Systematic Design of Unitary Space–Time Constellations

Bertrand M. Hochwald, *Member, IEEE*, Thomas L. Marzetta, *Senior Member, IEEE*, Thomas J. Richardson, Wim Sweldens, *Member, IEEE*, and Rüdiger Urbanke

***Abstract*—We propose a systematic method for creating constellations of unitary space–time signals for multiple-antenna communication links. Unitary space–time signals, which are orthonormal in time across the antennas, have been shown to be well-tailored to a Rayleigh fading channel where neither the transmitter nor the receiver knows the fading coefficients. The signals can achieve low probability of error by exploiting multiple-antenna diversity. Because the fading coefficients are not known, the criterion for creating and evaluating the constellation is nonstandard and differs markedly from the familiar maximum-Euclidean-distance norm.**

Our construction begins with the first signal in the constellation—an oblong complex-valued matrix whose columns are orthonormal—and systematically produces the remaining signals by successively rotating this signal in a high-dimensional complex space. This construction easily produces large constellations of high-dimensional signals. We demonstrate its efficacy through examples involving one, two, and three transmitter antennas.

***Index Terms*—Fading channels, multielement antenna arrays, receive diversity, transmit diversity, wireless communications.**

I. INTRODUCTION

RECENT theoretical treatments have shown that communication systems that employ multiple antennas can have very high channel capacities, especially in Rayleigh flat-fading environments [5], [16], [9]. In [5], a constructive approach to achieving some of this capacity is proposed under the assumption that the receiver knows the complex-valued Rayleigh fading coefficients. Under the same assumption, [14] presents a trellis-based approach for designing space–time codes, and [15] gives a space–time signaling method based on orthogonal designs. However, the known-channel assumption may not be realistic in a rapidly changing fading environment or with a large number of transmitter antennas.

A new class of *unitary space–time* signals is proposed in [10] that are well-tailored for flat-fading channels where neither the transmitter nor the receiver knows the fading coefficients. Suppose there are M transmitter antennas, and that we transmit signals in blocks of T time samples, over which interval the fading coefficients are approximately constant. Then a constellation of L unitary space–time signals $S_\ell = \sqrt{T}\,\Phi_\ell$, $\ell = 1, \ldots, L$, has the defining property that $\Phi_1, \ldots, \Phi_L$ are $T \times M$ complex-valued matrices obeying $\Phi_1^\dagger \Phi_1 = \ldots = \Phi_L^\dagger \Phi_L = I$. Of necessity, $M \le T$. The mth column of any S_ℓ contains the signal transmitted on antenna m as a function of time. Essentially, the directions, and not the lengths, of the orthonormal columns of Φ_ℓ (more precisely, the subspace spanned by its M columns) carry the message information.

Intuitive and theoretical arguments in [9] and [10] suggest that unitary space–time signals are not only simple to decode, but they also attain capacity when used in conjunction with coding in a multiple-antenna Rayleigh fading channel when either $T \gg M$ or the signal-to-noise ratio (SNR) is reasonably large and $M < T$. Hence, there is a strong motivation for designing good unitary space–time constellations. Some successful unitary space–time constellations are designed and demonstrated in [10] but the techniques used therein cannot be readily extended to large constellations or to signals of high dimension. This paper presents some simple algorithms for designing effective constellations of these signals.

The only *a priori* structure on a unitary space–time constellation is the time orthonormality of the signals. Constellation design is viewed in [10] as a difficult and cumbersome search and optimization problem. But, as we show, by imposing additional structure on these signals and requiring that their generation be systematic, we can construct some effective constellations with relatively little effort. We present the design in two disparate but ultimately equivalent ways. The first approach, Section III, is Fourier-based and uses ideas from signal processing theory. The second approach, Section IV, is algebraic and uses ideas from coding theory. Section V demonstrates the performance of these approaches on a multiple-antenna Rayleigh fading channel where neither the receiver nor the transmitter knows the propagation coefficients. The performances of constellations for use with one, two, and three transmitter antennas are compared.

Throughout the paper, we concentrate on modulation and constellation design, and do not address coding issues that lower error probability by adding redundancy. We focus, instead, on raw or uncoded signal and bit error probabilities.

The following notation is used throughout the paper. Two complex vectors, a and b, are *orthogonal* if $a^\dagger b = 0$, where the superscript † denotes "conjugate transpose." The zero-mean, unit-variance, circularly symmetric, complex Gaussian distribution is denoted by $\mathcal{CN}(0, 1)$.

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B. M. Hochwald, T. L. Marzetta, T. J. Richardson, and W. Sweldens are with Bell Laboratories, Lucent Technologies, Murray Hill, NJ 07974 USA (e-mail: hochwald@research.bell-labs.com; tlm@research.bell-labs.com; tjr@research.bell-labs.com; wim@research.bell-labs.com).

R. Urbanke was with Bell Laboratories, Lucent Technologies, Murray Hill, NJ 07974 USA. He is now with EPFL, Lausanne, Switzerland (e-mail: rudiger.urbanke@epfl.ch).

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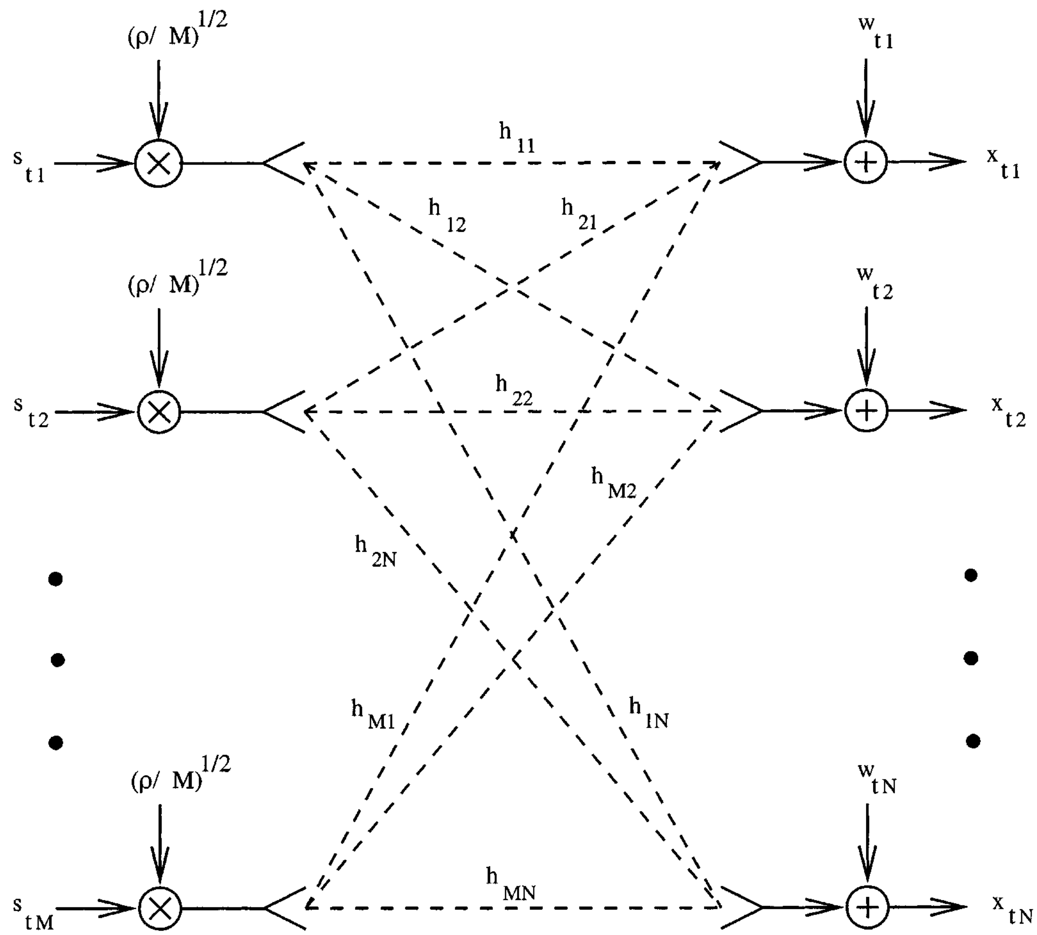

Fig. 1. Wireless link comprising M transmitter and N receiver antennas. Every receiver antenna is connected to every transmitter antenna through an independent, random, unknown propagation coefficient having Rayleigh-distributed magnitude and uniformly distributed phase. Normalization ensures that the total expected transmitted power is independent of M for a fixed ρ.

II. Channel Model; Unitary Space–Time Modulation

A. Rayleigh Flat Fading

Consider a communication link comprising M transmitter antennas and N receiver antennas that operates in a Rayleigh flat-fading environment. Each receiver antenna responds to each transmitter antenna through a statistically independent fading coefficient that is constant for T symbol periods. The received signals are corrupted by additive noise. We use complex baseband notation: during the T-symbol interval, we transmit the signal $\{s_{tm}, t=1,\ldots,T,\, m=1,\ldots,M\}$ on M antennas, and we receive the noisy signal $\{x_{tn}, t=1,\ldots,T, n=1,\ldots,N\}$ on N receivers

$$x_{tn}=\sqrt{\rho/M}\sum_{m=1}^{M}h_{mn}s_{tm}+w_{tn}, \quad t=1,\ldots,T,\ n=1\ldots N. \tag{1}$$

Here h_{mn} is the complex-valued fading coefficient between the mth transmitter antenna and the nth receiver antenna. The fading coefficients are constant for $t=1,\ldots,T$, and they are independent with respect to m and n and $\mathcal{CN}(0,\,1)$ distributed. The additive noise at time t and receiver antenna n is denoted w_{tn}, and is independent (with respect to both t and n) and identically distributed $\mathcal{CN}(0,\,1)$. The quantities in the signal model (1) are normalized so that ρ represents the expected SNR at each receiver antenna, independently of M. We assume that the realizations of $h_{mn},\, m=1,\,\ldots,\,M,\, n=1,\,\ldots,\,N$ are not known to the receiver or transmitter. See Fig. 1 and [9] for more details.

We assume that the fading coefficients change to new independent realizations every $T\geq 1$ symbol periods. This piecewise-constant fading process (also called a block-fading model [12], [2]) mimics, in a tractable manner, the approximate coherence interval of a continuously fading process. Furthermore, it is an accurate representation of many time-division multiple-access (TDMA), frequency hopping, or block-interleaved systems. Each channel use (consisting of a block of T transmitted symbols) is independent of every other.

Equation (1) can be written compactly as

$$X=\sqrt{\frac{\rho}{M}}\,SH+W \tag{2}$$

where X is the $T\times N$ complex matrix of received signals, S is the $T\times M$ matrix of transmitted signals, H is the $M\times N$ matrix of Rayleigh fading coefficients, and W is the $T\times N$ matrix of additive receiver noise. In this notation, the M columns of S represent the signals sent on the M transmitter antennas as functions of time.

B. Unitary Space–Time Signals

We use constellations of unitary space–time signals $S_1 = \sqrt{T}\Phi_1, \ldots, S_L = \sqrt{T}\Phi_L$ to transmit binary information over the multiple-antenna link. It is shown in [9] and [10] that the capacity-achieving distribution for $T \gg M$ and for a fixed ρ is $S = \sqrt{T}\Phi$, where $\Phi^\dagger\Phi = I$ and Φ is isotropically distributed. Details about the isotropic distribution may be found in [9], but it suffices to say that its defining characteristic is that Φ and $\Theta\Phi$ have the same distribution for any deterministic unitary Θ.

It is also shown in [10] that the maximum-likelihood (ML) decoder for a constellation of unitary space–time signals is

$$\Phi_{\rm ml} = \arg \max_{\Phi_\ell = \Phi_1, \ldots, \Phi_L} \operatorname{tr}\left\{X^\dagger \Phi_\ell \Phi_\ell^\dagger X\right\}. \tag{3}$$

This so-called noncoherent receiver has an equivalent interpretation as a generalized likelihood ratio test (GLRT)

$$\Phi_{\rm ml} = \arg \max_{\Phi_\ell = \Phi_1, \ldots, \Phi_L} \operatorname{tr}\left\{-\left[X - (\rho T/M)^{1/2}\Phi_\ell \hat{H}_\ell\right]^\dagger \cdot \left[X - (\rho T/M)^{1/2}\Phi_\ell \hat{H}_\ell\right]\right\} \tag{4}$$

which entails the use of the coherent receiver with the unknown value of H replaced by its ML estimate under the assumption that the ℓth signal was transmitted; hence

$$\hat{H}_\ell = \left(\frac{\rho T}{M}\right)^{-1/2} \Phi_\ell^\dagger X. \tag{5}$$

The maximum-likelihood interpretation for the noncoherent receiver (3) assumes that the propagation matrix has independent elements that are distributed as $\mathcal{CN}(0, 1)$, while the GLRT interpretation is less restrictive because it does not assume anything about the statistics of the propagation matrix. Built into the philosophy of the GLRT [17] is the notion that when the correct decision is made the associated ML estimate $\hat{H}_\ell$ is good. With this in mind, our case for using unitary space–time signals is further strengthened by the fact that these signals constitute optimal training signals [13], [8] for learning H. Specifically, if a known signal is transmitted from which the receiver obtains an ML estimate for H, the energy-constrained signal that minimizes the total error variance is a unitary space–time signal.

While our original motivation for using unitary space–time signals is information-theoretic, this paper focuses on modulation and on uncoded probability of error. These signals are of interest in their own right because they have a simple demodulator that also has a pleasing GLRT interpretation.

C. Constellations of Unitary Space–Time Signals

The task is to design a constellation of L unitary space–time signals that has a low probability of error. We note that the probability of error is invariant to two types of transformations: 1) left multiplication by a common $T \times T$ unitary matrix, $\Phi_\ell \to \Psi^\dagger \Phi_\ell$, $\ell = 1, \ldots, L$ and 2) right multiplication by individual $M \times M$ unitary matrices, $\Phi_\ell \to \Phi_\ell \Upsilon_\ell$, $\ell = 1, \ldots, L$; see [10]. We consider any two constellations to be equivalent if they are related by unitary transformations of this type.

We are unable to compute the block probability of error P_e for a general constellation of unitary space–time signals. However, the performance may be upper-bounded in terms of pairwise probabilities of error through the union bound

$$\begin{aligned} P_e &= \frac{1}{L}\sum_{\ell=1}^{L} P\{\text{error} \mid \Phi_\ell \text{ transmitted}\} \\ &\le \frac{1}{L}\sum_{\ell=1}^{L}\sum_{\ell' \ne \ell} P_{\ell,\ell'} \end{aligned} \tag{6}$$

where $P_{\ell,\ell'}$ is the pairwise (i.e., two-signal constellation) probability of mistaking Φ_ℓ for $\Phi_{\ell'}$ or vice versa, which has the closed-form expression [10]

$$\begin{aligned} P_{\ell,\ell'} &= P\{\text{choose } \Phi_{\ell'} | \Phi_\ell \text{ transmitted}\} \\ &= P\{\text{choose } \Phi_\ell | \Phi_{\ell'} \text{ transmitted}\} \\ &= \sum_j \operatorname{Res}_{\omega = i a_j} \left\{ -\frac{1}{\omega + i/2} \cdot \prod_{\substack{m=1 \\ d_m < 1}}^{M} \left[\frac{1 + \rho T/M}{(\rho T/M)^2 (1 - d_m^2)(\omega^2 + a_m^2)}\right]^N \right\} \end{aligned} \tag{7}$$

where $1 \ge d_1 \ge \cdots \ge d_M \ge 0$ are the singular values of the $M \times M$ correlation matrix $\Phi_\ell^\dagger \Phi_{\ell'}$, and

$$a_m \overset{\text{def}}{=} \sqrt{\frac{1}{4} + \frac{1 + \rho T/M}{(\rho T/M)^2 (1 - d_m^2)}}.$$

The singular values are a measure of the overlap of the two subspaces that are spanned by the column vectors of the signals. The exact pairwise probability of error is cumbersome to evaluate, requiring either the extraction of residues of high-order poles, or a one-dimensional numerical integration. The Chernoff bound is somewhat simpler [10]

$$P_{\ell,\ell'} \le \frac{1}{2}\prod_{m=1}^{M}\left[\frac{1}{1 + \frac{(\rho T/M)^2 (1 - d_m^2)}{4(1 + \rho T/M)}}\right]^N. \tag{8}$$

The probability of error (and Chernoff bound) is lowest when $d_1 = \cdots = d_M = 0$ and highest when $d_1 = \cdots = d_M = 1$. We obtain $d_1 = \cdots = d_M = 0$ when the columns of Φ_ℓ are all orthogonal to all the columns of $\Phi_{\ell'}$. The ideal constellation $\Phi_1, \ldots, \Phi_L$, therefore, has all the columns of Φ_ℓ orthogonal to all the columns of $\Phi_{\ell'}$ for $\ell' \ne \ell = 1, \ldots, L$. However, because the columns of each Φ_ℓ are within themselves orthogonal to one another, all the pairwise $d_1, \ldots, d_M$ cannot all be made zero if $L > T/M$. Conversely, Φ_ℓ and $\Phi_{\ell'}$ are indistinguishable, within the context of our model, when $d_1 = \cdots = d_M = 1$.

We can further simplify the bound (8) in terms of the average of squares of the singular values

$$\begin{aligned} \frac{1}{M}\sum_{m=1}^{M} d_m^2 &= \frac{1}{M}\operatorname{tr}\left\{\left(\Phi_\ell^\dagger \Phi_{\ell'}\right)^\dagger \left(\Phi_\ell^\dagger \Phi_{\ell'}\right)\right\} \\ &= \left\|\Phi_\ell^\dagger \Phi_{\ell'}\right\|^2 \end{aligned} \tag{9}$$

where (9) defines the matrix norm used in this paper (a scaled Frobenius norm). For both the pairwise probability of error and the Chernoff bound, it can be shown that the first and second derivatives with respect to the squares of the singular values are positive

$$\frac{\partial P_{\ell,\ell'}}{\partial (d_m^2)} > 0 \qquad \frac{\partial^2 P_{\ell,\ell'}}{\partial (d_m^2)^2} > 0.$$

This implies that for any two singular values that are contained in the open interval $(0, 1)$, if one increases the larger singular value while decreasing the smaller singular value such that their sum of squares is constant, thus maintaining constant norm (9), the pairwise probability of error (and its Chernoff bound) increases. Consequently, for a given norm (9), the probability of error is minimized when all the singular values are equal. Conversely, the probability of error is maximized when as many singular values as possible are equal to one. This implies that, in the worst case, about $M\|\Phi_\ell^\dagger \Phi_{\ell'}\|^2$ singular values are equal to one, and the remaining singular values are equal to zero, which gives an upper bound on the Chernoff bound

$$P_{\ell,\ell'} \le \frac{1}{2}\left[\frac{1}{1+\frac{(\rho T/M)^2}{4(1+\rho T/M)}}\right]^{N\cdot(M-\lceil M\|\Phi_{\ell'}^\dagger \Phi_\ell\|^2\rceil)}. \tag{10}$$

For a given constellation, let

$$\delta = \max_{1\le \ell<\ell'\le L} \left\|\Phi_\ell^\dagger \Phi_{\ell'}\right\|. \tag{11}$$

Then the combination of (6), (10), and (11) gives a bound on the block probability of error for the entire constellation in terms of δ

$$\begin{aligned} P_e &\le \frac{1}{L}\sum_{\ell=1}^{L}\sum_{\ell'\ne\ell}\frac{1}{2}\left[\frac{1}{1+\frac{(\rho T/M)^2}{4(1+\rho T/M)}}\right]^{N\cdot(M-\lceil M\|\Phi_\ell^\dagger \Phi_\ell\|^2\rceil)} \\ &\le \frac{L}{2}\left[\frac{1}{1+\frac{(\rho T/M)^2}{4(1+\rho T/M)}}\right]^{N\cdot(M-\lceil M\delta^2\rceil)}. \end{aligned} \tag{12}$$

Accordingly, we attempt to construct constellations that minimize δ in (11). This is a particularly simple performance measure to compute, and it does not depend on either the SNR or the number of receive antennas.[1] The definition of δ in (11) has a connection with the standard definition of distance between subspaces [6, Sec. 12.4.3]. Let F_ℓ and $F_{\ell'}$ be the M-dimensional subspaces of $\boldsymbol{C}^T$ spanned by the columns of Φ_ℓ and $\Phi_{\ell'}$, respectively. Then one can think of the singular values d_m as the cosines of the so-called *principal angles* θ_m between F_ℓ and $F_{\ell'}$. The L^2 distance between the two subspaces is now defined as $\max_m \sin(\theta_m) = \max_m \sqrt{1-d_m^2}$ while the chordal distance is $\sqrt{\sum_m (1-d_m^2)}$. The minimum chordal distance between any two subspaces F_ℓ and $F_{\ell'}$ for $(\ell' \ne \ell)$ is precisely $\sqrt{M(1-\delta^2)}$. This shows that our design problem is related to so-called packings in complex Grasmannian space. Some examples of packings in real Grasmannian space are given in [3].

The design criterion of minimizing δ is markedly different from the familiar maximum-Euclidean-distance criterion, and it arises entirely because the fading coefficients are unknown to the receiver. Because of this, antipodal pairs of signals $\pm\Phi$ are indistinguishable, for example.

An alternative criterion for constellation design that we do not pursue in this paper seeks to maximize the product

$$\prod_{m=1}^{M}(1-d_m^2)$$

[1]The performance of a given constellation always improves with increasing N. For the remainder of the paper we set $N=1$.

upon which the Chernoff bound depends dominantly for large SNRs.[2] We note simply that for small d_m

$$\prod_{m=1}^{M}(1-d_m^2) \approx 1 - \sum_{m=1}^{M} d_m^2$$

and, therefore, minimizing δ for small d_m is roughly the same as maximizing this product.

To transmit R bits per channel use, we need a constellation of at least $L = 2^{RT}$ signals. For example, if $R = 2$ bits/channel use and $T = 10$, then $L = 2^{20} \approx 10^6$. Generating and storing this many $T\times M$ complex matrices is cumbersome if the signals are not provided with some additional structure. Furthermore, it is not obvious how to ensure that the generated signals have low probability of error. In the next section, we describe a systematic approach to create signals with low probability of error and that requires storage of only Φ_1 and a $T\times T$ diagonal matrix with which to generate $\Phi_2, \ldots, \Phi_L$.

III. Fourier-Based Construction

In this section we present a Fourier-based construction of a constellation of unitary space–time signals. Section III-A gives the intuition behind the construction, which has a block-circulant signal correlation structure. Section III-B then proves that this construction yields all constellations having a block-circulant correlation structure.

We make no claim for the optimality of circulant correlation structure. However, this structure has the advantage that it significantly simplifies the design process.

A. Fourier-Based Construction Has Block-Circulant Correlation

We begin with $M = 1$ transmitter antenna; we, therefore, need L unit vectors in a T-dimensional complex space where, in general, $L \gg T$. Clearly, these vectors will form an overcomplete or linear dependent system. Overcomplete representations are becoming increasingly popular in signal representation and are often studied using the mathematical technique of *frames* [4]. Even though there is no immediate reason why frames would form good constellations, we draw inspiration from existing methods for building frames.

We say that a collection of L vectors Φ_ℓ in a T-dimensional space form a *tight frame* if all of the eigenvalues of the $T\times T$ matrix $\sum_{\ell=1}^{L} \Phi_\ell\Phi_\ell^\dagger$ are equal, implying that

$$\sum_{\ell=1}^{L} \Phi_\ell\Phi_\ell^\dagger = K\cdot I$$

where K is the frame constant. While the details of frame theory go beyond the scope of this paper, we use a well-known result that any tight frame with L vectors in T dimensions can be seen as the projection into a T-dimensional space of an orthogonal basis in L dimensions and vice versa, see, e.g., [7]. Balan and Daubechies construct tight frames by projecting an L-dimensional discrete Fourier transform (DFT) basis onto a T-dimensional space [1]. The projection simply retains the first

[2]This criterion was independently suggested by an anonymous reviewer.

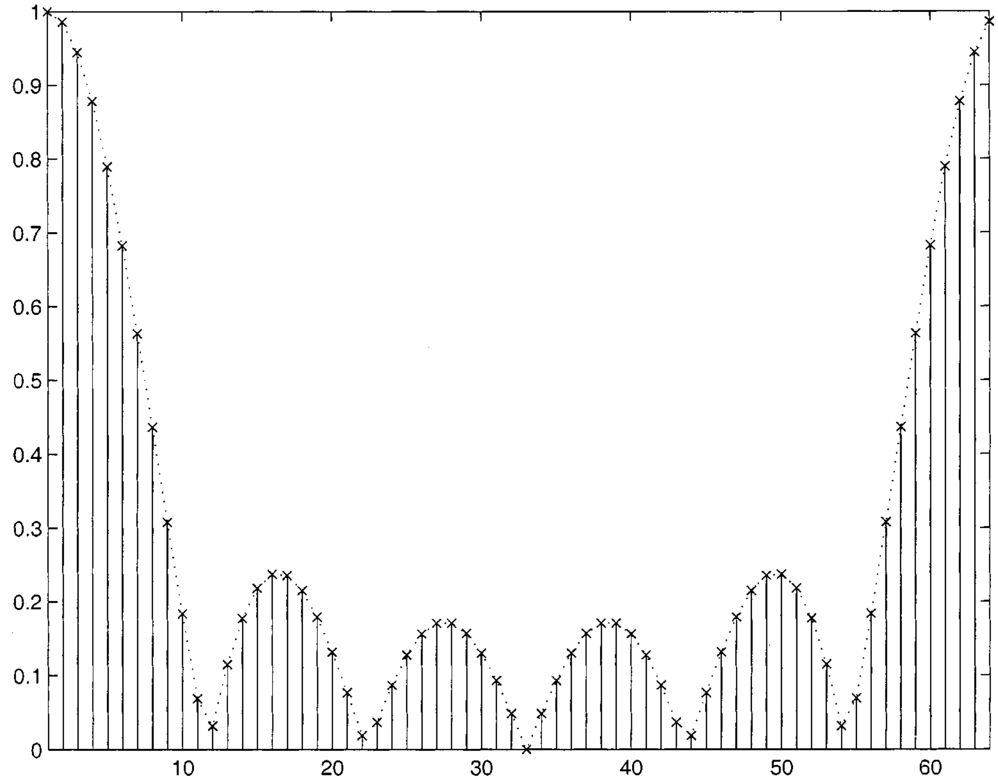

Fig. 2. Correlation structure of signals in (13) as a function of $\ell' - \ell$ when $T = 6$ and $L = 64$, which implies a transmission rate of $R = 1$-bit/channel use. We clearly see the sinc-like behavior. The maximum correlation δ as defined in (11) (which is achieved when $\ell' - \ell = 1$) is 0.986.

T components of the L-dimensional vectors. Inspired by this construction, we propose the one antenna constellation

$$\Phi_\ell = \frac{1}{\sqrt{T}} \begin{bmatrix} 1 \\ e^{i\frac{2\pi}{L}(\ell-1)} \\ e^{i\frac{2\pi}{L}2(\ell-1)} \\ \vdots \\ e^{i\frac{2\pi}{L}(T-1)(\ell-1)} \end{bmatrix}. \qquad (13)$$

For this choice, we obtain

$$\begin{aligned} d_1 &= \left|\Phi_\ell^\dagger \Phi_{\ell'}\right| \\ &= \begin{cases} 1, & (\ell' = \ell) \\ \dfrac{1}{T}\left|\displaystyle\sum_{t=1}^{T} e^{i\frac{2\pi}{L}(t-1)(\ell'-\ell)}\right| \\ \quad = \left|\dfrac{\sin(\pi(\ell'-\ell)T/L)}{T\sin(\pi(\ell'-\ell)/L)}\right|, & (\ell' \neq \ell). \end{cases} \end{aligned}$$

As shown in (7) and (8), the two-signal probability of error depends only on the correlation d_1 and decreases as d_1 decreases. We observe that

1) The correlation between Φ_ℓ and $\Phi_{\ell'}$ depends only on $(\ell' - \ell) \bmod L$; the correlation structure of the entire constellation is therefore circulant and it suffices to consider $|\Phi_1^\dagger \Phi_\ell|$ for $\ell = 2, \ldots, L$.

2) The correlation structure behaves roughly like a sinc function, and hence equation (11) yields
$$\delta = |\Phi_\ell^\dagger \Phi_{\ell+1}| = 1 - O(1/L^2)$$
as $L \to \infty$. For large L, (7) (with $M = 1$ and $d_1 = |\Phi_\ell^\dagger \Phi_{\ell+1}|$) therefore implies that the probability of mistaking Φ_ℓ for its immediate neighbors is high; this is decidedly undesirable. Fig. 2 shows the correlation structure for $T = 6$ and $L = 64$, for which $\delta = 0.986$.

Property 2) suggests that $\Phi_1, \ldots, \Phi_L$ given by (13) are a poor choice of signals, especially if L is large. However, we are not necessarily constrained to choose the first T rows of the $L \times L$ DFT matrix as is done in (13). To lower the correlation between neighbors, we may consider choosing another set of T components. We thus let

$$\Phi_\ell = \frac{1}{\sqrt{T}} \begin{bmatrix} e^{i\frac{2\pi}{L}u_1(\ell-1)} \\ e^{i\frac{2\pi}{L}u_2(\ell-1)} \\ \vdots \\ e^{i\frac{2\pi}{L}u_T(\ell-1)} \end{bmatrix} \qquad (14)$$

where, without loss of generality, $0 \le u_1, \ldots, u_T \le L - 1$. We still have a circulant correlation structure because

$$|\Phi_\ell^\dagger \Phi_{\ell'}| = \frac{1}{T}\left|\sum_{t=1}^{T} e^{i\frac{2\pi}{L}u_t(\ell'-\ell)}\right|.$$

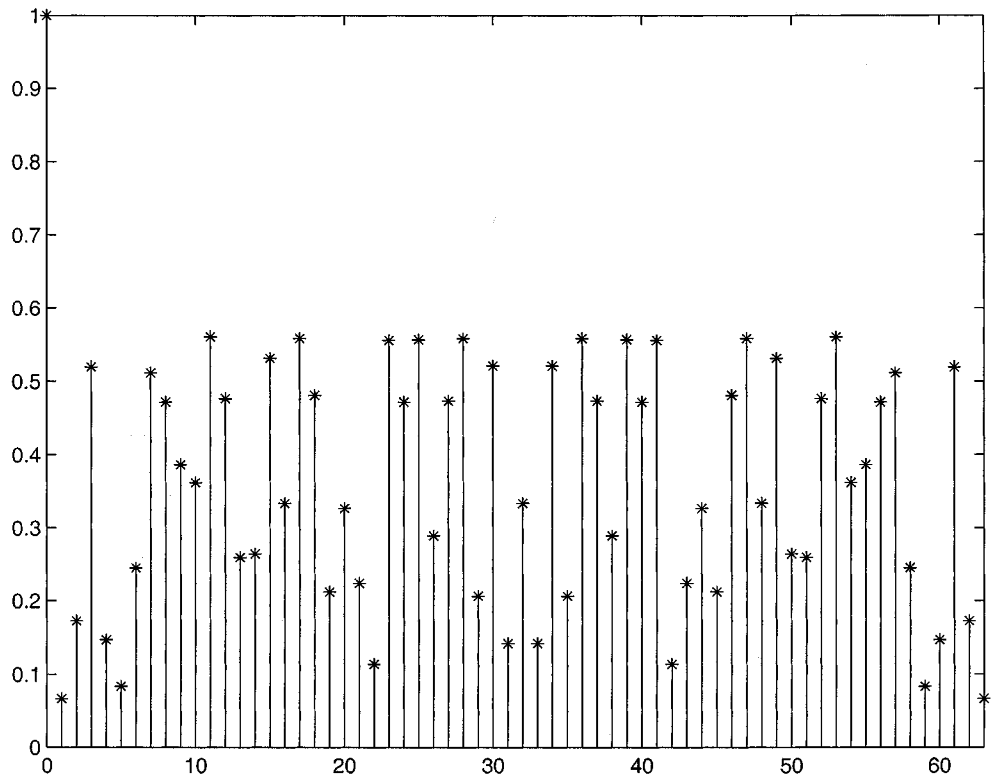

Fig. 3. Correlation as a function of $\ell' - \ell$ when choosing $u_1, \ldots, u_T$ in (14), with $T = 6$ and $L = 64$. Here $u = [1\ 18\ 23\ 39\ 46\ 57]$ and is found by minimizing δ using a random search, yielding $\delta = 0.5604$.

We can now choose the "frequencies" $u_1, \ldots, u_T$ to get the lowest possible correlations. As mentioned in Property 1, because of the circulant structure it suffices to look at

$$\left|\Phi_1^\dagger \Phi_\ell\right| = \left|\sum_{t=1}^{T} [\Phi_\ell]_t\right| = \frac{1}{T}\left|\sum_{t=1}^{T} e^{i\frac{2\pi}{L}u_t(\ell-1)}\right|, \qquad \ell = 2, \ldots, L, \quad (15)$$

where $[\cdot]_t$ denotes the tth component of $[\cdot]$. We wish to find $u_1, \ldots, u_T$ achieving

$$\min_{0\le u_1, \ldots, u_T \le L-1} \max_{\ell=2, \ldots, L} \frac{1}{T}\left|\sum_{t=1}^{T} e^{i\frac{2\pi}{L}u_t(\ell-1)}\right| = \min_{0\le u_1, \ldots, u_T\le L-1} \delta \quad (16)$$

where δ (given by (11)) depends on $u_1, \ldots, u_T$. Observe that $|\Phi_1^\dagger \Phi_\ell|$ can be interpreted as the modulus of the DFT of a length-L sequence with the value 1 at positions $u_1, \ldots, u_T$ and 0 elsewhere. Thus one can look at the minimization in (16) as a filter design problem, where the filter is sparse (i.e., only T out of a possible L filter coefficients are nonzero), the response at zero frequency is unity, and where we choose the locations of the T nonzero coefficients to minimize the response at frequencies that are multiples of $2\pi/L$.

The problem of sparse filter design is analogous to that of aperiodic antenna array design [11]. A conventional linear antenna array having T elements uses periodic half-wavelength spacing between its elements, and it has an angular frequency response having the sinc-like behavior shown in Fig. 2. The width of the central peak at zero frequency is inversely proportional to the physical length of the array. If one desires the narrower central peak associated with higher angular resolution for the *same* number of elements T, one has to use a longer array. Doubling the spacing to give a uniform spacing of one wavelength would reduce the width of the central peak by a factor of two, but with the penalty of replicating the angular frequency response at intervals of π (the so-called grating lobe effect). However, by using a longer aperiodic array, one can obtain a narrower central peak without introducing grating lobes. Despite much effort, there has never been a completely satisfactory way to design aperiodic arrays: for small arrays one can use exhaustive search, whereas, for large arrays, random search strategies seem to be the only resort. In our optimizations, we therefore also generally employ a random search. Fig. 3 shows the results of such a search. Observe how optimizing over $u_1, \ldots, u_T$ allows a much better correlation structure than in Fig. 2.

We now show how we can generalize this single-antenna construction to $M > 1$ antennas. In the single-antenna case, each signal can be written as

$$\Phi_\ell = \Theta^{\ell-1}\Phi_1 \quad (17)$$

where Θ is a $T \times T$ matrix whose diagonal elements are $e^{i2\pi u_1/L}, \ldots, e^{i2\pi u_T/L}$ and Φ_1 is $1/\sqrt{T}$ times a vector of all ones. Note that Θ is a unitary matrix and that $\Theta^L = I_L$. Geometrically, the construction can be interpreted as rotating an initial vector through T-dimensional complex space using a matrix which is the Lth root of unity. The matrix is chosen so that the resulting vectors have as little correlation as possible, and after L rotations the vector is brought back to its initial position.

For $M > 1$ transmitter antennas, let Φ_1 be a $T \times M$ matrix with $\Phi_1^\dagger \Phi_1 = I_M$ and form the constellation again by applying (17). Because Θ is an Lth root of unity, we have a block-circulant structure in the sense that the $M \times M$ matrix $\Phi_\ell^\dagger \Phi_{\ell'}$ only depends on $(\ell' - \ell) \bmod L$, and because Θ is unitary, $\Phi_\ell^\dagger \Phi_\ell = I_M$. Geometrically, this construction can be interpreted as rotating an initial M-dimensional subspace using an Lth root of unity to form L different M-dimensional subspaces.

As noted in Section II-C, a constellation with small probability of error generally has small δ. We may, therefore, choose $u_1, \ldots, u_T$ to achieve

$$\min_{0 \le u_1, \ldots, u_T \le L-1} \delta = \min_{0 \le u_1, \ldots, u_T \le L-1} \max_{\ell = 2, \ldots, L} \left\| \Phi_1^\dagger \Phi_\ell \right\|. \tag{18}$$

A simple method to build a starting matrix Φ_1 is to choose M distinct columns of a $T \times T$ DFT matrix. This ensures that $\Phi_1^\dagger \Phi_1 = I_M$. A secondary benefit is that the transmitted power never varies.

In the next section we show that the above construction generates all constellations with circulant correlation structure.

Remark: The starting unit vectors Φ_1 that we have used so far—either $1/\sqrt{T}$ times a vector of all ones, or the columns of a DFT matrix—have components all with modulus $1/\sqrt{T}$. There is no particular need to impose this constraint, and experiments indicate that optimizations that allow the moduli of the starting vector components to vary (but maintain unit norms for the columns of each Φ_ℓ) can yield even smaller values of δ. For simplicity, we do not pursue these optimizations.

B. Block-Circulant Correlation Has Fourier-Based Construction

In the previous section, we propose a constellation with a circulant correlation structure. This structure does not automatically guarantee that the constellation performs well. However, the structure simplifies performance testing since only $L-1$ rather than $L(L-1)/2$ correlations need to be checked. In this section, we investigate the restrictiveness of this condition by characterizing all constellations which yield a circulant correlation structure.

Let $\{\Phi_1, \ldots, \Phi_L\}$ be some constellation of unitary space–time signals. We impose the block-circulant correlation structure

$$\Phi_\ell^\dagger \Phi_{\ell'} = F_{(\ell' - \ell) \bmod L}, \qquad \ell = 1, \ldots, L, \ \ell' = 1, \ldots, L \tag{19}$$

where F_ℓ are $M \times M$ matrices and the orthonormality of the columns of each signal implies that $F_0 I_M$. It is also easy to see that

$$F_{-\ell} = F_{L-\ell} = \Phi_\ell^\dagger \Phi_L = F_{\ell-L}^\dagger = F_\ell^\dagger. \tag{20}$$

The block-circulant correlation structure implies that the exact conditional probability of error for deciding which of the L signals was transmitted is the same for all L signals.

We now take the double (i.e., both in ℓ and ℓ') DFT of both sides of (19) to obtain

$$\begin{aligned} \sum_{\ell=1}^{L} \sum_{\ell'=1}^{L} & \Phi_\ell^\dagger \Phi_{\ell'} e^{-i \frac{2\pi}{L} [(\ell'-1)(n'-1) - (\ell-1)(n-1)]} \\ &= \hat{\Phi}_n^\dagger \hat{\Phi}_{n'} \\ &= L \hat{F}_n \delta_{(n'-n) \bmod L} \end{aligned} \tag{21}$$

where the Fourier transforms, which are matrix-valued, are denoted by the hatted quantities

$$\hat{\Phi}_n = \sum_{\ell=1}^{L} \Phi_\ell e^{-i \frac{2\pi}{L} (\ell-1)(n-1)}$$

and

$$\hat{F}_n = \sum_{\ell=1}^{L} F_\ell e^{-i \frac{2\pi}{L} (\ell-1)(n-1)}. \tag{22}$$

Equation (21) is equivalent to the well-known result that a circulant matrix is diagonal in the Fourier domain.

According to (21), the L Fourier coefficients $\hat{\Phi}_n$, each a $T \times M$ matrix, are mutually orthogonal. Consequently, all but at most T of the coefficient matrices are zero. We denote the T possibly nonzero Fourier coefficient matrices by $\hat{\Phi}_{u_1}, \ldots, \hat{\Phi}_{u_T}$ where $0 \le u_1, \ldots, u_T \le L-1$. The signals are thus given by the inverse Fourier transform

$$\Phi_\ell = \frac{1}{L} \sum_{t=1}^{T} \hat{\Phi}_{u_t} e^{i \frac{2\pi}{L} u_t (\ell-1)}, \qquad \ell = 1, \ldots, L. \tag{23}$$

When exactly T coefficient matrices are nonzero, then orthogonality requires them all to have rank one, for there cannot be more than T linearly independent T-dimensional vectors. When only $T-1$ coefficients are nonzero, at most one of them can have rank two while the others have rank one. The rank-two matrix can always be written as the sum of two rank-one matrices; for example, take its singular value decomposition and write the two-element diagonal matrix as a sum of two one-element diagonal matrices. Then we again have a sum (23) with T terms where each coefficient matrix has rank one; the only difference is that the two coefficient matrices coming from the split have the same frequency term $e^{i \frac{2\pi}{L} u_t (\ell-1)}$. Similar arguments for $T-2$ or fewer nonzero coefficients yield the same conclusion that all coefficient matrices in (23) can be made to have rank one.

We now show that, without loss of generality, $\hat{\Phi}_{u_1}, \ldots, \hat{\Phi}_{u_T}$ can be nonzero in exactly one row. Consider the $T \times T$ matrix formed by taking the first column of each $\hat{\Phi}_{u_t}$, $t = 1, \ldots, T$. The columns of this matrix are then orthogonal, but not necessarily orthonormal. Thus this matrix can be written as ΨD, where Ψ is a $T \times T$ unitary matrix and D is diagonal. Now $\Psi^\dagger$ times the first column of $\hat{\Phi}_{u_t}$ is a vector with only the tth component nonzero. Because $\hat{\Phi}_{u_t}$ is rank-one, all its columns are scaled copies of one another. Hence $\Psi^\dagger \hat{\Phi}_{u_t}$ is a matrix with only its tth row nonzero. Recall that the error performance of a constellation does not change when applying the transformation

$$\Phi_\ell \mapsto \Psi^\dagger \Phi_\ell, \qquad \ell = 1, \ldots, L \tag{24}$$

for unitary Ψ. From (23) we see that this transformation is equivalently applied to the Fourier coefficient matrices: $\hat{\Phi}_{u_t} \mapsto \Psi^\dagger \hat{\Phi}_{u_t}$, $t = 1, \ldots, T$. After this transformation, $\hat{\Phi}_{u_t}$ is zero except in its tth row. The set of Fourier coefficients are therefore orthogonal by virtue of their disjoint row support.

The signal Φ_1 combines the different nonzero rows of the $\hat{\Phi}_{u_t}$ matrices

$$\Phi_1 = \frac{1}{L} \sum_{t=1}^{T} \hat{\Phi}_{u_t}.$$

Any other signal Φ_ℓ can be formed from Φ_1 by multiplying the tth row by $e^{i\frac{2\pi}{L}u_t(\ell-1)}$ as in (23). Hence, Φ_ℓ can be expressed more conveniently as a $T \times T$ diagonal unitary matrix Θ that is raised to the $(\ell-1)$th power, times the $T \times M$ matrix Φ_1

$$\Phi_\ell = \Theta^{\ell-1} \Phi_1 \tag{25}$$

where

$$\Theta = \begin{bmatrix} e^{i\frac{2\pi}{L}u_1} & 0 & \cdots \\ & \ddots & \\ 0 & \cdots & e^{i\frac{2\pi}{L}u_T} \end{bmatrix}, \qquad 0 \le u_1, \ldots, u_T \le L-1. \tag{26}$$

Since Φ_1 only underwent the unitary transformation (24), it still has the property that $\Phi_1^\dagger \Phi_1 = I_M$. By (25), the correlation matrix between any two signals has the block-circulant structure (19)

$$\Phi_\ell^\dagger \Phi_{\ell'} = \Phi_1^\dagger \Theta^{\ell'-\ell} \Phi_1. \tag{27}$$

We conclude that any unitary space–time constellation whose correlation matrix is block-circulant can be designed using the methods of Section III-A. We therefore have the following theorem.

Theorem 1: Any unitary space–time signal constellation of $T \times M$ matrices $\Phi_1, \ldots, \Phi_L$ with a block-circulant correlation structure is equivalent to one that can be written

$$\Phi_\ell = \Theta^{\ell-1} \Phi_1 \tag{28}$$

where Φ_1 is a $T \times M$ matrix obeying $\Phi_1^\dagger \Phi_1 = I_M$, and Θ is a $T \times T$ diagonal matrix whose diagonal elements are Lth roots of unity. Conversely, every constellation of the form (28), has a block-circulant correlation structure.

C. Multiple Index Block-Circulant Structures

The previous constellation construction may be extended to a doubly indexed construction

$$\{\Phi_{\ell_1\ell_2}, \ell_1 = 1, \ldots, L_1; \ell_2 = 1, \ldots, L_2\}$$

where $L_1 \cdot L_2 = L$, and where the constellation has the following correlation structure:

$$\Phi_{\ell_1\ell_2}^\dagger \Phi_{\ell_1'\ell_2'} = F_{(\ell_1'-\ell_1) \bmod L_1 (\ell_2'-\ell_2) \bmod L_2}.$$

It can be shown (we omit the details) that this construction yields a constellation that is generated by means of a separate rotation for each index

$$\Phi_{\ell_1\ell_2} = \Theta_1^{\ell_1-1}\Theta_2^{\ell_2-1}\Phi_1, \qquad \ell_1 = 1, \ldots, L_1;\ \ell_2 = 1, \ldots, L_2 \tag{29}$$

where Θ_1 and Θ_2 are diagonal unitary matrices that are th L_1th and L_2th roots of I_T, respectively. This construction involves choosing the diagonal elements of Θ_1 and Θ_2, which we label $0 \le u_{11}, \ldots, u_{1T} \le L_1 - 1$ and $0 \le u_{21}, \ldots, u_{2T} \le L_2 - 1$. The constellation is therefore completely determined by the $T \times M$ matrix Φ_1, and the $2 \times T$ matrix U whose entries are u_{kt}, $k = 1, 2$, $t = 1, \ldots, T$.

This construction extends readily to a K-indexed constellation in which $L = \prod_{k=1}^{K} L_k$ and U is a $K \times T$ matrix.

IV. Equivalent Algebraic Construction

The constellation construction described in the previous section can also be viewed algebraically, and in this section we create a constellation of signals by mapping a linear block code into complex signal matrices. The code is over the ring of integers modulo-q and the number of codewords is equal to the number of desired signals L. We will relate q to L shortly, and we begin by describing the construction for $M = 1$ transmitter antenna.

Let $\mathsf{R}_q = \{0, \ldots, q-1\}$ be the ring of integers modulo-q, and let $\mathcal{C} = \{c_1, \ldots, c_L\}$ denote a linear code over R_q of length T and containing L codewords. Each element c_ℓ of $\mathcal{C}$ is a vector of T integers in $\{0, \ldots, q-1\}$. Because the code is linear it contains the all-zero vector, and if c_ℓ and $c_{\ell'}$ are in $\mathcal{C}$ then so is $ac_\ell + bc_{\ell'}$ for any $a, b \in \mathsf{R}_q$.

We map these codewords into signals by mapping the T integers in a codeword to the T components of a complex signal using the function

$$\phi(j) = \frac{1}{\sqrt{T}} e^{i\frac{2\pi}{q}j}, \qquad j = 0, \ldots, q-1.$$

Note that addition modulo-q for the argument corresponds to complex multiplication for the function value. By letting the ϕ function work on vectors, we effectively obtain the one-antenna constellation

$$\Phi_\ell = \phi(c_\ell) = \frac{1}{\sqrt{T}} \begin{bmatrix} e^{i\frac{2\pi}{q}[c_\ell]_1} \\ e^{i\frac{2\pi}{q}[c_\ell]_2} \\ \vdots \\ e^{i\frac{2\pi}{q}[c_\ell]_T} \end{bmatrix}, \qquad 1 \le \ell \le L.$$

Let c_1 be the all zero codeword; then Φ_1 is $1/\sqrt{T}$ times a vector of all ones. We show that the maximum correlation of the resulting constellation is given by

$$\max_{\ell=2,\ldots,L} \left| \sum_{t=1}^{T} [\Phi_\ell]_t \right|$$

where $[\cdot]_t$ again denotes the tth component of $[\cdot]$ (and the arithmetic is in the field of complex numbers). To see this, pick two different signals Φ_ℓ and $\Phi_{\ell'}$. By definition, $\Phi_\ell = \phi(c_\ell)$ and $\Phi_{\ell'} = \phi(c_{\ell'})$ for some c_ℓ, $c_{\ell'} \in \mathcal{C}$. Thus

$$\begin{aligned}\Phi_\ell^\dagger \Phi_{\ell'} &= \sum_{t=1}^{T} \phi^*([c_\ell]_t)\phi([c_{\ell'}]_t)\\ &= \frac{1}{T}\sum_{t=1}^{T} e^{-i\frac{2\pi}{q}[c_\ell]_t} e^{i\frac{2\pi}{q}[c_{\ell'}]_t}\\ &= \frac{1}{T}\sum_{t=1}^{T} e^{i\frac{2\pi}{q}[c_{\ell'}-c_\ell]_t} = \frac{1}{T}\sum_{t=1}^{T} e^{i\frac{2\pi}{q}[c_{\ell''}]_t}\end{aligned}$$

for some ℓ'', where the last equality follows from the code's linearity. Therefore, as in (15), in searching for constellations that minimize their maximum correlation, we need to check only $L-1$ quantities.

So far, the codes $\mathcal{C}$ are restricted to be linear but are otherwise arbitrary. We further restrict our search by considering codes that have a $K \times T$ generator matrix U of elements in R_q, where K can be thought of as the dimension of the code. The code $\mathcal{C}$ represented by U is the linear span of the rows of U, i.e., every codeword can be written in the form

$$c_\ell = \ell \cdot U$$

for some $1 \times K$ vector $\ell = [\ell_1 \, \ldots \, \ell_K]$ whose K elements are all in R_q. We incorporate this restriction explicitly, because, unlike linear codes over finite fields, linear block codes over R_q do not necessarily have a generator matrix. It follows that the size of the constellation is $L = q^K$.

We may now call ℓ the multi-index (K-index) of the codewords of $\mathcal{C}$. Then the signals have a multi-index circulant correlation structure since

$$\begin{aligned}c_{\ell'} - c_\ell &= ([\ell'_1 \, \ldots \, \ell'_K] - [\ell_1 \, \ldots \, \ell_K])U\\ &= [\ell'_1 - \ell_1 \, \ldots \, \ell'_K - \ell_K]U\\ &= [\ell''_1 \, \ldots \, \ell''_K]U\\ &= c_{\ell''}\end{aligned}$$

where all arithmetic is modulo-q.

The connection to the constellation construction discussed in Section III-A becomes more apparent if we rewrite the codes in the following form. Given U, we let $\Theta_1, \Theta_2, \ldots, \Theta_K$ be diagonal $T \times T$ complex matrices with entries $[\Theta_k]_{tt} = \phi(U_{kt})$, $k = 1, \ldots, K, t = 1, \ldots, T$. Note that $\Theta_1^q = \ldots = \Theta_K^q = I$. The one-antenna constellation determined by the matrix U is then the set of all vectors of the form

$$\Theta_1^{\ell_1}\Theta_2^{\ell_2} \, \ldots \, \Theta_K^{\ell_K}\Phi_1.$$

For $K = 1$ and $K = 2$ these are exactly the forms suggested in (25) and (29). Thus the one-antenna constellation is the image of Φ_1 under the action of the discrete group generated by $\Theta_1, \ldots, \Theta_K$. We can extend this construction to admit multiple-antenna constellations by replacing the vector Φ_1 with a representation of a subspace of larger dimension in exactly the same way as is done in Section III-A.

The space of linear codes which do have a $K \times T$ generator matrix of elements in R_q is still quite large. Since we limit ourselves to finding codes that have low correlation by examining randomly chosen elements of the given space and keeping the one with the lowest correlation, it helps to restrict the class even further. In particular, we restrict U to have a systematic generator matrix of the form

$$U = [\, I \quad U' \,]$$

where I is the $K \times K$ identity matrix and U' is a $K \times (T-K)$ parity matrix with elements in R_q. Tables I and II list the best $M = 1$ and $M = 2$ antenna constellations for $T = 8$ we have found with our random search procedure. For each constellation, the maximum scaled Frobenius norm δ is given, as described in Section II in (11). The constellations all have a systematic representation and the rows of the parity matrix U' are listed. Hence, for a code of dimension K, K rows of $T-K$ elements in R_q are listed. The starting vector Φ_1 for $M = 1$ is $1/\sqrt{T}$ times a vector of all ones, and the starting matrix Φ_1 for

TABLE I

TABLE OF BEST FOUND $M = 1$ ANTENNA CONSTELLATIONS OF LENGTH $T = 8$ BASED ON LINEAR BLOCK CODES OVER R_q. THE NUMBER OF SIGNALS IN THE CONSTELLATION IS L, THE MAXIMUM CORRELATION IS δ, THE DIMENSION OF THE BLOCK CODE IS K, THE ARITHMETIC BASE IS q, AND THE ROWS OF THE PARITY MATRIX U' ARE GIVEN LAST. NOTE THAT $L = q^K$

L	δ	K	q	rows of U' (parity)
8	0.000000	1	8	[3 7 6 5 0 4 2]
16	0.306186	1	16	[0 3 14 15 11 10 8]
64	0.353553	3	4	[2 3 3 3 0]
				[2 0 3 1 1]
				[0 3 2 3 3]
133	0.534026	1	133	[48 98 104 72 38 123 4]
256	0.559017	4	4	[1 0 3 1]
				[3 1 1 2]
				[2 0 2 3]
				[1 1 3 2]
529	0.643485	2	23	[14 15 5 5 2 9]
				[11 2 11 4 13 19]
1296	0.695971	4	6	[2 0 1 5]
				[2 5 5 2]
				[2 3 0 3]
				[5 4 2 2]
2209	0.749396	2	47	[20 4 36 43 8 42]
				[44 8 34 6 12 1]

TABLE II

TABLE OF BEST FOUND $M = 2$ ANTENNA CONSTELLATIONS OF LENGTH $T = 8$ BASED ON LINEAR BLOCK CODES OVER R_q. THE NUMBER OF SIGNALS IN THE CONSTELLATION IS L, THE MAXIMUM CORRELATION IS δ, THE DIMENSION OF THE BLOCK CODE IS K, THE ARITHMETIC BASE IS q, AND THE ROWS OF THE PARITY MATRIX U' ARE GIVEN LAST. NOTE THAT $L = q^K$

L	δ	K	q	rows of U' (parity)
4	0.000000	2	2	[0 1 1 0 0 1]
				[0 1 0 1 0 1]
8	0.383533	1	8	[3 0 7 2 5 6 7]
17	0.475099	1	17	[12 11 9 14 6 10 0]
32	0.531944	1	32	[18 11 2 22 8 0 5]
67	0.588905	1	67	[7 31 15 3 29 20 0]
130	0.636015	1	130	[30 71 39 15 4 41 124]
257	0.669317	1	257	[7 60 79 187 125 198 154]
529	0.733934	2	23	[15 3 10 9 15 17]
				[22 16 14 4 21 21]
1024	0.76227	2	32	[26 22 1 3 7 26]
				[18 28 22 8 24 1]
2304	0.803542	2	48	[15 22 27 34 24 41]
				[18 1 38 29 33 25]

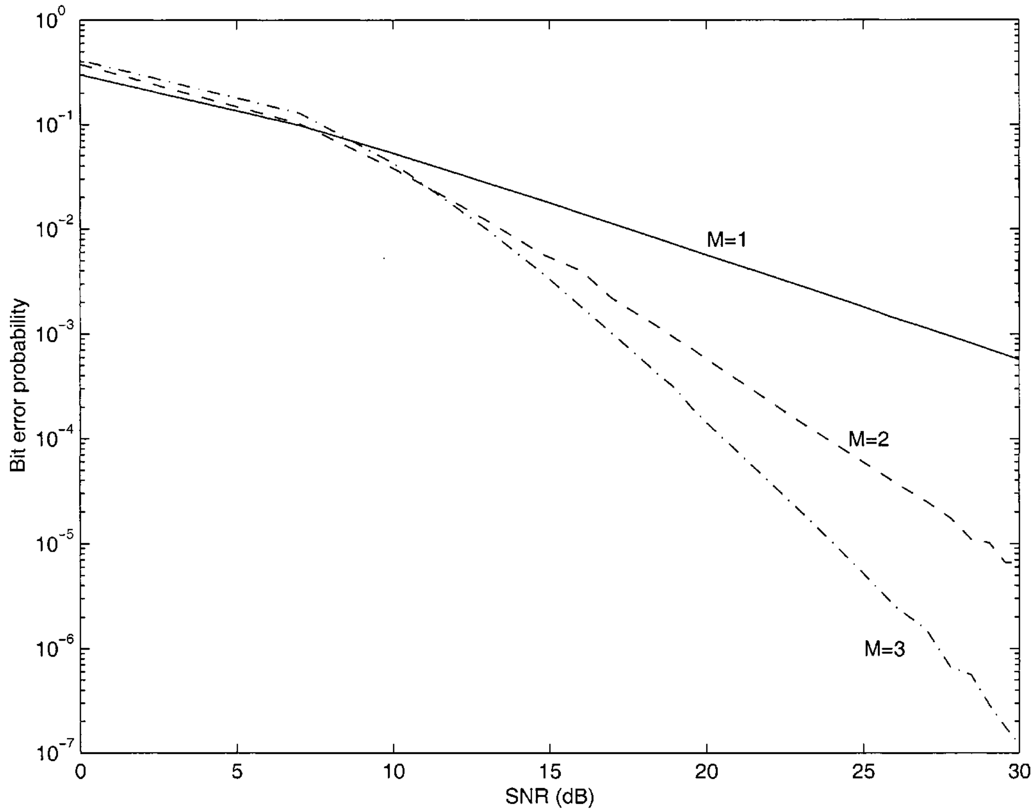

Fig. 4. Bit-error rate for $M = 1, 2$, and 3 transmitter antennas versus SNR with $N = 1$ receiver antenna on an unknown channel, $T = 8$, and $R = 1$ bit/channel use.

$M = 2$ is $1/\sqrt{T}$ times a matrix whose first column is all ones, and whose second column is

$$[1\ e^{i\frac{2\pi}{T}}\ \dots\ e^{i\frac{2\pi}{T}(T-1)}].$$

V. Application to Rayleigh Flat-Fading Channel

We now examine the performance of constellations designed using the methods of Sections III and IV on the multiple-antenna Rayleigh fading channel given in Section II. We look specifically at $M = 1$, 2, and 3 transmitter antennas and consider $N = 1$ receiver antenna. We choose typical parameters of $R = 1$ bit/channel use and we assume that the fading coefficients are constant for $T = 8$ channel uses. Thus we require a constellation of at least $L = 2^{RT} = 256$ signals, each an $M \times T$ matrix, for $M = 1$, 2, and 3.

The following constellations were used in the simulations.

- $M = 1$: The $L = 256$ constellation in Table I.
- $M = 2$: The first 256 signals from the $L = 257$ constellation in Table II.
- $M = 3$: The first 256 signals from an $L = 257$ constellation where

$$u = [220\ 191\ 6\ 87\ 219\ 236\ 173\ 170]$$

and Φ_1 comprises the first, sixth, and seventh columns of an 8×8 DFT matrix

$$\Phi_1 = \frac{1}{\sqrt{8}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{i\frac{2\pi}{8}5} & e^{i\frac{2\pi}{8}6} \\ 1 & e^{i\frac{2\pi}{8}2} & e^{i\frac{2\pi}{8}4} \\ 1 & e^{i\frac{2\pi}{8}7} & e^{i\frac{2\pi}{8}2} \\ 1 & e^{i\frac{2\pi}{8}4} & 1 \\ 1 & e^{i\frac{2\pi}{8}1} & e^{i\frac{2\pi}{8}6} \\ 1 & e^{i\frac{2\pi}{8}6} & e^{i\frac{2\pi}{8}4} \\ 1 & e^{i\frac{2\pi}{8}3} & e^{i\frac{2\pi}{8}2} \end{bmatrix}.$$

Here $\delta = 0.74355150$.

This code was found by the methods described in the previous section.

Fig. 4 shows the bit-error rate for the signal constellations designed for $M = 1$, 2, and 3 transmitter antennas. We see that the bit-error rate for larger M drops dramatically as the SNR ρ increases. To understand the reason for this, note from the Chernoff bound on pairwise error probability (8) that when $d_m < 1$ for all m, for high SNR and $N = 1$

$$P_{\ell,\ell'} \leq \frac{1}{2}\left(\frac{4M}{\rho T}\right)^M \prod_{m=1}^{M} \frac{1}{1-d_m^2}.$$

The probability of error therefore decays approximately as $1/\rho^M$. More generally, if some of the $d_m = 1$, then we have

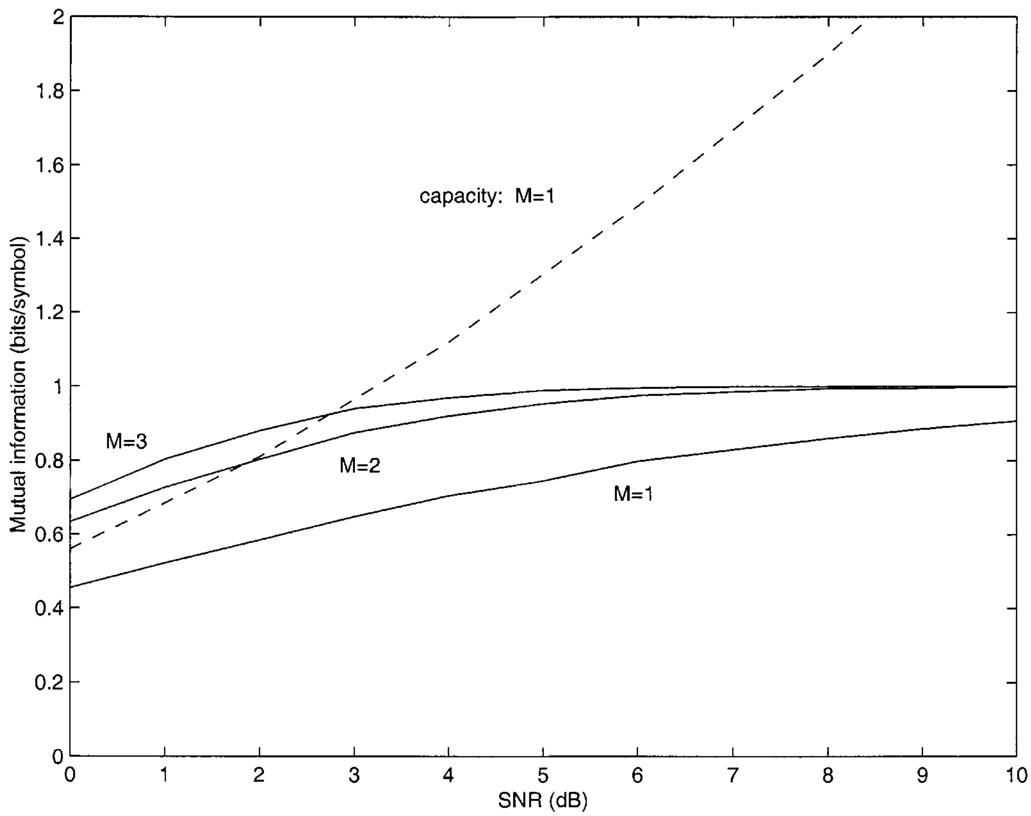

Fig. 5. Mutual information for the three constellations used to generate Fig. 4 versus SNR (solid curves); channel capacity versus SNR for $M = 1$ (dashed curve).

the pairwise probability of error bound (12), which for large ρ and $N = 1$ can be written

$$P_{\ell,\ell'} \leq \frac{1}{2}\left(\frac{4M}{\rho T}\right)^{M-\left\lceil \sum_{m=1}^{M} d_m^2 \right\rceil}.$$

In either case, the probability of error generally decreases more rapidly with ρ as M increases.

We also note that at low SNRs, the behavior of the unitary space–time signals with increasing M is reversed—the probability of error increases as M increases. A similar effect is noted in [10]. Fortunately, the decrease in performance at low SNR's is generally a fraction of a decibel.

By themselves, the simulations leading to Fig. 4 do not address the question of whether the constellations have good performance relative to some standard. Unfortunately, we are not aware of other unknown-channel designs with which comparisons may be made. We can, however, compute the mutual information of the constellations and compare their performance to signal designs for a channel that is known to the receiver.

Fig. 5 shows the mutual information as a function of SNR ρ for the three constellations ($M = 1, 2, 3$) that are used to generate Fig. 4. The dashed curve is the channel capacity when $M = 1$, which was computed by the methods described in [9]. (As in [9], we do not know how to compute the capacity for $M = 2$ or $M = 3$.) The constellations have rate $R = 1$, implying that for high SNRs, the mutual informations approach one. For SNRs below 3 dB, the mutual information of the $M = 1$ constellation is a significant fraction of the $M = 1$ channel capacity, which suggests that, in this regime, the $L = 256$ signals constitute a relatively efficient packing of the T-dimensional complex space. However, for higher SNRs we conclude that it should be possible, with a larger constellation, to transmit at much higher rates.

We can also examine the performance of the constellations when the channel is known to the receiver. Fig. 6 compares the block error rate for the constellations of Fig. 4 when the channel is known and unknown. Our constellations typically perform approximately 2–4 dB better when the channel is known. For $M = 2$ antennas, we also give the performance of an orthogonal design [15], which has an effective block size of $T = 2$ and is designed specifically for a known channel. As we can see, our block error rates compare favorably even though our constellations are designed for an unknown channel.

VI. Conclusions

Unitary space–time modulation is appropriate for flat-fading conditions where nobody knows the propagation coefficients. It requires the design of relatively large constellations of matrix-valued signals according to a criterion that differs markedly from the traditional maximum-Euclidean-distance criterion. We have introduced new design algorithms that easily produce large constellations of these signals in a systematic manner, by successive rotations of an initial signal. This entails the imposition of a circulant correlation structure on the constellation. Further research is needed to determine if significant improvements are possible by relaxing this structure.

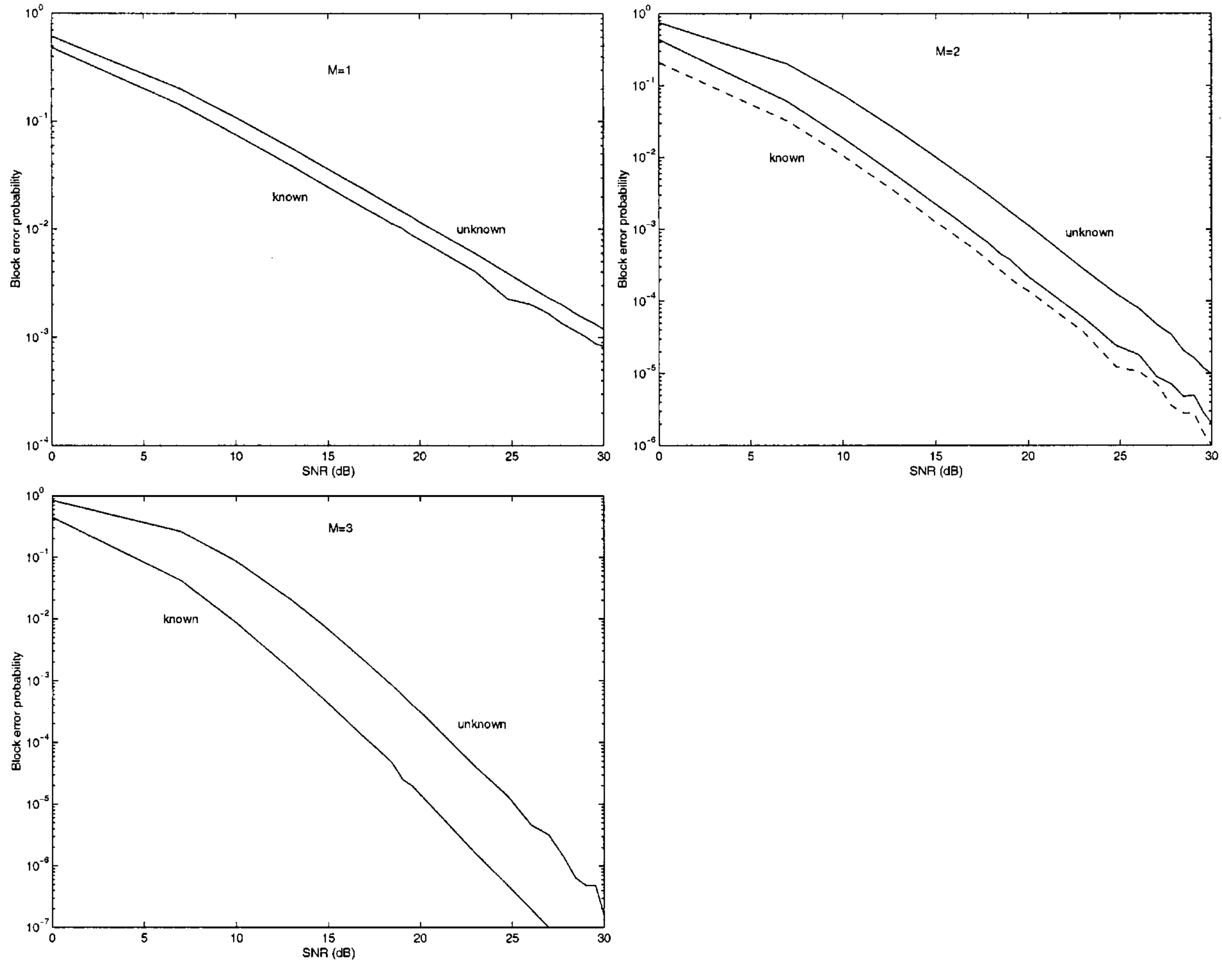

Fig. 6. Block-error rate comparison of unknown- and known-channel performance for $M = 1, 2$, and 3 transmitter antennas. The performance advantage when the known channel is approximately 2–4 dB. Also included for $M = 2$ is the performance of a rate-one orthogonal design (dashed line) with a known channel. (The orthogonal design has an effective block size of $T = 2$ and would be completely ineffective for all SNRs if the channel were unknown.)

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