Self-Scheduled $\mathcal{H}_\infty$ Control of Linear Parameter-Varying Systems: A Design Example

Pierre Apkarian
CERT-DERA,
31055 Toulouse Cedex, France
email: **apkarian@saturne.cert.fr**

Pascal Gahinet
INRIA Rocquencourt, BP 105
78153 Le Chesnay Cedex, France
email: **gahinet@colorado.inria.fr**

Greg Becker
Dept. of Mechanical Engineering
University of Cal. Berkeley
email: **beckerg@erg.berkeley.edu**

Abstract

This paper is concerned with the design of gain-scheduled controllers with guaranteed $\mathcal{H}_\infty$ performance for a class of Linear Parameter-Varying (LPV) plants. Here the plant state-space matrices are assumed to depend affinely on a vector θ of time-varying real parameters. Assuming real-time measurement of these parameters, they can be fed to the controller to optimize the performance and robustness of the closed-loop system. The resulting controller is therefore time-varying and automatically "gain-scheduled" along parameter trajectories.

Based on the notion of quadratic $\mathcal{H}_\infty$ performance, solvability conditions are obtained for continuous- and discrete-time systems. In both cases, the synthesis problem reduces to solving a system of Linear Matrix Inequalities (LMIs). The main benefit of this approach is to bypass most difficulties associated with more classical schemes such as gain interpolation or gain scheduling techniques.

The methodology presented in this paper is applied to the gain-scheduling of a missile autopilot. The missile has a large operating range and high angles of attack. The difficulty of the problem is reinforced by tight performance requirements as well as the presence of flexible modes that limit the control bandwidth.

1 Introduction

Following the terminology of (?), Linear Parameter-Varying Systems (**LPV**) are linear time-varying plants whose state-space matrices are fixed functions of some vector of varying parameters $\theta(t)$. Hence LPV systems are described by state-space equations of the form:

$$\begin{aligned} \dot{x} &= A(\theta(t))\, x + B(\theta(t))\, u \\ y &= C(\theta(t))\, x + D(\theta(t))\, u. \end{aligned}$$

From a practical point of view, LPV systems have at least two interesting interpretations:

- they can be viewed as linear time-invariant (**LTI**) plants subject to time-varying parametric uncertainty $\theta(t)$,
- they can be models of linear time-varying plants or result from the linearization of nonlinear plants along the trajectories of the parameter θ.

The first class of plants falls within the scope of the LTI robust control techniques described, e.g., in (?; ?; ?; ?; ?). For the second class of plants, the parameter θ is no longer uncertain and can often be measured in real time during system operation. Consequently, the control strategy can exploit the available measurements of θ to increase performance.

Until recently, hardly any theoretical framework existed for systematic gain-scheduling of LPV systems. A customary "heuristic" approach consisted of dividing the parameter space into areas of small variations where the plant was regarded as LTI. An LTI controller was then derived for each frozen value of the parameter θ, and the overall control law was constructed via gain scheduling or gain interpolation techniques. The main shortcoming of such schemes is to obliterate the time-varying nature of LPV plants (?; ?). As a result, there is no guaranty of satisfactory performance and robustness along all possible trajectories of $\theta(t)$. In fact, these gain-scheduled controllers are not even guaranteed to stabilize the LPV plant, except in the case of slowly-varying parameters (?).

An important and original contribution toward the elimination of this intrinsic weakness is found in (?; ?). The main thrust of this work is the development of a new controller structure dedicated to the gain scheduling task. This approach is restricted to LPV plants where measurements of $\theta(t)$ are available in real time, and constructs time-varying controllers with the same parameter dependence as the plant. That is,

$$\begin{aligned} \dot{x} &= A_K(\theta(t))\, x + B_K(\theta(t))\, y \\ u &= C_K(\theta(t))\, x + D_K(\theta(t))\, y \end{aligned}$$

where y denotes the vector of measurements and u the control inputs. By incorporating the parameter measurements, this controller adjusts to the variations in the plant dynamics in order to maintain stability and high performance along all trajectories $\theta(t)$. In other words, the controller is "self-scheduled," that is, automatically gain-scheduled with respect to θ.

A first technique for parameter-dependent controller synthesis is based on the Small Gain Theorem and applicable to LPV plants with an **LFT** (Linear Fractional Transformation) dependence on the parameter θ (?; ?). A drawback of the LFT formulation is that the variations of θ are allowed to be *complex*, thus introducing some conservatism when parameters are known to be real. Significant improvements can be obtained by using instead the notion of Quadratic $\mathcal{H}_\infty$ Performance. This notion is closely related to quadratic stability (?; ?) and seeks a single quadratic Lyapunov function to ensure $\mathcal{H}_\infty$-like performance for all possible trajectories of the LPV plant (?; ?). In this framework, the parameter is treated as real and should enter the state-space matrices of the LPV plant in an affine fashion. A detailed discussion of the conservatism of Small Gain and Quadratic $\mathcal{H}_\infty$ Performance approaches is provided in (?). The improvement essentially comes from the ability of the Quadratic $\mathcal{H}_\infty$ Performance formalism to handle *real* parameters. Note however that this approach remains conservative in the face of slowly varying parameters since quadratic Lyapunov techniques allow for arbitrarily fast parameter variations.

This paper considers the class of LPV plants where

- the state-space matrices depend affinely on the time-varying parameter θ,
- the measurements of θ are available in real time.

The focus is on the practical synthesis and applications of parameter-dependent controllers with Quadratic $\mathcal{H}_\infty$ Performance. A simple and unified framework is then proposed to handle both continuous- and discrete-time LPV systems. The derivation technique is an extension of (?) to LPV plants and makes extensive use of the Bounded Real Lemma (**BRL**) formulation of $\mathcal{H}_\infty$ performance. Under mild assumptions, the synthesis problem is reduced to solving a system of Linear Matrix Inequalities (**LMI**), which in turn falls within the scope of reliable and efficient convex optimization techniques (?; ?).

The remainder of the paper is organized as follows. Section ?? gives the notation and some definitions regarding LPV plants with an affine parameter dependence. Useful concepts and tools are recapped in Section ??, as well as the central notion of Quadratic $\mathcal{H}_\infty$ Performance. The synthesis problem is formally stated in Section ?? and solutions are characterized in Section ??. Finally, a physically motivated application is presented in Section ?? and solved using these techniques and the software package LMI-Lab, part of the *LMI Control Toolbox* for use with Matlab (?; ?).

2 Notations and Definitions

Throughout the paper, matrix transfer functions will be denoted $P(\sigma)$ where σ stands for the Laplace variable s in the continuous-time case and for the Z-transform variable z in the discrete-time case. Similarly, τ will stand for the time t ($\in \mathbb{R}^+$) in the continuous-time case and for the time samples k ($\in \mathbb{Z}^+$) in the discrete-time case. When sufficiently clear from the context, σ or the time-dependence τ will be omitted. The notation $\sigma.x$ stands for $\frac{dx}{dt}$ for continuous-time signals and for x_{k+1} for discrete-time signals.

For a stable real-rational transfer function $P(\sigma)$, the $\mathcal{H}_\infty$ norm is defined in the usual way:

- $\|P(s)\|_\infty = \sup_{\omega\in\mathbb{R}} \sigma_{max}(P(j\omega))$ for continuous-time systems
- $\|P(z)\|_\infty = \sup_{\phi\in[0,2\pi]} \sigma_{max}(P(e^{j\phi}))$ for discrete-time systems ,

where $\sigma_{max}(M)$ stands for the largest singular value of a matrix M. For real symmetric matrices M, the notation $M > 0$ stands for "positive definite" and indicates that all the eigenvalues of M are positive. Similarly, $M < 0$ means "negative definite", that is, all the eigenvalues of M are negative.

With these notations in mind, LPV systems are given in state-space form by the equations:

$$\sigma.x = A(\theta_\tau)x + B(\theta_\tau)u \tag{2.1}$$

$$y = C(\theta_\tau)x + D(\theta_\tau)u \tag{2.2}$$

where x, u, y denote the state, input, and output vectors, respectively, and θ_τ is a time-varying vector of real parameters. When "freezing" θ_τ to some given value θ, the LPV system (??)-(??) becomes an LTI system of transfer function:

$$G(\sigma) = D(\theta) + C(\theta)(\sigma I - A(\theta))^{-1}B(\theta).$$

Both LPV and LTI properties of such systems are interesting. LPV properties are *global* since they concern the behavior of the system along all possible trajectories θ_τ. In contrast, the LTI behavior is only *local* around some particular value θ of the parameters.

Matrix polytopes are defined as the convex hull of a finite number of matrices N_i with the same dimensions. That is,

$$\mathrm{Co}\ \{N_i\ :\ i = 1, ..., r\} := \{\sum_{i=1}^{r} \alpha_i N_i :\ \alpha_i \geq 0,\ \sum_{i=1}^{r} \alpha_i = 1\ \}$$

We restrict ourselves to LPV systems where

(a) the parameter dependence is affine, that is, the state-space matrices $A(\theta_\tau)$, $B(\theta_\tau)$, $C(\theta_\tau)$, $D(\theta_\tau)$ depend affinely on θ_τ,

(b) the time-varying parameter θ_τ varies in a polytope Θ of vertices $\theta_1, \theta_2, \ldots, \theta_r$. That is,

$$\theta_\tau \in\ \Theta := \mathrm{Co}\ \{\theta_1, \theta_2, \ldots, \theta_r\}.$$

These vertices correspond to all combinations of extremal parameter values.

Though not fully general, this description encompasses many practical situations. From (a)–(b), it is clear that the state-space matrices $A(\theta_\tau)$, $B(\theta_\tau)$, $C(\theta_\tau)$, $D(\theta_\tau)$ range in a polytope of matrices whose vertices are the images of the vertices $\theta_1, \theta_2, \ldots, \theta_r$. In other words,

$$\begin{pmatrix} A(\theta) & B(\theta) \\ C(\theta) & D(\theta) \end{pmatrix} \in \mathrm{Co} \left\{ \begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix} := \begin{pmatrix} A(\theta_i) & B(\theta_i) \\ C(\theta_i) & D(\theta_i) \end{pmatrix}\ :\ i = 1, ..., r \right\} \tag{2.3}$$

Because of this property, and with a slight abuse of language, we will refer to such LPV plants as "polytopic" in the sequel.

Definition 2.1 (Polytopic LPV Systems)
An LPV system is called "polytopic" when it can be represented by state-space matrices $A(\theta_\tau)$, $B(\theta_\tau)$, $C(\theta_\tau)$, $D(\theta_\tau)$ where the parameter vector θ_τ ranges over a fixed polytope, and the dependence of $A(.)$, $B(.)$, $C(.)$, $D(.)$ on θ is affine.

■

3 Useful Tools

A central tool in our formulation and derivation technique is the Bounded Real Lemma (**BRL**). Given an LTI system $G(\sigma)$ and a state-space realization $G(\sigma) = D + C(\sigma I - A)^{-1}B$ of G, we introduce the BRL map $\mathcal{B}^{\sigma}_{[A,B,C,D]}(.,.)$ defined for symmetric matrices X and positive scalars γ by:

$$\mathcal{B}^{s}_{[A,B,C,D]}(X, \gamma) := \begin{pmatrix} A^TX + XA & XB & C^T \\ B^TX & -\gamma\ I & D^T \\ C & D & -\gamma\ I \end{pmatrix} \quad \text{for } \sigma = s \tag{3.1}$$

$$\mathcal{B}^{z}_{[A,B,C,D]}(X, \gamma) := \begin{pmatrix} -X^{-1} & A & B & 0 \\ A^T & -X & 0 & C^T \\ B^T & 0 & -\gamma\ I & D^T \\ 0 & C & D & -\gamma\ I \end{pmatrix} \quad \text{for } \sigma = z. \tag{3.2}$$

With this notation in mind, the Bounded Real Lemma has the following general statement.

Theorem 3.1 (Bounded Real Lemma) *Given a continuous- or discrete-time transfer function $G(\sigma)$ of (not necessarily minimal) realization $G(\sigma) = D + C(\sigma I - A)^{-1}B$, the following statements are equivalent:*
(i) A is stable and $\|D + C(\sigma I - A)^{-1}B\|_\infty < \gamma$
(ii) there exists a positive definite solution X to the matrix inequality:

$$\mathcal{B}^\sigma_{[A,B,C,D]}(X, \gamma) < 0 \tag{3.3}$$

■

This theorem is only valid for LTI systems. However, the Bounded Real Lemma can be extended to LPV systems in conjunction with the notion of Quadratic $\mathcal{H}_\infty$ Performance.

Definition 3.2 (Quadratic $\mathcal{H}_\infty$ Performance) *The LPV system*

$$\begin{aligned} \sigma.x &= A(\theta_\tau)x + B(\theta_\tau)u && (3.4)\\ y &= C(\theta_\tau)x + D(\theta_\tau)u && (3.5) \end{aligned}$$

has Quadratic $\mathcal{H}_\infty$ Performance γ if and only if there exists a single matrix $X > 0$ such that

$$\mathcal{B}^\sigma_{[A(\theta),B(\theta),C(\theta),D(\theta)]}(X, \gamma) < 0 \tag{3.6}$$

for all admissible values of the parameter vector θ.

Then the Lyapunov function $V(x) = x^T X x$ establishes (global) stability and the $\mathcal{L}_2$ gain of the input/output map is bounded by γ. That is,

$$\|y\|_2 < \gamma\|u\|_2$$

along all possible parameter trajectories θ_τ.

■

For LTI systems (θ frozen), Quadratic Performance is equivalent to internal stability with an $\mathcal{H}_\infty$ bound γ on the transfer function $G(\sigma) = D(\theta) + C(\theta)(\sigma I - A(\theta))^{-1}B(\theta)$. However, this equivalence does not extend to general LPV systems since Quadratic $\mathcal{H}_\infty$ performance requires the existence of a fixed quadratic Lyapunov function for the entire operating range.

A difficulty with condition (??) is the infinite number of constraints it imposes. In the special case of polytopic LPV systems however, this condition can actually be reduced to a finite set of LMIs. Using convexity, it is easily shown that (??) will hold for all $(A(\theta), B(\theta), C(\theta), D(\theta))$ if and only if it holds at the vertices (A_i, B_i, C_i, D_i) for $i = 1, \ldots, r$. The following result formalizes this fact.

Theorem 3.3 (Vertex Property)
Consider a polytopic LPV plant described in state-space form by

$$\begin{aligned} \sigma.x &= A(\theta_\tau)x + B(\theta_\tau)u && (3.7)\\ y &= C(\theta_\tau)x + D(\theta_\tau)u && (3.8) \end{aligned}$$

with

$$\begin{pmatrix} A(\theta) & B(\theta) \\ C(\theta) & D(\theta) \end{pmatrix} \in \mathcal{P} := \mathrm{Co}\left\{ \begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix} : i = 1, ..., r \right\}. \tag{3.9}$$

The following statements are equivalent:

(i) this LPV system is stable with Quadratic $\mathcal{H}_\infty$ Performance γ,

(ii) there exists a single matrix $X > 0$ such that, for all $\begin{pmatrix} A(\theta) & B(\theta) \\ C(\theta) & D(\theta) \end{pmatrix} \in \mathcal{P}$,

$$\mathcal{B}^\sigma_{[A(\theta),B(\theta),C(\theta),D(\theta)]}(X, \gamma) < 0 \tag{3.10}$$

(iii) there exists $X > 0$ satisfying the system of LMIs:

$$\mathcal{B}^\sigma_{[A_i,B_i,C_i,D_i]}(X, \gamma) < 0, \qquad i = 1, 2, \ldots, r. \tag{3.11}$$

Proof: Statements (i) and (ii) are equivalent by definition, and the equivalence of (ii) and (iii) is a direct consequence of the fact that

$$\begin{pmatrix} A(\theta) & B(\theta) \\ C(\theta) & D(\theta) \end{pmatrix} = \sum_{i=1}^{r} \alpha_i \begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix}$$

with $\alpha_i \geq 0$ and $\sum_{i=1}^r \alpha_i = 1$.

■

4 Self-Scheduled $\mathcal{H}_\infty$ Control of LPV Systems

This section formulates an $\mathcal{H}_\infty$-like control problem for polytopic LPV systems. By $\mathcal{H}_\infty$ control, we mean control with a Quadratic $\mathcal{H}_\infty$ Performance as defined in Section ??. Assuming complete measurement of θ_τ, the controller is allowed to incorporate these measurements in the same LPV fashion as the plant. The resulting LPV controllers exploits all available information on θ_τ to adjust to the current plant dynamics (see Figure ??). This provides smooth and automatic gain-scheduling with respect to the varying parameters θ.

In the sequel, we consider LPV plants mapping exogenous inputs w and control inputs u to controlled outputs q and measured outputs y, i.e.,

$$\begin{array}{rcl} \sigma.x &=& A(\theta_\tau)x + B_1(\theta_\tau)w + B_2(\theta_\tau)u \\ q &=& C_1(\theta_\tau)x + D_{11}(\theta_\tau)w + D_{12}(\theta_\tau)u \\ y &=& C_2(\theta_\tau)x + D_{21}(\theta_\tau)w + D_{22}(\theta_\tau)u \end{array} \tag{4.1}$$

The plant is further assumed to be polytopic conformably to Definition ??. That is,

$$\begin{pmatrix} A(\theta_\tau) & B_1(\theta_\tau) & B_2(\theta_\tau) \\ C_1(\theta_\tau) & D_{11}(\theta_\tau) & D_{12}(\theta_\tau) \\ C_2(\theta_\tau) & D_{21}(\theta_\tau) & D_{22}(\theta_\tau) \end{pmatrix} \in \mathcal{P} := \mathrm{Co}\left\{ \begin{pmatrix} A_i & B_{1i} & B_{2i} \\ C_{1i} & D_{11i} & D_{12i} \\ C_{2i} & D_{21i} & D_{22i} \end{pmatrix},\ i = 1, 2, \ldots, r \right\} \tag{4.2}$$

where $A_i, B_{1i}, \ldots$ denote the values of $A(\theta_\tau), B_1(\theta_\tau), \ldots$ at the vertices $\theta_\tau = \theta_i$ of the parameter polytope. The problem dimensions are given by

$$A(\theta_\tau) \in \mathbb{R}^{n\times n},\ D_{11}(\theta_\tau) \in \mathbb{R}^{p_1\times m_1},\ D_{22}(\theta_\tau) \in \mathbb{R}^{p_2\times m_2} \tag{4.3}$$

We seek an LPV controller of the form:

$$\begin{array}{rcl} \sigma.x_K &=& A_K(\theta_\tau)x_K + B_K(\theta_\tau)y \\ u &=& C_K(\theta_\tau)x_K + D_K(\theta_\tau)y \end{array} \tag{4.4}$$

that guarantees some Quadratic $\mathcal{H}_\infty$ Performance γ for the closed-loop system of Figure ?? (see Definition ??). This will ensure that

- the closed-loop system is quadratically stable over $\mathcal{P}$,
- the $\mathcal{L}_2$-induced norm of the operator mapping w into q is bounded by γ for all possible trajectories θ_τ.

The controller order k is defined as the size of the matrix $A_K(\theta)$. With the notation

$$\Omega(\theta) := \begin{pmatrix} A_K(\theta) & B_K(\theta) \\ C_K(\theta) & D_K(\theta) \end{pmatrix},$$

the closed-loop system is described by the state-space equations:

$$\begin{array}{rcl} \sigma.x_{cl} &=& A_{c\ell}(\theta)x_{cl} + B_{c\ell}(\theta)w \\ q &=& C_{c\ell}(\theta)x_{cl} + D_{c\ell}(\theta)w \end{array} \tag{4.5}$$

where

$$\begin{array}{ll} A_{c\ell}(\theta) = A_0(\theta) + \mathcal{B}\,\Omega(\theta)\,\mathcal{C}, & B_{c\ell}(\theta) = B_0(\theta) + \mathcal{B}\,\Omega(\theta)\,\mathcal{D}_{21} \\ C_{c\ell}(\theta) = C_0(\theta) + \mathcal{D}_{12}\,\Omega(\theta)\,\mathcal{C}, & D_{c\ell}(\theta) = D_{11}(\theta) + \mathcal{D}_{12}\,\Omega(\theta)\,\mathcal{D}_{21} \end{array} \tag{4.6}$$

and

$$\begin{array}{c} A_0 = \begin{pmatrix} A(\theta) & 0 \\ 0 & 0_{k\times k} \end{pmatrix}, \qquad B_0 = \begin{pmatrix} B_1(\theta) \\ 0 \end{pmatrix}, \qquad C_0 = (C_1(\theta), 0) \\ \mathcal{B} = \begin{pmatrix} 0 & B_2 \\ I_k & 0 \end{pmatrix}, \qquad \mathcal{C} = \begin{pmatrix} 0 & I_k \\ C_2 & 0 \end{pmatrix}, \qquad \mathcal{D}_{12} = (0, D_{12}), \qquad \mathcal{D}_{21} = \begin{pmatrix} 0 \\ D_{21} \end{pmatrix}. \end{array} \tag{4.7}$$

The assumptions on the plant are as follows:

(A1) $D_{22}(\theta_\tau) = 0$ or equivalently $D_{22i} = 0$ for $i = 1, 2, \ldots, r$,

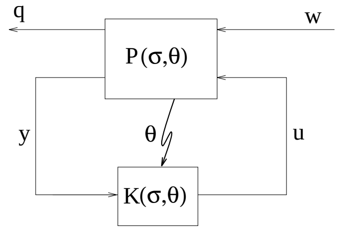

Figure 1: LPV Control of LPV Systems

(A2) $B_2(\theta_\tau), C_2(\theta_\tau), D_{12}(\theta_\tau), D_{21}(\theta_\tau)$ are parameter-independent or equivalently,

$$B_{2i} = B_2,\ C_{2i} = C_2,\ D_{12i} = D_{12},\ D_{21i} = D_{21} \text{for } i = 1, 2, \ldots, r. \tag{4.8}$$

(A3) The pairs $(A(\theta), B_2)$ and $(A(\theta), C_2)$ are quadratically stabilizable and quadratically detectable over Θ, respectively.

Quadratic detectability of $(A(\theta), C_2)$ is equivalent to the quadratic stabilizability of $(A(\theta)^T, C_2^T)$. By quadratic stabilizability of $(A(\theta), B_2)$ over Θ, we mean (for continuous-time systems) the existence of a matrix $X > 0$ such that

$$\mathcal{N}^T(A(\theta)^T X + XA(\theta))\mathcal{N} < 0 \ \text{ for all } \quad \theta \in \Theta, \tag{4.9}$$

where $\mathcal{N}$ denotes the null space of B_2^T. Using the affine parameter dependence and a convexity argument, this is equivalent to the existence of a matrix $X > 0$ satisfying

$$\mathcal{N}^T(A_i^T X + XA_i)\mathcal{N} < 0, \qquad i = 1, 2, \ldots, r. \tag{4.10}$$

The third assumption is necessary and sufficient to allow quadratic stabilization of the polytopic LPV plant by an output feedback LPV controller. Assumption **(A1)** can often be removed by redefining the plant output y. If Assumption **(A2)** is not satisfied, the computation of a solution requires solving a problem with an infinite number of constraints and is therefore not easily tractable (?). Yet, this difficulty can be alleviated by pre- and/or post-filtering of the control inputs u and/or the measured outputs y. Specifically, define a new control input $\tilde{u}$ and a new measured output $\tilde{y}$ by:

$$\left\{ \begin{array}{rcl} \sigma.x_u & = & A_u x_u + B_u \tilde{u} \\ u & = & C_u x_u \end{array} \right. \qquad \left\{ \begin{array}{rcl} \sigma.x_y & = & A_y x_y + B_y y \\ \tilde{y} & = & C_y x_y \end{array} \right. . \tag{4.11}$$

Assuming **(A2)**, the resulting LPV plant is described by

$$\begin{pmatrix} \sigma.x \\ \sigma.x_u \\ \sigma.x_y \end{pmatrix} = \begin{pmatrix} A(\theta_\tau) & B_2(\theta_\tau)C_u & 0 \\ 0 & A_u & 0 \\ B_y C_2(\theta_\tau) & 0 & A_y \end{pmatrix} \begin{pmatrix} x \\ x_u \\ x_y \end{pmatrix} + \begin{pmatrix} B_1(\theta_\tau) \\ 0 \\ B_y D_{21}(\theta_\tau) \end{pmatrix} w + \begin{pmatrix} 0 \\ B_u \\ 0 \end{pmatrix} \tilde{u} \tag{4.12}$$

$$q = \begin{pmatrix} C_1(\theta_\tau) & D_{12}(\theta_\tau)C_u & 0 \end{pmatrix} \begin{pmatrix} x \\ x_u \\ x_y \end{pmatrix} + D_{11}(\theta_\tau)w \tag{4.13}$$

$$\tilde{y} = \begin{pmatrix} 0 & 0 & C_y \end{pmatrix} \begin{pmatrix} x \\ x_u \\ x_y \end{pmatrix}. \tag{4.14}$$

Note that the control and the measurement matrices are now parameter-free. The filter bandwidth must be chosen larger than the desired system bandwidth. With this constraint, the proposed pre- and post-filtering will not significantly alter the original problem and preserve the conditions of assumption **(A3)**. Note also that whenever the plant model includes actuator and sensor dynamics then the control and measurement matrices are parameter-free. Hence, the proposed filtering operations are not restrictive in a practical perspective. In the sequel, we assume that these filtering operations have been performed beforehand so that **(A1)**-**(A3)** hold. Note also that similar definitions and transformations apply to the discrete-time case.

5 Characterization and Computation of Solutions

This section gives necessary and sufficient conditions for solvability of the Quadratic $\mathcal{H}_\infty$ Performance problem discussed in the previous section. As earlier, we assume that the LPV plant is polytopic subject to

(A1)-(A3) and that the parameter value θ_τ is measured in real time. If we restrict ourselves to LPV controllers, there is no loss of generality in assuming that the controller is polytopic as well. Indeed, if some LPV controller $\Omega(\theta)$ has quadratic performance γ, its values $\Omega_i := \Omega(\theta_i)$ at the vertices θ_i of the parameter box must satisfy the Bounded Real Lemma, which ensures that the *polytopic* LPV controller of vertices Ω_i yields the same performance as shown next.

Based on the Vertex Property (Theorem ??) of polytopic LPV systems, we devise the following constructive approach to LPV synthesis:

- first compute a matrix $X_{c\ell} > 0$ and adequate (LTI) $\mathcal{H}_\infty$ controllers Ω_i at the vertices θ_i of the parameter polytope
$$\Theta = \left\{ \sum_{i=1}^{r} \alpha_i \theta_i \;:\; \alpha_i \geq 0; \; \sum_{i=1}^{r} \alpha_i = 1 \right\}$$
- Define the LPV controller $\Omega(\theta)$ as an "interpolant" of the vertex controllers Ω_i. Here the interpolation is based on the position of θ in the polytope Θ (with respect to the vertices θ_i). More precisely, along some trajectory
$$\theta(\tau) = \sum_{i=1}^{r} \alpha_i(\tau)\theta_i$$
of the parameter, the state-space matrices $A_K(\theta), B_K(\theta), C_K(\theta), D_K(\theta)$ of $\Omega(\theta)$ will be given by
$$\begin{pmatrix} A_K(\theta) & B_K(\theta) \\ C_K(\theta) & D_K(\theta) \end{pmatrix} := \sum_{i=1}^{r} \alpha_i \Omega_i = \sum_{i=1}^{r} \alpha_i \begin{pmatrix} A_{Ki} & B_{Ki} \\ C_{Ki} & D_{Ki} \end{pmatrix}.$$

The first step enforces stability and $\mathcal{H}_\infty$ performance over the entire parameter polytope Θ and for arbitrary parameter variations (see Theorem ??). It must be emphasized that a naive interpolation of LTI controllers would generally fail to ensure stability and performance over Θ. Our approach is valid only because a single Lyapunov function $V(x) = x^T X_{c\ell} x$ is used over the entire operating range. While the vertex controllers Ω_i can be computed off-line, the LPV controller matrices $A(\theta), B(\theta), C(\theta), D(\theta)$ must be updated in real time based on the parameter measurement θ_τ. The notion of interpolating LPV controller is formalized in the next theorem.

Theorem 5.1 *Consider a continuous- or discrete-time LPV polytopic plant (??) and assume* **(A1)-(A3)**. *Given some positive scalar γ, the following statements are equivalent:*

(i) there exists a k-th order LPV controller solving the Quadratic $\mathcal{H}_\infty$ Performance problem with bound γ,

(ii) there exist some $(n+k) \times (n+k)$ positive definite matrix X_{cl} and k-th order LTI controllers $\Omega_i = \begin{pmatrix} A_{Ki} & B_{Ki} \\ C_{Ki} & D_{Ki} \end{pmatrix}$ such that
$$\mathcal{B}^{\sigma}_{[A_{c\ell}(\theta_i), B_{c\ell}(\theta_i), C_{c\ell}(\theta_i), D_{c\ell}(\theta_i)]}(X_{cl}, \gamma) < 0 \qquad (i = 1, 2, \ldots, r) \tag{5.1}$$
where $\theta_1, \ldots, \theta_r$ are the vertices of the parameter polytope and $A_{c\ell}(\theta_i) = A_0(\theta_i) + \mathcal{B}\Omega_i\mathcal{C}, \ldots$ with the notation (??).

If (i) or (ii) is satisfied, a possible choice of LPV controller is the polytopic controller given in state-space form by
$$\Omega(\theta) := \sum_{i=1}^{r} \alpha_i \Omega_i = \sum_{i=1}^{r} \alpha_i \begin{pmatrix} A_{Ki} & B_{Ki} \\ C_{Ki} & D_{Ki} \end{pmatrix} \tag{5.2}$$
where $(\alpha_1, \ldots, \alpha_r)$ is any solution of the convex decomposition problem:
$$\theta = \sum_{i=1}^{r} \alpha_i \theta_i. \tag{5.3}$$

Proof: (*Necessity part*) From Definition ??, Quadratic $\mathcal{H}_\infty$ Performance γ is equivalent to the existence of a positive definite matrix $X_{c\ell} \in \mathbb{R}^{(n+k)\times(n+k)}$ such that for all $\theta \in \Theta$:
$$\mathcal{B}^{\sigma}_{[A_{c\ell}(\theta), B_{c\ell}(\theta), C_{c\ell}(\theta), D_{c\ell}(\theta)]}(X_{c\ell}, \gamma) < 0. \tag{5.4}$$

Selecting $\theta := \theta_i$ and using the notation $\Omega_i := \Omega(\theta_i)$ immediately yields *(ii)*.

(*Sufficiency part*) Assume now that *(ii)* holds for $X_{c\ell} > 0$ and some Ω_i's, and consider the polytopic LPV controller $\Omega(\theta) = \sum_{i=1}^{r} \alpha_i \Omega_i$. Since this controller makes the closed-loop system (??) polytopic, the Vertex Property of Theorem ?? is applicable and guarantees for all $\theta \in \Theta$ that

$$\mathcal{B}^{\sigma}_{[A_{c\ell}(\theta), B_{c\ell}(\theta), C_{c\ell}(\theta), D_{c\ell}(\theta)]}(X_{c\ell}, \gamma) < 0. \tag{5.5}$$

This exactly says that the closed-loop system has Quadratic Performance γ over the parameter range Θ.

■

The core of the LPV synthesis problem is therefore to compute a single Lyapunov matrix $X_{c\ell} > 0$ and LTI controllers Ω_i which satisfy the system of LMIs (??). Here the difficulty lies in the fact that the same Lyapunov function should be used for all vertices. Fortunately, computing an adequate $X_{c\ell}$ (if any) reduces to solving some system of LMIs. Once the Lyapunov matrix $X_{c\ell}$ is determined, adequate vertex controllers Ω_i are easily deduced by solving the corresponding Bounded Real Lemma inequality (??) at each vertex. The LMI-based solvability conditions are given in the following theorem.

Theorem 5.2 (Convex Solvability Conditions)
Consider a continuous LPV polytopic plant (??) and assume **(A1)-(A3)**. *Let $\mathcal{N}_R$ and $\mathcal{N}_S$ denote bases of the null space of (B_2^T, D_{12}^T) and (C_2, D_{21}), respectively.*

There exists an LPV controller guaranteeing Quadratic $\mathcal{H}_\infty$ Performance γ along all parameter trajectories in the polytope

$$\Theta = \left\{ \sum_{i=1}^{r} \alpha_i \theta_i \; : \; \alpha_i \geq 0; \; \sum_{i=1}^{r} \alpha_i = 1 \right\}$$

if and only if there exist two symmetric matrices (R, S) in $\mathbb{R}^{n \times n}$ satisfying the system of $2r+1$ LMIs:

$$\left(\begin{array}{c|c} \mathcal{N}_R & 0 \\ \hline 0 & I \end{array}\right)^T \left(\begin{array}{cc|c} A_i R + R A_i^T & R C_{1i}^T & B_{1i} \\ C_{1i} R & -\gamma I & D_{11i} \\ \hline B_{1i}^T & D_{11i}^T & -\gamma I \end{array}\right) \left(\begin{array}{c|c} \mathcal{N}_R & 0 \\ \hline 0 & I \end{array}\right) < 0 \qquad (i = 1, \ldots, r) \tag{5.6}$$

$$\left(\begin{array}{c|c} \mathcal{N}_S & 0 \\ \hline 0 & I \end{array}\right)^T \left(\begin{array}{cc|c} A_i^T S + S A_i & S B_{1i} & C_{1i}^T \\ B_{1i}^T S & -\gamma I & D_{11i}^T \\ \hline C_{1i} & D_{11i} & -\gamma I \end{array}\right) \left(\begin{array}{c|c} \mathcal{N}_S & 0 \\ \hline 0 & I \end{array}\right) < 0 \qquad (i = 1, \ldots, r) \tag{5.7}$$

$$\begin{pmatrix} R & I \\ I & S \end{pmatrix} \geq 0. \tag{5.8}$$

Moreover, there exists k-th order LPV controllers solving the same problem if and only if R, S further satisfy the rank constraint

$$rank\,(\,I - RS\,) \leq k. \tag{5.9}$$

Proof: This is a straightforward application of Theorem ?? above and of the main results of (?). Note that Assumption **(A3)** is equivalent to the existence of R, S making the $(1,1)$ blocks in (??)-(??) negative subject to (??).

■

For the discrete-time LPV systems, the $2r$ constraints (??)-(??) should be replaced by discrete-time Riccati inequalities. The rank constraint (??) is immaterial in the full-order case ($k = n$). Hence R, S are only constrained by LMIs and the problem of finding a feasible pair (R, S) is convex as well as that of minimizing γ subject to (??)-(??). Once adequate matrices R, S are computed, the Lyapunov matrix $X_{c\ell}$ common to all inequalities (??) and the vertex controllers Ω_i are obtained along the lines of (?; ?). The construction of $X_{c\ell}$ from R, S proceeds as follows:

- compute full-rank matrices $M, N \in \mathbb{R}^{n \times k}$ such that

$$M N^T = I - RS, \tag{5.10}$$

- compute $X_{c\ell}$ as the unique solution of the linear matrix equation $\Pi_2 = X_{c\ell} \Pi_1$ where

$$\Pi_2 := \begin{pmatrix} S & I \\ N^T & 0 \end{pmatrix}; \qquad \Pi_1 := \begin{pmatrix} I & R \\ 0 & M^T \end{pmatrix}.$$

Given $X_{c\ell}$, a possible choice of vertex controllers $\Omega_i = \begin{pmatrix} A_{Ki} & B_{Ki} \\ C_{Ki} & D_{Ki} \end{pmatrix}$ is any solution of the matrix inequality

$$\mathcal{B}^{\sigma}_{[A_{c\ell}(\theta_i),B_{c\ell}(\theta_i),C_{c\ell}(\theta_i),D_{c\ell}(\theta_i)]}(X_{cl},\gamma) < 0. \tag{5.11}$$

This LMI in Ω_i can be solved by the same convex optimization algorithms. However, where no further constraint is placed on Ω_i it is more efficient to solve (??) by direct linear algebra techniques such as those discussed in (?).

6 Self-Scheduled $\mathcal{H}_\infty$ Control of a Missile

This section presents a realistic application of the LPV synthesis technique to the control of a missile pitch axis.

The missile dynamics are highly dependent on the angle of attack α, the air speed V and the altitude H. These three variables completely define the flight conditions (operating point) of the missile. They are assumed to be measured in real-time. Based on the linearization of the missile equations around its flight conditions, an LPV model can be developed for this problem. The gain-scheduling technique presented above is then readily applicable to the design of a self-scheduled autopilot.

We first discuss the modelling and the open-loop characteristics of the missile pitch axis dynamics. This preliminary analysis shows the relevance of LPV design for this system. Next we describe the control problem and outline the control law computation. Finally, the resulting gain-scheduled controller is validated by various simulations.

6.1 Open-loop analysis

In the sequel it is implicitly assumed that the pitch, yaw and roll axes are decoupled. Although this assumption ignores some coupling phenomena in the missile, it greatly simplifies the design procedure while retaining the main difficulties of the problem. The pitch axis model of the missile is depicted in Figure ??.

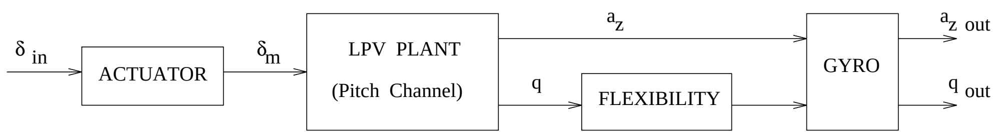

Figure 2: Missile pitch axis

The linearized dynamics of the missile (LPV part) are described by a state-space representation in the form

$$\left\{ \begin{array}{rcl} \begin{pmatrix} \dot\alpha \\ \dot q \end{pmatrix} & = & \begin{pmatrix} -Z_\alpha & 1 \\ -M_\alpha & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ q \end{pmatrix} + \begin{pmatrix} 0 \\ M_{\delta_m} \end{pmatrix} \delta_m \\ \begin{pmatrix} a_z \\ q \end{pmatrix} & = & \begin{pmatrix} -Z_\alpha V & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ q \end{pmatrix}, \end{array} \right. \tag{6.1}$$

where α denotes the angle of attack, q the pitch rate, a_z the vertical acceleration, δ_m the fin deflection, and V the air speed. The parameters Z_α, M_α and M_{δ_m} are functions of the flight condition (α, V, H) and are therefore available in real-time. The LPV model (??) can be further simplified by incorporating the parameter dependence of the B and C matrices into the self-scheduled controller. Hence, the corresponding entries are considered as normalized to 1, hereafter. In this simpler form, the LPV description of the missile satisfies Assumptions (A1)-(A3) and our synthesis technique is directly applicable. The parameter vector of the LPV plant (??) is denoted as $\theta(t)^T := (\, Z_\alpha(t) \quad M_\alpha(t)\,)$.

Though the parameter dependence of the original plant has been simplified, the control of the missile dynamics remains a hard task. Indeed, the parameter M_α and Z_α abruptly change as functions of the flight condition and range over a large parameter domain where the stability properties of the missile vary greatly. Moreover, the system can switch between stability and high instability regions. Analyzed as an LTI plant, the characteristic polynomial of the plant (??) reads $s^2 + Z_\alpha s + M_\alpha$. It follows that the plant is LTI unstable whenever M_α is negative. The parameter Z_α has a less dramatic impact and influences the damping.

The missile speed varies between $Mach$ 0.5 and $Mach$ 4. The altitude belongs to the interval $[0,\ 18000]$ $(m.)$ and the angle of attack evolves between 0 and 40 degrees. This wide variety of operating conditions results in a large range of parameter values. Moreover, a small increase in the angle of attack may induce large parameters variations. The parameter range Θ is a box defined by $[-365,\ 380]$ for M_α and $[0.35,\ 4.35]$ for Z_α.

6.1.1 Actuator, Gyro and Flexibility Descriptions

In addition to the LPV plant dynamics (??), tail-deflection actuators, gyros, and bending flexible modes must be incorporated to the model (see Figure ??). The gyros and actuators are adequately represented by second-order and third-order transfer functions, respectively. Meanwhile, flexible modes are modelled as a multiplicative output LTI perturbation affecting the measurement of the pitch rate q. The frequency responses of these components are plotted in Figure ??.

6.2 LPV control structure

We use the two-degree-of-freedom synthesis structure of Figure ??. This structure includes a feedforward part $K_2(\theta)$ and a feedback $K_1(\theta)$ and is potentially more powerful to achieve strong performance requirements than the usual unity feedback structure.

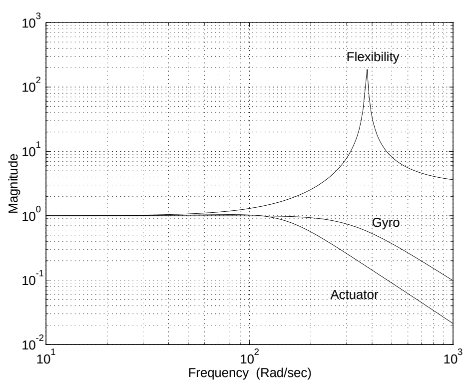

Figure 3: Actuator, Gyro and Flexible Modes Frequency Responses

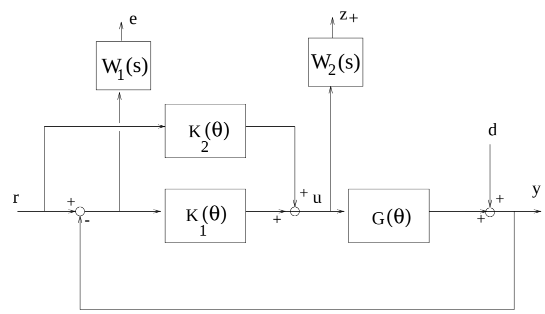

Figure 4: Two-Degrees-of-Freedom Control Structure

The design procedure parallels the customary $\mathcal{H}_\infty$-based loop shaping procedure except that the operators to be minimized are now parameter-dependent. We choose a mixed sensitivity criterion adapted to the control structure in Figure ??. The performance objectives are expressed through the sensitivity operator $S(\theta)$ while additive robustness is captured by the operator $K_1(\theta)S(\theta)$.

The self-scheduled $\mathcal{H}_\infty$ control problem consists in finding an LPV controller, denoted in operator form as

$$K(\theta) := (K_1(\theta), K_2(\theta)) ,$$

that satisfies the following objectives for *all admissible trajectories* $\theta(t)$ in Θ:

- internal stability of the closed-loop system of Figure ??,
- minimization of the $\mathcal{L}_2$-induced gain of the closed-loop operator between $\begin{pmatrix} r \\ d \end{pmatrix}$ and $\begin{pmatrix} e \\ z_+ \end{pmatrix}$, where W_1 and W_2 are linear time-invariant weights.

The full control structure together with the weights appears in Figure ??. It is readily shown that the unweighted LPV plant $P(\theta)$ is completely defined by the state-space equations:

$$\begin{pmatrix} \dot{x} \\ z_+ \\ e \\ r \end{pmatrix} = \begin{pmatrix} A(\theta(t)) & 0 & 0 & B \\ 0 & 0 & 0 & I \\ -C & I & -I & 0 \\ 0 & I & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ r \\ d \\ u \end{pmatrix}, \tag{6.2}$$

where the measurement vector is $\begin{pmatrix} e \\ r \end{pmatrix}$ with $e := r - y$ and $A(\theta)$ ranges in the polytope $\mathrm{Co}\{A_i\ i = 1, \ldots, 4\}$. The vertices A_i's are the values of $A(\theta)$ at the four vertices of the parameter box:

$$\theta_1 := \begin{pmatrix} {Z_\alpha}^{min} \\ {M_\alpha}^{min} \end{pmatrix}, \quad \theta_2 := \begin{pmatrix} {Z_\alpha}^{max} \\ {M_\alpha}^{min} \end{pmatrix}, \quad \theta_3 := \begin{pmatrix} {Z_\alpha}^{min} \\ {M_\alpha}^{max} \end{pmatrix}, \quad \theta_4 := \begin{pmatrix} {Z_\alpha}^{max} \\ {M_\alpha}^{max} \end{pmatrix}.$$

The synthesis structure of Figure ?? is the weighted version of $P(\theta)$. It is also LPV and ranges in a polytope easily obtained as the convex hull of the weighted vertices of $P(\theta)$.

6.3 Weight selection and synthesis

The selection of weights is based on a frozen-time analysis of the LPV system and follows the same lines as classical $\mathcal{H}_\infty$ synthesis. They must ensure adequate settling-time (0.25 sec.) and high frequency gain attenuation. Finally, based on different closed-loop analyses the following filters were adopted:

- the sensitivity weight is described as

$$W_1(s) = \begin{pmatrix} W_{1\alpha}(s) & 0 \\ 0 & W_{1q}(s) \end{pmatrix},$$

  where $W_{1\alpha}(s)$ is a second-order low-pass filter and $W_{1q}(s)$ is a simple static gain.

- the robustness weight $W_2(s)$ is a 6th-order Chebycheff high-pass filter.

The corresponding frequency responses are shown in Figure ??.

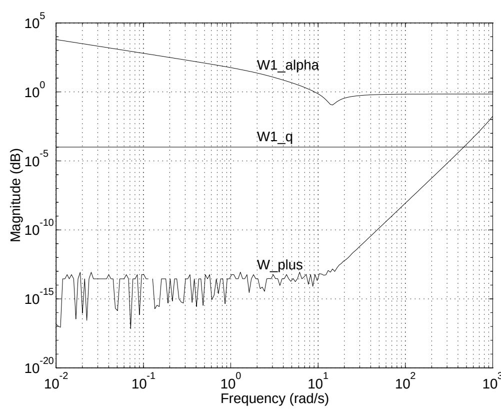

Figure 5: Weighting Functions

The LPV synthesis structure of Figure ?? being fixed, the LMIs (??)-(??) are solved using LMI-LAB. Solving (??)-(??) for R, S yielded a performance level of $\gamma = 1.1$ after 100 iterations of the algorithm. Given a solution (R, S), an LPV polytopic controller $\{\Omega_1, \ldots, \Omega_4\}$ can be constructed as described in (??). The vertex controllers Ω_i's are obtained as solutions to the convex problem (??). Due to a rank loss of 1 in (??), a 9th order controller was computed. The Ω_i's are $(9+1, 9+4)$ matrices with the partitioning (??). Finally, a formal expression of the LPV controller is derived by solving the convex decomposition problem (??) for the α_i's in the case of a 2-dimensional box Θ. The following formulas for the state-space data of the LPV controller are readily obtained:

$$\begin{pmatrix} A_K \begin{pmatrix} Z_\alpha(t) \\ M_\alpha(t) \end{pmatrix} & B_K \begin{pmatrix} Z_\alpha(t) \\ M_\alpha(t) \end{pmatrix} \\ C_K \begin{pmatrix} Z_\alpha(t) \\ M_\alpha(t) \end{pmatrix} & D_K \begin{pmatrix} Z_\alpha(t) \\ M_\alpha(t) \end{pmatrix} \end{pmatrix} := \sum_{i=1}^{r} \alpha_i(t) \begin{pmatrix} A_{Ki} & B_{Ki} \\ C_{Ki} & D_{Ki} \end{pmatrix}, \tag{6.3}$$

where

$$\alpha_1 = xy\,, \qquad \alpha_2 = (1-x)y\,, \qquad \alpha_3 = x(1-y)\,, \qquad \alpha_4 = (1-x)(1-y) \tag{6.4}$$

with

$$x := \frac{Z_\alpha{}^{max} - Z_\alpha}{Z_\alpha{}^{max} - Z_\alpha{}^{min}}\ , \qquad y := \frac{M_\alpha - M_\alpha{}^{min}}{M_\alpha{}^{max} - M_\alpha{}^{min}}.$$

It is easy to check that the α_i's in (??) are convex coordinates as they satisfy $0 \le \alpha_i \le 1$ and $\sum_{i=1}^{4} \alpha_i = 1$. Now recalling that the parameters Z_α and M_α are known functions of the flight condition through (α, V, H), the controller dynamics can be easily updated in real-time according (??)-(??).

6.4 Assessment of the LPV controller

The resulting LPV controller is now tested through different time-domain simulations. Various parameter trajectories are considered in Figure ??. Trajectory #1 is a smooth trajectory ranging over the (frozen-time) stable and unstable regions of the missile. In contrast, Trajectory #2 is non-smooth and is intended to test the reaction of the LPV control law in the face of abrupt parameter changes. For trajectory #3, the second coordinate of the parameter is corrupted by noise up to 10 percent of its nominal value. This last test is of fundamental importance for potential real-world implementations of LPV controllers. Indeed, noise or uncertainties are always present and gain-scheduled controllers must be insensitive to small parameter fluctuations.

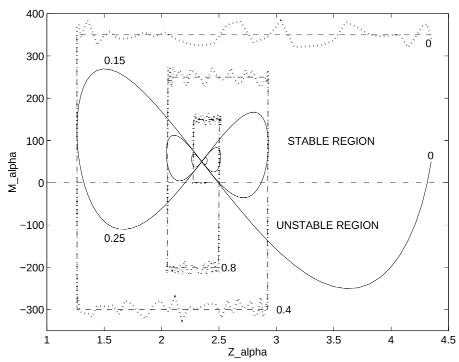

Figure 6: Parameter Trajectories
#1: solid - #2: dashed - #3: dotted

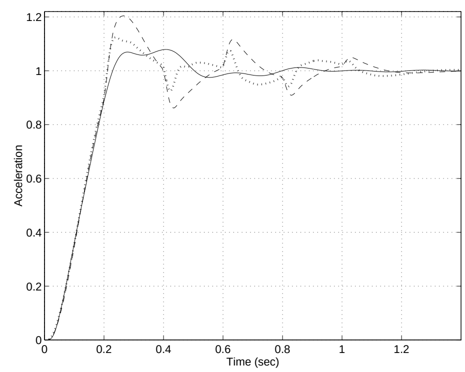

Figure 7: LPV Simulations
#1: solid - #2: dashed - #3: dotted

The corresponding LPV simulations are presented in Figures ??. It is seen that the LPV controller performs well and withstands transitions between stable, badly damped, and unstable regions of the parameter box without any loss of stability or performance. The settling-time and overshoot specifications are clearly met for Trajectory #1. A slight overshoot degradation occurs for Trajectory #2. This degradation essentially stems from the corner points of the parameter trajectory where one has an infinite derivative of the parameter. However, from a practical point of view, the response is still satisfactory since infinite derivatives are not realistic. Surprisingly, the overshoot is improved when 10 % of relative noise is introduced on the parameter measurement M_α. Accurate measurements of the varying parameter are therefore not required for the efficiency of the LPV controller. Since in practical implementations perfect measurement or estimation are never achieved, this encouraging result motivates other applications of the proposed technique.

Summing up, the LPV controller behaved as predicted by the theory and managed to meet tight specifications with little apparent conservatism.

7 Conclusion

This paper has considered the design of self-scheduled controllers with guaranteed $\mathcal{H}_\infty$ performance. The proposed technique applies to LPV plants (continuous- or discrete-time) assuming an affine parameter dependence of their state-space representation. Under mild assumptions, the overall synthesis problem amounts to solving a convex program. The major advantage of the approach is to bypass most critical aspects associated with gain interpolation or gain scheduling techniques in

- accounting for the time-varying nature of the plant,
- handling the whole parameter range of the plant in one "shot", that is, without extensive simulations.

The potential and performance of the approach has been demonstrated on a realistic missile pilot problem. The design procedure is very similar in spirit to classical $\mathcal{H}_\infty$ designs. Adequate filter selections allow to meet tight robustness and performance specifications over the missile operating domain. More importantly, such specifications were shown to be maintained even for rapidly changing parameters.

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