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Unsteady Flow Calculations by Numerical Methods[1]

A review of methods of handling unsteady flow problems in metal pipes by numerical methods is undertaken. The characteristic method, typifying explicit methods, and the centered implicit method are developed, including the manner various boundary conditions are introduced into the solutions. High velocity flow is briefly reviewed, i.e., flow cases with the velocity of flow of the same order of magnitude as the pulse wave speed. Three complex boundary conditions are examined: turbomachinery, column separation, and the compressed gas accumulator.

THE calculation of unsteady flow of liquids in pipes by numerical methods is reviewed in this paper. The first definitive treatment of the subject is that of Joukowsky [1][2] in 1898 which not only developed the theory, but confirmed the findings by large scale experiments. Another early pioneer in the field is Allievi [2]. Until about 1930, waterhammer calculations were made by the arithmetic method which neglected friction, and applied reflection coefficients at the ends of the piping elements. Graphical methods of calculation came into practical use about 1930, and were developed to a high degree by many practitioners in the field [3, 4, 5, 6]. In the early 1960's, the digital computer was applied to unsteady flow problems and is now in the process of supplanting the graphical and arithmetic methods. Early work leading to application of the digital computer methods include Gray [7]; Perkins, Tedrow, Eagleson, and Ippen [8]; and Wood, Dörsch, and Lightner [9].

The basic partial differential equations of continuity and motion, including the equation of state for the liquid and the elastic properties of the pipe, are placed into algebraic finite-difference equations by use of either implicit or explicit formulations. The most popular explicit method utilizes the method of characteristics and is the only explicit method discussed in this treatment. The centered-implicit method is treated as representative of the implicit approach.

[1] Symposium Paper: State-of-the-Art: Fluid Transients.
[2] Numbers in brackets designate References at end of paper.

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Nomenclature

A = pipe cross-sectional area
A_R = reservoir area
a = pulse wave speed (acoustic velocity) in pipe
$B1$–$B4$ = constants
D = inside dia of pipe
E = Young's modulus for pipe wall material
e = pipe wall thickness
f = Darcy-Weisbach friction factor
g = acceleration due to gravity
H = piezometric head (elevation of hydraulic gradeline)
H_{bar} = barometric head
HCM = constant in $c-$ characteristic equation
HCP = constant in $c+$ characteristic equation
H_{valve} = head loss across valve (steady state)
K = bulk modulus of elasticity
k = loss coefficient
l = length of riser pipe
M = number of units in operation
M_0 = number of units running at constant speed
M_f = number of units changing speed
m = exponent on friction term
N = number of reaches in a pipe; rotational speed of turbomachine
$P1$–$P6$ = known functions of p
P = point on xt-diagram
p = pressure
Q = discharge (volume per unit time flowing)
R = steady-state fluid friction coefficient
TDH = total dynamic head
T_L = wave travel time through pipe
t = time
V = average velocity
$\forall$ = volume
x = distance along pipe, measured positively downstream
z = elevation above datum
γ = specific weight of liquid
ρ = mass density of liquid
θ = constant in implicit method
τ = dimensionless valve opening

The basic equations are first presented, followed by the methods of solution for the basic flow element, first by the method of characteristics and then by the centered-implicit method. Questions of stability and accuracy are discussed. Boundary conditions are introduced in terms of algebraic and ordinary differential equations. Applications to high velocity flow are considered, followed by a discussion of frequently encountered boundary conditions, including turbomachines, column separation, and the gas accumulator.

Equations of Continuity and Motion

Unsteady flow theory of liquids in metal pipes assumes that the liquid is compressible and that the pipe wall is extensible, but limited to an extent that the liquid density ρ and the pipe diameter D do not change by more than, say, one percent. For these conditions the bulk modulus of elasticity K may be considered to be a constant (as well as Young's modulus E for the pipe walls). Hence, the area A of a cross section is taken as constant, but it is recognized that the small changes in A and in ρ determine the acoustic wave speed a in the systems. When high velocity flows are considered, with velocity V varying from zero up to a, the density must be considered to change appreciably, but strength considerations require small changes only in the pipe areas.

Poisson ratio effects alter the relationship between the internal pressure p in the pipe and its cross-sectional area, which changes the acoustic wave speed by a few percent. In technical applications liquids usually contain some dissolved air and other gases which may come out of solution in the form of microscopic bubbles when pressure is reduced. These bubbles do not immediately return to the dissolved state upon return to higher pressure. The bubbles reduce the density very slightly, but greatly reduce the bulk modulus of elasticity of the liquid. One tenth percent of air content in bubbles in water reduces the acoustic wave speed by more than fifty percent [10]. Because of the uncertainty as to the microscopic bubble content of liquid during a transient, the smaller effects of Poisson's ratio are neglected.

Continuity Equation. With reference to Fig. 1 showing a control volume within a short segment of pipe, the net mass inflow per unit time is equated to the time rate of increase of mass within the control volume,

$$-(\rho Q)_x \delta x = (\rho A)_t \delta x \tag{1}$$

x and t are the independent variables distance along the pipe, measured positively downstream, and time, respectively. $Q(Q = AV)$ is the discharge (volume per unit time flowing). The independent variable subscripts represent partial differentiation with respect to the subscript. Equation (1) may be written as

$$V_x + \frac{\dot{\rho}}{\rho} + \frac{\dot{A}}{A} = 0 \tag{1a}$$

which is also the proper equation when Poisson effects are included.

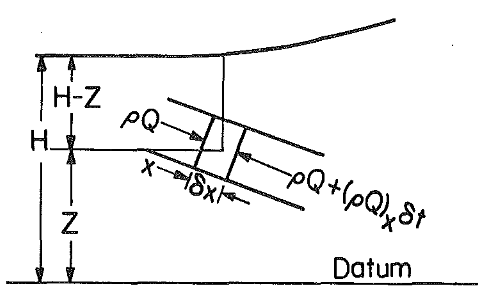

Fig. 1 Fluid element in a conduit

Bulk Modulus of Elasticity. For small changes in density the bulk modulus is the constant K given by

$$K = \rho \frac{dp}{d\rho} \tag{2}$$

p is the pressure. Equation (2) may also be written as

$$K = \rho \frac{\dot{p}}{\dot{\rho}} \tag{3}$$

with the dot indicating total differentiation with respect to time.

Elastic Pipe Wall Equation. In terms of the pipe wall thickness e and the Young's modulus E, the rate of change of area is given by

$$\frac{\dot{A}}{A} = \frac{\dot{p}D}{eE} \tag{4}$$

considering thin wall pipe stresses and neglecting Poisson's ratio effects. By use of $p = \rho g(H - z)$ from Fig. 1, equations (1), (3), and (4) may be combined to yield the continuity equation in a more useful form

$$H_t + \frac{a^2}{gA} Q_x = 0 \tag{5}$$

Some small terms have been neglected, g is the acceleration of gravity, and a is defined as the collection of constants

$$a = \sqrt{\frac{K/\rho}{1 + \frac{K}{E}\frac{D}{e}}} \tag{6}$$

H is the piezometric head, or elevation of the hydraulic gradeline above an arbitrary fixed datum, Fig. 1.

Equation of Motion. The equation of motion for a small segment of liquid in a conduit may take the form

$$H_x + \frac{1}{gA} Q_t + RQ|Q|^m = 0 \tag{7}$$

R is the steady-state resistance coefficient per unit length of conduit. If one wishes to use the Darcy-Weisbach friction factor f, $m = 1$, and

$$R = \frac{f}{2DA^2g} \tag{8}$$

Equations (5) and (7) are the controlling equations for unsteady flow of liquid in a closed conduit. As they are quasilinear equations, there is no closed form solution. The method of characteristics solution and the centered-implicit solution are discussed in the next section.

Methods of Solution for the Flow Element

To solve an unsteady flow problem, the transmission of piezometric head and discharge through a pipe, or segment of pipe, must be formulated in algebraic form, and methods must be available to bring into the problem the boundary conditions arising at the ends of the pipes.

By use of the method of characteristics, equations (5) and (7) become transformed into the following two algebraic equations:

$$C^+: \quad H_P - H_A + \frac{a}{gA}(Q_P - Q_A) + R\Delta x Q_A|Q_A|^m = 0 \tag{9}$$

$$C^-: \quad H_P - H_B - \frac{a}{gA}(Q_P - Q_B) - R\Delta x Q_B|Q_B|^m = 0 \tag{10}$$

Equation (9) is valid along a line in the xt-plane, Fig. 2, having the slope $\Delta t/\Delta x = 1/a$, and equation (10) is valid along a line having the slope $\Delta t/\Delta x = -1/a$. In the equations $\Delta x = a\Delta t$, and the friction terms have been evaluated by using values of Q at the beginning of the time increment Δt. The characteristics method shows a to be the pulse wave speed (acoustic velocity).

For a single pipe, having constant wave speed a, the solution may be visualized on an xt-plot, Fig. 3, using a rectangular grid whose intersections represent specific locations along the pipe at specified times. By selecting N, the number of reaches in the pipe, as an even number, then $\Delta x = L/N$, with L the pipe length, and $\Delta t = \Delta x/a$. In general, steady, or known, conditions prevail at $t = t_0$, and the transient is to be calculated for later times, when particular boundary conditions are imposed on each end of the pipe. Calculations are made at the intersections of the diagonals of Fig. 3, and the boundary calculations are made at time increments of $2\Delta t$.

To calculate the piezometric head H and flow Q at point P, H_A, Q_A, H_B, and Q_B are known in equations (9) and (10). After eliminating Q_p in the two equations, the first of the following two equations is obtained:

$$H_P = 0.5\left(H_A + H_B + \frac{a}{gA}(Q_A - Q_B) + R\Delta x(Q_B|Q_B|^m - Q_A|Q_A|^m)\right) \qquad (11)$$

$$Q_P = Q_A - \frac{gA}{a}(H_P - H_A + R\Delta x Q_A|Q_A|^m) \qquad (12)$$

Equation (12) is equation (9) solved for Q_p.

To calculate the boundary conditions, the C^+ equation (9) is available as a linear equation in Q_P and H_P at the downstream end, and similarly equation (10) applies at the upstream end. One piece of information is needed from outside the piping element at each end that states the value of Q_P, H_P, or some relation between the two.

Evangelisti [11] has extended the concept of the rectangular grid to apply to a pipe or pipes in series where a may vary along the length of line, but is a known function of x. The cross-sectional area may also change somewhat, by steps, along the pipe. He first calculates the time T_L for a wave to travel the full length of the system, then divides this time by an even integer to obtain Δt, the time increment for vertical spacing of the grid. T_L is given by

$$T_L = \int_0^L \frac{dx}{a} \qquad (13)$$

The horizontal spacings Δx_i may now vary across the grid. Starting at the upstream end Δx_1 is given by

$$\frac{T_L}{N} = \int_0^{\Delta x_1} \frac{dx}{a} \qquad (14)$$

and Δx_2 is determined by

$$2\frac{T_L}{N} = \int_0^{\Delta x_1 + \Delta x_2} \frac{dx}{a} \qquad (15)$$

Δx_1 is the only unknown in equation (14) and then Δx_2 is the unknown in equation (15). Continuing with these integrals all Δx_i are found including Δx_N. When the integral is carried across a change in cross section, the value of R and a/gA must be adjusted to yield a fair approximation of the whole segment. It should be mentioned that there is no value of a/gA for the change in cross section case that will yield the proper natural frequency of the system (this frequency has different values when the flow direction sense is reversed). Fig. 4 indicates the form of this staggered grid.

When two or more pipes comprise a system, such as a parallel, branching or network system of pipes, it is essential that the same Δt be used for all calculations, so that the boundary conditions can be worked out. To find the proper Δx_i throughout the system may require that either the lengths of the pipes be altered somewhat, or that the estimated wave speeds be adjusted. In some situations neither of these means are satisfactory; it is then necessary to use interpolations [12, pp. 36–38], to use lumped segments of pipelines (treated as incompressible), or to use a combined implicit-characteristic method [13].

Implicit Method. In the centered-implicit method equations (5) and (7) are used to solve for Q and H at the end points of a segment of pipe, Fig. 5, but since each of the two equations has the four unknowns H_{PA}, Q_{PA}, H_{PB}, and Q_{PB}, the solution of the

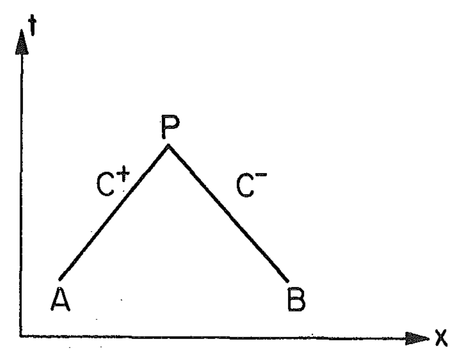

Fig. 2 Characteristic lines in the *xt*-plane

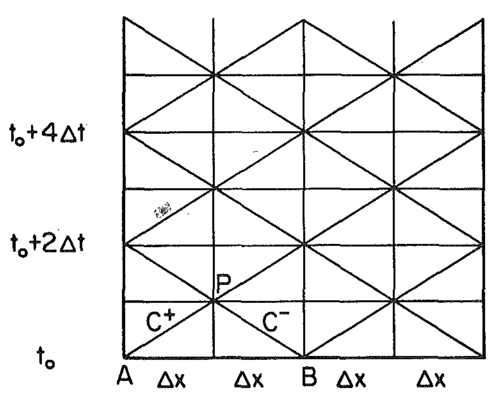

Fig. 3 Staggered rectangular grid

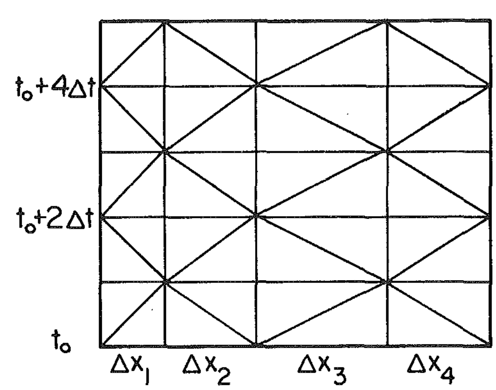

Fig. 4 Staggered grid for series pipeline, or for variable wave speeds

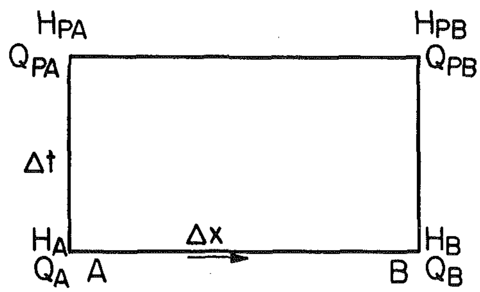

Fig. 5 Centered-implicit cell for a segment of pipe of length Δx

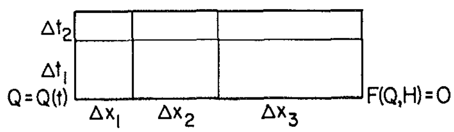

Fig. 6 3-cell implicit model of pipeline

whole system, including boundary conditions, must be accomplished simultaneously.

By the centered-implicit method, the terms in equations (5) and (7) become, in finite different notation

$$H_t = \frac{H_{PA} + H_{PB} - H_A - H_B}{2\Delta t} \tag{16}$$

$$Q_x = \frac{Q_{PB} + Q_B - Q_{PA} - Q_A}{2\Delta x} \tag{17}$$

$$H_x = \frac{H_{PB} + H_B - H_{PA} - H_A}{2\Delta x} \tag{18}$$

$$Q_t = \frac{Q_{PA} + Q_{PB} - Q_A - Q_B}{2\Delta t} \tag{19}$$

$$Q = 0.25(Q_A + Q_B + Q_{PA} + Q_{PB}) \tag{20}$$

Equation (5) becomes

$$H_{PA} + H_{PB} + C1(Q_{PB} - Q_{PA}) + C2 = 0 \tag{21}$$

in which

$$C1 = \frac{a^2}{gA}\frac{\Delta t}{\Delta x} \quad C2 = C1(Q_B - Q_A) - H_A - H_B \tag{22}$$

and equation (7) becomes

$$H_{PB} - H_{PA} + C3(Q_{PA} + Q_{PB}) + C4 + R\frac{\Delta x}{2}(Q_{PA} + Q_{PB} + Q_A + Q_B) \times |0.25(Q_{PA} + Q_{PB} + Q_A + Q_B)|^m = 0 \tag{23}$$

in which

$$C3 = \frac{\Delta x}{gA\Delta t} \quad C4 = H_B - H_A - C3(Q_A + Q_B) \tag{24}$$

Equations (21) and (23) are the algebraic equivalents of equations (5) and (7). The implicit method has no restrictions on the ratio $\Delta x/\Delta t$ (as in the case of the characteristics method) which makes it much more flexible in applying to systems of pipes of various lengths. The time increment must be the same for all portions of the system in the simultaneous solution, but Δt may change for the next set of cells, as indicated in Fig. 6. To solve the problem of Fig. 6, a series piping system having 3 cells, there are two equations for each cell, plus one equation for boundary condition at each end, yielding eight equations for the eight unknowns, Q_P, H_P at each section. As the equations of motion (23) are nonlinear, the eight equations are usually solved by the Newton-Raphson method.

Boundary Conditions

The algebraic equations relating head and discharge at each of the ends of pipe reaches have been developed for the characteristics and the centered-implicit methods. To solve actual problems the boundary conditions must be expressed in suitable form. In the characteristics method, one equation is always available from the adjacent pipe reach (C^+ equation downstream and C^- equation upstream). This equation in effect brings to the boundary all information from the system and may be referred to as the system response equation at the boundary. This equation has the form

$$Q_P = B_1 H_P + B_2 \tag{25}$$

with B_1 and B_2 known constants at time of application of the equation. Q_P and H_P are the unknown discharge and piezometric head at the boundary at the end of the time increment Δt under consideration. A few very simple boundary conditions follow:

Reservoir at End of Pipe. H_P is a known constant, so Q_P may be found directly from the characteristic equation.

Waves on a Reservoir. H_P may be expressed as a known function of time, from information about the waves, and Q_P then may be calculated from the characteristic equation.

Valve at Downstream End of Pipe. The valve is treated as an orifice of variable area. In dimensionless form

$$\frac{Q_P}{Q_0} = \tau\sqrt{\frac{H_P}{H_0}} \tag{26}$$

in which τ is the dimensionless valve opening, equal to unity for initial steady state for Q_0 and piezometric head H_0 (the datum for elevation of hydraulic gradeline must be through the centerline of the valve for this simple form of equation). In general, τ is a known function of time which describes the motion of the valve. The C^+ characteristic equation for the downstream pipe reach is solved simultaneously with equation (26), requiring the solution of a quadratic equation.

Simple Constant Speed Centrifugal Pump at Upstream End of Pipe. By taking the elevation datum as the suction reservoir surface, the head-discharge curve may be approximated by a quadratic equation

$$H_P = H_0 + b_1 Q_P + b_2 Q_P^2 \tag{27}$$

in which H_0 is the shutoff head. This equation is solved with the C^- equation for the upstream reach of pipe to find H_P and Q_P.

One big advantage of the characteristic method is the explicit solution for Q_P and H_P at every computing section.

Implicit method boundary conditions. The boundary condition equations may be expressed in the same analytic form for either the implicit or the characteristic solutions. The solutions for all unknowns, however, must be made simultaneously in the implicit method, which usually requires an iterative solution by the Newton-Raphson method.

Simple Differential Equation Boundary Condition. If a small storage tank of cross-sectional area A_R supplies liquid to a piping system, the change in surface elevation affects the transient flow, Fig. 7. The differential equation for flow from the reservoir is given by

$$-A_R dz = Q dt \tag{28}$$

As the piezometric head is the liquid surface, this equation may be approximated by

$$-A_R(H_P - H_1) = \frac{Q_P + Q_1}{2}\,2\Delta t \tag{29}$$

in which H_1 and Q_1 are head and discharge at the boundary from the previous boundary calculation $2\Delta t$ seconds earlier. As H_P and Q_P are the only unknowns, the solution consists of solving the C^- equation and equation (29), two linear equations in two unknowns. The simultaneous solution for the whole system is required in the implicit method.

A second order procedure for solving a differential equation boundary condition is discussed for turbomachines.

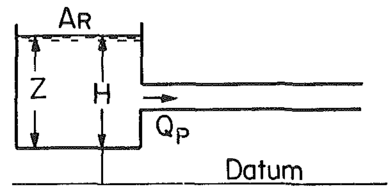

Fig. 7 Small reservoir at upstream end of pipeline

Comparison of Implicit and Characteristics Methods

The characteristics method is unconditionally stable for all Δx, so long as $\Delta t = \Delta x/a$. The longer the reach Δx for the basic computation unit, the less well the friction is modeled. When friction is very important, as in long pipelines, a trapezoidal approximation for friction may be made, yielding the following characteristic equations:

$$C^+: H_P - H_A + \frac{a}{gA}(Q_P - Q_A) + \frac{R}{2}\Delta x(Q_A|Q_A|^m + Q_P|Q_P|^m) \quad (30)$$

$$C^-: H_P - H_B - \frac{a}{gA}(Q_P - Q_B) - \frac{R}{2}\Delta x(Q_B|Q_B|^m + Q_P|Q_P|^m) \quad (31)$$

These equations are to be compared with equations (9) and (10). The equations are now nonlinear, and require an iterative-type (Newton-Raphson) solution for each interior point and at the boundaries.

For the same values of Δx and Δt, and using equations (9) and (10), the computer program is usually more simple to construct, and easier to debug, than the implicit one; it also requires less computing time. The implicit method has no restriction on the ratio $\Delta x/\Delta t$, which permits various sizes of Δx_i to be accommodated within a system for given Δt. It also permits Δt to be large to handle slow, long duration transient computations economically. The implicit scheme has neutral stability as written in equations (17) and (18). After generalizing these equations, they become

$$Q_x = \frac{\theta(Q_{PB} - Q_{PA}) + (1 - \theta)(Q_B - Q_A)}{\Delta x} \quad (32)$$

$$H_x = \frac{\theta(H_{PB} - H_{PA}) + (1 - \theta)(H_B - H_A)}{\Delta x} \quad (33)$$

When $\theta = 0.5$, equations (32) and (33) become equations (17) and (18). For most piping systems with changing boundary conditions $\theta = 0.5$ has yielded satisfactory results. A complex piping system implicit program for a pumped-storage project, which contained the details of the penstock bifurcations was found to yield satisfactory results until steady-state conditions started to be re-established. Then, with continued running of the program, the implicit scheme became unstable. By increasing θ to some value between 0.5 and 1.0, the instabilities disappeared. Open channel applications [14, 15] first indicated the instabilities in the centered-implicit method. Recent unpublished work indicates a considerable difference in results as θ is varied between 0.5 and 1.0 for open channel calculations.

The implicit method has also been noted to yield unsatisfactory results for very sudden, sharp transients. Physically, the imposition of a sudden pressure increase cannot be noticed at distance Δx away in less than $\Delta x/a$ sec. However, in the simultaneous solution required by the implicit method, an appreciable change in conditions (H and Q) can be observed at distances greater than a Δt. In severe cases the resulting transient calculations are unreliable.

The implicit scheme models the friction in a very satisfactory manner, probably better than the trapezoidal scheme of the characteristics method, because it uses the average discharge obtained from the four corners of the cell, Fig. 5.

In some physical systems being modeled, it appears advantageous to use both methods. The inclusion of characteristic reaches tends to reduce the size of matrices needed in the implicit solution, and large values of Δt may be employed by including the short reaches in the implicit portion.

High Velocity Flow

The characteristic equations in finite difference form, equations (9) and (10) are for those situations where velocity is very small compared with pulse wave speed. For very severe transients, with pressure changes of tens of thousands of psi, the velocity may become of the same order of magnitude as the pulse wave speed. Small terms in the derivations of continuity and the equation of motion, which previously were neglected, may now be of importance. Not only must these terms be considered, but the method of solution of the characteristic equations must be altered, either by using interpolations (which leads to inaccuracies) or by use of the characteristic grid.

The piping must be extremely strong to withstand the large pressure changes, and this means that the diameter changes do not increase much more than with the low velocity transient cases. The liquid density may change, however, by as much as 20 percent, so that it is appropriate to consider the bulk modulus of elasticity as a linear function of p rather than as a constant,

$$K = K_0 + k_1 p \quad (34)$$

With equation (34) and the retention of small terms, the continuity equation and the equation of motion become

$$F_1 = M_x + P6P_t = 0$$

$$F2 = p_x + P1M|M| + P2MM_x + P3M_t + P4M^2P_x + P5Mp_t = 0$$

in which $P1$ through $P6$ are known functions of p only. M is the mass flow. These two equations may be converted to four ordinary differential equations by the method of characteristics, in a similar manner to the previous low-velocity case. The equations are quite complicated and are not reproduced here. The four equations, in finite difference form, are solved simultaneously for point P, say, of Fig. 8, when conditions at A and B are known. As the equations are nonlinear, the Newton-Raphson method may be used to solve for x, t, p, and M. This procedure is then repeated to complete the characteristics grid.

The centered-implicit method gave unsatisfactory results for the sharp transient cases, as illustrated in Fig. 9. This case involved the rise of pressure at the upstream end of a 200 ft pipe from 1000 psi to 2000, 4000, and 8000 psi at 0.0, 0.01, 0.02, and 0.03 sec, respectively. The wave, physically, could only reach the 150-ft location, but the implicit method showed a substantial change at the downstream end of the pipe.

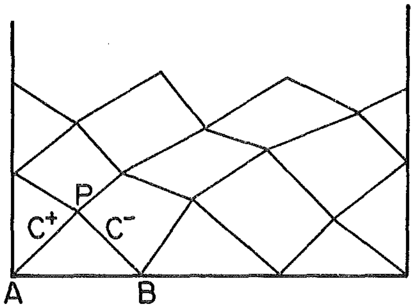

Fig. 8 Characteristics grid for high-velocity flow

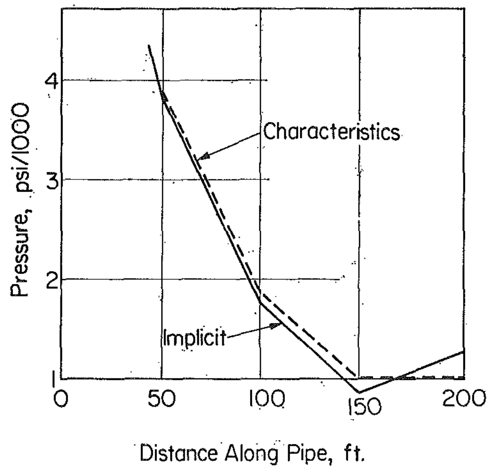

Fig. 9 High velocity transient case

Series Pipes, Branching Pipes, and Networks

It has been noted that a series pipe system may be treated on a single x-t diagram by using Evangelisti's method of a rectangular grid. For complex systems it is usually necessary to consider a separate xt-diagram for each pipe comprising the system. A series connection is shown in Fig. 10. If minor losses and changes in velocity head are neglected, then $Q_{P1} = Q_{P2}$ and $H_{P1} = H_{P2}$ and the two end characteristics equations are solved for the two unknowns. If it is desired to take minor losses and velocity-head changes into account, then the energy equation (steady state) is written from point 1 to point 2 of Fig. 10, including the loss term for a sudden expansion

$$H_{P_1} + \frac{Q_P{}^2}{2gA_1{}^2} = H_{P_2} + \frac{Q_P{}^2}{2gA_2{}^2} + \frac{KQ_P{}^2}{2gA_1{}^2} \qquad (35)$$

This equation in the three unknowns Q_P, H_{P_1}, and H_{P_2} is solved simultaneously with the two characteristics equations.

For a general branching junction, such as that shown in Fig. 11, with two inflow pipes, two outflow pipes and a known flow demand $Q(t)$, the solution is quite simple when minor losses are neglected or are accounted for by increasing the resistance terms for each pipe. The continuity equation for the function is

$$Q_{P_1} + Q_{P_2} = Q_{P_3} + Q_{P_4} + Q(t) \qquad (36)$$

In addition, the four characteristic equations may be expressed as

$$H_P = \text{HCP1} - B1Q_{P_1} \qquad (37)$$

$$H_P = \text{HCP2} - B2Q_{P_2} \qquad (38)$$

$$H_P = \text{HCM3} + B3Q_{P_3} \qquad (39)$$

$$H_P = \text{HCM4} + B4Q_{P_4} \qquad (40)$$

with H_P, Q_{P_1}, Q_{P_2}, Q_{P_3}, and Q_{P_4} the unknowns. By substituting for the Q_P's into equation (36)

$$H_P = \frac{\text{HCP1}/B1 + \text{HCP2}/B2 + \text{HCM3}/B3 + \text{HCM4}/BB - Q(t)}{1/B1 + 1/B2 + 1/B3 + 1/B4} \qquad (41)$$

This equation is easily changed to any number of inflow or outflow pipes. The actual flow directions, of course, do not have to be in the direction of the arrows.

One may solve a transient network by solving for each branch junction plus other boundary conditions, as well as for internal sections along the pipes. This method, however, would require a special computer program for each network. More sophisticated methods are available in which the program is quite general and the network to be solved is specified in the input data for the program [16, 17].

Complex Boundary Conditions

Three complex boundary conditions are considered in this section, all based on the characteristics method. They are turbomachines, column separation, and the compressed gas surge tank.

Turbomachines. The turbomachines may be pumps or turbines, and they may be connected to a reservoir or to suction and discharge pipes, with a discharge valve as shown in Fig. 12. The transient problems are startup of pumps, loss of power (or shutoff) of pumps, and loss of generator load for turbines. A single machine is first considered, or several identical machines in parallel, acting in identical fashion. The case of several identical parallel units, some of which lose power or load, is next considered.

For one or more identical machines, the two characteristic equations for pipelines are available, plus the head balance equation over the units and discharge valves, and the differential equation relating torque to rotational acceleration of the units. The characteristic equations are

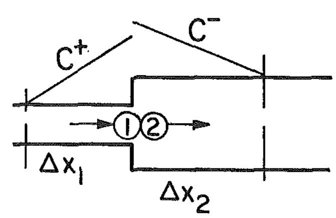

Fig. 10 Series connection

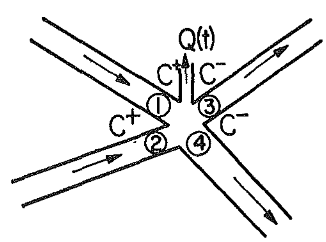

Fig. 11 Branching of pipes

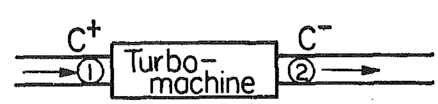

Fig. 12 Turbomachine with discharge valve

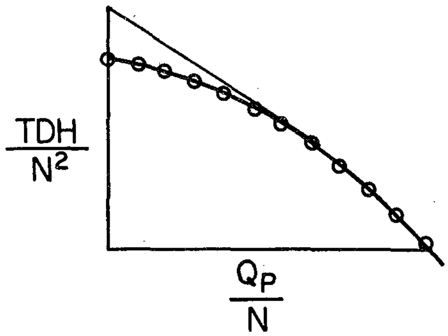

Fig. 13 Head-discharge curve for normal zone of pump operation

$$H_{P_1} = \text{HCP} - B1Q_PM \tag{42}$$

$$H_{P_2} = \text{HCM} + B2Q_PM \tag{43}$$

with HCP, HCM being known at time of application of the equations. M is the number of units and Q_P the flow through a unit. The head balance equation is

$$H_{P_1} + \text{TDH} - \frac{H_{\text{valve}}}{\tau^2 Q_R^{\,2}} Q_P|Q_P| = H_{P_2} \tag{44}$$

in which TDH is the total dynamic head and H_{valve} is the loss across the valve when in its steady state position($\tau = 1$) discharging the rated flow Q_R. As the valve closes, τ is considered to be a known function of time, varying from 1.0 to zero. The turbomachine characteristic curves may be represented by straight-line equations through points on the curves at the current points of interest. In Fig. 13, for a portion of the curve, the straight line

$$\frac{\text{TDH}}{N^2} = A_0 + A_1 \frac{Q_P}{N} \tag{45}$$

represents the curve; N is the rotational speed of machine; A_0 and A_1 are calculated for each time increment, based on the extrapolated value of Q_P/N for end of the calculation period Δt. When N is small the homologous curve may be replaced by the straight line

$$\frac{\text{TDH}}{Q_P^{\,2}} = A_0 + A_1 \frac{N}{Q_P} \tag{46}$$

It is essential that good characteristic curve data be obtained for the units to ensure reliable transient calculations for all zones of operation. In a similar manner the torque T data may be represented either by

$$\frac{T}{N^2} = C_0 + C_1 \frac{Q_P}{N} \tag{47}$$

or by

$$\frac{T}{Q_P^{\,2}} = C_0 + C_1 \frac{N}{Q_P} \tag{48}$$

From the torque equation

$$T = \frac{-WR^2\pi}{g30} \frac{dN}{dt} \tag{49}$$

the rate of change of speed is related to torque, N is in rpm, and WR^2 is the product of weight of rotating parts (including entrained liquid) and the square of the radius of gyration.

By Substitution into equation (44) for H_{P_1}, H_{P_2} and TDH

$$\text{HCP} - B1Q_PM + N^2(A_0 + A_1Q_PN) - \frac{H_{\text{valve}}}{\tau^2 Q_R^{\,2}} Q_P|Q_P| = \text{HCM} + B_2Q_PM \tag{50}$$

This algebraic equation contains two unknowns Q_P and N for the end of the time period Δt. Equation (49) solved for ΔN yields (with equation (47))

$$\Delta N = -\frac{30g\Delta t}{WR^2\pi} N^2(C_0 + C_1Q_P/N) \tag{51}$$

To solve equations (50) and (51) a second order procedure is utilized. Since N and Q are known at the beginning of the period Δt, from equation (51) a first order approximation of ΔN is found. By using this, a first order value N_1 of the speed is found at the end of Δt. By use of N_1 in equation (50), it is solved for Q_P. This value of Q_P and N_1 are now used in equation (51) to find a new ΔN. The two ΔN's are averaged to find the speed change and the N is then calculated for end of Δt. Equation (50) is again solved for Q_P with the new N. With Q_P and N known at end of Δt, all other quantities are easily calculated.

For the case of parallel units with M_0 units operating and M_f units failing, two new equations are needed (and equation (50) is rewritten). The continuity equation

$$MQ_P = M_0Q_{P_0} + M_fQ_{Pf} \tag{52}$$

in which Q_{P_0} is flow through an operating unit, and Q_{Pf} is flow through a failing unit. The head balance for operating pumps is

$$\text{HCP} - B1Q_PM + N_R^{\,2}(A_{00} + A_{10}Q_{P_0}/N_R) - \frac{H_{\text{valve}}}{Q_R^{\,2}} Q_{P_0}|Q_{P_0}| = \text{HCM} + B_2Q_PM \tag{53}$$

Equation (50) becomes, for the failing units

$$\text{HCP} - B1Q_PM + N^2(A_{0f} + A_{1f}Q_{Pf}/N) - \frac{H_{\text{valve}}}{\tau^2 Q_R^{\,2}} Q_{Pf}|Q_{Pf}| = \text{HCM} + B_2Q_PM \tag{54}$$

The procedure for solving is to find N_1, as before, then solve equations (52), (53), and (54) for Q_{P_0}, Q_{Pf}, and Q_P. Then equation (51) is solved again using Q_{Pf} and M_1 to find a new N. The two ΔN's are averaged and N is computed. Equations (52), (53), and (54) are then again solved for Q_P, Q_{Pf}, and Q_P the final values for end of period Δt.

The outlined procedure applies to failing pumps or to turbine trips outs. If the turbine gates are moving as a known function of time, this requires an extra interpolation of the turbine characteristic curves for the proper gate position.

Column Separation. With the usual technical liquids carried in pipelines, vapor pockets will form as soon as the pressure in the pipe is reduced to vapor pressure. Upon occurrence of vapor pressure at an internal computing section, the hydraulic grade line is then known at the section and computation is treated as a boundary condition with known head. The volume of the vapor pocket is calculated, assuming the liquid does not have vapor pockets between sections. The procedure is continued until the volume of the vapor pocket becomes negative (or zero), then the section is treated as an ordinary internal section again with $HP(I)$ and $QP(I)$ both as unknowns. In general the size of vapor pocket is very small compared with the volume of liquid between sections, so each reach is treated as if it were completely filled with liquid. The cavity is assumed to stay at the section.

The procedure to use in computing is to check after each section piezometric head calculation to see if it is below vapor pressure. If so, an index is set to indicate an internal boundary condition is in effect. With reference to Fig. 14, $QPP(I)$ is calculated from the C^+ characteristic equation and $QP(I)$ from the C^- equation; then, the first cavity size is

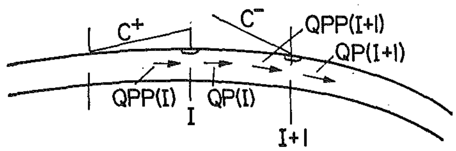

Fig. 14 Notation for calculation of cavity volume

$$CAV(I) = 0.5(QP(I) - QPP(I))\Delta T \qquad (55)$$

Each calculation $QPP(I)$ is stored as $QQP(I)$, and the next cavity calculation is

$$CAV(I) = CAV(I) + 0.5(Q(I) + QP(I) - QQP(I) - QPP(I))\Delta t \qquad (56)$$

After each cavity calculation a check is made to see if it is negative. If so, the index is changed for an internal calculation and the section is recalculated for $HP(I)$ and $QP(I)$. No special calculation of pressure rise upon collapse of the pocket is needed. It is important that transient calculations be continued for, say, $2L/a$ sec after collapse, as the worst situation may develop at another section of the system. Laboratory experiments indicate that the pressure rise is actually less than the value obtained from this procedure.

Compressed Gas Surge Tanks. If of the proper size and location, a compressed gas accumulator or surge tank can be most helpful in alleviating sharp transients in piping systems. The gas, being much more compressible than liquid, can absorb energy, or release it to the system with relatively slow changes in pressure. An accumulator just downstream from a reciprocating pump may greatly reduce the pressure variations at the pump and throughout the system. In order to determine the effectiveness of the accumulator, the system may be analyzed for transient behavior.

In Fig. 15 an accumulator is shown attached to a pipeline by a short reach of pipe of length l. For quick response l must be small. The gage pressure at 4 is $P_4 = \gamma(H_4 - z)$ with H_4 the hydraulic grade line elevation at the liquid gas interface, γ the specific weight of liquid, and z the elevation of liquid surface above datum. If the barometer reading, $\mathrm{H_{Bar}}$, is in feet of liquid, then the polytropic expansion equation for the compressed gas of volume $\forall_4$ becomes

$$(H_{P_4} + \mathrm{H_{Bar}} - z_P)\forall_{P_4}{}^n = \text{const} \qquad (57)$$

n is usually taken as 1.2. The equation of motion for the vertical riser of length l is (neglecting compressibility of the riser liquid)

$$\gamma A_3\left(\frac{P_{P_4} + H_4 - H_P - H}{2} - R\,Q_3|Q3|\right) = \frac{\gamma A_3 l}{g A_3}\,\frac{(QP3 - Q3)}{\Delta t} \qquad (58)$$

in which R is the resistance coefficient for the liquid column. The two characteristic equations are

$$H_P = \mathrm{HCP} - B\,QP1 \qquad (59)$$

$$H_P = \mathrm{HCM} + B\,QP2 \qquad (60)$$

and the continuity equation for the junction is

$$QP1 - QP2 = QP3 \qquad (61)$$

The change in z is given by

$$\frac{OP3 + Q3}{2} = \frac{z_P - z}{\Delta t} = -\frac{\Delta\forall_4}{\Delta t} = \frac{\forall_4 - \forall_{P_4}}{\Delta t} \qquad (62)$$

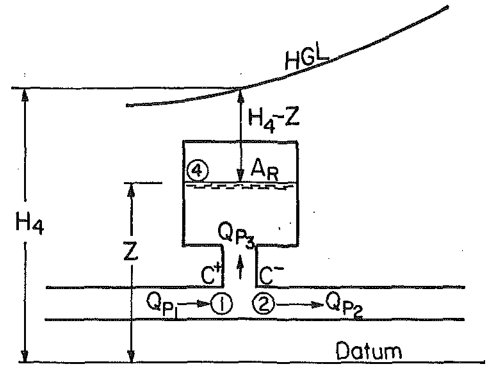

Fig. 15 Notation for an accumulator

By eliminating $QP1$ and $QP2$ from equations (59), (60), and (61), then substituting the resulting H_P into equation (58), it will contain as unknowns $QP3$ and $HP4$. By use of equation (62) $\forall_{P_4}$ may be eliminated from equation (57), and then equations (57) and (58) contain HP_4 and QP_3 from which a solution is easily obtained.

This procedure is carried on for each transient period Δt as one of the boundary conditions of the system.

Summary

Transient calculations for liquids in metal pipes may be carried out by digital computer using either explicit or implicit methods. The explicit methods were represented by the method of characteristics, which is stable and accurate. The main limitation is the length Δx of computing reach which is tied to the time increment of the calculation Δt. It is a very detailed method, and may require excessive calculations by virtue of the small Δt required. The implicit methods were illustrated by the centered implicit method which has no restriction on the ratio $\Delta x/\Delta t$. It requires a simultaneous solution of all unknowns for any given time. For very sharp transients some error results in the initial calculations that carry through to succeeding calculations and yield questionable results. Instabilities have also been encountered in the method.

The calculation of transients in high velocity flow was discussed in terms of the characteristics grid.

Boundary conditions were examined for the characteristics method, including turbomachinery, column separation, and the gas accumulator.

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