Nonmonotonic Inference Based on Expectations

Peter Gärdenfors and David Makinson

Abstract

We show how nonmonotonic inferences may elegantly be interpreted in terms of underlying *expectations*. The fundamental idea is that when we reason, we make use of not only the information that we firmly believe, but also expectations that guide our beliefs without quite being part of them. We propose two ways of modelling the expectations used in nonmonotonic reasoning: by expectation sets, equipped with selection functions, and by expectation relations. For each of these we prove representation theorems and establish relations with several other modellings in the area, including Poole systems and preferential models.

We also show that by using the notion of expectation, one can unify the treatment of the theory of belief revision and that of nonmonotonic inference relations. This is accomplished by viewing the relation of 'epistemic entrenchment' used in Gärdenfors (1988) and Gärdenfors and Makinson (1988) as a kind of expectation ordering. Thus we see belief revision and nonmonotonic reasoning as basically the same process, albeit used for two different purposes.

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Peter Gärdenfors

Cognitive Science, Department of Philosophy, University of Lund

S-223 50 Lund, Sweden

David Makinson

Les Etangs B2, La Ronce

F-92410 Ville d'Avray, France

1. Introduction

1.1. Motivation

In this paper we want to explore ways in which the concept of an *expectation* can be used to give a general framework for nonmonotonic reasoning. The fundamental idea is that when we reason, we make use of not only the information that we firmly believe, but also the expectations that guide our beliefs without quite being part of them. Such expectations may take diverse forms, of which the two most salient are *expectation sets* and *expectation relations*.

On the one hand, our expectations may manifest themselves as propositions, drawn from the same language as our firm beliefs and indeed not differing from them in kind but at most in the ways in which we are prepared to use them. So understood, expectations include not only our firm beliefs as limiting case, but also other propositions that are regarded as plausible enough to be used as a basis for inference so long as they do not give rise to inconsistency. The key idea for using this set of propositions can be put informally as follows:

α nonmonotonically entails β iff β follows logically from α together with 'as many as possible' of the set of our expectations as are compatible with α.

The technical problem is essentially one of unpacking the innocent-looking 'as many as possible'.

On the other hand, our expectations may appear as an ordering between propositions, and in particular among those that we do not believe. The key idea for using this relation may be expressed informally thus:

α nonmonotonically entails β iff β follows logically from α together with all those propositions that are 'sufficiently well expected' in the light of α.

The technical problems here are essentially those of unpacking the 'sufficiently well expected', and of ascertaining the most appropriate conditions on the expectation ordering itself.

These two ideas are evidently closely related, but not quite the same. One difference that suggests itself already from the above summary formulations is that on the former we will tend to have a multiplicity of possible sets of auxiliary premises (due to the existence of many maximal consistent sets), and their elements will be determined 'globally'. By contrast, the latter formulation points towards a unique set of auxiliary premises, whose elements are determined 'locally'. Our formal development will clarify the exact relationship between the two manifestations of the underlying notion of expectation, as well as their relationship to other approaches, such as that in terms of preferential models. (For a third analysis of expectations in terms of activities in neural networks, see Balkenius and Gärdenfors (1991)).

But before turning to technicalities, let us illustrate the gist of the first conceptualization by a simple example. 'α nonmonotonically entails β' will be denoted $\alpha \mid\!\sim \beta$ as usual.

Example 1.1. Let the language L contain the following predicates:

Sx: x is a Swedish citizen

Ix: x has Italian parents

Px: x is a protestant

Assume that the set of expectations contains $Sb \to Pb$ and $Sb \wedge Ib \to \neg Pb$, for all individuals b. Assuming that the set of expectations is closed under logical consequences it also contains $Sb \to \neg Ib$ and, of course, the logical truth $Sb \wedge Ib \to Sb$. If we now learn that b is a Swedish citizen, that is Sb, this piece of information is consistent with the expectations. Thus, according to the recipe above, we can conclude that $Sb \mid\!\sim Pb$.

On the other hand, if we learn both that b is a Swedish citizen and has Italian parents, that is $Sb \wedge Ib$, then this information is *inconsistent* with the set of expectations and so we cannot use all expectations when determining which inferences can be drawn from $Sb \wedge Ib$. The most natural expedient is to give up the expectation $Sb \to Pb$ and the consequence $Sb \to \neg Ib$. The contracted set of expectations which contains $Sb \wedge Ib \to \neg Pb$ and its logical consequences contains 'as many as possible' (in a sense to be made precise below) of the sentences in the set of expectations that are compatible with $Sb \wedge Ib$. So, by the general rule above, we have $Sb \wedge Ib \mid\!\sim \neg Pb$, illustrating the nonmontonicity of $\mid\!\sim$.

Our treatment of expectation sets in Section 2 can be seen as generalizing work of Poole (1988), whilst our treatment of expectation orderings in Section 3 is a prolongation of work of a number of authors, including in particular Dubois and Prade (see e.g. Dubois (1986) and Dubois and Prade (1988), (1991a), (1991b)). At the same time, our formal work is influenced by recent developments in the logic of belief change, where the concept of expectation has been extensively investigated. In particular, our treatment of expectation sets in Section 2 is inspired by the theory of 'partial meet' contractions and revisions in the logic of belief change, as developed by Alchourrón, Gärdenfors, and Makinson (1985) and also set out in the book Gärdenfors (1988). Likewise our treatment of expectation orderings in Section 3 takes its departure from the theory of epistemic entrenchment as a determinant of belief revision, as set out in Gärdenfors and Makinson (1988) and again in book form in Gärdenfors (1988).

Such assistance from the logic of belief change is itself only to be expected, especially in view of the close connections, noted in Makinson and Gärdenfors (1990), between properties that may hold of belief revision operators, and those that may hold of nonmonotonic inference relations. Nevertheless, our exposition will be such as to make the present paper self-contained, without requiring familiarity with the belief revision literature. Connections with that literature are explained subsequently, in Section 4.

Here is should also be noted that our choice of the word ‘expectation’ should not be confused with the notion of ‘expected utility’ in decision theory. ‘Expected utility’ has to do with expectations of the *values* of various outcomes, whilst our notion of expectation concerns *beliefs* about the world. Our use of ‘expectation’ thus comes closer to the everyday use.

1.2. Technical assumptions

We shall work with a language L which is based on propositional logic. It will be assumed that L is closed under applications of the *boolean connectives* $\neg$ (negation), $\wedge$ (conjunction), $\vee$ (disjunction), and $\rightarrow$ (implication). We will use α, β, γ, etc. as variables over sentences in L. It is also convenient to introduce the symbols $\top$ and $\bot$ for the two sentential constants ‘truth’ and ‘falsity’.

All the different expectations will be formulated in L. In contrast to many other theories of nonmonotonic reasoning there are thus no default rules or other additions to the basic language, such as modal operators, that will be used to express the defeasible forms of information.

We will assume that the underlying logic includes *classical propositional logic* and that it is compact. Recall that a logic is said to be compact iff whenever α is a logical consequence of a set A of sentences, then there is a *finite* subset A' of A such that α is a logical consequence of A'. If A logically entails α we will write this as $A \vdash \alpha$. We assume that $\vdash$, like classical entailment, satisfies the deduction theorem and ‘disjunction in the premises’, i.e. that $A \cup \{\beta \vee \gamma\} \vdash \alpha$ whenever both $A \cup \{\beta\} \vdash \alpha$ and $A \cup \{\gamma\} \vdash \alpha$. Other details of L are left open. Where A is a set of sentences we shall use the notation Cn(A) for the set of all logical consequences of A, i.e. $Cn(A) = \{\alpha: A \vdash \alpha\}$.

Several kinds of nonmonotonic inference relations will be studied. The inference relation $\mathrel{|}\!\sim$ will sometimes be written with a subscript to mark that it belongs to a particular family of inference relations. We will also introduce the notation $C(\alpha)$ for the set of all nonmonotonic conclusions that can be drawn from α, that is, $\beta \in C(\alpha)$ iff $\alpha \mathrel{|}\!\sim \beta$. Again, the operation C will sometimes be decorated with a subscript. All proofs are reserved for Appendix II.

1.3. Postulates for nonmonotonic inference operations

In this section we shall briefly review some postulates for nonmonotonic inference relations. We shall consider only inferences from *finite* sets of premises, in this respect following Kraus, Lehmann, and Magidor (1990) rather than Makinson (1989) which considers inferences from arbitrary sets of premises. All of the postulates to be presented, with one exception, are familiar from the above two papers — the exception being that of ‘Consistency Preservation’ which is introduced in Makinson and Gärdenfors (1990) and discussed in the general review Makinson (to appear). However, we shall *group* the postulates rather differently than in any of the above references, so as to bring out Lindström's (1990) concept of an ‘inference operation’ and our own concept of ‘basic’ postulates for nonmonotonic inference, corresponding to the ‘basic’ postulates for belief revision, following Gärdenfors (1988) and Makinson and Gärdenfors (1990).

Following Lindström (1990) we say that a relation $\mathrel{|}\!\sim$ between propositions is an *inference relation* iff it satisfies the four conditions:

If $\alpha \vdash \gamma$, then $\alpha \mathrel{|}\!\sim \gamma$ (*Supraclassicality*)

If $\vdash \alpha \leftrightarrow \beta$ and $\alpha \mathrel{|}\!\sim \gamma$, then $\beta \mathrel{|}\!\sim \gamma$ (*Left Logical Equivalence*)

If $\vdash \beta \rightarrow \gamma$ and $\alpha \mathrel{|}\!\sim \beta$, then $\alpha \mathrel{|}\!\sim \gamma$ (*Right Weakening*)

If $\alpha \mathrel{|}\!\sim \beta$ and $\alpha \mathrel{|}\!\sim \gamma$, then $\alpha \mathrel{|}\!\sim \beta \wedge \gamma$ (*And*)

Clearly, Supraclassicality implies:

$\alpha \mathrel{|}\!\sim \alpha$ (*Reflexivity*) and Right Weakening and And together imply, using the compactness of $\vdash$:

If $\alpha \mid\sim \beta_i$ for all $\beta_i \in$ B and B $\vdash \gamma$, then $\alpha \mid\sim \gamma$ (*Closure*)

Conversely, Reflexivity, Left Logical Equivalence, and Closure together imply Supra-classicality, Right Weakening and And, and so constitute an equivalent definition of an inference relation (in fact, the one used by Lindström).

By the *basic postulates* for nonmonotonic inference we mean the above four for the concept of an inference relation plus the following, where $\mid\sim \alpha \rightarrow \beta$ is an abbreviation for $\top \mid\sim \alpha \rightarrow \beta$:

If $\alpha \mid\sim \beta$, then $\mid\sim \alpha \rightarrow \beta$ (*Weak Conditionalization*)

If $\not\mid\sim \neg\alpha$ and $\mid\sim \alpha \rightarrow \beta$, then $\alpha \mid\sim \beta$ (*Weak Rational Monotony*)

If $\alpha \mid\sim \bot$, then $\alpha \vdash \bot$ (*Consistency Preservation*)

These basic postulates correspond, under the translation of Makinson and Gärdenfors (1990), to the ‘basic postulates’ of the logic of belief revision. The key idea of that translation is that a statement of the form $\beta \in K^*_\alpha$, where K^*_α is the revision of a belief state K by a sentence α, is seen as a nonmonotonic inference from α to β given the set K of sentences as *background* expectations. So the statement $\beta \in K^*_\alpha$ for belief revision is translated into the statement $\alpha \mid\sim \beta$ for nonmonotonic logic (or into $\alpha \mid\sim_K \beta$, if one wants to emphasize the role of the background beliefs). It turns out, as shown there, that the translations of the postulates (K*1) - (K*6) from Gärdenfors (1988) correspond respectively precisely to Closure, Reflexivity, Weak Conditionalization, Weak Rational Monotony, Consistency Preservation, and Left Logical Equivalence, and thus collectively to the basic postulates for $\mid\sim$.

By the *extended* set of postulates for nonmonotonic inference we mean the basic postulates plus the following three:

If $\alpha \mathrel{|\!\sim} \beta$ and $\beta \vdash \alpha$, then $\alpha \mathrel{|\!\sim} \gamma$ iff $\beta \mathrel{|\!\sim} \gamma$ (*Cumulativity*)

If $\alpha \mathrel{|\!\sim} \gamma$ and $\beta \mathrel{|\!\sim} \gamma$, then $\alpha \vee \beta \mathrel{|\!\sim} \gamma$ (*Or*)

If $\alpha \mathrel{|\!\not\sim} \neg\beta$ and $\alpha \mathrel{|\!\sim} \gamma$ then $\alpha \wedge \beta \mathrel{|\!\sim} \gamma$ (*Rational Monotony*)

We recall that Cumulativity is equivalent (given the basic postulates) to the conjunction of the following two postulates:

If $\alpha \mathrel{|\!\sim} \beta$ and $\alpha \wedge \beta \mathrel{|\!\sim} \gamma$, then $\alpha \mathrel{|\!\sim} \gamma$ (*Cut*)

If $\alpha \mathrel{|\!\sim} \beta$ and $\alpha \mathrel{|\!\sim} \gamma$, then $\alpha \wedge \beta \mathrel{|\!\sim} \gamma$ (*Cautious Monotony*)

and that it is also equivalent to:

If $\alpha \mathrel{|\!\sim} \beta$ and $\beta \mathrel{|\!\sim} \alpha$, then $\alpha \mathrel{|\!\sim} \gamma$ iff $\beta \mathrel{|\!\sim} \gamma$ (*Reciprocity*)

Strictly speaking, Cumulativity is redundant in our extended set of postulates, as follows from observations of Kraus, Lehmann and Magidor (1990), and Freund, Lehmann and Morris (1991). For Cut follows easily from Conditionalization, And, and Right Weakening, whilst Cautious Monotony follows easily from Rational Monotony, And, Right Weakening, Consistency Preservations and Supraclassicality.

The postulate Or is also known as ‘Disjunction in the Premises’ and also, especially in its infinitary form, as ‘Distribution’. It is equivalent to the following, given the basic postulates for inference relations:

If $\alpha \wedge \beta \mathrel{|\!\sim} \gamma$, then $\alpha \mathrel{|\!\sim} \beta \to \gamma$ (*Conditionalization*)

Conditionalization is the translation in Makinson and Gärdenfors (1990) of the postulate (K*7) for belief revision. Weak Conditionalization is the special case when $\alpha = \top$. Similarly, for Rational Monotony and its weak version. Rational Monotony is equivalent to the translation in Makinson and Gärdenfors (1990) of the postulate (K*8) for belief revision.

The following is an easy consequence of the extended set of postulates:

If $\alpha \lor \beta \mid\sim \gamma$, then $\alpha \mid\sim \gamma$ or $\beta \mid\sim \gamma$ (*Disjunctive Rationality*)

This principle will be useful in Section 2 in proving a representation theorem for the extended set of postulates. We recall the verification (cf. Lehmann and Magidor (1990) or Makinson (to appear)). Suppose $\alpha \lor \beta \mid\sim \gamma$, but $\alpha \not\mid\sim \gamma$. From the latter we have by Left Logical Equivalence that $(\alpha \lor \beta) \land \alpha \not\mid\sim \gamma$ so that by Rational Monotony $\alpha \lor \beta \mid\sim \neg\alpha$, so by Reflexivity, And and Right Weakening $\alpha \lor \beta \mid\sim \beta$. Putting this together with $\alpha \lor \beta \mid\sim \gamma$ gives us by Cautious Monotony that $(\alpha \lor \beta) \land \beta \mid\sim \gamma$, so by Left Logical Equivalence $\beta \mid\sim \gamma$ as desired.

2. Expectation inference operations based on sets

2.1. Maximal subsets and selection functions

Any set of propositions in L that are, intuitively, playing the role of 'expectations' will be denoted Δ. The problem of this section is how to determine which elements of the set Δ of expectations to give up when adding a new piece of information α that is inconsistent with Δ. A general idea is to start from Δ and then give some recipe for choosing which propositions to delete from Δ to form a subset of Δ which does not contain α as a logical consequence. According to the general criterion we should look at as large a subset of Δ as possible.

The following notion is useful: A set D is a *maximal subset of Δ that fails to imply α* if and only if (i) $D \subseteq \Delta$, (ii) $\alpha \notin Cn(D)$, and (iii) $\alpha \in Cn(D')$ for every D' with $D \subset D' \subseteq \Delta$. The set of all maximal subsets of Δ that fail to imply α will be denoted $\Delta\perp\alpha$. Using the assumption that $\vdash$ is compact it is easy to show that this set is nonempty, unless α is logically valid.

We now turn to a first solution to the problem of determining when $\alpha \mid\sim \beta$ holds, that is, determining when α nonmonotonically implies β. The idea is to use the sets in $\Delta\perp\neg\alpha$ for the construction since these sets are maximally consistent with α. As is notorious, it is in general not possible to select a unique maximal subset of Δ that fails to imply $\neg\alpha$. To take a trivial example, if Δ is the set of all logical consequences of $\beta \rightarrow \delta$ and $\gamma \rightarrow \neg\delta$, and we put

$\alpha = \beta \wedge \gamma$ then there are two maximal subsets of Δ consistent with α, one containing $\beta \rightarrow \delta$ and the other containing $\gamma \rightarrow \neg\delta$. One must thus be content with picking out the 'most relevant' maximal subsets.

Technically, this can be done with the aid of a *selection function* S_Δ depending on Δ, or more briefly S when Δ is taken as fixed. This is a function of one argument, and the elements of its domain are sets of the form $\Delta\bot\alpha$ for some formula α. They are thus subsets of Δ. A selection function is required to satisfy $\varnothing \neq S(\Delta\bot\alpha) \subseteq \Delta\bot\alpha$ in the principal case that $\Delta\bot\alpha$ is non-empty, and to satisfy $S(\Delta\bot\alpha) = \{\Delta\}$ in the limiting case that $\Delta\bot\alpha$ is empty. Note that as a particular application of the principal case we have that if $\Delta\bot\alpha = \{\Delta\}$, then also $S(\Delta\bot\alpha) = \{\Delta\}$. Note also that whenever $Cn(\alpha) = Cn(\beta)$, then $\Delta\bot\alpha = \Delta\bot\beta$ so that $S(\Delta\bot\alpha) = S(\Delta\bot\beta)$.

Definition 2.1. An *expectation inference operation* $C_{\Delta,S}$ is defined, for all $\alpha \in L$, by the equation $C_{\Delta,S}(\alpha) = \bigcap\{Cn(\{\alpha\} \cup D): D \in S(\Delta\bot\neg\alpha)\}$, where Δ is a non-empty default set and S is a selection function. $C_{\Delta,S}$ is *closed* when $\Delta = Cn(\Delta)$, and is *consistently generated*, when Δ is consistent.

This definition is one way of making the general rule above technically precise. Intuitively, when we want to determine whether $\alpha \mid\!\sim_{\Delta,S} \beta$, that is, whether $\beta \in C_{\Delta,S}(\alpha)$, we consider the 'most relevant' maximal subsets of Δ that fail to imply $\neg\alpha$, which are picked out by the selection function S. If $\alpha \rightarrow \beta$ belongs to *all* of these maximal subsets, then β belongs to $Cn(\{\alpha\} \cup D)$ for all D in $S(\Delta\bot\neg\alpha)$ and so, by the definition, $\alpha \mid\!\sim_{\Delta,S} \beta$. In other words, we take with a grain of salt the initial idea, mentioned in Section 1.1, of adding to α 'as many as possible' of the sets of our expectations as are compatible with α. We do not choose a single maximal α-consistent subset of Δ, but intersect certain among them.

Before turning to the theoretical aspects of the definition, a potential objection must be considered. Since we express defaults by formulas in an ordinary first order language ruled by classical logic, *contrapositions* of expectations in a closed Δ are also in Δ. Assume, for example that 'Computer scientists typically don't know about nonmonotonic logic' (write this as $C \rightarrow \neg N$) is in Δ. If we assume that Δ is closed (or even just closed under logical equivalence), both $C \rightarrow \neg N$ and $N \rightarrow \neg C$ are then in Δ, where the latter formula corresponds to ‘People who know about nonmonotonic logic are normally not computer scientists’. Intuitively, these two sentences do not function in the same way in default reasoning: From the fact that somebody is a computer scientist we want to conclude that she does not know anything about nonmonotonic logic, but we don’t want to conclude that somebody is not a computer scientist from the fact that he know about nonmonotonic logic. In other words, we don’t want contraposition to be valid of nonmonotonic inference relations.

However, the fact that $C \to \neg N$ and $N \to \neg C$ are both in Δ does not mean that they must be used in the same way in nonmonotonic reasoning. In particular it does not follow that $C \mathrel{|\!\sim}_{\Delta,S} \neg N$ iff $N \mathrel{|\!\sim}_{\Delta,S} \neg C$. The reason for this is that the two ‘premises’ C and N are not symmetrical – the class of people satisfying N is very small. In order words, it is natural to assume in this example that $\neg N$ belongs to Δ. And, by logical closure, if $\neg N$ is in Δ, so is $N \to C$ (as well as $N \to \neg C$). If we now want to check what N nonmonotonically entails, we have to select some maximal subsets of Δ. And the most natural selection function picks out those subsets where $N \to C$ are included, and consequently N nonmonotonically entails C rather than $\neg C$! On the other hand, if C is consistent with Δ, which is reasonable in the example, then we immediately have that C nonmonotonically entails $\neg N$, so contraposition is indeed not valid as a general rule for nonmonotonic inferences.

Clearly, in terms of the customary distinction, Definition 2.1 is a *sceptical* approach to nonmonotonic reasoning, rather than a ‘choice’ or ‘liberal’ one. A ‘liberal’ definition would use union rather than intersection; a ‘choice’ definition would choose one of the items over which the intersection is performed. It should be noted that expectation inference operations are here only defined for *finite* sets of premises, which can always be replaced by their conjunction α, which is the single formula in the definition above. However, as shown by Freund, Lehmann and Makinson (1990), there is a canonical way of extending any such finitary relation to cover infinite sets of premises.

When Δ is closed under logical consequence, i.e., when $\Delta = Cn(\Delta)$, another way of interpreting Definition 2.1 becomes available when we consider the sets $Cn(\{\alpha\} \cup D)$. It can easily be shown (using the assumption that $\Delta = Cn(\Delta)$) that in the principal case when $\neg\alpha \in \Delta$ it holds that for any $D \in \Delta\perp\neg\alpha$ and for any $\beta \in L$, we have either $\alpha \to \beta \in D$ or

$\alpha \to \neg\beta \in D$. This means that when $\neg\alpha \in \Delta$ we have either $\beta \in Cn(\{\alpha\} \cup D)$ or $\neg\beta \in Cn(\{\alpha\} \cup D)$, for any $\beta \in L$. Thus, for any $D \in \Delta\perp\neg\alpha$, $Cn(\{\alpha\} \cup D)$ can be identified with an *interpretation* or a *world* which makes α true (this is the terminology used by Shoham (1988) and Kraus, Lehmann, and Magidor (1990)). In this sense the selection function S, if suitably constrained, can be seen as indirectly picking out a set of 'preferred' α-worlds and $\alpha \mid\!\sim_{\Delta,S} \beta$ holds when β is true in all the preferred α-worlds.

We are thus quite close to the *preferential models* of Shoham (1988), generalized in Makinson (1989), Kraus, Lehmann and Magidor (1990), and Lindström (1990) – cf. also Katsuno and Mendelzon (1991). Technically, Shoham's models are considerably less general than ours: they assume that the preferred models are determined by minimalization of all classical models under a partial ordering. Those of Makinson (1989) and Kraus, Lehmann and Magidor (1990) are in some ways less general than ours, and in other ways more so. Less general, in that they still require that preferred models are determined by minimization under a relation; more general in that the relation holds between 'states' that need not be well-behaved as classical models. Lindström's constructions are in all respects at least as general as ours, and in some respects more – covering, for example, infinite premise sets. Beyond these technical considerations of levels of generality there is also a basic difference of *gestalt* between the present models ans the others mentioned. In this paper we are seeing preferences between worlds as merely a byproduct, and we are taking nonmonotonic inference relations to be generated directly out of a set, usually far from complete, of propositions that serve as 'expectations'.

Definition 2.1 offers a wide class of inference operations because no particular constraints are put on the selection function. In order to select a particular nonmonotonic inference rule, one must provide some way of describing the underlying selection function. This would be the natural step to take if one wants to implement an inference operation in a theorem proving program. We shall not pursue this line here, but instead strive for a meta-theoretic description of the class of expectation inference operations by examining which general conditions on inference operations they satisfy. The conclusion of this investigation will be a representation theorem, closely related to that of Lindström (1990) but rather simpler in its formulation. Section 2.3 reviews the relevant syntactic conditions for $\mid\!\sim$, and

Section 2.4 states and proves the representation in terms of expectation inference relations $\mid\!\sim_{\Delta,S}$.

2.2. Poole systems

It is also interesting to compare the general definition of an expectation inference operation with Poole's approach to nonmonotonic reasoning as presented in his (1988). A *Poole system* can be described as a pair $\langle\Delta,K\rangle$ where Δ and K are both sets of sentences (not necessarily closed under Cn). Poole calls Δ the set of ‘default propositions’, and K is called the set of ‘constraints’ of the system. He too works with the general idea that when determining the nonmonotonic consequences of α we should look at the maximal subsets of Δ that are consistent with α and K. So given a Poole system $\langle\Delta,K\rangle$, and using the terminology from the exposition of Poole's idea given in Makinson (to appear) we can define its associated ‘extension family function’ by the rule that for every α

> $e(\alpha) = \{Cn(\alpha \cup D$: D is maximal among the subsets D' of such Δ that $\alpha \cup D' \cup K$ is consistent$\}$.

Now Poole in effect defines that α ‘explains’ β if there is *some* extension E in $e(\alpha)$ such that $\beta \in E$. (Poole's definition can be found on p. 29 of his (1988) and the result connecting this definition to extensions on p. 30.) This is evidently a ‘liberal’ conception, using existential generalization. Nevertheless, if we adopt the ‘sceptical’ approach, we can define a *Poole inference operation* $C_{\Delta,K}$ by the rule that $C_{\Delta,K}(\alpha) = \cap e(\alpha)$. Now, it is easy to see that in the case when the set K of constraints is empty, the sceptical Poole inference is the special case of an expectation inference operation where the selection function is defined by the rule $S(\Delta\bot\neg\alpha) = \Delta\bot\neg\alpha$. In other words, for this special case the selection function makes no selection at all among the elements of $\Delta\bot\neg\alpha$ but considers them as equally relevant. However, as shown in Makinson and Gärdenfors (1990), if the set Δ of expectations is assumed, as in what follows, to be closed under logical consequence, then the special case becomes a degenerate case: $C_{\Delta,K}(\alpha)$ reduces to $Cn(\alpha)$ whenever α is inconsistent with Δ, and to $Cn(\Delta \cup \{\alpha\})$ when α is consistent with Δ. Thus the Poole inference relation is of interest only when the set Δ of expectations is taken on the level of ‘presentation’ rather than on the level of ‘content’. An analogous phenomenon holds, of course, for the logic of belief revision, as was shown by Alchourrón and Makinson (1982).

2.3. Representation theorems using expectation sets

Lindström (1990) proves a representation theorem for inference relations in terms of selection functions. A difference of presentation between his result and the theorems to be presented below is that his selection functions do not operate on maximal subsets of Δ that do not entail $\neg\alpha$, but rather on maximal theories containing α. There are also some differences of content: he makes a distinction between ‘states’ and maximal theories (which he calls ‘worlds’); he also operates in a more general setting where one allows *infinite* sets of premises for the nonmonotonic inference operation.

In this section it will be shown that the set of *basic postulates* for nonmonotonic inference relations exactly characterizes the class of expectation inference relations. The proof of the following representation theorem is a kind of translation of the corresponding theorem for belief contraction (Observation 2.5 in Alchourrón, Gärdenfors, and Makinson (1985), Theorem 4.13 in Gärdenfors (1988)). The formal structures of the two areas are very similar, even if the translation itself involves some tricky technical details. On the conceptual level, the representation theorem provides a new perspective on nonmonotonic reasoning that is based on natural and independently motivated constructions.

Before formulating the representation theorem, we recall the following very useful lemma (Lemma 2.4 of Alchourrón, Gärdenfors, and Makinson (1985). For the sake of making the exposition self-contained we include its proof, along with those of the other lemmas and theorems, in Appendix II.

Lemma 2.2. Suppose that $\Delta = Cn(\Delta)$. If $D \in \Delta\perp\alpha$, then $D \in \Delta\perp\beta$ for all $\beta \in \Delta$ such that $\beta \notin D$.

Theorem 2.3. A nonmonotonic inference relation $\mid\!\sim$ satisfies the set of basic postulates *if and only if* there exists a closed and consistently generated expectation inference relation $\mid\!\sim_{\Delta,S}$ such that $\alpha \mid\!\sim \beta$ iff $\alpha \mid\!\sim_{\Delta,S} \beta$, for all α and β.

For Theorem 2.3, no restrictions are put on the selection function. The theorem shows that for nonmonotonic inference operations based on such selection functions we can in general only expect the basic postulates to be satisfied, but not, for example, Cumulativity. A very natural condition on a selection function is the following:

(SC) If $S(\Delta\perp\neg\alpha) \subseteq \Delta\perp\neg\beta \subseteq \Delta\perp\neg\alpha$, then $S(\Delta\perp\neg\beta) = S(\Delta\perp\neg\alpha)$

The interpretation is that if the set of maximal subsets in $\Delta\perp\neg\beta$ is included in the set $\Delta\perp\neg\alpha$ and the 'preferred' maximal subsets in $\Delta\perp\neg\alpha$ are all members of $\Delta\perp\neg\beta$, then these are also the best in $\Delta\perp\neg\beta$. Note that for $\neg\alpha, \neg\beta \in \Delta$ the condition that $\Delta\perp\neg\beta \subseteq \Delta\perp\neg\alpha$ holds just if $\neg\alpha \vdash \neg\beta$, that is $\beta \vdash \alpha$ (cf. the condition called 'Aizerman' in Lindström (1990)).

It is now perhaps not surprising that adding this requirement on the selection function corresponds to making the generated nonmonotonic inference relation cumulative, and *vice versa*:

Theorem 2.4. A nonmonotonic inference relation $\mid\!\sim$ satisfies the set of basic postulates and Cumulativity *if and only if* there exists a closed, consistently generated expectation inference relation $\mid\!\sim_{\Delta,S}$ where S satisfies (SC) such that $\alpha \mid\!\sim \beta$ iff $\alpha \mid\!\sim_{\Delta,S} \beta$, for all α and β.

Additional connections between conditions on selection functions and properties of the generated nonmonotonic inference relations are studied in Lindström (1990), albeit in a slightly more general setting.

A further strengthening of the requirements for a selection function would be to demand that it is generated by some underlying 'preference' relation in the following sense:

Definition 2.5. A selection function S is *relational* over Δ iff there is a relation $\angle$ over the subsets of Δ such that for all α with $\neg\alpha \notin Cn(\emptyset)$, it holds that $S(\Delta\perp\neg\alpha) = \{D \in \Delta\perp\neg\alpha: D \angle D' \text{ for all } D' \in \Delta\perp\neg\alpha\}$. S is *transitively relational* iff S is relational under some transitive relation $\angle$.

Theorem 2.6. Any relational closed expectation inference relation $\mid\!\sim_{\Delta,S}$ satisfies Or.

Theorem 2.7. Any transitively relational closed expectation inference relation $\mid\!\sim_{\Delta,S}$ satisfies Rational Monotony (as well as Or and thus also Cumulativity).

It is possible to prove also the converse of Theorem 2.7. For this representation theorem, we need some preparatory lemmas.

Lemma 2.8. Suppose that $\Delta = Cn(\Delta)$ and $\alpha, \beta \in \Delta$. Then $\Delta \perp \alpha \wedge \beta = \Delta \perp \alpha \cup \Delta \perp \beta$.

Lemma 2.9. Suppose that $\Delta = Cn(\Delta)$ and $D \in \Delta \perp \alpha$. Then $\Delta \subseteq Cn(D \cup \{\alpha\})$.

Lemma 2.10. Suppose that $\Delta = Cn(\Delta)$, $\neg\alpha \in \Delta$ and $D \in \Delta \perp \neg\alpha$. Then:

(a) $D = Cn(D \cup \{\alpha\}) \cap \Delta$

(b) $C(\alpha) \subseteq Cn(D \cup \{\alpha\})$ iff $C(\alpha) \cap \Delta \subseteq D$, whenever C satisfies Right Weakening.

Lemma 2.11. Let $\mid\!\sim$ be any inference relation satisfying the extended set of postulates. If $\alpha \mid\!\sim \gamma$ and $\neg\alpha \vdash \gamma$ then $\alpha \vee \beta \mid\!\sim \gamma$ for any β.

Theorem 2.12. An inference relation $\mid\!\sim$ satisfies the extended set of postulates *iff* there is a closed, consistently generated, and transitively relational expectation inference relation $\mid\!\sim_{\Delta,S}$ with $\mid\!\sim\ = \ \mid\!\sim_{\Delta, S}$.

The proofs of the lemmas and the theorem, which parallels the representation theorem 4.4 for belief contraction in Alchourrón, Gärdenfors, and Makinson (1985), are given in Appendix II.

3. Expectation inference operations based on orderings

3.1. Expectation orderings

Although we have been able to prove some representation results in the previous section, the use of selection functions as a mechanism for generating nonmonotonic inferences is not very satisfactory from a computational perspective. One reason is that it is extremely costly to compute the maximal subsets in $\Delta \perp \neg\alpha$, especially when Δ is assumed to be closed under logical consequence. Another reason is that a functional description of a selection function is, in general, not available, even in the most favourable situation that the selection function is relational. A general algorithm for computing an expectation inference relation would have to solve the multiplied effects of these two problems.

From a computational point of view it would be much more natural to work with an ordering of the *sentences* in Δ rather than with an ordering of the maximal subsets of Δ, let alone a general selection function defined on Δ. And from the epistemological perspective it seems intuitively plausible that our expectations about the world do not all have the same strength. For example, we consider some rules to be almost universally valid, so that an exception to the rule would be extremely unexpected; while other rules are better described as rules of thumb that we use for want of more precise information. An exception to the latter type of rule is not unexpected to the same degree as in the former case. In brief, our expectations are all defeasible (unless logically valid), but they exhibit varying *degrees of defeasibility*. An alternative way of phrasing the idea is to speak of the degrees of *firmness* of our expectations in some sense that need not be assumed to correspond to degrees of probability. In fact, as we shall see, the postulates to be presented in this section for such an ordering are incompatible with a simple 'threshold probability' interpretation although, as shown by Lehmann and Magidor (1990), building on work of Adams, Pearl and others, it is compatible with an account in terms of 'limiting values' of probabilities.

It will also be seen that on this approach, unlike that of Section 2, there is no need to fix a set Δ of expectations in advance. The expectation ordering will cover *all* sentences, and the set of expectations, or 'real possibilities' as one might wish to call them, can be constructed out of the ordering in a natural way, as made up of those sentences that are strictly more to be expected than a contradiction.

In order to make these ideas more precise, we shall now assume that there is an ordering $\leq$ of the sentences in L. '$\alpha \leq \beta$' should be interpreted as 'β is at least as expected as α' or 'α is at least as surprising as β'. '$\alpha < \beta$' will be written as an abbreviation for 'not $\beta \leq \alpha$' and '$\alpha \approx \beta$' is an abbreviation for '$\alpha \leq \beta$ and $\beta \leq \alpha$'. Note that the relation $\leq$ is *not* part of the object language, but is used on the meta-level to compare formulas from the object language. This is in contrast to Lewis's (1973) notion of 'comparative possibility' which otherwise shows many formal similarities with $\leq$. The relation $\leq$ will be assumed to satisfy the following postulates:

(E1) If $\alpha \leq \beta$ and $\beta \leq \gamma$, then $\alpha \leq \gamma$ (*Transitivity*)

(E2) If $\alpha \vdash \beta$, then $\alpha \leq \beta$ (*Dominance*)

(E3) For any α and β, $\alpha \leq \alpha \wedge \beta$ or $\beta \leq \alpha \wedge \beta$ (*Conjunctiveness*)

The first postulate on the expectation ordering is very natural for an ordering relation. The second postulate says that a logically stronger sentence is always less expected. From this it follows that the relation $\leq$ is reflexive. The third constraint is crucial for the results to come, but presumably the one that is most open to query. It concerns the relation between the degree of expectation of a conjunction $\alpha \wedge \beta$ and the corresponding degrees of α and β.

From (E2) it follows immediately that $\alpha \wedge \beta \leq \alpha$ and $\alpha \wedge \beta \leq \beta$, so (E3) entails that $\alpha \wedge \beta \approx \alpha$ or $\alpha \wedge \beta \approx \beta$. Clearly we cannot interpret the degrees of expectation directly in terms of their *probabilities*, since (E3) is violated by any probability measure, as observed by Dubois (1986). Appendix I.2 comments on the relation between expectation orderings and the 'qualitative necessity measures' of Dubois and Prade.

Note that the three conditions imply *connectivity*: either $\alpha \leq \beta$ or $\beta \leq \alpha$. For by (E3) and (E2) either $\alpha \leq \alpha \wedge \beta \leq \beta$ or $\beta \leq \alpha \wedge \beta \leq \alpha$ and we conclude by (E1). The dominance condition also immediately implies that $\alpha \wedge \neg\alpha \leq \beta$, and thus the three conditions together imply that for all $\alpha \in L$, either $\alpha \leq \beta$ for all $\beta \in L$ or $\neg\alpha \leq \beta$ for all $\beta \in L$. By way of comparison, (E1) to (E3) are three of the five conditions used in Gärdenfors (1988) and Gärdenfors and Makinson (1988) to define a notion of 'epistemic entrenchment' for the logic of theory change. Section 4 comments more closely on the relationship between the two concepts.

Let us now return to how the ordering $\leq$ can be used to determine when α nonmonotonically imples β. According to the key idea of this section $\alpha \mid\!\sim \beta$ means that β follows from α together with all the propositions that are 'sufficiently well' expected in the light of α. How well is 'sufficiently well'? A natural idea is to require that the added sentences be strictly more expected than $\neg\alpha$ in the ordering. This idea was already used by Rott (1991) in the context of the logic of belief revision; see also Appendix I.2. It motivates the following:

Definition 3.1. $\mid\!\sim$ is a *comparative expectation* inference relation iff there is an ordering $\leq$ satisfying (E1) - (E3) such that the following condition holds:

(C$\mid\!\sim$) $\alpha \mid\!\sim \gamma$ iff $\gamma \in \mathrm{Cn}(\{\alpha\} \cup \{\beta: \neg\alpha < \beta\})$

Recalling that by the three conditions on expectation orderings we have $\neg\alpha \leq \beta_i$ for all $i \leq n$ iff $\neg\alpha \leq \beta_1 \wedge \ldots \wedge \beta_n$, it is immediate, using the compactness of Cn, that (C$\mid\!\sim$) is equivalent to:

$$\alpha \mid\!\sim \gamma \text{ iff either } \alpha \vdash \gamma \text{ or there is a } \beta \in L \text{ with } \alpha \wedge \beta \vdash \gamma \text{ and } \neg\alpha < \beta$$

and henceforth we use these two formulations interchangeably. Further equivalent formulations will be established in Theorem 3.5 below.

Note that the definition of $\mid\!\sim$ makes sense only on the finitary level. For it makes essential use of the negation $\neg\alpha$ of the proposition α serving as premise, and the negation of an infinite set of propositions is not defined unless we admit infinitely long disjunctions into our language. On the other hand, as in Section 2, it is possible to proceed to the infinitary level indirectly, by first defining the finitary relation $\mid\!\sim$ as above and then taking its ‘canonical extension’ to the infinitary level, as defined in Freund, Lehmann and Makinson (1990). Such passage to canonical extensions preserves satisfaction of the extended set of postulates, mentioned in Theorem 3.2 below. It may perhaps also be possible to proceed to the infinitary level directly, by appropriately lifting the relation $\leq$ to one between subsets of L and generalizing the definition (C$\mid\!\sim$) by using inconsistency under Cn as a substitute for negation. However, in this section we shall, for simplicity, remain at the finitary level.

We refer again to the syntactic postulates on inference relations that were listed in Section 1.3. Here in Section 3 we are no longer particularly interested in dividing them into subgroups, e.g. Lindström's postulates for an inference relation, the ‘basic’ postulates, and the remaining conditions. We wish to consider them all together. There are of course various ways of making a reduced list from which the extended set of postulates, and thus all the conditions mentioned in Section 1.3, follow. One handy such list, which we shall use is: Supraclassicality, Left Logical Equivalence, And, Consistency Preservation, Cut, Or, and Rational Monotony. All the postulates of the extended set follow from this reduced list. The only one for which this is not immediately obvious is Cautious Monotony. We can derive it using Rational Monotony and Consistency Preservation, as follows. Suppose that $\alpha \mid\!\sim \gamma$ and $\alpha \mid\!\sim \beta$; we need to show $\alpha \wedge \gamma \mid\!\sim \beta$. On the one hand, if $\alpha \not\mid\!\sim \neg\gamma$ we have by Rational Monotony that $\alpha \wedge \gamma \mid\!\sim \beta$ as desired. On the other hand if $\alpha \mid\!\sim \neg\gamma$ we have by And that $\alpha \mid\!\sim \gamma \wedge \neg\gamma$ so by Consistency Preservation $\alpha \vdash \gamma \wedge \neg\gamma$ so that by classical logic $\alpha \wedge \gamma \vdash \beta$ and we may conclude that $\alpha \wedge \gamma \mid\!\sim \beta$.

Theorem 3.2. Let $\leq$ be an expectation ordering over L. Then the inference relation $\mid\!\sim_{\leq}$ that it determines by (C$\mid\!\sim$) satisfies the extended set of postulates of Section 1.3.

It is also possible to prove a converse of Theorem 3.2, serving as a representation theorem:

Theorem 3.3. Let $\mid\!\sim$ be any inference relation on L that satisfies the extended set of postulates. Then $\mid\!\sim$ is a comparative expectation inference relation, i.e., there is an expectation ordering $\leq$ over L such that $\mid\!\sim \; = \; \mid\!\sim_{\leq}$.

The proof is based on the following definition of the expectation ordering: $\alpha \leq \beta$ iff either $\alpha \wedge \beta \in Cn(\emptyset)$ or $\neg(\alpha \wedge \beta) \not\mid\!\sim \alpha$. This definition, and the verification that it yields the desired properties, parallel those used in Gärdenfors and Makinson (1988) to represent contraction operations in terms of epistemic entrenchment relations.

For the sake of comparison, we note that the second disjunct of the definition of $\alpha \leq \beta$ is closely connected with a relation *R* introduced by Lehmann and Magidor (1990) in the course of proving a different representation theorem for nonmonotonic inference relations satisfying all of the listed postulates other than Consistency Preservation. They put $\alpha \, R \, \beta$ iff $\alpha \vee \beta \not\mid\!\sim \neg\alpha$ (Definition 16 and text just before Definition 30). Clearly, we have by Left Logical Equivalence and Right Weakening that our second disjunct $\neg(\alpha \wedge \beta) \not\mid\!\sim \alpha$ holds iff $\neg\alpha \vee \neg\beta \not\mid\!\sim \neg\neg\alpha$ iff $\neg\alpha \, R \, \neg\beta$.

To some extent the construction presented in this section reminds one of Brewka's (1989, 1990, 1991) use of ‘preferred subtheories’ as a way of handling default reasoning. There are, however, major differences, as explained in Appendix I.2.

3.2. Belief valuations

In this subsection we show how the notion of an expectation ordering may be given an alternative formulation in terms of ‘belief valuations’. Formally, the two are trivially equivalent, but the latter provides a quite different *gestalt* and a connection with probabilistic approaches to nonmonotonic reasoning.

We define a *belief scale* to be any pair $(S,\leq)$ where S is a non-empty set and $\leq$ is a total ordering of S (i.e. transitive, connected and hence also reflexive, and antisymmetric). Intuitively, we think of this as a generalization of the real interval [0,1] with its familiar total ordering.

By a *belief valuation* into $(S,\leq)$ we mean any function $f: L \to S$ satisfying the following two conditions for all $\alpha,\beta \in L$:

(F1) $Cn(\alpha) = Cn(\beta)$ implies $f(\alpha) = f(\beta)$,

(F2) $f(\alpha \wedge \beta) = min(f(\alpha),f(\beta))$,

where Cn is as explained in Section 1.2, and *min* is minimality with respect to the relation $\leq$ over S. Belief valuations may be thought of as akin to probability distributions. The condition (F1) holds, of course, for any probability distribution, whilst (F2) clearly does not. The condition (F2) requires that f is a homomorphism between conjunction in the language and the *min* operation in the belief scale. However the conditions do not explicitly require, or even imply, any similar homomorphism for disjunction, nor for negation. In this respect, belief valuations are quite unlike the usual evaluation functions for many-valued logics, which are homomorphisms for all logical connectives of their languages. It is also a further difference from probability distributions, which make negation homomorphic with subtraction from unity. On the other hand, there are close connections with work of Shackle (1961), Spohn (1987), and Dubois and Prade (1986, 1988, 1991a, 1991b). These connections are discussed in Appendix I.

Note that whenever $\alpha \vdash \beta$ then $f(\alpha) \leq f(\beta)$, for any belief valuation f. For if $\alpha \vdash \beta$ then $Cn(\alpha) = Cn(\alpha \wedge \beta)$ so by (F1) and (F2), $f(\alpha) = f(\alpha \wedge \beta) = min(f(\alpha),f(\beta)) \leq f(\beta)$. This in turn implies that the image f(L) of L under f, which will in general be a *proper* subset of S, will have a unique greatest element 1_f under $\leq$, and a unique least element 0_f under $\leq$.

Moreover, instead of a homomorphism property, negation satisfies the following condition: for all $\alpha \in L$, either $f(\alpha) = 0_f$ or $f(\neg\alpha) = 0_f$. For $\alpha \wedge \neg\alpha \vdash \beta$ for all $\beta \in L$, so f(α ∧ ¬α)⁄f(β), so $f(\alpha \wedge \neg\alpha) = 0_f$, so by (F2) either $f(\alpha) = 0_f$ or $f(\neg\alpha) = 0_f$.

Given a belief valuation f into a belief scale (S⁄) we can define a nonmonotonic inference relation $\mid\!\sim_f$ or more briefly $\mid\!\sim$ when the context is clear, in a manner parallel to (C$\mid\!\sim$):

(C$\mid\!\sim_f$) $\quad \alpha \mid\!\sim_f \gamma$ iff $\gamma \in Cn(\{\alpha\} \cup \{\beta \in L\colon f(\neg\alpha)$ Œ $f(\beta)\})$

or equivalently:

$\alpha \mid\!\sim_f \gamma$ iff either $\alpha \vdash \gamma$ or there is a $\beta \in L$ with $\alpha \wedge \beta \vdash \gamma$ and $f(\neg\alpha)$ Œ $f(\beta)$.

It is thus possible to generate inference relations from belief valuations. Clearly, if (S⁄) is a belief scale and f is a valuation into it, then the generated inference relation $\mid\!\sim_f$ does not depend on any elements of S outside f(L). In other words, every inference relation determined by a belief valuation f into a scale (S⁄) is determined by the same evaluation *onto* the scale (f(L)⁄).

It is not surprising that this approach to nonmonotonic reasoning is equivalent to that described in Section 3.1 in terms of expectation orderings. In other words:

Theorem 3.4. Expectation orderings and belief valuations generate precisely the same class of nonmonotonic inference relations.

The essential idea of the verification is simply to take quotient structures on propositions, determined by the relation $\alpha \approx \beta$ holding iff both $\alpha \leq \beta$ and $\beta \leq \alpha$.

3.3. Expressing defaults by expectation orderings

As an argument in favour of using expectation orderings for nonmonotonic reasoning we want to show in this section that an expectation ordering contains enough information to express, in a very simple way, what we require with respect to default information. The principal idea is that a default statement of the type ‘F's are normally G's’ can be expressed by saying that ‘if something is an F then it is less expected that it is non-G than that it is G’.

This formulation is immediately representable in an expectation ordering by assuming that the relation $Fb \to \neg Gb < Fb \to Gb$ holds for all individuals b.

Before we turn to an illustration of the mechanisms, we prove a simple theorem that provides us with several reformulations of the condition in ($C\mid\!\sim$) which will be useful in analyzing examples. Let us call a set Γ of sentences a *cut* of an expectation ordering $\leq$ iff $\beta \in \Gamma$ whenever $\alpha \in \Gamma$ and $\alpha \leq \beta$, for all α and β.

Theorem 3.5. Let $\leq$ be any expectation ordering. Then for all sentences α, β, γ, the following are equivalent:

(1) $\gamma \in Cn(\{\alpha\} \cup \{\beta: \neg\alpha < \beta\})$

(2) $\alpha \vdash \gamma$ or $\neg\alpha < \alpha \to \gamma$

(3) $\alpha \vdash \gamma$ or $\alpha \to \neg\gamma < \alpha \to \gamma$

(4) $\alpha \vdash \gamma$ or $\alpha \to \gamma$ is in the greatest cut of $\leq$ that does not contain $\neg\alpha$.

To illustrate the general idea of expressing defaults of the form ‘F's are normally G's’ as a set of relations $Fb \to \neg Gb < Fb \to Gb$ for all individuals b, assume that all we know about b is that Fb. We want to decide the nonmonotonic consequences of this fact. It can be determined immediately, via ($C\mid\!\sim$) and (3) of Theorem 3.5, that $Fb \mid\!\sim Gb$. It can also be determined that $Fb \not\mid\!\sim \neg Gb$. For on the one hand applying (E2) to our assumption we have $\nvdash Fb \to \neg Gb$, i.e., $Fb \nvdash \neg Gb$; and on the other hand, by the asymmetry of $<$, we have not $Fb \to \neg Gb < Fb \to Gb$, so we may again apply part (3) of Theorem 3.5. Further information about b, for example that Hb, will mean that we no longer need to check whether $Fb \to \neg Gb < Fb \to Gb$, but rather whether $Fb \wedge Hb \to \neg Gb < Fb \wedge Hb \to Gb$, which may give a different answer (cf. Example 3.7 below). This is exactly how we want a default rule to operate.

Example 3.6. Let us suppose that L contains the following predicates:

Sx: x is Sicilian

Bx: x is blond

Hx: x is hot-tempered

Assume that we have the default rules ‘Sicilians are normally hot-tempered’ and ‘Blond persons are normally not hot-tempered’. According to the rule given above, we express these defaults by a number of ordering relations of the form $Sb \to \neg Hb < Sb \to Hb$ and $Bb \to Hb < Bb \to \neg Hb$, respectively, for various individuals b.

From this we conclude, as above, that if all we know about Fiora is that she is a Sicilian, then we expect her to be hot-tempered (and we don’t expect her to be blond); and if all we know about Lucia is that she is blond, then we expect her to be cool (and don’t expect her to be a Sicilian). Now, suppose that, contrary to our expectations, Amadeo is a blond Sicilian, that is $Sa \wedge Ba$. What can be concluded concerning his temper? (This example is a variation of the so called ‘Nixon Diamond’).

If we know that $Ba \wedge Sa$ and we want to decide whether Ha or ¬Ha follows nonmonotonically, this can be determined, via ($C\mathrel{|\!\sim}$) and (3) of Theorem 3.5, by looking for the strictly greater of $Ba \wedge Sa \to Ha$ and $Ba \wedge Sa \to \neg Ha$ in the expectation ordering. Three cases are possible: (1) $Ba \wedge Sa \to Ha < Ba \wedge Sa \to \neg Ha$. In this case, we conclude that $Ba \wedge Sa \mathrel{|\!\sim} \neg Ha$. (2) $Ba \wedge Sa \to \neg Ha < Ba \wedge Sa \to Ha$. For similar reasons, we conclude that $Ba \wedge Sa \mathrel{|\!\sim} Ha$. (3) $Ba \wedge Sa \to \neg Ha \approx Ba \wedge Sa \to Ha$. In this case, neither $Ba \wedge Sa \mathrel{|\!\sim} Ha$, nor $Ba \wedge Sa \mathrel{|\!\sim} \neg Ha$ will hold.

None of these three possibilities is ruled out by the two ordered pairs $Sa \to \neg Ha < Sa \to Ha$ and $Ba \to Ha < Ba \to \neg Ha$. The reason is that it follows from (E2) that $Sa \to Ha \leq Ba \wedge Sa \to Ha$ and that $Ba \to \neg Ha \leq Ba \wedge Sa \to \neg Ha$. Consequently, the maximum of $Ba \wedge Sa \to Ha$ and $Ba \wedge Sa \to \neg Ha$ will be at least as high as each of $Sa \to Ha$ and $Ba \to \neg Ha$ in the expectation ordering. But on the other hand, the two given comparisons do not suffice to determine which, if any, of $Ba \wedge Sa \to Ha$ and $Ba \wedge Sa \to \neg Ha$ is the greater. So, the information available does not permit us to conclude anything concerning Ha or ¬Ha.

To sum up, the nonmonotonic consequences one can draw from the premise that $Ba \wedge Sa$ depends on which is chosen to be the maximal element of $Ba \wedge Sa \to Ha$ and $Ba \wedge Sa \to \neg Ha$ in the expectation ordering. The default relations $Sa \to \neg Ha < Sa \to Ha$ and $Ba \to Ha < Ba \to \neg Ha$ are not sufficient to determine this choice.

A general principle about nonmonotonic reasoning is that more *specific* information about an individual should override less specific when it comes to applying various kinds of default information. Let us call this idea the *specificity principle*. Suppose we know that a certain individual is a bird and indeed, more specifically, is an emu. Now, birds fly by default, but emus don't. The specificity principle requires that only the more specific default information that emus don't fly is applicable when reasoning about the properties of this individual (this information can, of course, be overridden by some still more specific facts or laws).

Another well known principle is that of using all the relevant information available. Let us call this the *full information principle*. If we know that b is an emu and, hence, that b is a bird, we should use both of these facts when drawing inferences. Note that neither the specificity principle nor the full information principle governs the power or behaviour of the inference relation itself. They are pragmatic guides for choosing what parts of the information at our disposal to consider on the left hand side of the inference relation. In this way, they tell us what questions to ask, rather than what answers to give.

The specificity principle tells us that we should use the most specific information in our premises, whereas the full information principle tells us that we should use all relevant information. Does this make a difference? Is there any clash? In a system such as that of expectation relations – and indeed in any system in which Left Logical Equivalance is satisfied and there is flexibility in expanding the background consequence operation to take account of known specificities – the two principles give the same results. In effect, their equivalence is a weak form of Cumulativity: If $\alpha \vdash \beta$, then $\alpha \mid\!\sim \gamma$ iff $(\alpha \wedge \beta) \mid\!\sim \gamma$. This rather abstract feature may be illustrated by a simple example.

Example 3.7. Let Bb stand for the fact that a given individual b is a bird, Eb for the fact that it is an emu, and Fb for the fact that it flies. We assume that being an emu is more specific information than being a bird and that this is reflected in the background consequence operation, i.e., Eb $\vdash$ Bb. We also assume the two default rules that birds normally fly and that emus normally don't fly which are expressed by the following two expectation relations:

$$Bb \rightarrow \neg Fb < Bb \rightarrow Fb$$

$$Eb \to Fb < Eb \to \neg Fb$$

Note that these imply, using (E2), that $Bb \not\vdash \neg Fb$ and $Eb \not\vdash Fb$.

We are given a b and told that it is an emu. We know from our background consequence operation that it is a bird. The question is: Should we conclude that it flies? Now it is easy to calculate from the assumptions of the example, using definition ($C\mid\!\sim$) and Theorem 3.5 (any of the four criteria there will do), that:

$Bb \mid\!\sim Fb$ whilst $Bb \not\mid\!\sim \neg Fb$

$Eb \mid\!\sim \neg Fb$ whilst $Eb \not\mid\!\sim Fb$

$Eb \wedge Bb \mid\!\sim \neg Fb$ whilst $Eb \wedge Bb \not\mid\!\sim Fb$

Our question thus becomes: Given Eb and Bb, which should be the premise of the nonmonotonic inference: Eb alone, Bb alone, or the conjunction $Eb \wedge Bb$? The answer according to the specificity principle is Eb; according to the full information principle it is $Eb \wedge Bb$. As can be seen above, these give the same result under our approach if $\vdash$ is appropriately chosen to reflect the specificity principle.

Another way of representing a default statement "F's are normally G's" would be as $Fb \wedge \neg Gb < Fb \wedge Gb$, for all individual constants b, which is reminiscent of Lewis' representation of counterfactuals in terms of "comparative possibility" (see Lewis (1973), Section 2.5). Provided the ordering is properly understood, this is not a conflict, but a duality. If we define the dual $\leq_d$ of the expectation ordering $\leq$ by the rule $\alpha \leq_d \beta$ iff $\neg\beta \leq \neg\alpha$ (so that postulates (E1) and (E2) continue to hold of $\leq_d$ but (E3) becomes dualized to $\alpha \vee \beta \leq_d \alpha$ or $\alpha \vee \beta \leq_d \beta$), then our representation $Fb \to \neg Gb < Fb \to Gb$ is equivalent to $Fb \wedge \neg Gb <_d Fb \wedge Gb$.

3.4. Equivalence with nice preferential models

One of the best known semantic approaches to nonmonotonic reasoning is that of preferential models, as devised by Shoham (1988) and generalized and studied in depth by Makinson (1989), Kraus, Lehmann and Magidor (1990) and Lehmann and Magidor (1990), with an overview in Makinson (to appear). There are slight differences of presentation and of detail, compared in Dix and Makinson (1991); here we shall follow the formulation of Makinson (1989; to appear).

We recall that on this formulation, a preferential model is a triple $\mathcal{M} = \langle M, \models, < \rangle$ where M is an arbitrary non-empty set (whose elements are called 'states'), $<$ is an arbitrary relation between elements of M (called the 'preference relation') and $\models$ is an arbitrary relation between elements of M and sentences (called the 'satisfaction relation'). In this version, there is no distinction drawn between 'states' and 'worlds' as in the rather more complex (but less general) formulation of Kraus, Lehmann, and Magidor (1990). Let us say that a state m 'preferentially satisfies' a formula α, denoted $m \models_< \alpha$, iff $m \models \alpha$ and there is no $m' \in M$ with $m' < m$ and $m' \models \alpha$. A preferential model $\mathcal{M} = \langle M, \models, < \rangle$ generates an inference relation $\mid\!\sim_{\mathcal{M}}$ defined by putting $\alpha \mid\!\sim_{\mathcal{M}} \gamma$ iff for every $m \in M$, if $m \models_< \alpha$ then $m \models \gamma$. In words, $\alpha \mid\!\sim_{\mathcal{M}} \gamma$ holds when γ is satisfied in all the 'preferred' states satisfying α.

The question arises whether there is any close relationship between preferential models and our expectation orderings (alias belief valuations). As has been shown above, the latter satisfy Supraclassicality, Cumulativity and Rational Monotony. In order to cover these properties, preferential models need to satisfy some further constraints. In the terminology of Makinson (to appear), they need to be classical, stoppered, and ranked (to be defined below). Moreover, as preferential models do not in general satisfy Consistency Preservation, a further special constraint will need to be imposed on them.

We therefore consider the class of all preferential models $\mathcal{M} = \langle M, \models, < \rangle$ that are

(1) *classical* : for all $m \in M$ and $\alpha, \beta \in L$, $m \models \neg\alpha$ iff $m \not\models \alpha$, and $m \models \alpha \wedge \beta$ iff both $m \models \alpha$ and $m \models \beta$ (this property is satisfied for all standard satisfaction relations $\models$);

(2) *ample* with respect to Cn: for every proposition α consistent under Cn, there is an $m \in M$ with $m \models \alpha$;

(3) based on a relation $<$ which *irreflexive, transitive*, and *ranked* (the last also known as 'modular' and defined: whenever $m < n$ and not $n' < n$ then $m < n'$);

(4) *finitarily stoppered* (alias 'finitarily smooth'). This last condition means: whenever $m \in M$ and $m \models \alpha$ then there is an $n \in M$ with either $n = m$ or $n < m$ such that $n \models_< \alpha$. In words, for every formula α that is non-preferentially satisfied by some state in $\mathcal{M}$, there

exists a better state n that preferentially satisfies α. (This condition is essentially the same as the 'limit assumption' in Lewis (1973)).

A preferential model M satisfying (1) - (4) will be called *nice*.

We want to show that an inference relation is determined by some expectation ordering via definition (C$\mid\!\sim$) iff it is determined by some nice preferential model via the definition given just above. The *shortest* proof of this is indirect, making use of another representation theorem in the theory of preferential models due to Lehmann and Magidor. For we know by Theorems 3.2 and 3.3 above that $\mid\!\sim$ is determined by an expectation ordering iff it satisfies the extended set of postulates. And we know from Theorem 5 of Lehmann and Magidor (1990) that $\mid\!\sim$ satisfies those postulates (other than Consistency Preservation with respect to the background consequence operation Cn) iff it is determined by some preferential model that has the properties listed above for niceness (other than that of being ample). So it will suffice to show that when a preferential model is ample with respect to Cn, the inference relation that it determines satisfies Consistency Preservation with respect to Cn (which is trivial), and that if $\mid\!\sim$ satisifes Consistency Preservation with respect to Cn then the canonical preferential model built by Lehmann and Magidor is ample with respect to Cn (which is not difficult).

However, we shall also give a direct proof, for two reasons. First, our purpose is not only to compare the induced inference relations but also to discover the relationship between the expectation ordering ≤ that determines $\mid\!\sim$ and the preference relation ‹ that determines it. This is not easily extracted from the indirect argument, but will be explicit in the constructions below. Second, our constructions are rather simpler than those of Lehmann and Magidor.

We begin by mapping expectation orderings into equivalent nice preferential models. For the special case that L is finite (modulo equivalence under Cn) Theorem 3.8 below was in effect established by Dubois and Prade (1991b) using a different construction.

Theorem 3.8. For every expectation ordering ≤ over L, there is a nice preferential model $M = \langle M, \models, \prec \rangle$ such that $\mid\!\sim_M \; = \; \mid\!\sim_{\leq}$.

The proof is based on the following construction: Put M to be the collection of all maximally consistent (under Cn) sets of propositions of L. For each $m \in M$, define $m \models \alpha$ to hold for propositions $\alpha \in L$, iff $\alpha \in m$. The key definition is that of the relation $\prec$ over M. (For preliminary orientation, remember that in a preferential model down is 'better', in contrast to an expectation ordering, where up is 'better'.) Intuitively, the construction we use means: $m \prec n$ iff there is a proposition that is falsified by n but is more expected that any proposition falsified by m. Formally, this is expressed as follows: for $m,n \in M$ we put $m \prec n$ iff for some proposition $\alpha \in L$ we have $\beta \in m$ for every $\beta \in L$ with $\alpha \leq \beta$, but $\alpha \notin n$. In other words, writing α^+ for $\{\beta \in L: \alpha \leq \beta\}$, iff for some proposition α we have $\alpha^+ \subseteq m$ but $\alpha \notin n$. On this construction, some of the required properties of $\mathcal{M}$ are immediate: that $\mathcal{M}$ is ample (by compactness of Cn), that $\models$ is classical, and that $\prec$ is irreflexive. The remainder are verified in Appendix II.

Next, we map nice preferential models into equivalent expectation orderings, giving the converse of Theorem 3.8.

Theorem 3.9. Let $\mathcal{M} = \langle M,\models,\prec\rangle$ be any nice preferential model. Then there is an expectation ordering $\leq$ over L such that $\mid\!\sim_{\leq} = \mid\!\sim_{\mathcal{M}}$.

The key idea for the proof is to define, for each proposition $\alpha \in L$, $g(\alpha) = \{m \in M: n \models \alpha$ for all $n \in M$ with $n \prec m\}$. Note that $\{m: m \models_{\prec} \neg\alpha\} \subseteq g(\alpha)$ though not conversely. Then the relation $\leq$ over L is defined by the rule: $\alpha \leq \beta$ iff $g(\alpha) \subseteq g(\beta)$. It can now be verified that $\leq$ is an expectation ordering over L, with $\mid\!\sim_{\leq} = \mid\!\sim_{\mathcal{M}}$.

Putting Theorems 2.12, 3.2 to 3.4, and 3.8 to 3.9 together, we can express the main results as follows:

Theorem 3.10. Let $\mid\!\sim$ be any relation between formulas in L. Then the following five conditions are mutually equivalent:

(1) $\mid\!\sim$ is determined by some closed, consistently generated, and transitively relational expectation inference relation $\mid\!\sim_{\Delta,S}$.

(2) $\mid\!\sim$ is determined by some expectation relation $\leq$ over L, under definition (C$\mid\!\sim$).

(3) $\mathrel{|\!\sim}$ is determined by some belief valuation from L into some belief scale, under definition (C$\mathrel{|\!\sim}_f$).

(4) $\mathrel{|\!\sim}$ is determined by some nice preferential model.

(5) $\mathrel{|\!\sim}$ satisfies the conditions of Supraclassicality, Left Logical Equivalence, And, Cumulativity, Or, Rational Monotony and Consistency Preservation.

In principle it should also be possible to construct maps between the transitively relational expectation set systems of Section 2 and nice preferential models, as well as between the former and expectation orderings, to complement the maps between the latter two established here. We know that they all determine the same postulates for nonmonotonic inference.

4. A unified treatment of nonmonotonic logic and belief revision

Nonmonotonic inference relations have in Section 3 been analysed in terms of an ordering $\leq$ over a language. Although the notion of a set Δ of 'expected sentences' played no explicit part in that analysis (in contrast to that of Section 2) it can, as we promised at the beginning of Section 3.1, be constructed out of the ordering $\leq$. Recall that definition (C$\mathrel{|\!\sim}$) puts $\alpha \mathrel{|\!\sim} \gamma$ iff $\gamma \in Cn(\{\alpha\} \cup \{\beta: \neg\alpha < \beta\})$. Now clearly a sentence β can occur as an element of a set $\{\beta: \neg\alpha < \beta\}$ for some α, only if $\bot < \beta$. Thus if we introduce the set Δ of 'expected sentences' by the definition $\Delta = \{\beta: \bot < \beta\}$, definition (C$\mathrel{|\!\sim}$) may equivalently be expressed thus: $\alpha \mathrel{|\!\sim} \gamma$ iff $\gamma \in Cn(\{\alpha\} \cup \{\beta \in \Delta: \neg\alpha < \beta\}$.

The elements of Δ may thus be seen as 'expected sentences', 'defeasible beliefs', or 'real possibilties' according to whichever terminology we prefer. They are in principle available to help us draw conclusions, but are such that when a premise α of a nonmonotonic inference is used, some of the elements of Δ yield in case of a conflict with α.

The ordering $\leq$ is used to 'chop off' elements from Δ in case of a conflict with α so that it can be decided whether $\alpha \mathrel{|\!\sim} \gamma$ according to the recipe that $\alpha \mathrel{|\!\sim} \gamma$ iff $\gamma \in Cn(\{\alpha\} \cup \{\beta \in \Delta: \neg\alpha < \beta\})$. The higher up in the ordering a proposition is found, the less vulnerable it is to deletion; in other words, the *less defeasible* is the proposition.

This way of describing how nonmonotonic inferences are determined may be compared to the situation in *belief revision theory*. In Gärdenfors (1988) an account is developed of how a belief state K is revised in the light of new input α. In this theory, states of belief are modelled by *belief sets* which are sets of sentences from L. Belief sets are assumed to be closed under logical consequence. Belief sets model the statics of epistemic states. For their *dynamics* we need methods for modifying belief sets. Three kinds of change are central:

(i) Expansion: A new sentence together with its logical consequences is *added* to a belief set K. The belief set that results from expanding K by a sentence α is denoted K^{+}_{α}.

(ii) Revision: A new sentence that is *inconsistent* with a belief set K is added, but in order that the resulting belief set be consistent some of the old sentences of K are deleted. The result of revising K by a sentence α is denoted K^{*}_{α} .

(iii) Contraction: Some sentence in K is retracted without adding any new beliefs. In order that the resulting belief set be closed under logical consequence some other sentences from K must be given up. The result of contracting K with respect to α is denoted K^{-}_{α}.

There are two methods of attacking the problem of specifying revision and contraction operations. One is to present *rationality postulates* for the processes. Such postulates are introduced in Gärdenfors (1984), Alchourrón, Gärdenfors and Makinson (1985) and discussed extensively in Gärdenfors (1988). A guiding idea for these postulates is that changes should be *minimal*, so that when changing beliefs in response to new evidence, one should continue to believe as many of the old beliefs as possible. As was mentioned in Section 2.2, Makinson and Gärdenfors (1990) showed that the postulates for belief revision can be *translated* into postulates for nonmonotonic logic, and vice versa. The key idea for the translation from belief revision to nonmonotonic logic is that a statement of the form $\beta \in K^{*}_{\alpha}$ is seen as a nonmonotonic inference from α to β given the set K of sentences as the set of background *expectations*. So the statement $\beta \in K^{*}_{\alpha}$ for belief revision is translated into the statement $\alpha \mid\!\sim \beta$ for nonmonotonic logic (or into $\alpha \mid\!\sim_{K} \beta$, if one wants to emphasize the role of the background beliefs). Conversely, a statement of the form $\alpha \mid\!\sim \beta$ for nonmonotonic logic is translated in to a statement of the form $\beta \in K^{*}_{\alpha}$ for belief revision, where K is introduced as a *fixed* belief set. Clearly in this translation K plays the role of Δ in Section 2 of this paper.

Using this recipe it is possible to translate all the postulates (K*1) - (K*8) for belief revision from into conditions for nonmonotonic logic. It turns out that every postulate translates into a condition on $\mid\!\sim$ that is valid in some kinds of nonmonotonic inference in the literature. Conversely, every postulate on $\mid\!\sim$ that has been presented above in Section 1.3 translates into a condition on belief revision that is a consequence of (K*1) - (K*8). For example, Cautious Monotony translates into 'if $\beta \in K^*_{\alpha}$ and $\gamma \in K^*_{\alpha}$, then $\beta \in K^*_{\alpha\wedge\gamma}$' which follows from (K*1) - (K*8). To sum up, using the proposed translation it has been established that there is a very tight connection between postulates for belief revision and those for nonmonotonic logic.

The second method within belief revision theory of solving the problems of revision and contraction is to build *models* of belief revision which can take a belief set (or some representation of such a set) together with a sentence to be added as input and which gives a revised belief set as output. One such approach, in terms of partial meet contractions and revisions was developed in Alchourrón, Gärdenfors, and Makinson (1985) and provides the inspiration for the account of nonmonotonic inference relations based on expectation sets in Section 2 of the present paper. However, it is not very computationally oriented. A more 'constructive' or computation-friendly approach was adopted in Gärdenfors (1988) and Gärdenfors and Makinson (1988). It is based on the notion of the *epistemic entrenchment* of the propositions in a belief set, ordering the sentences in K. The interpretation of this ordering is very closely related to that of the expectation orderings in Section 3.1.

The postulates for epistemic entrenchment introduced in Gärdenfors (1988) *include* the postulates (E1) - (E3) for expectation orderings of section 3.1 of this paper (with the notational difference that K replaces Δ). A relation ≤ over a language L is called one of epistemic entrenchment with respect to a theory K iff it satisfies (E1) - (E3), together with two further conditions, as follows.

(E4) If K is consistent, then $\alpha \notin K$ iff $\alpha \leq \beta$ for all β (*Minimality*)

In words, all sentences not in K have the same degree of entrenchment, which is the lowest of all degrees. A formulation that is equivalent, given (E1) to (E3), is: If K is consistent then for all $\alpha \in L$, $\alpha \notin K$ iff $\alpha \leq \neg\alpha$. Another equivalent formulation is: If K is consistent then $K = \{\alpha: \bot < \alpha\}$.

(E5) $\beta \leq \alpha$ for all β, only if $\vdash \alpha$ (*Maximality*)

This postulate says that the only sentences with a maximal degree of expectation are the logically valid sentences. Equivalently given (E1) - (E3): If $\beta \vee \neg\beta \leq \beta$ then $\beta \in Cn(\emptyset)$. Of course, the converse of (E5) is already an immediate consequence of (E2).

It was shown by Gärdenfors and Makinson (1988) that given a theory K closed under Cn, each relation $\leq$ of epistemic entrenchment for K determines an operation K^{-}_{α} of contraction by the following definition (which rather awkwardly and unintuitively involves a disjunctive formula):

(C-) $\beta \in K^{-}_{\alpha}$ iff $\beta \in K$ and either $\alpha \in Cn(\emptyset)$ or $\alpha < \alpha \vee \beta$.

From this, via the so called Levi identity $K^{*}_{\alpha} = Cn((K^{-}_{\neg\alpha}) \cup \{\alpha\})$, the entrenchment relation determines a revision operation K^{*}_{α}. Both the contraction operation and the revision operation thus generated satisfy the relevant postulates from Alchourrón, Gärdenfors and Makinson (1985). Conversely, it was shown in the same paper that each contraction or revision operation satisfying those postulates can be generated from such an epistemic entrenchment relation.

From the correspondence between postulates for belief revision and those for nonmonotonic inferences, developed by Makinson and Gärdenfors (1990), it is thus natural to expect that the notion of epistemic entrenchment may also be used to generate nonmonotonic inference relations, and it was such an expectation that provided the original motivation for the work reported in this paper (also see some preliminary ideas in Gärdenfors 1990). However, it turns out that the passage from the context of belief revision to the context of nonmonotonic inference permits two simplifications: the postulates (E4) and (E5) both become superfluous. As we have remarked, (E4) was needed in the logic of belief revision in order to relate the entrenchment relation $\leq$ to the particular belief set undergoing contraction or revision. In the context of nonmonotonic inference such a connection is not necessary. The condition (E5) was needed in the logic of belief revision in order to give an adequate account of *contraction*, specifically in order to generate an operation of contraction that satisfies the postulate of 'recovery': $K \subseteq Cn((K^{-}_{\alpha}) \cup \{\alpha\})$. On the other hand, as noted in Makinson (1987), recovery is not needed to satify the postulates of Alchourrón, Gärdenfors and Makinson (1985) for *revision*; and as we have seen in Section 3.2 of this paper, (E5) is not needed in order to generate well-behaved nonmonotonic inference relations.

This last point can also be put in another way: Expectation orderings and epistemic entrenchment relations generate the same classes of inference relations. Let $\leq$ be any expectation relation over L, i.e. satisfying conditions (E1) - (E3), and let $\mid\sim$ be the inference relation that it generates by the definition (C$\mid\sim$) in Section 3.1 of this paper. Then there is an epistemic entrenchment relation $\leq'$ over L, i.e. satisfying all of (E1) - (E5) for some consistent belief set K, such that the inference relation $\mid\sim'$ generated from it under the same definition is equal to $\mid\sim$.

To see this first put $K = \{\alpha \in L: \neg\alpha < \alpha\}$. Then as Dubois and Prade (1991a) have in effect observed, if $\leq$ satisfies (E1) - (E3), then K is a consistent belief set and (E4) is also satisfied. To get all of (E1) - (E5), however, we need to massage the relation $\leq$ a little. Define $\leq'$ by putting $\alpha \leq' \beta$ iff both $\alpha \leq \beta$ and it is not the case that $\alpha \in Cn(\emptyset)$ whilst $\beta \notin Cn(\emptyset)$. Then it is easy to check that $\leq'$ satisfies all of (E1) to (E5) and determines, under definition (C$\mid\sim$) of Section 3.1, the same inference relation as does $\leq$.

Hence we may add to the list of equivalent conditions in Theorem 3.10, the following:

(6) $\mid\sim$ is generated from some relation $\leq$ that is an epistemic entrenchment relation with respect to some consistent belief set K, via definition (C$\mid\sim$).

Indeed, it is possible to add a further equivalent condition to the list in Theorem 3.10, illustrating again the close relationships between nonmonotonic inference and belief revision.

(7) $\mid\sim$ is generated from some relation $\leq$ that is an epistemic entrenchment relation with respect to some consistent belief set K, by the sequence of definitions beginning with

definition (C-) for contraction given above, followed by the Levi definition of revision in terms of contraction also given above, followed by the definition of $\alpha \mid\!\sim \beta$ to hold iff $\beta \in K^*_\alpha$.

We omit the verification, which is tedious but not particularly difficult.

In summary, there is a very close *formal* correspondence between belief revisions based on orderings of epistemic entrenchment and nonmonotonic inference relations based on expectation orderings. Given a belief set K (a set Δ of expectations) and a proposition α to be used as the basis for a revision (as a premise for nonmonotonic inference), we use the ordering to determine which beliefs (expectations) may legitimately be saved (accompany α as additional premises) for subsequent closure under logical consequence.

Epistemologically, the difference between belief sets and expectations lies only in our attitude to them, i.e., what we are willing to do with them. For so long as we are *using* a belief set K, its elements function as full beliefs. But as soon as we seek to *revise* K, thus putting its elements into question, they lose the status of full belief and become merely expectations, some of which may have to go in order to make consistent place for beliefs introduced in the revision process.

5. Conclusion

In this article we have argued that nonmonotonic inferences may elegantly be interpreted in terms of underlying *expectations*. These are propositions, just as are our ‘ordinary’ beliefs, and indeed they include the latter among them. On this approach, there is no need for a special formalism to express default beliefs. We have proposed two ways of modelling the expectations used in nonmonotonic reasoning, to wit, by expectation sets equipped with selection functions (the latter possibly being relationally determined in the sense of Section 2), and by expectation relations. For each of these models we have proved a number of representation theorems and established their relations to other models in the area, notably Poole systems and preferential models.

We have also argued that by using the notion of expectation, one can give a unified treatment of the theory of *belief revision* and that of nonmonotonic inference relations. This is accomplished by viewing the relation of ‘epistemic entrenchment’ used in Gärdenfors (1988) and Gärdenfors and Makinson (1988) as a kind of expectation ordering. Thus we view belief revision and nonmonotonic reasoning as basically *the same process*, albeit used for two different purposes.

Expectations, in the form of sets or orderings, have been treated as primitive concepts. But where do they come from? One answer is to define an expectation ordering by using a nice preferential model as in the proof of Theorem 3.9. However, in our opinion, this is like putting the cart in front of the horse, since orderings of *models* seems epistemologically more advanced than orderings of *sentences*.

A better answer is to view expectations as emerging from *learning processes*. Expectations can be regarded as a way of summarising previous experience in a cognitively economical way. For example, Balkenius and Gärdenfors (1991) show that, under a particular interpretation, a large class of neural networks can be seen as performing nonmonotonic inferences based on the ‘expectations’ of the network. The upshot is that we propose that the question of the genesis of expectations should be delegated to cognitive science.

Appendix I. Belief valuations compared

In Section 4 we compared our use of expectation orderings for nonmonotonic inference with our earlier (1988) use of epistemic entrenchment orderings for belief revision. In this appendix we briefly review a number of other ideas and constructions in the literature that are also closely related to expectation orderings.

It is difficult to assign credits here because of the multiplicity of independent contributions differing more in point of departure and in terminology than in destination, but we shall attempt to do justice to the earlier work of Shackle, Levi, Cohen, Shafer and Zadeh, as well as the later but important technical developments of Dubois and Prade, Spohn, Brewka, and a key idea of Rott. For the sake of the comparisons, it is convenient to take our constructions in the form that they were given in Section 3.2, i.e., as valuations into belief scales with conjunction homomorphic to minimality and with nonmonotonic inference relations generated via rule ($C{\mid\!\sim}_f$).

I.1. Plausibility story

The idea of a plausibility grading of some kind, with valuations into it that to some extent resemble ordinary probability distributions, yet still behave rather differently from them, goes back a long way. In particular, the use of the minimum operation to evaluate the plausibility of compound propositions was suggested by the economist G.L.S. Shackle in a series of publications culminating in his (1961).

Shackle's background was not that of a logician. Notions such as that of a language closed under truth-functional connectives, or classical logical consequence, are quite absent from his apparatus, and his writings are very discursive in style. For this reason it is often rather difficult to see what the contents and implications of his proposals are. But if we are willing to do a little interpretation, we can see him as in effect working with two dual gradings.

Both of them take the real interval [0,1] as their unique domain, and are thus more highly structured than the arbitrary total orderings that we have been considering in the present paper. One of his gradings can be thought of as representing degree of belief (or confidence, credence, etc), in which *conjunction* is treated as homomorphic to minimality: $f(\alpha \wedge \beta) = \min(f(\alpha),f(\beta))$. The other grading, to which Shackle gives much more attention, may be thought of as one of degrees of 'potential surprise' (or disbelief, information, etc), in which *disjunction* is treated as homomorphic to minimality: $k(\alpha \vee \beta) = \min(k(\alpha),k(\beta))$ for any valuation k. The two gradings appear to be understood as dual to each other. For example, given a valuation $k: L \to [0,1]$ of potential surprise, we may form a belief valuation $f: L \to [0,1]$ by putting $f(\alpha) = k(\neg\alpha)$ – which, in general, will not be the same as $1 - k(\alpha)$. Then clearly, by a calculation that is quite foreign to Shackle's mode of presentation, $f(\alpha \wedge \beta) = k(\neg(\alpha \wedge \beta) = k(\neg\alpha \vee \neg\beta) = \min(k(\neg\alpha),k(\neg\beta)) = \min(f(\alpha),f(\beta))$, under the background assumption that $k(\alpha) = k(\beta)$ whenever $Cn(\alpha) = Cn(\beta)$. An early reconstruction of Shackle's ideas can be found in Levi (1966, 1967). In Levi (1967) potential surprise is called 'degree of confidence in rejection'.

Essentially the same idea was suggested by Cohen (1970, 1973), also in rather informal terms, and again in a much more rigorous setting by Shafer (1976). Any function f from a Boolean algebra into the real interval [0,1] is called by Shafer a 'consonant belief function' iff it satisfies the extremal conditions that $f(0) = 0$ and $f(1) = 1$, where the arguments 0 and 1 are the zero and unit of the Boolean algebra, and the condition $f(\alpha \wedge \beta) = \min(f(\alpha),f(\beta))$. If we translate from the language of Boolean algebras to that of propositional languages, we must add of course the condition that whenever $Cn(\alpha) = Cn(\beta)$, then $f(\alpha) = f(\beta)$. This concept is introduced in Chapter 10 of Shafer (1976) as a special case of a more general notion of 'belief function', which forms the main subject of the book, and which also covers ordinary probability distributions as another special case. Shafer studies some of the properties of his consonant belief functions, but the concept of a nonmonotonic inference relation is not on his agenda, nor on that of Zadeh (1978), who defined dual notions motivated by the perspective of fuzzy sets. These authors do not raise the question of defining such relations from belief functions.

I.2. Snakes from scales

Scales are one thing; their use is another. Shackle, Shafer and Zadeh had uses for their gradings, but as we have mentioned, these did not include the generation of nonmonotonic inference relations. On the other hand, Spohn (1987) uses a kind of grading to define a process of belief revision which can be linked with that of nonmonotonic inference in a manner akin to that described in Section 4.

Spohn's gradings are also more constrained than those of this paper, but in a different way. Instead of the real interval, he considers sets that are not only totally but also well-ordered, so that they may be identified with initial segments of the class of all ordinal numbers. Moreover, they are, like Shackle's principal gradings, presented dually as representing degrees of distance from the expected or given. Lesser ordinals are thus taken as 'better' than larger ones.

Valuation functions k, called by Spohn 'ordinal conditional functions', are understood as acting in the first instance on a set of 'all possible worlds' for L, with values in the ordinals, with at least one world getting the 'best' value 0. Derivatively, for propositions $\alpha \in L$, $k(\alpha)$ is defined to be the 'best', i.e. the least value of k(w) for worlds w satisfying α (in the principal case that α is consistent). Belief sets are identified with such valuations, and so are more complex objects than mere sets of propositions closed under logical consequence as in the Alchourrón, Gärdenfors, and Makinson (1985) approach to belief revision.

Every Spohn grading is (with order reversed) trivially a belief scale in our sense (although not conversely). Moreover, if k is an ordinal conditional function then the function f defined by the Shackle identity $f(\alpha) = k(\neg\alpha)$ for all $\alpha \in L$, is a belief valuation in our sense. To verify (F1), note that if $Cn(\alpha) = Cn(\beta)$ then $Cn(\neg\alpha) = Cn(\neg\beta)$ so that the worlds satisfying $\neg\alpha$ are just those satisfying $\neg\beta$, so the least such worlds are the same, so $f(\alpha) = k(\neg\alpha) = k(\neg\beta) = f(\beta)$ as required. As for (F2), we need only recall Spohn's definition of $k(\alpha)$, for propositions α, as the minimum value of k(w) for all the worlds w satisfying α, from which it follows that $k(\alpha \vee \beta) = \min(k(\alpha),k(\beta))$, and then calculate as above with Shackle's gradings.

Spohn then employs arithmetic functions on the ordinals to define a revision operation that takes us from a valuation (alias belief set) k to another valuation $k^*(\alpha,a)$ understood as 'the result of revising a theory k so as to introduce with degree of firmness $a > 0$ a proposition α'. We shall not reproduce his definition here, but note that it is possible to relate it to our construction of a nonmonotonic inference relation. Following Spohn, define the set [k] of all propositions ‘accepted’ by a valuation k to be $\{\beta \in L: k(\neg\beta) > 0\}$. This is easily checked to be a set of propositions closed under logical consequence. It is then possible to show that $[k^*(\alpha,a)] = \{\beta \in L: \alpha \mid\!\sim_f \beta\}$, where f is constructed out of k by the Shackle identity. Thus we can say: the ‘propositional part’ of any Spohn revision function can be expressed in terms of the nonmonotonic inference relation generated by a suitable belief valuation. It is not clear whether the converse is also true. We omit the details of the verification, which of course depend upon those of the arithmetic definition of Spohn’s revision function *.

In a number of publications beginning with Dubois (1986), Dubois and Prade have studied, on a formal level, concepts of ‘qualitative possibility’ and dually ‘qualitative necessity’. As in the present paper, they do so from two perspectives: that of a relation between propositions, and that of a valuation into a grading scale.

For relations of ‘qualitative necessity’, Dubois' postulates are indeed equivalent to our conditions (E1) - (E3) for expectation orderings – with the addition of a nontriviality condition: $\bot < \top$. For evaluations, Dubois and Prade follow Shackle, Shafer and Zadeh. Indeed, their ‘qualitative necessity measures’ are exactly the consonant belief functions of Shafer. The interesting thing, from our point of view, is what they do with the measures.

Following an idea that can be traced back to Hisdal (1978), Dubois and Prade employ their qualitative necessity measures to develop a notion of ‘conditional possibility’ and ‘conditional necessity’ which, as the names suggest, are intended to serve as an analogy to the familiar notion of conditional probability. Given a qualititative necessity measure f, the conditional necessity of γ given α, written $N(\gamma|\alpha)$, is defined to be equal to $f(\neg\alpha \vee \gamma)$ in the case that $f(\neg\alpha) < f(\neg\alpha \vee \gamma)$ and to be zero otherwise. In a recent paper (1991b) they draw attention to the fact that given such a measure of conditional necessity, one may generate a nonmonotonic inference relation $\mid\!\sim$ by putting $\alpha \mid\!\sim \gamma$ iff $N(\gamma|\alpha) > 0$. So defined, however, $\mid\!\sim$ does not quite satisfy Reflexivity. When $f(\neg\alpha) = 1$ we have $N(\alpha|\alpha) = 0$ and thus $\alpha \not\mid\!\sim \alpha$. However, by modifying the treatment of this limiting case, putting $\alpha \mid\!\sim \gamma$ iff either $N(\gamma|\alpha) > 0$ or $\alpha \vdash \gamma$, one obtains an inference relation that can easily be shown, using criterion (2) of Theorem 3.5, to be identical that given by definition $(C\mid\!\sim)$ or $(C\mid\!\sim_f)$ of Section 3 of this paper.

The theory of nonmonotonic inference relations based on belief valuations, as set out in this paper, is thus closely related to that of Dubois and Prade. It differs in five respects: (1) Our scales are arbitrary total orderings, rather than only the real interval [0,1]; (2) we do not exclude the trivial valuation f, i.e. we allow the possibility that $f(\bot) = f(\top)$; (3) we generate the inference relation $\mid\sim$ *directly* from the valuation, rather than via a notion of conditional necessity, which is not only indirect but also contains a certain 'surplus content' – as far as $\mid\sim$ is concerned one doesn't ever need to know the value of $N(\gamma|\alpha)$, but only whether or not it differs from zero; (4) we modify the treatment of a limiting case in the generation of $\mid\sim$ so as to ensure satisfaction of Reflexivity and Supraclassicality; and (5) we extend a representation result (Theorem 3.8) established in the finite case only by Dubois and Prade, and give two further representation results (Theorems 3.3 and 3.9) that they do not consider.

Rott (1991, Section 5), in a study of belief contraction and revision, observed that the definition of those operations from an epistemic entrenchment relation as given in Gärdenfors and Makinson (1988), which makes rather unintuitive use of a disjunction (see rule (C-) in Section 4 of this paper) can be simplified, so that $K*\alpha$ may be defined as $Cn(\{\alpha\} \cup \{\beta: \neg\alpha < \beta\})$ where < is an epistemic entrenchment relation with respect to K. It is the translation of this into the language of nonmonotonic reasoning that provides the key rule ($C\mid\sim$) for generating nonmonotonic inference relations out of expectation orderings in Section 3.1 of this paper. The idea of definition ($C\mid\sim$) can also be found, in dual form in the context of the logic of counterfactual conditionals, in Lewis (1973) Section 2.6.

Finally, it may be noted that to a limited extent, our work with expectation ordering is reminiscent of Brewka's (1989, 1990, 1991) use of 'preferred subtheories' as a way of handling default reasoning. He too works only with first order formulas, instead of using a special formalism for default rules as in Reiter (1980). Corresponding to degrees of epistemic entrenchment, he introduces 'levels of reliability' to order the formulas representing available information and he uses these levels to define nonmonotonic inferences in much the same way as we do. Poole's (1988) theory can be seen as a special case of Brewka's where there are only two levels of formulas.

However, there are some crucial differences betwen Brewka's approach and ours. Most importantly, his formulas are only supposed to be ordered (he also considers a generalization where the formulas are only supposed to be partially ordered). There is thus nothing in his account that corresponds to the postulates (E2) and (E3), which are, of course, essential for theorems 3.2 and 3.3. There are no corresponding representation results in Brewka's work. Furthermore, what is provable in Brewka's system depends on the syntactic form of the premises. Finally, on his definition of nonmonotonic inference it is impossible to derive inconsistent conclusions from any set of premises, whereas on our approach this happens as soon as the premises are inconsistent themselves.

Appendix II: Proofs of lemmas and theorems

Lemma 2.2. Suppose that $\Delta = Cn(\Delta)$. If $D \in \Delta\perp\alpha$, then $D \in \Delta\perp\beta$ for all $\beta \in \Delta$ such that $\beta \notin D$.

Proof. The following verification is a little more direct that that carried out in lemmas 2.1 and 2.4 for the same result in Alchourrón, Gärdenfors, and Makinson (1985). Suppose $D \in \Delta\perp\alpha$, $\beta \in \Delta$, and $\beta \notin D$. First note that since $\Delta = Cn(\Delta)$, $D = Cn(D)$, so $\beta \notin Cn(D)$. Now suppose $D \subset D' \subseteq \Delta$; to complete the proof we need to show that $\beta \in Cn(D')$. For this it will suffice to show that both $\alpha \in Cn(D')$ and $\neg\alpha \vee \beta \in Cn(D')$. The former is immediate from the fact that $D \in \Delta\perp\alpha$. For the second, we indeed have $\neg\alpha \vee \beta \in D \subseteq Cn(D')$, for otherwise $D \subset D \cup \{\neg\alpha \vee \beta\} \subseteq \Delta$ since $\Delta = Cn(\Delta)$, so again $\alpha \in Cn(D \cup \{\neg\alpha \vee \beta\})$, so $\neg\alpha \vee \beta \rightarrow \alpha \in Cn(D)$. Since this last sentence is classically equivalent to α, we have $\alpha \in Cn(D)$ contradicting $D \in \Delta\perp\alpha$.

Theorem 2.3. A nonmonotonic inference relation $\mid\!\sim$ satisfies the set of basic postulates *if and only if* there exists a closed, consistently generated, expectation inference relation $\mid\!\sim_{\Delta,S}$ such that $\alpha \mid\!\sim \beta$ iff $\alpha \mid\!\sim_{\Delta,S} \beta$, for all α and β.

Proof. *From right to left.* Suppose $\mid\!\sim\ = \mid\!\sim_{\Delta,S}$ is a closed and consistently generated expectation inference relation. We want to show that it satisfies the set of basic postulates.

Supraclassicality and Left Logical Equivalence are trivial. For Right Weakening it is sufficient to note that if $\vdash \beta \rightarrow \gamma$ and $\alpha \mid\!\sim \beta$, then $\alpha \rightarrow \beta \in$ D, for all D $\in$ S($\Delta\bot\neg\alpha$) and so $\alpha \rightarrow \gamma \in$ D, for all D $\in$ S($\Delta\bot\neg\alpha$). Hence $\alpha \mid\!\sim \gamma$. Similarly, for And it follows from $\alpha \mid\!\sim \beta$ and $\alpha \mid\!\sim \gamma$ that $\alpha \rightarrow \beta \in$ D and $\alpha \rightarrow \gamma \in$ D, for all D $\in$ S($\Delta\bot\neg\alpha$) and so $\alpha \rightarrow \beta \wedge \gamma \in$ D, for all D $\in$ S($\Delta\bot\neg\alpha$). Hence $\alpha \mid\!\sim \beta \wedge \gamma$.

For Consistency Preservation, suppose $\alpha \mid\!\sim \bot$. this means that for all D $\in$ S($\Delta\bot\neg\alpha$) it holds that $\neg\alpha \in$ D, which can only occur when $\Delta\bot\neg\alpha$ is empty. Thus $\neg\alpha \in$ Cn(Ø) and hence $\alpha \vdash \bot$.

Before verifying Weak Rational Monotony and Weak Conditionalization, we observe that since Δ = Cn(Δ) we have that whenever $\neg\alpha \notin \Delta$ then $\Delta\bot\neg\alpha$ = {Δ}= S($\Delta\bot\neg\alpha$) so that C(α) = $\bigcap${Cn({α} $\cup$ D): D $\in$ S($\Delta\bot\neg\alpha$)} = Cn({α} $\cup$ Δ). In particular, since Δ is assumed consistent $\neg\top \notin \Delta$ and thus C($\top$) = Cn({$\top$} $\cup$ Δ) = Δ.

For Weak Rational Monotony, suppose $\not\mid\!\sim \neg\alpha$ and $\alpha \not\mid\!\sim \beta$, i.e., $\neg\alpha \notin$ C($\top$) = Δ and $\beta \notin$ C(α). We want to show that $\not\mid\!\sim$ $\alpha \rightarrow \beta$, i.e., $\alpha\rightarrow\beta \notin$ C($\top$). But since $\neg\alpha \notin \Delta$ we have C(α) = Cn({α} $\cup$ Δ), so $\alpha\rightarrow\beta \notin$ Cn(Δ) = Δ and we are done.

For Weak Conditionalization finally, suppose $\not\mid\!\sim$ $\alpha \rightarrow \beta$, i.e., $\alpha \rightarrow \beta \notin$ C($\top$) = Δ. We want to show that $\alpha \not\mid\!\sim \beta$, i.e., that $\beta \notin$ C(α). But since $\alpha\rightarrow\beta \notin \Delta$ = Cn(Δ) clearly $\neg\alpha \notin \Delta$ so C(α) = Cn({α} $\cup$ Δ) and again since $\alpha\rightarrow\beta \notin \Delta$ = Cn(Δ) we have $\beta \notin$ Cn({α} $\cup$ Δ) as desired.

From left to right: Suppose $\mid\!\sim$ satisfies the set of basic postulates. We want to show that there is a closed, consistently generated expectation inference relation $\mid\!\sim_{\Delta,S}$ such that $\alpha \mid\!\sim \beta$ iff $\alpha \mid\!\sim_{\Delta,S} \beta$, for all α and β.

First of all, put Δ = C($\top$). By Closure it follows that Δ = Cn(Δ) and by Reflexivity $\top \in$ C($\top$), so Δ is non-empty. Moreover, if Δ is inconsistent under Cn, then $\bot \in$ Cn(Δ) = Δ = C($\top$) so $\top \mid\!\sim \bot$, which is impossible by Consistency Preservation. Thus Δ is consistent under Cn. We define a selection function S as follows. In the limiting case that $\Delta\bot\neg\alpha$ is empty (i.e., when $\neg\alpha \in$ Cn(Ø)) we put S($\Delta\bot\neg\alpha$) = {Δ}. In the case that $\Delta\bot\neg\alpha$ = {Δ} (i.e., when $\neg\alpha \notin \Delta$) we also put S($\Delta\bot\neg\alpha$) = {Δ}. Finally, when $\Delta\bot\neg\alpha$ is nonempty and distinct from Δ (i.e., when $\neg\alpha \in \Delta$ but $\neg\alpha \notin$ Cn(Ø)) we put D $\in$ S($\Delta\bot\neg\alpha$) iff D $\in$ $\Delta\bot\neg\alpha$ and C(α) $\sqsubseteq$ Cn({α} $\cup$ D). Note that in the last case, S($\Delta\bot\neg\alpha$) is indeed well-defined, in that its identity does not depend upon the choice of α. For if $\Delta\bot\neg\alpha = \Delta\bot\neg\beta$ and $\neg\alpha, \neg\beta \in \Delta$, then it is easy to show, using compactness of Cn, that $Cn(\alpha) = Cn(\beta)$ and thus the inclusion $C(\alpha) \subseteq Cn(\{\alpha\} \cup D)$ holds iff the inclusion $C(\beta) \subseteq Cn(\{\beta\} \cup D)$ holds.

We need to show that $S(\Delta\bot\neg\alpha)$ cannot be empty if $\Delta\bot\neg\alpha$ is not. If $\Delta\bot\neg\alpha \neq \emptyset$, then not $\alpha \vdash \bot$, and hence, by Consistency Preservation, $\alpha \not\mid\!\sim \bot$, i.e., $\neg\alpha \notin C(\alpha)$. Let $\Gamma = \{\alpha \rightarrow \gamma : \gamma \in C(\alpha)\}$. Now $\Gamma \subseteq \Delta$, because if $\alpha \rightarrow \gamma \in \Gamma$, then $\gamma \in C(\alpha)$, so, by Weak Conditionalization, $\alpha \rightarrow \gamma \in C(\top) = \Delta$. Hence we have $Cn(\Gamma) \subseteq \Delta$. We claim that $\neg\alpha \notin Cn(\Gamma)$. For if $\neg\alpha \in Cn(\Gamma)$, there are $\gamma_1, ..., \gamma_n \in C(\alpha)$ such that $(\alpha \rightarrow \gamma_1) \wedge ... \wedge (\alpha \rightarrow \gamma_n) \vdash \neg\alpha$ by compactness. It follows that $\neg\alpha \vee (\gamma_1 \wedge ... \wedge \gamma_n) \vdash \neg\alpha$ and so $(\gamma_1 \wedge ... \wedge \gamma_n) \vdash \neg\alpha$. But from this we have $C(\alpha) \vdash \neg\alpha$ contradicting that $\alpha \not\mid\!\sim \bot$. Since $\neg\alpha \notin Cn(\Gamma)$ and $\Gamma \subseteq \Delta$ it follows by compactness there there is some $D \in \Delta\bot\neg\alpha$ with $\Gamma \subseteq D$. In order to show that $D \in S(\Delta\bot\neg\alpha)$ it remains to show that $C(\alpha) \subseteq Cn(\{\alpha\} \cup D)$. But if $\gamma \in C(\alpha)$, then $\alpha \rightarrow \gamma \in \Gamma \subseteq D$, so clearly $\gamma \in Cn(\{\alpha\} \cup D)$.

This defines a closed, consistently generated expectation inference relation $\mid\!\sim_{\Delta,S}$. We want to show that $\alpha \mid\!\sim \beta$ iff $\alpha \mid\!\sim_{\Delta,S} \beta$, for all α and β, which is the same as showing that $C(\alpha) = C_{\Delta,S}(\alpha)$ for all α.

$C(\alpha) \subseteq C_{\Delta,S}(\alpha)$: Suppose that $\beta \in C(\alpha)$. If $\Delta\bot\neg\alpha$ is empty, it follows that $\neg\alpha \dashv\vdash \top$ and hence $C_{\Delta,S}(\alpha) = Cn(\{\neg\top\} \cup \Delta) = L$ and thus $\beta \in C_{\Delta,S}(\alpha)$. If $\Delta\bot\neg\alpha$ is not empty, then, by the definition of the selection function S, $\beta \in Cn(\{\alpha\} \cup D)$ for all $D \in S(\Delta\bot\neg\alpha)$ and hence $\beta \in C_{\Delta,S}(\alpha\}$.

$C_{\Delta,S}(\alpha) \subseteq C(\alpha)$: Suppose that $\beta \notin C(\alpha)$. We want to show that $\beta \notin C_{\Delta,S}(\alpha)$. Suppose first that $\neg\alpha \notin \Delta$. Then $\Delta\bot\neg\alpha = \{\Delta\}$ and hence $C_{\Delta,S}(\alpha) = Cn(\{\alpha\} \cup \Delta)$. So clearly it suffices to show that $\alpha \rightarrow \beta \notin \Delta$. Since $\Delta = C(\top)$, it follows that $\neg\alpha \notin C(\top)$ and since $\beta \notin C(\alpha)$ we have by Weak Rational Monotony that $\alpha \rightarrow \beta \notin C(\top) = \Delta$, as desired.

Suppose for the principal case that $\neg\alpha \in \Delta$. If $\Delta\bot(\neg\alpha \vee \beta)$ is empty, then $\beta \in Cn(\alpha)$ and then $\beta \in C(\alpha)$ contradicting our hypothesis. So suppose that $\Delta\bot(\neg\alpha \vee \beta)$ is non-empty. We have assumed that $\beta \notin C(\alpha)$ and hence $\neg\alpha \vee \beta \notin C(\alpha)$. It follows that $\neg\alpha \vee \beta \notin C(\top) \cap C(\alpha) = \Delta \cap C(\alpha) \subseteq \Delta$, by the definition of Δ. Since both $C(\top)$ and $C(\alpha)$ are closed under Cn, so too is their intersection, and so $\neg\alpha \vee \beta \notin Cn(C(\top) \cap C\alpha))$. Hence there is a $D' \in \Delta\bot\neg\alpha \vee \beta$ such that $C(\top) \cap C(\alpha) \subseteq D'$. It follows, using Weak Conditionalization and Right Weakening that $C(\alpha) \subseteq Cn(\{\alpha\} \cup D')$. This is one requirement for D' to belong to $S(\Delta\bot\neg\alpha)$; we must also show that $D' \in \Delta\bot\neg\alpha$, which is easily done using Lemma 2.2. Finally, since $\neg\alpha \vee \beta \notin D'$, it follows that $\beta \notin Cn(\{\alpha\} \cup D')$ and hence $\beta \notin C_{\Delta,S}(\alpha)$.

Theorem 2.4. A nonmonotonic inference relation $\mid\!\sim$ satisfies the set of basic postulates and Cumulativity *if and only if* there exists a closed, consistently generated expectation inference relation $\mid\!\sim_{\Delta,S}$ where S satisfies (SC) such that $\alpha \mid\!\sim \beta$ iff $\alpha \mid\!\sim_{\Delta,S} \beta$, for all α and β.

Proof. *From right to left.* Assume that $\mid\!\sim\; = \;\mid\!\sim_{\Delta,S}$ is an expectation inference relation where S satisfies (SC). To show that $\mid\!\sim$ satisfies Cumulativity suppose that $\alpha \mid\!\sim \beta$ and $\beta \vdash \alpha$. We want to show that $\beta \mid\!\sim \gamma$ iff $\alpha \mid\!\sim \gamma$. The assumption $\alpha \mid\!\sim \beta$ amounts to $\beta \in Cn(\{\alpha\} \cup D)$ for all $D \in S(\Delta\bot\neg\alpha)$. We consider separately the cases that $\neg\alpha \notin \Delta$ and that $\neg\alpha \in \Delta$.

First, suppose that $\neg\alpha \notin \Delta$. Then $\{\Delta\} = \Delta\bot\neg\alpha = S(\Delta\bot\neg\alpha)$ so by the assumption $\alpha \mid\!\sim \beta$ we have $\neg\alpha \vee \beta \in \Delta$ so that $\neg\beta \notin \Delta$. This also means that $\alpha \mid\!\sim \gamma$ iff $\neg\alpha \vee \gamma \in \Delta$, and likewise $\beta \mid\!\sim \gamma$ iff $\neg\beta \vee \gamma \in \Delta$. Since Δ is closed under Cn, it will suffice to show that $\neg\alpha \vee \beta$ and $\neg\beta \vee \alpha$ are both in Δ. But we already have the former, and the latter follows from the hypothesis that $\beta \vdash \alpha$.

Next, suppose $\neg\alpha \in \Delta$. Since $\beta \vdash \alpha$ we have that $\neg\beta \in \Delta$ and also using Lemma 2.2 we get $\Delta\bot\neg\beta \subseteq \Delta\bot\neg\alpha$, which is one half of the antecedent of the (SC) condition. To get the other half, note that if $D \in S(\Delta\bot\neg\alpha)$, then, by our supposition that $\alpha \mid\!\sim \beta$, $\neg\alpha \vee \beta \in D$ and hence $\neg\beta \notin D$, because otherwise $\neg\alpha \in D$.Since $\neg\beta \in \Delta$, we can apply Lemma 2.2 to conclude $D \in \Delta\bot\neg\beta$ and thus $S(\Delta\bot\neg\alpha) \subseteq \Delta\bot\neg\beta$. The fact that S satisfies (SC) then gives $S(\Delta\bot\neg\beta) = S(\Delta\bot\neg\alpha)$. Hence if $\alpha \mid\!\sim \gamma$, i.e., $\gamma \in Cn(\{\alpha\} \cup D)$ for all $D \in S(\Delta\bot\neg\alpha)$, then $\gamma \in Cn(\{\beta\} \cup D)$ for all $D \in S(\Delta\bot\neg\beta)$, i.e., $\beta \mid\!\sim \gamma$. Conversely, if $\beta \mid\!\sim \gamma$, i.e., $\gamma \in Cn(\{\beta\} \cup D)$ for all $D \in S(\Delta\bot\neg\beta)$, then $\gamma \in Cn(\{\beta\} \cup D)$ for all $D \in S(\Delta\bot\neg\alpha)$. It then follows from the assumption $\alpha \mid\!\sim \beta$, which amounts to $\beta \in Cn(\{\alpha\} \cup D)$ for all $D \in S(\Delta\bot\neg\alpha)$, that $\gamma \in Cn(\{\alpha\} \cup D)$ for all $D \in S(\Delta\bot\neg\alpha)$, i.e., $\alpha \mid\!\sim \gamma$.

From left to right: Assume that $\mathrel{|\!\sim}$ satisfies the set of basic postulates and Cumulativity. Define Δ, S as in the proof of Theorem 2.3 and the expectation inference relation $\mathrel{|\!\sim}_{\Delta,S}$ as in Definition 2.1. From that theorem it follows that $\alpha \mathrel{|\!\sim} \beta$ iff $\alpha \mathrel{|\!\sim}_{\Delta,S} \beta$, for all α and β. We need to show that S also satisfies (SC). Suppose that $S(\Delta\bot\neg\alpha) \subseteq \Delta\bot\neg\beta \subseteq \Delta\bot\neg\alpha$. If $\neg\alpha \notin \Delta$, then $S(\Delta\bot\neg\alpha) = \{\Delta\}$, so by the former inclusion and the definition of $\Delta\bot\neg\beta$ we have $S(\Delta\bot\neg\beta) = \Delta\bot\neg\beta = \{\Delta\} = S(\Delta\bot\neg\alpha)$ as desired. If $\neg\beta \notin \Delta$ then $\Delta\bot\neg\beta = \{\Delta\}$ so by the first inclusion and the definition of $S(\Delta\bot\neg\alpha)$ we have $S(\Delta\bot\neg\alpha) = \{\Delta\} = S(\Delta\bot\neg\beta)$ again as desired. So suppose that $\neg\alpha,\neg\beta \in \Delta$. From the inclusion $\Delta\bot\neg\beta \subseteq \Delta\bot\neg\alpha$ it then follows that $\beta \vdash \alpha$. For any $D \in S(\Delta\bot\neg\alpha)$, it holds that $D \in \Delta\bot\neg\beta$ and thus $\neg\beta \notin D$ from which it follows that $\neg\beta \rightarrow \neg\alpha \in D$ by the definition of $\Delta\bot\neg\alpha$. Thus $\beta \in Cn(\{\alpha\} \cup D)$ for all $D \in S(\Delta\bot\neg\alpha)$, i.e., $\alpha \mathrel{|\!\sim} \beta$. From this and $\beta \vdash \alpha$ it follows by Cumulativity that $C(\alpha) = C(\beta)$. If $D \in S(\Delta\bot\neg\alpha)$, then $C(\alpha) \subseteq Cn(\{\alpha\} \cup D)$ by the definition of S. But $C(\alpha) = C(\beta)$ and $Cn(\{\alpha\} \cup D) \subseteq Cn(\{\beta\} \cup D)$ because $\beta \vdash \alpha$. Hence $D \in S(\Delta\bot\neg\beta)$, again by the definition of S. Conversely, suppose $D \in S(\Delta\bot\neg\beta)$. Since by hypothesis $S(\Delta\bot\neg\alpha) \subseteq \Delta\bot\neg\beta$ and by definition the former cannot be empty, $\Delta\bot\neg\beta$ is not empty. Hence we may say $D \in \Delta\bot\neg\beta$ and hence $D \in \Delta\bot\neg\alpha$. It follows that $\alpha \rightarrow \beta \in D$, by the definition of $\Delta\bot\neg\alpha$. Thus $Cn(\{\beta\} \cup D) \subseteq Cn(\{\alpha\} \cup D)$. Hence, using the definition of S, $C(\alpha) = C(\beta) \subseteq Cn(\{\beta\} \cup D) \subseteq Cn(\{\alpha\} \cup D)$. We conclude by the definition of S again that $D \in S(\Delta\bot\neg\alpha)$.

Theorem 2.6. Any relational closed expectation inference relation $\mathrel{|\!\sim}_{\Delta,S}$ satisfies Or.

Proof. (The proofs of this and next theorem are basically translations of the corresponding parts in Observation 4.3 in Alchourrón, Gärdenfors, and Makinson (1985)). Suppose that $\gamma \notin C(\alpha \vee \beta)$. We want to show that $\gamma \notin C(\alpha)$ or $\gamma \notin C(\beta)$. In the limiting case that $\neg(\alpha \vee \beta) \notin \Delta$ we have $C(\alpha \vee \beta) = Cn(\{\alpha \vee \beta\} \cup \Delta) = Cn(\{\alpha\} \cup \Delta) \cap Cn(\{\beta\} \cup \Delta)$ which includes $C(\alpha) \cap C(\beta)$ and we are done. So we may suppose $\neg(\alpha \vee \beta) \in \Delta$ so that $\neg\alpha, \neg\beta \in \Delta$. From $\gamma \notin C(\alpha \vee \beta)$ it follows that there is some $D \in S(\Delta\bot\neg(\alpha \vee \beta))$ such that $\alpha \vee \beta \rightarrow \gamma \notin D$. Hence $\alpha \rightarrow \gamma \notin D$ or $\beta \rightarrow \gamma \notin D$. Assume the former, the latter is parallel. Since $\alpha \rightarrow \gamma \notin D$ we know that $\neg\alpha \notin D$ and hence $D \in \Delta\bot\neg\alpha$ by Lemma 2.2 since $\neg\alpha \in \Delta$. We want to show that $D \in S(\Delta\bot\neg\alpha)$. Suppose D' is any set in $\Delta\bot\neg\alpha$.

Since $\neg(\alpha \vee \beta) \notin D'$, it follows by Lemma 2.2 again that $D' \in \Delta\perp\neg(\alpha \vee \beta)$. By relationality $D \,/\, D'$ and since D' is an arbitrary set in $\Delta\perp\neg\alpha$ it follows that $D \in S(\Delta\perp\neg\alpha)$ and hence that $\gamma \notin C(\alpha) = \bigcap\{Cn(\{\alpha\} \cup D): D \in S(\Delta\perp\neg\alpha)\}$.

Theorem 2.7. Any transitively relational closed expectation inference relation $\mid\!\sim_{\Delta,S}$ satisfies Rational Monotony (as well as Or and thus also Cumulativity).

Proof. Assume that $\neg\alpha \notin C(\beta)$ and $\gamma \in C(\beta)$, but $\gamma \notin C(\alpha\wedge\beta)$; we want to derive a contradiction. We need to divide the argument into two cases.

Case 1: Suppose $\neg\beta \notin \Delta$. Then $\Delta\perp\neg\beta = \{\Delta\}$ so $S(\Delta\perp\neg\beta) = \{\Delta\}$; hence $C(\beta) = Cn(\Delta \cup \{\beta\})$, so $\neg\alpha \notin Cn(\Delta \cup \{\beta\})$, i.e., $\beta \rightarrow \neg\alpha \notin Cn(\Delta) = \Delta$. It follows that $\Delta\perp\neg(\alpha\wedge\beta) = \{\Delta\}$ and hence $S(\Delta\perp\neg(\alpha\wedge\beta)) = \{\Delta\}$, so $C(\beta) \subseteq Cn(\Delta \cup (\alpha \wedge \beta\}) = C(\alpha \wedge \beta)$. Since $\gamma \in C(\beta)$ we have $\gamma \in C(\alpha \wedge \beta)$ giving a contradiction.

Case 2: Suppose $\neg\beta \in \Delta$. Then since $\Delta = Cn(\Delta)$ we have $\neg(\alpha\wedge\beta) \in \Delta$ too. From the fact that $\neg\alpha \notin C(\beta)$ it follows that there is some $D \in S(\Delta\perp\neg\beta)$ such that $\beta \rightarrow \neg\alpha \notin D$. Hence $\neg(\alpha\wedge\beta) \notin D$ and Lemma 2.2 gives us $D \in \Delta\perp\neg(\alpha\wedge\beta)$. From the assumption that $\gamma \notin C(\alpha\wedge\beta)$ it follows that there is some $D' \in S(\Delta\perp\neg(\alpha\wedge\beta))$ such that $\alpha\wedge\beta \rightarrow \gamma \notin D'$. Relationality gives us $D' \,/\, D$. But since $\alpha\wedge\beta \rightarrow \gamma \notin D'$ it follows that $\neg\beta \notin D'$ and hence by Lemma 2.2 that $D' \in \Delta\perp\neg\beta$. But since $D' \,/\, D$ it follows by transitivity that $D' \in S(\Delta\perp\neg\beta)$. From the fact that $\gamma \in C(\beta)$ we then conclude that $\beta \rightarrow \gamma \in D'$ and hence $\alpha\wedge\beta \rightarrow \gamma \in D'$ which gives us the desired contradiction.

Lemma 2.8. Suppose that $\Delta = Cn(\Delta)$ and $\alpha, \beta \in \Delta$. Then $\Delta\perp\alpha\wedge\beta = \Delta\perp\alpha \cup \Delta\perp\beta$.

Proof. This is lemma 4.1 of Alchourrón, Gärdenfors and Makinson (1985). For completeness, we recall the proof, which is an easy application of Lemma 2.2. If $D \in \Delta\perp\alpha\wedge\beta$ then $\alpha\wedge\beta \notin D$ so $\alpha \notin D$ or $\beta \notin D$ so by Lemma 2.2 either $D \in \Delta\perp\alpha$ or $D \in \Delta\perp\beta$. Conversely, if $D \in \Delta\perp\alpha$ or $D \in \Delta\perp\beta$ then $D \nvdash \alpha\wedge\beta$ and by Lemma 2.2 again $D \in \Delta\perp\alpha\wedge\beta$.

Lemma 2.9. Suppose that $\Delta = Cn(\Delta)$ and $D \in \Delta\perp\alpha$. Then $\Delta \subseteq Cn(D \cup \{\alpha\})$.

Proof. We recall the verification from Alchourrón and Makinson (1982). If $\beta \in \Delta = Cn(\Delta)$ then $\neg\alpha\vee\beta \in \Delta$. To show $\beta \in Cn(D \cup \{\alpha\})$ it will clearly suffice to show $\neg\alpha\vee\beta \in D$. But if $\neg\alpha\vee\beta \notin D$ then since $\neg\alpha\vee\beta \in \Delta$ we have $D \cup \{\neg\alpha\vee\beta\} \vdash \alpha$ so by assumed properties of the background consequence operation (Section 1.2) we have $D \vdash \alpha$ contradicting $D \in \Delta\bot\alpha$.

Lemma 2.10. Suppose that $\Delta = Cn(\Delta)$, $\neg\alpha \in \Delta$ and $D \in \Delta\bot\neg\alpha$. Then:

(a) $D = Cn(D \cup \{\alpha\}) \cap \Delta$

(b) $C(\alpha) \subseteq Cn(D \cup \{\alpha\})$ iff $C(\alpha) \cap \Delta \subseteq D$, whenever C satisfies Right Weakening.

Proof. For (a) the left to right inclusion is immediate. For its converse, suppose $\beta \in \Delta$ and $\beta \in Cn(D \cup \{\alpha\})$. Since $D \in \Delta\bot\neg\alpha$ and $\beta \in \Delta$ we have by Lemma 2.9 that also $\beta \in Cn(D \cup \{\neg\alpha\})$ so, by assumed properties of Cn, $\beta \in Cn(D) = D$ as required.

For (b), the left to right implication is immediate from (a). For the converse implication, suppose $C(\alpha) \cap \Delta \subseteq D$, where C satisfies Right Weakening, and that $\beta \in C(\alpha)$. Then clearly $\neg\alpha\vee\beta \in C(\alpha) \cap \Delta \subseteq D$ so that $\beta \in Cn(D \cup \{\alpha\})$ as required.

Lemma 2.11. Let $\mid\!\sim$ be any inference relation satisfying the extended set of postulates. If $\alpha \mid\!\sim \gamma$ and $\neg\alpha \vdash \gamma$ then $\alpha\vee\beta \mid\!\sim \gamma$ for any β.

Proof. This lemma parallels observation 3.3 of Alchourrón, Makinson and Gärdenfors (1985) and its proof is similar. Suppose $\alpha \mid\!\sim \gamma$ and $\neg\alpha \vdash \gamma$. Now $\alpha\vee\beta \dashv\vdash \alpha \vee (\beta\wedge\neg\alpha)$ so it will suffice to show $\alpha \vee (\beta\wedge\neg\alpha) \mid\!\sim \gamma$. By Or it will suffice to show both $\alpha \mid\!\sim \gamma$ and $\beta\wedge\neg\alpha \mid\!\sim \gamma$. We have the former by supposition. For the latter, our supposition $\neg\alpha \vdash \gamma$ gives us $\beta\wedge\neg\alpha \vdash \gamma$ so by Reflexivity $\beta\wedge\neg\alpha \mid\!\sim \gamma$ as desired.

Theorem 2.12. An inference relation $\mid\!\sim$ satisfies the extended set of postulates iff there is a closed, consistently generated, and transitively relational expectation inference relation $\mid\!\sim_{\Delta,S}$ with $\mid\!\sim = \mid\!\sim_{\Delta,S}$.

Proof. Right to left is already given by theorems 2.3, 2.6, 2.7, so we need only show the left to right implication. Suppose $\mid\!\sim$ satisfies the extended set of postulates. Define Δ, S as in the proof of Theorem 2.3. Then by that theorem, $\mid\!\sim = \mid\!\sim_{\Delta,,S}$ and $\mid\!\sim_{\Delta,S}$ is closed and consistently generated. The additional point that remains to be proven is that $\mid\!\sim_{\Delta,S}$ is transitively relational, i.e. that there is a transitive relation $/$ over the subsets of $\Delta = Cn(\Delta)$ such that for all α with $\neg\alpha \notin Cn(\emptyset)$:

(*) $S(\Delta\bot\neg\alpha) = \{D \in \Delta\bot\neg\alpha: D / D' \text{ for all } D' \in \Delta\bot\neg\alpha\}$

We define $/$ as follows: put D / D' iff D, D' are subsets of Δ and either $D = D' = \Delta$ or else the following three conditions all hold:

(i) $D' \in \Delta\bot\neg\psi$ for some $\neg\psi \in \Delta$,

(ii) $D \in \Delta\bot\neg\varphi$ for some $\neg\varphi \in \Delta$ with $C(\varphi) \subseteq Cn(D \cup \{\varphi\})$

(iii) For all γ, if $D, D' \in \Delta\bot\neg\gamma$ and $C(\gamma) \subseteq (D' \cup \{\gamma\})$ then $C(\gamma) \subseteq Cn(D \cup \{\gamma\})$.

This definition of $/$, and the argument that follows, are essentially translations of those used in the context of belief contraction in observation 4.4 of Alchourrón, Gärdenfors and Makinson (1985). Nevertheless, we write the proof out in full, as the details are rather tricky. We need to show that the identity (*) holds and that the relation $/$ is transitive.

Verification of the identity (*). Suppose $\neg\alpha \notin Cn(\emptyset)$. In the limiting case that $\neg\alpha \notin \Delta$, clearly the left and right hand sides of (*) are both equal to $\{\Delta\}$, using the initial part of the definition of $/$. So we suppose without loss of generality that $\neg\alpha \in \Delta$.

For the left to right inclusion, suppose $D \in S(\Delta\bot\neg\alpha)$. Then by the definition of S (see proof of Theorem 2.3) we have $D \in \Delta\bot\neg\alpha$ and $C(\alpha) \subseteq Cn(D \cup \{\alpha\})$. Now let $D' \in \Delta\bot\neg\alpha$; we need to show D / D', i.e. we need to show conditions (i), (ii), (iii) above. Our suppositions give us (i) and (ii) directly, putting $\psi = \varphi = \alpha$. For (iii), let γ be any formula and suppose $D, D' \in \Delta\bot\neg\gamma$, $C(\gamma) \subseteq Cn(D' \cup \{\gamma\})$ and $\gamma \mid\!\sim \delta$ whilst $\delta \notin Cn(D \cup \{\gamma\})$; we seek a contradiction.

First note that from the last hypothesis $\neg\gamma\vee\delta \notin Cn(D) = D$. Also note that $\neg\gamma\vee\delta \in \Delta$, for $\neg\alpha \in \Delta$ and $D \in \Delta\bot\neg\alpha$ gives $D \neq \Delta$ and so since $D \in \Delta\bot\neg\gamma$ we have $\neg\gamma \in \Delta$ so $\neg\gamma\vee\delta \in \Delta$. Thus we may apply Lemma 2.2 to conclude that $\Delta\bot\neg\alpha = \Delta\bot\neg\gamma\vee\delta$, so $D \in \Delta\bot\neg\gamma\vee\delta$ so by Lemma 2.9, $D \cup \{\neg\gamma\vee\delta\} \vdash \neg\alpha$. We now split the argument into two cases, according to whether $\gamma \mid\!\sim \neg\alpha$.

In the case that $\gamma \not\mid\!\sim \neg\alpha$ we can apply Rational Monotony to the supposition $\gamma \mid\!\sim \delta$ to conclude $\alpha \wedge \gamma \mid\!\sim \delta$ so by Conditionalization $\alpha \mid\!\sim \neg\gamma \vee \delta$, i.e., $\neg\gamma \vee \delta \in C(\alpha) \subseteq Cn(D \cup \{\alpha\})$ so by the logic of Cn we have $D \cup \{\neg(\neg\gamma \vee \delta)\} \vdash \neg\alpha$. Putting this together with $D \cup \{\neg\gamma \vee \delta\} \vdash \neg\alpha$, already established, gives us $D \vdash \neg\alpha$ contradicting $D \in \Delta\perp\neg\alpha$.

In the case that $\gamma \mid\!\sim \neg\alpha$ we recall the supposition $C(\gamma) \subseteq Cn(D' \cup \{\gamma\})$ to conclude $\neg\alpha \in Cn(D' \cup \{\gamma\})$. But since by supposition $D' \in \Delta\perp\neg\gamma$ and $\neg\alpha \in \Delta$ we also have by Lemma 2.9 that $\neg\alpha \in Cn(D' \cup \{\neg\gamma\})$. Putting these two together gives us again $D' \vdash \neg\alpha$ contradicting $D' \in \Delta\perp\neg\alpha$.

This completes the verification of the left to right inclusion of (*). For the right to left, suppose $D \in \Delta\perp\neg\alpha$, but $D \notin S(\Delta\perp\neg\alpha)$. We need to find a $D' \in \Delta\perp\neg\alpha$ such that not D / D'. By the definition of S, since $\neg\alpha \notin Cn(\emptyset)$ and $\neg\alpha \in \Delta$ and the supposition just made we have $C(\alpha) \not\subseteq Cn(D \cup \{\alpha\})$. To construct an appropriate D', first put $X = \{\neg\alpha \vee \beta : \alpha \mid\!\sim \beta\}$. Since $\neg\alpha \in \Delta = Cn(\Delta)$, $X \subseteq \Delta$. Also $X \nvdash \neg\alpha$: otherwise by compactness and classical logic $\neg\alpha \vee (\beta_1 \wedge \ldots \wedge \beta_n) \vdash \neg\alpha$ where $\alpha \mid\!\sim \beta_i$ for all $i \leq n$, so $\beta_1 \wedge \ldots \wedge \beta_n \vdash \neg\alpha$ so by And and Right Weakening for $\mid\!\sim$, $\alpha \mid\!\sim \neg\alpha$ so by Consistency Preservation $\neg\alpha \in Cn(\emptyset)$ contradicting $D \in \Delta\perp\neg\alpha$.

Since $X \nvdash \neg\alpha$ and $X \subseteq \Delta$ we have by compactness of Cn that there is a $D' \in \Delta\perp\neg\alpha$ with $X \subseteq D'$. We claim that not D / D'. Since $D \in \Delta\perp\neg\alpha$ and $\neg\alpha \in \Delta$ we know that $D' \neq \Delta$. So to show that not D / D' we need only show that condition (iii) fails. Since $C(\alpha) \not\subseteq Cn(D \cup \{\alpha\})$ it will suffice to show that $C(\alpha) \subseteq Cn(D' \cup \{\alpha\})$. But whenever $\alpha \mid\!\sim \beta$ then by construction $\neg\alpha \vee \beta \in X \subseteq D'$ so $\beta \in Cn(D' \cup \{\alpha\})$ as desired.

Verification that / is transitive. Suppose $D / D' / D''$; we want to show D / D''. If $D = D' = D'' = \Delta$ we are done. Suppose one of D, D', D'' is distinct from Δ. Then it is clear from the hypothesis and the definition of / that they all are. We need to show that (i) holds for D'', (ii) holds for D, and (iii) holds for the pair D, D''.

That (i) holds for D'' is immediate from D' / D'' and $D'' \neq \Delta$. That (ii) holds for D is likewise immediate from D / D' and $D' \neq \Delta$. For condition (iii), let γ be any formula and suppose $D, D'' \in \Delta\perp\neg\gamma$ and $C(\gamma) \subseteq Cn(D'' \cup \{\gamma\})$; we need to show that $C(\gamma) \subseteq$

$Cn(D \cup \{\gamma\})$. Since $D \neq \Delta$ clearly $\neg\gamma \in \Delta$. Hence by lemma 2.10 (b) we have $C(\gamma) \cap \Delta \subseteq D''$ and we need only show that $C(\gamma) \cap \Delta \subseteq D$.

Since D' / D'' and $D'' \neq \Delta$ we know from condition (ii) for D' that there is a $\neg\beta \in \Delta$ with $D' \in \Delta\bot\neg\beta$ and $C(\beta) \subseteq Cn(D' \cup \{\beta\})$. Note that since $\neg\beta, \neg\gamma \in \Delta$ we have $\neg\beta\wedge\neg\gamma \in \Delta$ so by Lemma 2.8, $\Delta\bot(\neg\beta\wedge\neg\gamma) = \Delta\bot\neg\beta \cup \Delta\bot\neg\gamma$. Hence all of D, D', D'' $\in \Delta\bot(\neg\beta\wedge\neg\gamma) = \Delta\bot\neg(\beta\vee\gamma)$. Note also that to show $C(\gamma) \cap \Delta \subseteq D$ and complete the proof, it will suffice to show $C(\beta\vee\gamma) \cap \Delta \subseteq D$. For then, if $\gamma \mid\!\sim \delta$ and $\delta \in \Delta$ then $\gamma \mid\!\sim \neg\gamma\vee\delta$ whilst of course $\neg\gamma \vdash \neg\gamma\vee\delta$ so, by Lemma 2.11, $\beta\vee\gamma \mid\!\sim \neg\gamma\vee\delta$ so $\neg\gamma\vee\delta \in D$ and thus since $D \in \Delta\bot\neg\gamma$ we can conclude $\delta \in D$ as needed.

Indeed, it will suffice to show $C(\beta\vee\gamma) \cap \Delta \subseteq D'$. For then by Lemma 2.10 (b), $C(\beta\vee\gamma) \subseteq Cn(D' \cup \{\beta\vee\gamma\})$ and since D / D' we may apply condition (iii) to the pair D, D' to conclude that $C(\beta\vee\gamma) \subseteq Cn(D \cup \{\beta\vee\gamma\})$ and thus by Lemma 2.10 (b) again that $C(\beta\vee\gamma) \cap \Delta \subseteq D$.

Now either $C(\beta\vee\gamma) \subseteq C(\beta)$ or $C(\beta\vee\gamma) \subseteq C(\gamma)$: otherwise there are φ, ψ with $\beta\vee\gamma \mid\!\sim \varphi$, $\beta \not\mid\!\sim \varphi$, $\beta\vee\gamma \mid\!\sim \psi$, $\gamma \not\mid\!\sim \psi$ so that $\beta\vee\gamma \mid\!\sim \varphi\wedge\psi$ whilst $\beta \not\mid\!\sim \varphi\wedge\psi$ and $\gamma \not\mid\!\sim \varphi\wedge\psi$ contradicting Disjunctive Rationality, which is a consequence of the extended set of postulates (see section 1.3). We consider the two cases in turn, showing that in each case $C(\beta\vee\gamma) \cap \Delta \subseteq D'$ and thus completing the proof.

Suppose for the first case that $C(\beta\vee\gamma) \subseteq C(\beta)$. Recall that $C(\beta) \subseteq Cn(D' \cup \{\beta\})$ (go back three paragraphs) whilst $\neg\beta \in \Delta$ and $D' \in \Delta\bot\neg\beta$ so by Lemma 2.10 (b) $C(\beta) \cap \Delta \subseteq D'$ so by the condition of the case $C(\beta\vee\gamma) \cap \Delta \subseteq D'$ and we are done.

Suppose for the second case that $C(\beta\vee\gamma) \subseteq C(\gamma)$. Recall that $C(\gamma) \cap \Delta \subseteq D''$ (go back five paragraphs) so by the condition of the case $C(\beta\vee\gamma) \cap \Delta \subseteq D''$. But since D' / D'' we may apply condition (iii) to the pair D', D'' and the formula $\beta\vee\gamma$ to conclude that $C(\beta\vee\gamma) \cap \Delta \subseteq D'$ and once again we are done.

Theorem 3.2. Let $\leq$ be an expectation ordering over L. Then the inference relation $\mid\!\sim$ that it determines by (C$\mid\!\sim$) satisfies the extended set of postulates of Section 1.3.

Proof. We verify the reduced list given in Section 1.3.The verifications are all quite straightforward, but we give them in full. For the proof we use (C$\mid\!\sim$) in the form $\alpha \mid\!\sim \gamma$ iff either $\alpha \vdash \gamma$ or there is a $\beta \in L$ with $\alpha \wedge \beta \vdash \gamma$ and $\neg\alpha < \beta$

Supraclassicality is immediate from the definition. For Left Logical Equivalence, suppose $Cn(\alpha) = Cn(\alpha')$, and suppose $\alpha \mid\!\sim \gamma$; we need to show that $\alpha' \mid\!\sim \gamma$. If $\alpha \vdash \gamma$ then clearly $\alpha' \vdash \gamma$ and it follows from (C$\mid\!\sim$) that $\alpha' \mid\!\sim \gamma$ so we are done. Suppose for the principal case that there is a $\beta \in L$ with $\alpha \wedge \beta \vdash \gamma$ and $\neg\alpha < \beta$, i.e., not $\beta \leq \neg\alpha$. Since $Cn(\alpha) = Cn(\alpha')$ we have $\alpha' \wedge \beta \vdash \gamma$; it remains to check that $\neg\alpha' < \beta$, i.e. that not $\beta \leq \neg\alpha'$. Now, since $Cn(\alpha) = Cn(\alpha')$ we have $\neg\alpha' \vdash \neg\alpha$ so that by (E2) $\neg\alpha' \leq \neg\alpha$ so by transitivity since not $\beta \leq \neg\alpha$ we have not $\beta \leq \neg\alpha'$ as desired.

For And, suppose $\alpha \mid\!\sim \gamma$ and $\alpha \mid\!\sim \delta$; we want to show $\alpha \mid\!\sim \gamma \wedge \delta$. In the case $\alpha \vdash \gamma$, $\alpha \vdash \delta$ we have $\alpha \vdash \gamma \wedge \delta$ and so $\alpha \mid\!\sim \gamma \wedge \delta$ as desired. In the case $\alpha \vdash \gamma$, $\alpha \wedge \beta \vdash \delta$, $\neg\alpha < \beta$ we have $\alpha \wedge \beta \vdash \gamma \wedge \delta$ and we are done. The third case is similar. In the principal case that $\alpha \wedge \beta \vdash \gamma$, $\neg\alpha < \beta$, $\alpha \wedge \beta' \vdash \delta$, $\neg\alpha < \beta'$ we have $\alpha \wedge (\beta \wedge \beta') \vdash \gamma \wedge \delta$, and by (E3) we have either $\beta \leq \beta \wedge \beta'$ or $\beta' \leq \beta \wedge \beta'$ so that $\neg\alpha < \beta \wedge \beta'$, and thus $\alpha \mid\!\sim \gamma \wedge \delta$ as required.

For Or, suppose $\alpha \mid\!\sim \gamma$ and $\alpha' \mid\!\sim \gamma$; we want to show that $\alpha \vee \alpha' \mid\!\sim \gamma$. In the case $\alpha \vdash \gamma$, $\alpha' \vdash \gamma$ we clearly have $\alpha \vee \alpha' \vdash \gamma$ and so $\alpha \vee \alpha' \mid\!\sim \gamma$. In the case $\alpha \vdash \gamma$, $\alpha' \wedge \beta \vdash \gamma$, $\neg\alpha' < \beta$ we have $(\alpha \vee \alpha') \wedge \beta \vdash \gamma$ and $\neg(\alpha \vee \alpha') \vdash \neg\alpha'$ so $\neg(\alpha \vee \alpha') \leq \neg\alpha' < \beta$ and we are done. The third case is similar. In the case $\alpha \wedge \beta \vdash \gamma$, $\neg\alpha < \beta$, $\alpha' \wedge \beta' \vdash \gamma$, $\neg\alpha' < \beta'$ we have $(\alpha \vee \alpha') \wedge (\beta \wedge \beta') \vdash \gamma$ and $\neg(\alpha \vee \alpha') \vdash \neg\alpha, \neg\alpha'$ so that $\neg(\alpha \vee \alpha') \leq \neg\alpha, \neg\alpha'$ so $\neg(\alpha \vee \alpha') < \beta, \beta'$ so using (E3) $\neg(\alpha \vee \alpha') < \beta \wedge \beta'$ and we are done.

For Rational Monotony, suppose $\alpha \mid\!\sim \gamma$ and $\alpha \not\mid\!\sim \neg\delta$; we need to show $\alpha \wedge \delta \mid\!\sim \gamma$. In the case that $\alpha \vdash \gamma$ we have $\alpha \wedge \delta \vdash \gamma$ and so $\alpha \wedge \delta \mid\!\sim \gamma$ as desired. In the case $\alpha \wedge \beta \vdash \gamma$, $\neg\alpha < \beta$, we have $(\alpha \wedge \delta) \wedge \beta \vdash \gamma$ so that we need only check that $\neg(\alpha \wedge \delta) < \beta$. Now noting that $\alpha \wedge \neg(\alpha \wedge \delta) \vdash \neg\delta$, we conclude from our negative hypothesis that not $\neg\alpha < \neg(\alpha \wedge \delta)$, i.e., that $\neg(\alpha \wedge \delta) \leq \neg\alpha$. Thus using the hypotheses of the case, $\neg(\alpha \wedge \delta) \leq \neg\alpha < \beta$ and we conclude by transitivity.

For Consistency Preservation, suppose $\alpha \mid\!\sim \gamma \wedge \neg\gamma$, we need to show that $\alpha \vdash \gamma \wedge \neg\gamma$. Suppose $\alpha \nvdash \gamma \wedge \neg\gamma$; we derive a contradiction. By the definition of $\mid\!\sim$ there is a β with $\alpha \wedge \beta \vdash \gamma \wedge \neg\gamma$ and $\neg\alpha < \beta$. From the former we have $\beta \vdash \neg\alpha$ so by (E2), $\beta \leq \neg\alpha$ contradicting $\neg\alpha < \beta$.

Finally for Cut we need to show that if $\alpha \mid\!\sim \gamma$ and $\alpha \wedge \gamma \mid\!\sim \delta$ then $\alpha \mid\!\sim \delta$.Suppose that $\alpha \mid\!\sim \gamma$ and $\alpha \wedge \gamma \mid\!\sim \delta$. In the case that $\alpha \vdash \gamma$ we have $Cn(\alpha) = Cn(\alpha \wedge \gamma)$ so by Left Logical Equivalence it follows that $\alpha \mid\!\sim \delta$ as desired. Suppose then that $\alpha \wedge \beta \vdash \gamma$, $\neg\alpha < \beta$. If $\alpha \wedge \gamma \vdash \delta$ we have $\alpha \wedge \beta \vdash \delta$, $\neg\alpha < \beta$ so $\alpha \mid\!\sim \delta$ as desired. So suppose that $(\alpha \wedge \gamma) \wedge \varepsilon \vdash \delta$, $\neg(\alpha \wedge \gamma) < \varepsilon$. Then $\alpha \wedge (\beta \wedge \varepsilon) \vdash \delta$ so we need only check that $\neg\alpha < \beta \wedge \varepsilon$, for which it suffices to have $\neg\alpha < \beta$, $\neg\alpha < \varepsilon$. We have the former by the conditions of the case. As for the latter, we have by (E2) and our suppositions that $\neg\alpha \leq \neg(\alpha \wedge \gamma) < \varepsilon$ and we are done.

Theorem 3.3. Let $\mid\!\sim$ be any inference relation on L that satisfies the extended set of postulates. Then $\mid\!\sim$ is a comparative expectation inference relation, i.e., there is an expectation ordering $\leq$ over L such that $\mid\!\sim \; = \; \mid\!\sim_{\leq}$.

Proof. Define $\alpha \leq \beta$ iff either $\alpha \wedge \beta \in Cn(\varnothing)$ or $\neg(\alpha \wedge \beta) \not\mid\!\sim \alpha$.

Before beginning, we note that Consistency Preservation can also be expressed in the form: if $\neg\alpha \mid\!\sim \alpha$ then $\alpha \in Cn(\varnothing)$. For clearly $\neg\alpha \vdash \neg\alpha$ so that by Supraclassicality $\neg\alpha \mid\!\sim \neg\alpha$ so if $\neg\alpha \mid\!\sim \alpha$ then by And $\neg\alpha \mid\!\sim \alpha \wedge \neg\alpha$ so by Consistency Preservation $\neg\alpha \vdash \alpha \wedge \neg\alpha$ and thus $\alpha \in Cn(\varnothing)$.

To verify the dominance condition (E2), suppose $\alpha \vdash \beta$ and $\neg(\alpha \wedge \beta) \mid\!\sim \alpha$; we need to show that $\alpha \wedge \beta \in Cn(\varnothing)$. By the first supposition, $\neg(\alpha \wedge \beta) \vdash \neg\alpha$ so by Supraclassicality $\neg(\alpha \wedge \beta) \mid\!\sim \neg\alpha$. Hence by the second supposition using And, $\neg(\alpha \wedge \beta) \mid\!\sim \alpha \wedge \neg\alpha$ so that by Consistency Preservation $\alpha \wedge \beta \in Cn(\varnothing)$ as desired.

For the conjunction property (E3), suppose that $\neg(\alpha \wedge (\alpha \wedge \beta)) \mid\!\sim \alpha$ and $\neg(\beta \wedge (\alpha \wedge \beta)) \mid\!\sim \beta$; it will suffice to show that $(\alpha \wedge (\alpha \wedge \beta) \in Cn(\varnothing)$. The hypotheses give, by Left Logical Equivalence and And, that $\neg(\alpha \wedge \beta) \mid\!\sim \alpha \wedge \beta$ so by Consistency Preservation $\alpha \wedge \beta \in Cn(\varnothing)$ and thus $(\alpha \wedge (\alpha \wedge \beta) \in Cn(\varnothing)$ as required.

The tricky condition is transitivity (E1). Suppose for reductio ad absurdum that not $\alpha \leq \gamma$ whilst $\alpha \leq \beta$ and $\beta \leq \gamma$. Unpacking the definition of $\leq$ we thus have the hypotheses that $\neg(\alpha \wedge \gamma) \mid\!\sim \alpha$ and $\alpha \wedge \gamma \notin Cn(\varnothing)$; whilst either $\alpha \wedge \beta \in Cn(\varnothing)$ or $\neg(\alpha \wedge \beta) \not\mid\!\sim \alpha$, and either $\beta \wedge \gamma \in Cn(\emptyset)$ or $\neg(\beta \wedge \gamma) \not\mid\!\sim \beta$. We break the argument into three cases, of which the last is the principal, and delicate, one.

Case 1: Suppose $\alpha \wedge \beta \in Cn(\emptyset)$. Then clearly $\neg(\beta \wedge \gamma) \vdash \beta$ so by Supraclassicality $\neg(\beta \wedge \gamma) \mid\!\sim \beta$ so by the last hypothesis $\beta \wedge \gamma \in Cn(\emptyset)$, so we have $\alpha \wedge \gamma \in Cn(\emptyset)$ contradicting the second hypothesis.

Case 2: Suppose $\beta \wedge \gamma \in Cn(\emptyset)$. Then clearly $Cn(\neg\alpha) = Cn(\neg(\alpha \wedge \gamma))$ so by the first hypothesis and Left Logical Equivalence we have $\neg\alpha \mid\!\sim \alpha$, so by Consistency Preservation $\alpha \in Cn(\emptyset)$, and thus again $\alpha \wedge \gamma \in Cn(\emptyset)$ contradicting the second hypothesis.

Case 3: Suppose for the principal case that $\alpha \wedge \beta \notin Cn(\emptyset)$ and $\beta \wedge \gamma \notin Cn(\emptyset)$, so that $\neg(\alpha \wedge \beta) \not\mid\!\sim \alpha$ and $\neg(\beta \wedge \gamma) \not\mid\!\sim \beta$. First we observe that $\neg\alpha \vee \neg\beta \vee \neg\gamma \mid\!\sim \alpha$. For clearly we have $\alpha \wedge \neg\beta \vdash \alpha$, so $\alpha \wedge \neg\beta \mid\!\sim \alpha$ by Supraclassicality, so by Or, using the hypothesis of the proof that $\neg(\alpha \wedge \gamma) \mid\!\sim \alpha$, we have $\neg(\alpha \wedge \gamma) \vee (\alpha \wedge \neg\beta) \mid\!\sim \alpha$, so that by Left Logical Equivalence $\neg\alpha \vee \neg\beta \vee \neg\gamma \mid\!\sim \alpha$. From this we see that also $\neg\alpha \vee \neg\beta \vee \neg\gamma \mid\!\sim \beta$. For the hypothesis of the case that $\neg(\alpha \wedge \beta) \not\mid\!\sim \alpha$ tells us by Left Logical Equivalence that $(\neg\alpha \vee \neg\beta \vee \neg\gamma) \wedge \neg(\alpha \wedge \beta) \not\mid\!\sim \alpha$; so we may apply Rational Monotony to conclude that $\neg\alpha \vee \neg\beta \vee \neg\gamma \mid\!\sim \neg\neg(\alpha \wedge \beta)$ and so, by Right Weakening, $\neg\alpha \vee \neg\beta \vee \neg\gamma \mid\!\sim \beta$. From this we see that in turn $\neg\alpha \vee \neg\beta \vee \neg\gamma \mid\!\sim \gamma$. For the hypothesis of the case that $\neg(\beta \wedge \gamma) \not\mid\!\sim \beta$ tells us by Left Logical Equivalence that $(\neg\alpha \vee \neg\beta \vee \neg\gamma) \wedge \neg(\beta \wedge \gamma) \not\mid\!\sim \beta$ so we may again apply Rational Monotony to conclude that $\neg\alpha \vee \neg\beta \vee \neg\gamma \mid\!\sim \neg\neg(\beta \wedge \gamma)$ so that by Right Weakening $\neg\alpha \vee \neg\beta \vee \neg\gamma \mid\!\sim \gamma$. Putting these three points together by And we get $\neg\alpha \vee \neg\beta \vee \neg\gamma \mid\!\sim \alpha \wedge \beta \wedge \gamma$, so that by Consistency Preservation $\alpha \wedge \beta \wedge \gamma \in Cn(\emptyset)$ contradicting the hypothesis $\alpha \wedge \gamma \notin Cn(\emptyset)$.

It remains to show $\mid\!\sim \; = \; \mid\!\sim_{\leq}$. We recall the definition ($C\mid\!\sim$) of the latter: $\alpha \mid\!\sim_{\leq} \gamma$ iff either $\alpha \vdash \gamma$ or there is a β with $\alpha \wedge \beta \vdash \gamma$ and $\neg\alpha < \beta$, i.e. not $\beta \leq \neg\alpha$, i.e., using the definition of $\leq$ above, $\beta \wedge \neg\alpha \notin Cn(\emptyset)$ and $\neg(\beta \wedge \neg\alpha) \mid\!\sim \beta$.

Suppose first $\alpha \mid\!\sim \gamma$; we want to show $\alpha \mid\!\sim_{\leq} \gamma$. If $\neg\alpha \in Cn(\emptyset)$, then $\alpha \vdash \gamma$ and we are done. So suppose $\neg\alpha \notin Cn(\emptyset)$. Put $\beta = \neg\alpha \vee \gamma$. Clearly $\alpha \wedge \beta \vdash \gamma$. Also clearly $\beta \wedge \neg\alpha = (\neg\alpha \vee \gamma) \wedge \neg\alpha \dashv\vdash \neg\alpha \notin Cn(\emptyset)$ by hypothesis. Finally, $\neg(\beta \wedge \neg\alpha) \dashv\vdash \alpha \mid\!\sim \gamma \vdash \neg\alpha \vee \gamma = \beta$ so by hypothesized properties of $\mid\!\sim$, $\neg(\beta \wedge \neg\alpha) \mid\!\sim \beta$ as desired.

For the converse, suppose $\alpha \mid\!\sim_{\leq} \gamma$; we want to show $\alpha \mid\!\sim \gamma$. Now if $\alpha \vdash \gamma$ we have $\alpha \mid\!\sim \gamma$ by Supraclassicality and we are done. So we may suppose that there is a β with $\alpha \wedge \beta \vdash \gamma$, $\beta \wedge \neg\alpha \notin Cn(\emptyset)$ and $\neg(\beta \wedge \neg\alpha) \mid\!\sim \beta$. Noting that $\alpha \dashv\vdash (\alpha \vee \neg\beta) \wedge \alpha$ it will suffice to show $(\alpha \vee \neg\beta) \wedge \alpha \mid\!\sim \gamma$. By Rational Monotony, it will thus suffice to show both $\alpha \vee \neg\beta \mid\!\sim \gamma$ and $\alpha \vee \neg\beta \not\mid\!\sim \neg\alpha$.

For the former, the hypothesis $\neg(\beta \wedge \neg\alpha) \mid\!\sim \beta$ gives by Left Logical Equivalence that $\alpha \vee \neg\beta \mid\!\sim \beta$, whilst the hypothesis $\alpha \wedge \beta \vdash \gamma$ gives $(\alpha \vee \neg\beta) \wedge \beta \vdash \gamma$ so by Supraclassicality $(\alpha \vee \neg\beta) \wedge \beta \mid\!\sim \gamma$. Putting the two together with Cut gives $\alpha \vee \neg\beta \mid\!\sim \gamma$ as needed.

For the latter, i.e., $\alpha \vee \neg\beta \not\mid\!\sim \neg\alpha$, it will clearly suffice to show that $\alpha \vee \neg\beta \mid\!\sim \alpha$ but $\alpha \vee \neg\beta \not\mid\!\sim \alpha \wedge \neg\alpha$. By Consistency Preservation the latter follows from $\alpha \vee \neg\beta$ ª $\alpha \wedge \neg\alpha$, i.e. from the hypothesis that $\beta \wedge \neg\alpha \notin Cn(\emptyset)$. As for $\alpha \vee \neg\beta \mid\!\sim \alpha$ we already have $\alpha \vee \neg\beta \mid\!\sim \beta$ and also of course $\alpha \vee \neg\beta \mid\!\sim \alpha \vee \neg\beta$, so by And and Right Weakening, $\alpha \vee \neg\beta \mid\!\sim \alpha$, completing the proof.

Theorem 3.4. Expectation orderings and belief valuations generate precisely the same class of nonmonotonic inference relations.

Proof: Let f be any belief valuation into a belief scale (S,⁄). For all $\alpha,\beta \in L$, define $\alpha \leq_f \beta$, or more briefly $\alpha \leq \beta$ when the context is clear, to hold iff f(α)⁄f(β). Then $\leq_f$ is an expectation ordering over L. Clearly it satisfies (E1), i.e. it is transitive. It satisfies dominance, (E2), for if $\alpha \vdash \beta$ we have f(α)⁄f(β) as already shown in Section 3.2 so that $\alpha \leq \beta$ by the definition of $\leq$. For the conjunction property (E3) we have either f(α) = *min*(f(α),f(β)) = f(α ∧ β) or f(β) = *min*(f(α),f(β)) = f(α ∧ β) so either f(α)⁄ f(α ∧ β) or f(β)⁄f(α ∧ β) and thus either $\alpha \leq \alpha \wedge \beta$ or $\beta \leq \alpha \wedge \beta$ as desired. Moreover, it is clear from the definitions ($C\mid\!\sim$) and ($C\mid\!\sim_f$) that f generates the same inference relation as does the associated expecta-tion ordering $\leq_f = \leq$, for we have f(¬α) **Œ** f(β) iff $\neg\alpha < \beta$, so that $\alpha \mid\!\sim_f \gamma$ iff $\alpha \mid\!\sim_{\leq} \gamma$.

Conversely, let $\leq$ be an expectation ordering over L. We define a belief scale (S,⁄) and valuation f by taking a quotient structure over $(L,\leq)$ as follows. Put $\alpha \approx \beta$, for $\alpha,\beta \in L$, iff both $\alpha \leq \beta$ and $\beta \leq \alpha$. It is immediate from the conditions (E1) to (E3) that $\approx$ is an equivalence relation well-behaved under $\leq$ (i.e. $\alpha' \approx \alpha$ and $\beta \approx \beta'$ implies $\alpha \leq \beta$ iff

$\alpha' \leq \beta'$), so we may put S to be the set of all equivalence classes under ≈, put f: L → S to be the canonical valuation f(α) = |α| = {α' ∈ L: α ≈ α'}, and finally put f(α)⁄f(β) iff α ≤ β, observing that⁄ is thus well-defined. It is trivial to verify that⁄ is transitive, connected and antisymmetric, and that both (F1) and (F2) hold, so that (S,⁄) is indeed a belief scale and f is indeed a belief valuation into it. Moreover, it is again clear that ¬α < β iff f(¬α) **Œ** f(β), so that the inference relations $\mid\!\sim_{\leq}$ and $\mid\!\sim_{f}$ generated under definitions (C$\mid\!\sim$) and (C$\mid\!\sim_{f}$) respectively, are identical.

Theorem 3.5. Let ≤ be any expectation ordering. Then for all sentences α, γ, the following are equivalent:

(1) γ ∈ Cn({α} ∪ {β: ¬α < β})

(2) either α ⊢ γ or ¬α < α → γ

(3) either α ⊢ γ or α → ¬γ < α → γ

(4) either α ⊢ γ or α → γ is in the greatest cut of ≤ that does not contain ¬α.

Proof. We first prove that (1) entails (2) Suppose γ ∈ Cn({α} ∪ {β: ¬α < β}). Since Cn satisfies the deduction theorem we have α → γ ∈ Cn({β: ¬α < β}). But it is easy to see, using compactness of Cn and properties of ≤, that {β: ¬α < β} is either empty or else closed under Cn. So either α ⊢ γ or ¬α < α → γ as desired for (2).

Next we show that (2) implies (3). In the case α ⊢ γ we are done. So suppose ¬α < α → γ, and suppose for reductio ad absurdum that α → γ ≤ α → ¬γ. Then we have ¬α < α → γ ≤ α → ¬γ so by properties of ≤, ¬α < (α → γ) ∧ (α → ¬γ) ⊢ ¬α, so by properties of ≤, ¬α < ¬α, giving a contradiction.

Next step is to show that (3) entails (1). If α ⊢ γ, then (1) is immediate. In the other case, we recall that ¬α ⊢ α → ¬γ so by (E2) ¬α ≤ α → ¬γ < α → γ, so that α → γ ∈ {β: ¬α < β}and it follows that γ ∈ Cn({α} ∪ {β: ¬α < β}) as required for (1).

Finally, we need to show the equivalence of (4) to e.g. (2). It suffices to show that {β: ¬α < β} is in fact the greatest cut of ≤ that does not contain ¬α. Clearly it is a cut of ≤ that does not contain ¬α. And any superset of it most contain some β with β ≤ ¬α and so any greater cut must contain ¬α.

Theorem 3.8. For every expectation ordering $\leq$ over L, there is a nice preferential model $\mathcal{M} = \langle M, \models, \prec \rangle$ such that $\mid\!\sim_{\mathcal{M}} = \mid\!\sim_{\leq}$.

Proof. Put M to be the collection of all maximally consistent (under Cn) sets of propositions of L. For each $m \in M$, define $m \models \alpha$ to hold for propositions $\alpha \in L$, iff $\alpha \in m$. The key definition is that of the relation $\prec$ over M. For $m,n \in M$ we put $m \prec n$ iff for some proposition $\alpha \in L$ we have $\beta \in m$ for every $\beta \in L$ with $\alpha \leq \beta$, but $\alpha \notin n$. In other words, writing α^+ for $\{\beta \in L: \alpha \leq \beta\}$, iff for some proposition α we have $\alpha^+ \subseteq m$ but $\alpha \notin n$.

We need to verify that $\mathcal{M}$ has all the required properties. Some are immediate: that $\mathcal{M}$ is ample (by compactness of Cn), that $\models$ is classical, and that $\prec$ is irreflexive.

For the transitivity of $\prec$, suppose $m \prec n \prec p$. Then by the definition of $\prec$ there are $\alpha,\beta \in L$ such that $\alpha^+ \subseteq m$, $\alpha \notin n$, and $\beta^+ \subseteq n$, $\beta \notin p$. Now by the conditions on an expectation ordering, either $\alpha \leq \beta$ or $\beta \leq \alpha$. But the latter is impossible since $\beta^+ \subseteq n$ whilst $\alpha \notin n$. Since $\alpha \leq \beta$ we clearly have $\beta^+ \subseteq \alpha^+$, so $\beta^+ \subseteq m$ whilst $\beta \notin p$ which gives us $m \prec p$ as desired.

For the rankedness of $\prec$, suppose $m \prec n$ and not $p \prec n$; we need to show that $m \prec p$. Since $m \prec n$, there is an $\alpha \in L$ with $\alpha^+ \subseteq m$, $\alpha \notin n$. Since not $p \prec n$ we have that for every $\beta \in L$, if $\beta^+ \subseteq p$ then $\beta \in n$, so in particular since $\alpha \notin n$ we have $\alpha^+ \not\subseteq p$. Hence there is a $\gamma \in \alpha^+$ with $\gamma \notin p$. But since $\gamma \in \alpha^+$, clearly $\gamma^+ \subseteq \alpha^+$, so $\gamma^+ \subseteq m$. Thus $\gamma^+ \subseteq m$ whilst $\gamma \notin p$, so that $m \prec p$ as required.

Before verifying finitary stoppering and $\mid\!\sim_{\mathcal{M}} = \mid\!\sim_{\leq}$, we make the following remark: Let α be any proposition in L, and consider the set $C_{\leq}(\alpha) = \{\gamma \in L: \alpha \mid\!\sim_{\leq} \gamma\}$, i.e., recalling definition ($C\mid\!\sim$) in Section 3.1, $C_{\leq}(\alpha) = Cn(\{\alpha\} \cup \{\beta \in L: \neg\alpha < \beta)\}$. From Theorem 3.2 we know that $\mid\!\sim_{\leq}$ satisfies both Consistency Preservation and And, hence by the compactness of Cn we know that whenever α is consistent under Cn so is $C_{\leq}(\alpha)$.

For finitary stoppering, suppose $m \models \alpha$. We want to find an $n \in M$ with $n \models \alpha$, either $n \prec m$ or $n = m$, and such that there is no $p \in M$ with both $p \models \alpha$ and $p \prec n$. Since $m \models \alpha$ and $\models$ is classical, α is consistent under Cn and thus by the remark above so is $C_{\leq}(\alpha)$, so since M is ample there is an $m' \in M$ with $C_{\leq}(\alpha) \subseteq m'$. Clearly $\alpha \in m'$, i.e. $m' \models \alpha$. We define n thus: if m' ‹ m we put n = m'; if not m' ‹ m we put n = m. Then $n \vDash \alpha$ and either n ‹ m or n = m. It remains to show that there is no $p \in M$ with p ‹ n and $p \vDash \alpha$. Suppose there is such a p. Then by the definition of ‹, there is a γ with $\gamma^+ \sqsubseteq p$ and $\gamma \notin n$. Hence $\neg\alpha \notin \gamma^+$, i.e., not $\gamma \le \neg\alpha$, so $\neg\alpha < \gamma$, so $\gamma^+ \sqsubseteq C_{\le}(\alpha)$, so since $C_{\le}(\alpha) \sqsubseteq m'$ we have $\gamma^+ \sqsubseteq m'$.

In the case that m' ‹ m we have put n = m', which gives us $\gamma^+ \sqsubseteq n$, so $\gamma \in n$ giving a contradiction.

In the case that not m' ‹ m we have put n = m, and by the definition of ‹, for every proposition δ, $\delta^+ \sqsubseteq m'$ implies $\delta \in m$, so in particular since $\gamma^+ \sqsubseteq m'$ we have $\gamma \in m = n$, again giving a contradiction and completing the verification of stoppering.

It remains to show that $\mid\!\sim_{\le} = \mid\!\sim_M$. Recall the definitions: $\alpha \mid\!\sim_{\le} \gamma$ iff $\gamma \in Cn(\{\alpha\} \cup \{\beta \in L: \neg\alpha < \beta\})$ whilst $\alpha \mid\!\sim \gamma$ iff $m \vDash \gamma$ for all $m \in M$ with $m \vDash_{‹} \alpha$. Given that M contains all maximal sets of propositions consistent under Cn, it will suffice to show that for all $m \in M$, $m \vDash_{‹} \alpha$ iff both $m \vDash \alpha$ and also $m \vDash \beta$ for all β with $\neg\alpha < \beta$.

Suppose $m \vDash_{‹} \alpha$. Then clearly $m \vDash \alpha$. Suppose $\neg\alpha < \beta$; we want to show that $m \vDash \beta$. Since $m \vDash \alpha$, α is consistent under Cn, so by our earlier remark the set $C_{\le}(\alpha)$ is consistent under Cn and so is included in some maximally consistent n which, by the definition of M is in M. Now clearly $n \vDash \alpha$ and so since $m \vDash_{‹} \alpha$ we have not n ‹ m. Hence for all $\gamma \in L$, $\gamma^+ \sqsubseteq n$ implies $\gamma \in m$. But since $\neg\alpha < \beta$ we have $\neg\alpha < \delta$ for all δ with $\beta \le \delta$, which is to say that $\beta^+ \sqsubseteq C_{\le}(\alpha) \sqsubseteq n$, so that setting $\gamma = \beta$ we have $\beta \in m$, i.e., $m \vDash \beta$ as desired.

Conversely, suppose $m \vDash \alpha$ and $m \vDash \beta$ for all β with $\neg\alpha < \beta$. Suppose p ‹ m; we need to show $p \nvDash \alpha$. Since p ‹ m, there is a $\gamma \in L$ with $\gamma^+ \sqsubseteq p$, $\gamma \notin m$. From the latter, not $\neg\alpha < \gamma$, i.e., $\gamma \le \neg\alpha$, so from the former $\neg\alpha \in p$, so $p \nvDash \alpha$ and we are done.

Theorem 3.9. Let $\mathcal{M} = \langle M, \vDash, ‹ \rangle$ be any nice preferential model. Then there is an expectation ordering $\le$ over L such that $\mid\!\sim_{\le} = \mid\!\sim_{\mathcal{M}}$.

Proof. For each proposition $\alpha \in L$, define $g(\alpha) = \{m \in M: n \vDash \alpha$ for all $n \in M$ with n ‹ m$\}$. Note that $\{m: m \vDash_{‹} \neg\alpha\} \sqsubseteq g(\alpha)$ though not conversely. Define the relation $\le$ over L by the rule: $\alpha \le \beta$ iff $g(\alpha) \sqsubseteq g(\beta)$. We need to verify that $\le$ is an expectation ordering over L, with $\mid\!\sim_{\le} = \mid\!\sim_{\mathcal{M}}$.

Condition (E1), i.e. transitivity of $\leq$, is immediate from the definition. For (E2), dominance, suppose $\alpha \vdash \beta$. Then, by the definition of g, we have immediately that $g(\alpha) \subseteq g(\beta)$, so $\alpha \leq \beta$ as required. For (E3), the conjunction property, we need to show that either $g(\alpha) \subseteq g(\alpha \wedge \beta)$ or $g(\beta) \subseteq g(\alpha \wedge \beta)$. Suppose for reductio ad absurdum that neither holds. Then from the former there are $m,n \in M$ with $n \prec m$, $n \not\models \alpha \wedge \beta$, and for all $p \prec m$, $p \models \alpha$. And from the latter there are $m',n' \in M$ with $n' \prec m'$, $n' \not\models \alpha \wedge \beta$, and for all $p' \prec m'$, $p' \models \beta$. Hence from the former we have $n \not\models \beta$ and from the latter we have $n' \not\models \alpha$. Hence not $n \prec m'$ and not $n' \prec m$. The former gives us $m' \prec m$; for if not $m' \prec m$ then by rankedness using $n \prec m$ we get $n \prec m'$ giving a contradiction. Similarly, the latter gives us $m \prec m'$. Thus $m \prec m' \prec m$ so by transitivity $m \prec m$, contradicting irreflexivity of $\prec$.

It remains to show that $\mathrel{|\!\sim}_{\leq} = \mathrel{|\!\sim}_{M}$. Suppose first that $\alpha \mathrel{|\!\sim}_{\leq} \gamma$. Let $m \in M$ and suppose $m \models_{\prec} \alpha$. To show $\alpha \mathrel{|\!\sim}_{M} \gamma$ it will suffice to show $m \models \gamma$. Since $m \models_{\prec} \alpha$ we have $m \models \alpha$. Since $\models$ is classical, it follows from the definition of $\mathrel{|\!\sim}_{\leq}$ that to show $m \models \gamma$ it will suffice to show $m \models \beta$ for all $\beta \in L$ with $\neg\alpha < \beta$. Suppose $\neg\alpha < \beta$, i.e., not $\beta \leq \neg\alpha$, i.e., $g(\beta) \not\subseteq g(\neg\alpha)$. Then by the definition of g there is an $n \in M$ such that for some $p \prec n$, $p \models \alpha$ whilst for all $q \prec n$, $q \models \beta$. To show $m \models \beta$, it will thus suffice to show $m \prec n$. But if not $m \prec n$ then since $p \prec n$, rankedness gives $p \prec m$ which combined with $p \models \alpha$ contradicts $m \models_{\prec} \alpha$.

For the converse, suppose $\alpha \mathrel{|\!\not\sim}_{\leq} \gamma$. Then, since M is ample, there is an $m \in M$ with $m \models \alpha$, $m \models \beta$ for all β with $\neg\alpha < \beta$, but $m \not\models \gamma$. Hence $m \not\models \neg\alpha \vee \gamma$, so not $\neg\alpha < \neg\alpha \vee \gamma$, i.e., $\neg\alpha \vee \gamma \leq \neg\alpha$, so by the definition of $\leq$, $g(\neg\alpha \vee \gamma) \subseteq g(\neg\alpha)$. Clearly $m \models \alpha \wedge \neg\gamma$, so by stoppering there is a $p \in M$ with $p \models_{\prec} \alpha \wedge \neg\gamma$, i.e. $p \models_{\prec} \neg(\neg\alpha \vee \gamma)$, so by the definition of g, $p \in g(\neg\alpha \vee \gamma) \subseteq g(\neg\alpha)$ and also $p \models \alpha$. But since $p \in g(\neg\alpha)$, we have $q \models \neg\alpha$ for all $q \prec p$. Putting these two together gives us $p \models_{\prec} \alpha$, whilst since $p \models_{\prec} \alpha \wedge \neg\gamma$ we also have $p \not\models \gamma$, so finally $\alpha \mathrel{|\!\not\sim}_{M} \gamma$ as required.

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