's theorem for homotopy coends)**.** *Suppose $\mathscr{I}$ and $\mathscr{K}$ are small categories and*

$$X : e(\mathscr{I} \times \mathscr{K}) \to \mathscr{C}$$

is a diagram. Then there is a natural weak equivalence

$$\int_{\mathbf{L}}^{i,k\in\mathscr{I}\times\mathscr{K}} X_{i,k}^{i,k} \simeq \int_{\mathbf{L}}^{i\in\mathscr{I}} \int_{\mathbf{L}}^{k\in\mathscr{K}} X_{i,k}^{i,k}. \tag{3.7}$$

Proposition 3.5. *Suppose $X : e\mathscr{I} \to \mathscr{C}^{\mathscr{K}}$ is a diagram. There is a weak equivalence*

$$\left(\int_{\mathbf{L}}^{i\in\mathscr{I}} X_i^i \right)_k \simeq \int_{\mathbf{L}}^{i\in\mathscr{I}} X_{i,k}^i \tag{3.8}$$

natural in $k \in \mathscr{K}$ and X.

Proposition 3.6. *Suppose $F : \mathscr{C} \to \mathscr{D}$ is a left Quillen functor and $X : e\mathscr{I} \to \mathscr{C}$ a diagram. Write $\mathbf{L}F$ for the left derived functor of F; then there is a natural weak equivalence*

$$\int_{\mathbf{L}}^{i\in\mathscr{I}} \mathbf{L}F(X_i^i) \simeq \mathbf{L}F \int_{\mathbf{L}}^{i\in\mathscr{I}} X_i^i \tag{3.9}$$

The proofs of most of these are routine. Propositions 3.4 and 3.5 follow from immediately from Lemma 2.1.

3.1. **Universality of the coend and enriched homotopy coends.** With the language of enriched category theory, coends can be expressed as an *indexed* colimit [17]: the weight for a diagram $X : e\mathscr{I}^{\mathrm{op}} \to \mathscr{C}$ is the Hom-functor $\mathscr{I}(-,-) : e\mathscr{I} \to$ **Set**. This is related to the fact that the lower twisted arrow category $\mathrm{ar}_\tau\, \mathscr{I}$ is the opposite of the Grothendieck construction applied to $\mathscr{I}(-,-)$. In enriched category theory, one is forced to use the language of weights rather than $\mathrm{ar}_\tau\, \mathscr{I}$.

Homotopy colimit may be defined in terms of a universal property, namely $\mathrm{hocolim} : \mathrm{Ho}\,\mathscr{C}^{\mathscr{I}} \to \mathrm{Ho}\,\mathscr{C}$ is the right Kan extension of $\mathrm{colim} : \mathscr{C}^{\mathscr{I}} \to \mathrm{Ho}\,\mathscr{C}$ along the localization map $\mathscr{C}^{\mathscr{I}} \to \mathrm{Ho}\,\mathscr{C}^{\mathscr{I}}$. Although Theorem 3.2 and the subsequent Propositions give some tools for computing homotopy coends, they are not a universal characterization of homotopy coends. Such a characterization arises from the language of indexed colimits—as Michael Shulman has pointed out to the author, a homotopy coend is an example of an indexed homotopy colimit. In a precise sense, the Grothendieck construction used to define the twisted arrow category is standing in for the weight. In enriched homotopy theory, the Grothendieck construction is unavailable; we may define homotopy coends either as a weighted homotopy colimit, or use the cyclic bar resolution (described below) to first define homotopy coends as a fundamental construction. There is a discussion of enriched homotopy coends along these lines in [23, §21]. In a subsequent paper with Mark Behrens, the author will discuss applications of enriched homotopy coends to categorified Fourier analysis and Goodwillie calculus [1].

4. A proof of Theorem 3.2

Let $\mathscr{C}$ be a small-cocomplete model category. Recall we are trying to construct a natural weak equivalence

$$\int_{\mathbf{L}}^{i\in\mathscr{I}} \mathscr{K}(fi, D) \times F_i \to (\mathbf{L}f_!F)_D,$$

where $F : \mathscr{I} \to \mathscr{C}$ is objectwise cofibrant, $f : \mathscr{I} \to \mathscr{K}$ is a functor, and $D \in \mathscr{K}$. The difficulty in proving Theorem 3.2 arises from the fact that the functor

$$(i \to j) \mapsto \mathscr{K}(fi, D) \times F_j \tag{4.1}$$

is unlikely to be cofibrant as a $\mathrm{ar}_\tau \mathscr{I}$-diagram, even if F is cofibrant. We will start by proving Theorem 3.2 in the special case that $\mathscr{K} = *$.

As promised, we will first show the following result.

Proposition 4.1. *The source functor* $s : \mathrm{ar}_\tau \mathscr{I} \to \mathscr{I}$ *is right homotopy cofinal. As a result, if* $F : \mathscr{I} \to \mathscr{C}$ *is a diagram,* $\int_{\mathbf{L}}^{i\in\mathscr{I}} F_i$ *is naturally weakly equivalent to the homotopy colimit* $\operatorname{hocolim}_{\mathscr{I}} F$.

Proof. That s is right homotopy cofinal is a consequence of our description of $\mathrm{ar}_\tau \mathscr{I}$ as the opposite of a Grothendieck construction. To avoid confusion about variance, we'll give an explicit proof. Let $i \in \mathscr{I}$. Define a functor $i \downarrow s \to (i \downarrow \mathscr{I})^{\mathrm{op}}$ sending the data

$$\begin{array}{ccc} i & \xrightarrow{g} & j \\ & & \downarrow{\scriptstyle h} \\ & & j' \end{array} \quad \text{to} \quad i \xrightarrow{hg} j'.$$

This has a left adjoint sending

$$i \xrightarrow{k} j' \quad \text{to} \quad \begin{array}{ccc} i & \xrightarrow{\mathrm{id}} & i \\ & & \downarrow{\scriptstyle k} \\ & & j'. \end{array}$$

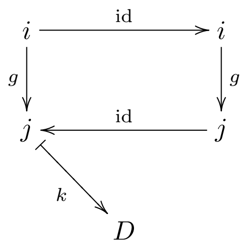
Hence $i \downarrow s \to (i \downarrow \mathscr{I})^{\mathrm{op}}$ induces an equivalence on nerves [24]; since $(i \downarrow \mathscr{I})^{\mathrm{op}}$ has a terminal object, $\mathrm{N}(i \downarrow s)$ is weakly contractible, so s is right homotopy cofinal. By Theorem 2.3, $\operatorname{hocolim} s^*F$ and $\operatorname{hocolim} F$ are weakly equivalent. □

Before we prove Theorem 3.2, we record a similar computation. Let $D \in \mathscr{K}$. Write $\Pi : \mathrm{ar}_\tau(f \downarrow D) \to \mathrm{ar}_\tau \mathscr{I}$ be the functor induced by projection $\pi : f \downarrow D \to \mathscr{I}$. We introduce a bit of shorthand: we'll write

$$i \overset{h}{\longmapsto} D$$

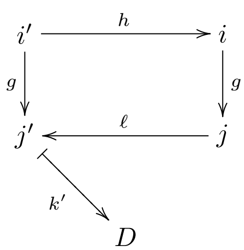
to denote the pair $i, h : fi \to D$ in $f \downarrow D$.

Lemma 4.2. *Suppose $g : i \to j$ is an arrow in $\mathscr{I}$. Regard the set $\mathscr{K}(fj, D)$ as a discrete category and define a functor $\mathscr{K}(fj, D) \to \Pi \downarrow g$ sending $k : fj \to D$ to the diagram*



The functor $\mathscr{K}(fj, D) \to \Pi \downarrow g$ is a right adjoint and hence homotopy right cofinal.

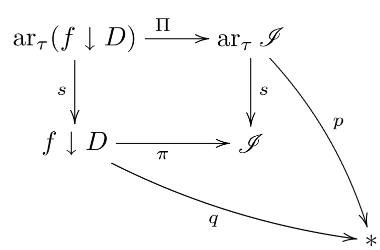
Proof. The left adjoint sends



to $k'\ell : j \mapsto D$. In general, a functor $r : \mathscr{A} \to \mathscr{B}$ is a right adjoint precisely when $b \downarrow r$ has an initial object for all $b \in \mathscr{B}$ [19], so all right adjoints are homotopy right cofinal. □

Proof of Theorem 3.2. Let $D \in \mathscr{K}$ and consider the commutative diagram

(4.2)



The base-change formula (2.2) for homotopy left Kan extensions gives a weak equivalence

$$\mathbf{L}q_! \pi^* F \simeq (\mathbf{L}f_! F)_D$$

Since s is right homotopy cofinal, there is a weak equivalence $\mathbf{L}s_! s^* \pi^* F \to \pi^* F$; hence

$$\mathbf{L}q_! \mathbf{L}s_! s^* \pi^* F \to \mathbf{L}q_! \pi^* F$$

is a weak equivalence. But since derived functors compose and (4.2) is commutative, there is a further equivalence

$$\mathbf{L}p_! \mathbf{L}\Pi_! \Pi^* s^* F \to \mathbf{L}q_! \mathbf{L}s_! s^* \pi^* F.$$

It suffices to find an equivalence between the $\mathrm{ar}_\tau\, \mathscr{I}$-diagrams

$$\mathbf{L}\Pi_! \Pi^* s^* F \to (t \times s)^* \big(\mathscr{K}(f-, D) \times F_-\big).$$

Weak equivalences are determined pointwise, so this reduces to producing a weak equivalence

$$(\mathbf{L}\Pi_!\Pi^* s^* F)_g \to \mathscr{K}(fj, D) \times F_i.$$

natural in $g : i \to j$. Using base change again, we have a weak equivalence

$$\operatorname*{hocolim}_{\Pi \downarrow g} \rho^* s^* F \to (\mathbf{L}\Pi_!\Pi^* s^* F)_g$$

where $\rho : \Pi \downarrow g \to \mathrm{ar}_\tau \mathscr{I}$ is the projection. By Lemma 4.2 and Theorem 2.3, we obtain the required weak equivalence

$$\coprod_{k \in \mathscr{B}(fj,D)} F_i \to \operatorname*{hocolim}_{\Pi \downarrow g} \rho^* s^* F.$$

Note that we have implicitly used the cofibrance of F_i here, since the coproduct on the left is really a homotopy colimit indexed on a discrete category. ☐

5. The cyclic bar construction

So far, we have avoided using any explicit cofibrant resolutions for homotopy colimits or coends. In this section we'll describe a well-known modification of the classical bar resolution that computes homotopy coends (see, e.g., [23, §21]). Assume $\mathscr{C}$ is a *simplicial* model category, i.e., that $\mathscr{C}$ is enriched over the category $s\mathbf{Set}$ of simplicial sets and satisfies Quillen's axiom SM7 [21]. Let $s\mathscr{C} = \mathscr{C}^{\Delta^{\mathrm{op}}}$ denote the category of simplicial objects in $\mathscr{C}$.

Definition 5.1. Let $X : e\mathscr{I} \to \mathscr{C}$ be a diagram in $\mathscr{C}$. Define the *circular bar resolution* of X to be the simplicial object $B^{\mathrm{cyc}}_\cdot(\mathscr{I}, X)$ whose n-simplices are

$$B^{\mathrm{cyc}}_n(\mathscr{I}, X) = \coprod_{\iota:[n] \to \mathscr{I}} X^{\iota(n)}_{\iota(0)}.$$

Now suppose $F : \mathscr{I}^{\mathrm{op}} \to \mathbf{Set}$ and $G : \mathscr{I} \to \mathscr{C}$ are functors. By taking copowers we can define a functor $F \times G : e\mathscr{I} \to \mathscr{C}$ with $(F \times G)^i_j = F^i \times G_j$. The *classical bar resolution* is the simplicial object $B_\cdot(F, \mathscr{I}, G) = B^{\mathrm{cyc}}_\cdot(\mathscr{I}, F \times G)$.

The enrichment of $\mathscr{C}$ over $s\mathbf{Set}$ yields an adjoint pair

$$|-| : s\mathscr{C} \rightleftarrows \mathscr{C} : \mathrm{Sing}$$

where $|Y_\cdot|$ is the coend $\int^{[n] \in \Delta} \Delta[n] \otimes Y_n$. We call $|Y_\cdot|$ the *geometric realization* of $Y_\cdot$ and quote without proof the following Proposition, originally due to Reedy [22] (see also [13, 15]):

Proposition 5.2. *The adjunction* $|-| \dashv \mathrm{Sing}$ *is a Quillen pair when* $s\mathscr{C}$ *is equipped with the Reedy model structure. Furthermore, if* $X_\cdot \in s\mathscr{C}$ *is Reedy cofibrant, then there is a weak equivalence* $|X_\cdot| \simeq \operatorname{hocolim}_{[n] \in \Delta^{\mathrm{op}}} X_n$.

We now state the main result of this section.

Theorem 5.3. *Suppose* $X : e\mathscr{I} \to \mathscr{C}$ *is an objectwise cofibrant diagram. Then there is a weak equivalence*

$$\left|B^{\mathrm{cyc}}_\cdot(\mathscr{I}, X)\right| \simeq \int_{\mathbf{L}}^{i \in \mathscr{I}} X^i_i$$

natural in X.

Recall that if $G : \mathscr{I} \to \mathscr{C}$ is an objectwise cofibrant diagram,

$$\big|B_\cdot(*, \mathscr{I}, G)\big| \simeq \operatorname*{hocolim}_{\mathscr{I}} G. \tag{5.1}$$

This is a classical fact—in fact, some authors define the homotopy colimit via the bar resolution (see, for example [2, 13]). The equivalence of the two definitions follows from the following facts: first, we may define a functor $B(\mathscr{I}, \mathscr{I}, G) : \mathscr{I} \to \mathscr{C}$ given by

$$B(\mathscr{I}, \mathscr{I}, G)_j = \big|B_\cdot(\mathscr{I}(-, j), \mathscr{I}, G)\big|. \tag{5.2}$$

This maps to the constant simplicial functor at G via the $\mathscr{I}$-functoriality of G. The resulting map

$$\varepsilon : \big|B(\mathscr{I}, \mathscr{I}, G)\big| \to G \tag{5.3}$$

is a weak equivalence. Since $B(\mathscr{I}, \mathscr{I}, G)$ is Reedy cofibrant in $s\mathscr{C}$, its geometric realization is cofibrant, so ε is a cofibrant resolution. Furthermore, there is an isomorphism

$$\operatorname*{colim}_{\mathscr{I}} \big|B(\mathscr{I}, \mathscr{I}, G)\big| \cong \big|B_\cdot(*, \mathscr{I}, G)\big| \tag{5.4}$$

In fact, more generally, if $f : \mathscr{I} \to \mathscr{K}$ is a functor and $D \in \mathscr{K}$, then

$$(\mathbf{L}f_! G)_D \simeq \big|B_\cdot(\mathscr{K}(f-, D), \mathscr{I}, G)\big|,$$

so Theorem 5.3 furnishes an alternative proof of Theorem 3.2 in the case that $\mathscr{C}$ is simplicially enriched.

We only need a modest modification of these steps to verify Theorem 5.3. This is all more-or-less standard, but we'll reprise the proofs.

Lemma 5.4. *Suppose $X : e\mathscr{I} \to \mathscr{C}$ is objectwise cofibrant. Then $B_\cdot^{\text{cyc}}(\mathscr{I}, X)$ is Reedy cofibrant.*

Proof. The nth latching map for $B_\cdot^{\text{cyc}}(\mathscr{I}, X)$ is the map

$$\coprod_{\iota \in (\operatorname{sk}_{n-1} \mathrm{N}\mathscr{I})_n} X_{\iota(0)}^{\iota(n)} \to \coprod_{\iota \in (\mathrm{N}\mathscr{I})_n} X_{\iota(0)}^{\iota(n)}.$$

By assumption, this is a coproduct of cofibrations. □

Corollary 5.5. *The functor $\big|B_\cdot^{\text{cyc}}(\mathscr{I}, -)\big|$ preserves weak equivalences of objectwise cofibrant diagrams $e\mathscr{I} \to \mathscr{C}$.*

Lemma 5.6. *Suppose $X \in s^2\mathscr{C}$ is a bisimplicial object in $\mathscr{C}$. Then there is a natural isomorphism*

$$\int^{[n],[m] \in \Delta} (\Delta[n] \times \Delta[m]) \otimes X_{n,m} \cong \int^{[n] \in \Delta} \Delta[n] \otimes X_{n,n}.$$

Proof of Theorem 5.3. This proof is a slightly modified version of Hollender and Vogt's proof of [14, Proposition 6.2]. We'll actually prove the result for $\tau^* X$, where $\tau : e\mathscr{I} \to e\mathscr{I}$ is the symmetry (two applications of a symmetry are unavoidable in this method of proof). Define a bisimplicial object $B_{\cdot,\cdot} \in s^2\mathscr{C}$ as follows with

$$B_{m,n} = \coprod_{\iota : [m] \to \mathscr{I}^{\mathrm{op}}} \coprod_{\kappa : [n] \to \mathscr{I}} \mathscr{I}(\kappa(n), \iota(m)) \times X_{\kappa(0)}^{\iota(0)}$$

Note that we can identify $B_{m,n} = \coprod X^{\iota(0)}_{\kappa(0)}$ with the coproduct indexed on diagrams

$$\begin{array}{ccccccc} \kappa(0) & \longrightarrow & \kappa(1) & \longrightarrow & \cdots & \longrightarrow & \kappa(n) \\ & & & & & & \downarrow \\ \iota(0) & \longleftarrow & \iota(1) & \longleftarrow & \cdots & \longleftarrow & \iota(m) \end{array} \tag{5.5}$$

in $\mathscr{I}$. Now, $\operatorname{diag} B_{\cdot,\cdot}$ is the classical bar resolution

$$B_\cdot(\mathscr{I}(-,-), e\mathscr{I}, X) \cong B_\cdot(*, \operatorname{ar}_\tau \mathscr{I}, (t \times s)^* X), \tag{5.6}$$

so $|\operatorname{diag} B_{\cdot,\cdot}| \simeq \int_{\mathbf{L}}^{i\in\mathscr{I}} X_i^i$. Geometrically realize $B_{m,n}$ in the variable n: for fixed $[m] \in \Delta$, we have

$$\begin{aligned} \int^{[n]\in\Delta} \Delta[n] \otimes B_{m,n} &\cong \coprod_{\iota:[m]\to\mathscr{I}^{\mathrm{op}}} \int^{[n]\in\Delta} \coprod_{\kappa:[n]\to\mathscr{I}} \mathscr{I}(\kappa(n), \iota(m)) \times X^{\iota(0)}_{\kappa(0)} \\ &\cong \coprod_{\iota:[m]\to\mathscr{I}^{\mathrm{op}}} \left| B_\cdot(\mathscr{I}(-,\iota(m)), \mathscr{I}, X_\cdot^{\iota(0)}) \right| \\ &\cong B_m^{\mathrm{cyc}}\big(\mathscr{I}^{\mathrm{op}}, \left| B_\cdot(\mathscr{I}(-,\iota(m)), \mathscr{I}, X_\cdot^{\iota(0)}) \right|\big). \end{aligned}$$

Since X is objectwise cofibrant, there is a weak equivalence

$$\left| B_\cdot(\mathscr{I}(-,\iota(m)), \mathscr{I}, X_\cdot^{\iota(0)}) \right| \to X^{\iota(0)}_{\iota(m)}$$

natural in $\iota(m)$ and $\iota(0)$. Furthermore, both the source and target of this arrow are cofibrant in $\mathscr{C}$. Now geometrically realize in the m direction; we find there is a weak equivalence

$$\begin{aligned} \int^{[m],[n]\in\Delta} (\Delta[m] \times \Delta[n]) \otimes B_{m,n} &= \left| [m] \mapsto B_m^{\mathrm{cyc}}\big(\mathscr{I}^{\mathrm{op}}, \left| B_\cdot(\mathscr{I}(-,\iota(m)), \mathscr{I}, X_\cdot^{\iota(0)}) \right|\big) \right| \\ &\simeq \left| B_\cdot^{\mathrm{cyc}}(\mathscr{I}^{\mathrm{op}}, \tau^* X) \right|. \end{aligned}$$

Here $\tau : e\mathscr{I} \to e\mathscr{I}$ is the symmetry. Assembling all this together and recalling Lemma 5.6 and Proposition 3.3, we obtain a chain of weak equivalences

$$\left| B_\cdot^{\mathrm{cyc}}(\mathscr{I}^{\mathrm{op}}, \tau^* X) \right| \simeq \int_{\mathbf{L}}^{i\in\mathscr{I}} X_i^i \simeq \int_{\mathbf{L}}^{i\in\mathscr{I}^{\mathrm{op}}} (\tau^* X)_i^i. \qquad \square$$

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