MODEL REDUCTION OF NONLINEAR DIFFERENTIAL-ALGEBRAIC EQUATIONS

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Abstract:
In this work, a computational method to compute balanced realizations for nonlinear differential-algebraic equation systems is derived. The work is a generalization of an earlier work for nonlinear control-affine systems, and is based on analysis of the controllability and observability functions.

Keywords: Differential-Algebraic Equations, Model Reduction, Computational methods

1. INTRODUCTION

During the last decades, Differential-Algebraic Equations (DAEs) have been extensively studied, see for example (Dai, 1989; Brenan *et al.*, 1996; Kumar and Daoutidis, 1999; Kunkel and Mehrmann, 2006) and references therein. One reason is the natural formulation of many applications using this kind of system models. The growing use of objected-oriented modeling languages such as MODELICA also increases the interest in these models, since most often the output from such tools is equations in this form.

The topic of this paper is model reduction of nonlinear DAEs. The basic idea in model reduction is to analyze the system and decide which states of the system are most important for the input-output behavior. The analysis is based on the Hankel operator and studies the energy in the input and output signals. States difficult to control, *i.e.*, requiring a lot of control effort, while not contributing so much to the output energy, will affect the input-output behavior less than other states and can therefore be removed. To analyze the energy in the input and output signals, the controllability and observability functions will be used.

The controllability function is the minimum amount of control energy required to reach a specific state in infinite time, and hence it measures how hard a given state is to reach. It is defined as the solution to an optimal control problem. The observability function measures the energy in the output signal when the system is released from a given state and the control input is equal to zero.

For control-affine state-space systems the background and theory can be found in (Scherpen, 1994; Gray and Mesko, 1999; Fujimoto and Scherpen, 2005) and references therein. For linear time-invariant state-space systems the controllability function is equal to the controllability gramian multiplied from the left and right by the state, while the observability function equals the observability gramian post- and pre-multiplied by the state. Computation of both the controllability and the observability function in the nonlinear case includes solving a partial differential equation. For the controllability function it is nonlinear while for the observability function it is linear. In both cases, it can be difficult to find a solution in closed form. For control-affine nonlinear systems some methods have been studied. One method finds a local solution expressed as a power series expansion, see (Scherpen, 1994), while another is based on stochastics (Newman and Krishnaprasad, 2000).

In (Stykel, 2004), linear time-invariant DAE systems are considered and methods to compute the controllability and observability functions are derived. A phenomenon that may occur for DAE systems is that some combinations of the variables are not allowed due to inherent constraints. Allowed combinations are denoted consistent and in (Stykel, 2004) only consistent variables are considered.

In (Sjöberg and Glad, 2006; Sjöberg, 2006), nonlinear DAE systems are considered. There some methods to compute the controllability and the observability functions are derived. One of these methods is similar to a method derived in (Scherpen, 1994), yielding local solutions expressed as power series. This method will be used in this paper. All methods in (Sjöberg and Glad, 2006; Sjöberg, 2006) follows the line of (Stykel, 2004) and considers only consistent variables and we will also do the same in this paper.

The research concerning nonlinear model reduction is rather extensive. Some references are (Scherpen, 1994; Newman and Krishnaprasad, 2000; Lall *et al.*, 1999; Fujimoto and Scherpen, 2003*b*; Fujimoto and Scherpen, 2003*a*; Fujimoto and Scherpen, 2005). In

the last three of these references, the Hankel operator has been studied. Based on this operator it has been shown that natural nonlinear extensions to results for linear models can be derived. The computation of a balanced realization and a reduced order model is however rather involved to do analytically. Therefore a method for nonlinear control-affine systems was derived in (Fujimoto and Tsubakino, 2006) which is based on power series expansions of the functions involved to find a local balanced realization. In this paper this work will be generalized to handle also nonlinear time-invariant DAE systems.

Notation: The notation in this paper is fairly standard. The Jacobian matrix of V_h, *i.e.*, $\frac{\partial V_h}{\partial x}$ is denoted $V_{h;x}$.

2. PRELIMINARIES

The work in this paper will mostly consider nonlinear DAEs in semi-explicit form

$$\dot{x}_1 = F_1(x_1, x_2, u) \tag{1a}$$
$$0 = F_2(x_1, x_2, u) \tag{1b}$$
$$y = h(x_1, x_2) \tag{1c}$$

where $x_1 \in \mathbb{R}^{n_1}$, $x_2 \in \mathbb{R}^{n_2}$, $u \in \mathbb{R}^m$, $F_1 : \mathbb{R}^{n_1+n_2+m} \to \mathbb{R}^{n_1}$, $F_2 : \mathbb{R}^{n_1+n_2+m} \to \mathbb{R}^{n_2}$ and $h : \mathbb{R}^{n_1+n_2} \to \mathbb{R}^p$. Throughout this paper it will be assumed that the DAE system satisfies the following assumption.

Assumption 1. The system (1) has an equilibrium at the origin and the corresponding output is zero, *i.e.*,

$$F_1(0,0,0) = 0, \quad F_2(0,0,0) = 0, \quad h(0,0) = 0$$

Furthermore, it holds that $F_{2;x_2}(0,0,0)$ is nonsingular.

From this assumption, it is known from the implicit function theorem that it is possible to solve F_2 for x_2 as

$$x_2 = \varphi(x_1, u) \tag{2}$$

on a small neighborhood of the origin. The challenge lies in the fact that in most practical cases, φ is not possible to express explicitly.

It may seem very restrictive to just consider this class of systems, but in (Kunkel and Mehrmann, 2006) it is shown how rather general nonlinear DAEs

$$\bar{F}(\dot{x}, x, u) = 0 \tag{3}$$

can be put into semi-explicit form satisfying the assumptions above, possibly after introduction of an integrator chain and control by using the highest derivative of the input signal. Furthermore, it should be noted that semi-explicitness is mostly assumed to lessen the notation. The method derived can also handle systems not being semi-explicit. For details, see (Sjöberg, 2006).

One further assumption made in order to be able to compute the reduced model will be that the system is analytic.

Assumption 2. The functions $F_1(x_1, x_2, u)$, $F_2(x_1, x_2, u)$ and $h(x_1, x_2, u)$ are real analytic functions in some open set around the origin.

From this assumption it follows that the system can be expressed by power series

$$F(x_1, x_2, u) = A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + Bu + F_h(x_1, x_2, u) \tag{4a}$$

$$h(x_1, x_2, u) = C \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + h_h(x_1, x_2, u) \tag{4b}$$

where the matrices A, B, and C are constant matrices partitioned as

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \quad C = (C_1 \ C_2)$$

and F_h and h_h are the higher order terms of each series expansion. For convenience later, we make the following notational definition

$$\hat{A} = A_{11} - A_{12}A_{22}^{-1}A_{21} \quad \hat{B} = B_1 - A_{12}A_{22}^{-1}B_2$$
$$\hat{C} = C_1 - A_{12}A_{22}^{-1}C_2$$

Based on this notation two more assumptions that will be made throughout the paper can be stated.

Assumption 3. The matrix $\hat{A}$ is Hurwitz.

Assumption 4. $(\hat{A}, \hat{B})$ is controllable.

That is, we assume that the linear part of the system is stable and controllable.

2.1 Controllability Function

The controllability function basically measures the minimal amount of energy in the control signal $u(t)$ required to reach a specific state x. It is defined as the solution to the following optimal control problem

$$L_c\big(x_1(0)\big) = \min_{u(\cdot)} \frac{1}{2} \int_{-\infty}^{0} u(t)^T u(t)\, dt \tag{5}$$

subject to the system dynamics

$$\dot{x}_1 = F_1(x_1, x_2, u) \tag{6a}$$
$$0 = F_2(x_1, x_2, u) \tag{6b}$$

and the boundary conditions

$$x_1(0) = x_{1,0} \in \Omega_x$$
$$\lim_{t \to -\infty} x_1(t) = 0$$

where Ω_x is a neighborhood of $x_1 = 0$.

The controllability function must satisfy the condition $L_c(0) = 0$. The reason is the interpretation of $L_c(x_{1,0})$ as the minimum amount of input energy required to drive the system from zero at $t = -\infty$ to $x_1(0) = x_{1,0}$ at $t = 0$. According to the assumptions, the system (6) has an equilibrium at the origin and since no control effort is needed to keep $(x_1, x_2) = 0$, the energy then equals zero. We also use a convention that if x_0 cannot be asymptotically reached from 0, *i.e.*, if there exists no control input such that $x_1(-\infty) = 0$ and $x_1(0) = x_{1,0}$, $L_c(x_{1,0})$ is infinite.

We will only consider variables in the set

$$\mathcal{N} = \{x_1 \in \Omega_x,\, x_2 \in \mathbb{R}^{n_2} \,|\, x_2 = \varphi(x_1, u),\, u \in \Omega_u\}$$

where Ω_u is a neighborhood of $u = 0$ such that the implicit function theorem is valid. This means that only points not violating the constraints are considered.

The optimal control problem presented above leads to the task of finding a solution to the equations

$$0 = u^T - L_{c;x_1}\big(F_{1;u} - F_{1;x_2}F_{2;x_2}^{-1}F_{2;u}\big) \tag{7a}$$
$$0 = \tfrac{1}{2}u^T u - L_{c;x_1}F_1 \tag{7b}$$
$$0 = F_2 \tag{7c}$$

Sufficient conditions for the existence of a locally unique solution to (7), which then is the controllability function, will be shown below.

Theorem 5. Consider a nonlinear DAE system (1) satisfying Assumptions 1–4. Then the system has a local controllability function around the origin which can be expressed as

$$L_c(x_1) = \frac{1}{2}x_1^T G_c x_1 + L_{ch}(x_1) \tag{8}$$

where G_c is a positive definite matrix and $L_{ch}(x_1)$ is a uniformly convergent series expansion beginning with terms of order three.

PROOF. See (Sjöberg, 2006).

The matrix G_c is the unique positive definite solution to the Lyapunov equation

$$0 = G_c^{-1}\hat{A}^T + \hat{A}G_c^{-1} + \hat{B}\hat{B}^T \tag{9}$$

such that $-\hat{A} - \hat{B}\hat{B}^T G_c$ is Hurwitz. The higher order terms in $L_{ch}(x_1)$ can be computed recursively to any degree. The corresponding control signal, achieving this controllability function, is also given as a power series

$$u_*(x_1) = \hat{B}^T G_c x_1 + u_{h,*}(x_1) \tag{10}$$

where the higher order terms can be computed recursively as higher orders of $L_c(x_1)$ become available. Please see (Sjöberg, 2006) for details about the computation.

2.2 Observability Function

The observability function reflects the energy in the output signal when the system is released from a certain initial state. Only the energy corresponding to the initial state is of interest and therefore the control signal is set to zero. The observability function $L_o(x_1)$ is then defined as

$$L_o\big(x_1(0)\big) = \frac{1}{2}\int_0^\infty y(t)^T y(t)\,dt \tag{11}$$

subject to

$$\dot{x}_1 = F_1(x_1, x_2, 0) \tag{12a}$$
$$0 = F_2(x_1, x_2, 0) \tag{12b}$$
$$y = h(x_1, x_2) \tag{12c}$$

and the initial conditions $x_1(0) = x_{1,0} \in \Omega_x$.

The observability function is found by solving

$$0 = \frac{1}{2}h(x_1, x_2)^T h(x_1, x_2) + L_{o;x_1}(x_1)F_1(x_1, x_2, 0)$$
$$0 = F_2(x_1, x_2, 0)$$

and a theorem giving sufficient conditions for the existence of a local observability function is given below.

Theorem 6. Consider a nonlinear DAE system given in the form (12). Assume that it satisfies Assumptions 1–3. Then a local observability function exists, and can be expressed as

$$L_o(x_1) = \frac{1}{2}x_1^T G_o x_1 + L_{oh}(x_1) \tag{13}$$

where G_o is positive semi-definite and $L_{oh}(x_1)$ is a uniformly convergent series consisting of terms of order three and higher. Moreover, if $\big(\hat{A}, \hat{C}\big)$ is observable, G_o is positive definite.

PROOF. See (Sjöberg, 2006).

The first term G_o, is given as the positive semi-definite solution to the Lyapunov equation

$$0 = G_o\hat{A} + \hat{A}^T G_o + \hat{C}^T\hat{C}$$

and higher order terms in $L_o(x_1)$ can be computed recursively as described in (Sjöberg, 2006).

2.3 Model Reduction for State-Space Systems

In this section, theory for model reduction of state space systems, *i.e.*, when F_2 is absent, will be discussed briefly. For a more thorough discussion see (Scherpen, 1994; Fujimoto and Scherpen, 2005; Fujimoto and Scherpen, 2003*b*; Fujimoto and Scherpen, 2003*a*).

The considered systems can be written as

$$\dot{x}_1 = F_1(x_1, u) \tag{14a}$$
$$y = h(x_1) \tag{14b}$$

where $x_1 \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $F_1 : \mathbb{R}^{n+m} \to \mathbb{R}^n$ and $h : \mathbb{R}^n \to \mathbb{R}^p$.

Before a theorem presenting conditions under which a system can be formulated as an input-normalized/ output-diagonalized realization, an assumption needs to be made.

Assumption 7. The eigenvalues of $G_c^{-1}G_o$ are distinct.

Having this assumption and the assumptions made earlier, a theorem can be stated as follows.

Theorem 8. Suppose that Assumptions 3, 4, and 7 are satisfied. Then there exists a neighborhood $X_0 \subset \Omega_x$ of the origin and a coordinate transformation

$$x_1 = \Phi(z), \quad \Phi(0) = 0 \tag{15}$$

on X_0, converting the system into input-normal/output-diagonal form, *i.e.*, there exist n smooth functions $\rho_i : S_0 \to R_+$, $i = \{1, 2, \ldots, n\}$, such that

$$L_c\big(\Phi(z)\big) = \frac{1}{2}z^T z \tag{16a}$$
$$L_o\big(\Phi(z)\big) = \frac{1}{2}\sum_{i=1}^{n} z_i^2\rho_i(z_i)^2 \tag{16b}$$

for $i = \{1, 2, \ldots, n\}$.

PROOF. Follows by combining the proofs in (Fujimoto and Scherpen, 2003*b*) and (Fujimoto and Scherpen, 2003*a*).

Note that we will normally assume that Ω_x is chosen small enough so that $X_0 = \Omega_x$.

From (16) it is then possible, using one further coordinate transformation, to obtain the balanced realization given as

$$L_c\big(\Psi(q)\big) = \frac{1}{2}\sum_{i=1}^{n}\frac{q_i^2}{\sigma_i(q_i)} \tag{17a}$$

$$L_o\big(\Psi(q)\big) = \frac{1}{2}\sum_{i=1}^{n} q_i^2\sigma_i(q_i) \tag{17b}$$

The coordinate transformation is defined to give

$$\sigma_i(q_i) = \rho_i\big(\psi_i(z_i)\big) \tag{18}$$

and is described by

$$z = \Psi(q) = \big(\psi(q_1), \psi_2(q_2), \ldots, \psi_n(q_n)\big),\ \Psi(0) = 0 \tag{19}$$

where ψ_i, $i \in \{1, 2, \ldots, n\}$ is given as the solution to

$$q_i = \psi_i(q_i)\sqrt{\rho_i\big(\psi_i(q_i)\big)},\ \forall q_i \in \Omega \tag{20}$$

Having the coordinate transformations we want to pick out the parts of the system that affect the input-output behavior most. For this end, consider the transformed version of system (14)

$$\dot{q} = \bar{F}(q, u), \quad y = \bar{h}(q) \tag{21a}$$

i.e., where the coordinate transformation $x_1 = \Phi \circ \Psi(q)$ has been applied. Assume that the Hankel singular value functions are in order, *i.e.*,

$$\min_{q_i\in[-c,c]} \sigma_i(q_i) > \max_{q_i\in[-c,c]} \sigma_{i+1}(q_i), \quad i = 1, 2, \ldots, n$$

where c is a small number determining the interesting interval for the norm of u. Assume that σ_j is much larger than σ_{j+1}. It then implies that the first j states of (21) affect the input-output behavior much more than the last $n - j$ states. Hence, divide the system into two subparts $q = (q_a, q_b)$ with $q_a = (q_1, \ldots, q_j)$ and $q_b = (q_{j+1}, \ldots, q_n)$. The system

$$\dot{q}_a = \bar{F}(q_a, 0, u), \quad y = \bar{h}(q_a, 0) \tag{22a}$$

is then a j:th order reduced model of (14) in the sense that (22) preserves several important properties of the original system, such as controllability, observability and stability. For a more thorough discussion, see (Fujimoto and Scherpen, 2003*b*; Fujimoto and Tsubakino, 2006).

3. MODEL REDUCTION FOR DAES

In this section, model reduction for nonlinear DAEs will be discussed. For nonlinear DAEs satisfying Assumption 1, it is known that at least in some small neighborhood of the origin it is possible to rewrite the system (1) as

$$\dot{x}_1 = \hat{F}(x_1, u) = F_1\big(x_1, \varphi(x_1, u), u\big) \tag{23a}$$

$$y = \hat{h}(x_1, u) = h\big(x_1, \varphi(x_1, u), u\big) \tag{23b}$$

Since this is a state-space system, it is from a theoretical point of view possible to use the results in Section 2.3. However, the challenge comes from the fact that φ normally is not explicitly given. This means that $\hat{F}$ and $\hat{h}$ cannot be written in closed form and therefore the results in Section 2.3 can only be used in very special cases to compute the balanced realization for nonlinear DAEs in practice.

Also for state-space models, the methods in Section 2.3 are not always easy to use computationally. Therefore, a method based on power series expansion is derived in (Fujimoto and Tsubakino, 2006). In this paper this method is generalized to also handle nonlinear DAEs given in the form (1). However, as mentioned earlier there is no problem to also handle systems not in semi-explicit form, except for a more involved notation.

The basic idea in the power series method is to assume that the transformations $\Phi(z)$, $\Psi(q)$ and the singular value functions $\rho_i(z_i)$, $\sigma_i(z_i)$ can be expressed as convergent power series

$$x_1 = \Phi(z) = T_\Phi z + \Phi_h(z) \tag{24a}$$

$$z = \Psi(q) = T_\Psi q + \Psi_h(q) \tag{24b}$$

$$\rho_i(z_i) = \rho_{0i} + \rho_{hi}(z_i), \quad i = 1, \ldots, n_{x_1} \tag{24c}$$

$$\sigma_i(z_i) = \sigma_{0i} + \sigma_{hi}(z_i), \quad i = 1, \ldots, n_{x_1} \tag{24d}$$

where T_Ψ will be diagonal as can be seen from (19).

If the DAE system satisfies the required assumptions, we know from Theorem 5 and 6 that we can compute the controllability and observability functions, respectively. This can be done locally in the vincinity of the origin and the functions will be expressed as power series. The key point will therefore be that even though closed expressions for $\hat{F}$ and $\hat{h}$ are not known, their series expansions are still possible to compute, see (Sjöberg, 2006) for details.

If Φ and ρ in (24) are substituted into the controllability and observability functions we want the result to be in the form (16). The equations formed then become

$$\begin{aligned} 0 &= \frac{1}{2}z^T(T_\Phi^T G_c T_\Phi - I)z + \frac{1}{2}z^T T_\Phi^T G_c \Phi_h(z) \\ &\quad + \frac{1}{2}\Phi_h^T(z) G_c \Phi(z) + L_c\big(\Phi(z)\big) \\ 0 &= \frac{1}{2}z^T\big(T_\Phi^T G_o T_\Phi - \rho(z)^2\big)z + \frac{1}{2}z^T T_\Phi^T G_o \Phi_h(z) \\ &\quad + \frac{1}{2}\Phi_h^T(z) G_o \Phi(z) + L_o\big(\Phi(z)\big) \end{aligned}$$

where

$$\rho(z) = \operatorname{diag}\big(\rho_1(z_1), \rho_2(z_2), \ldots, \rho_{n_{x_1}}(z_{n_{x_1}})\big)$$

Since these equations are supposed to be valid for all z in a neighborhood, the coefficients corresponding to different degrees in z must equal zero. The second order terms yield the eigenvalue problem

$$G_c^{-1} G_o T_\Phi = T_\Phi \operatorname{diag}(\rho_{01}^2, \rho_{02}^2, \ldots, \rho_{0n_{x_1}}^2) \tag{25}$$

and solving this equation then gives the first order terms of Φ and the zeroth order terms of ρ_i. The higher order terms of Φ are obtained by solving the coefficients in the equations

$$0 = \Phi_h^T(z) G_c T_\Phi + \frac{1}{2}\Phi_h^T(z) G_c \Phi_h(z) + L_{ch}\big(\Phi(z)\big) \tag{26a}$$

$$\begin{aligned} 0 &= \frac{1}{2}z^T \rho_h(z) z + \frac{1}{2}z^T T_\Phi^T G_o \Phi_h(z) \\ &\quad + \frac{1}{2}\Phi_h^T(z) G_o \Phi(z) + L_{oh}\big(\Phi(z)\big) \end{aligned} \tag{26b}$$

where

$$\rho_h(z) = \text{diag}\left(2\rho_{o1}\rho_{h1}(z_1) + 2\rho_{on_{x_1}}\rho_{hn_{x_1}}(z_{n_{x_1}}) + \rho_{h1}(z_1)^2, \ldots, \rho_{hn_{x_1}}(z_{n_{x_1}})^2\right)$$

A problem is that $\rho_i(z_i)$ are unknown, but it is shown in (Fujimoto and Tsubakino, 2006) that it is always possible to just solve the equations corresponding to coefficients not including parameters in $\rho_i(z_i)$. The system of equations will be linear in the unknown parameters and for cases with one or two states it will also be well-determined. However, in cases with more than two states, the system of equations will be underdetermined. This means that there will be several different coordinate changes, achieving different input-normal/output-diagonalized realizations, and which to choose is a design choice.

Remark 9. Note that it might seem like Φ_h is uniquely determined by (26a), since $G_c T_\Phi$ is nonsingular (compare with the discussion in for example (Lukes, 1969)). However, this is not the case since it will be the m:th order terms of (26a) that determine the $m-1$:th order terms in Φ. This leads to a system of equations not uniquely determined.

Since the parameters in $\rho_i(z_i)$ are not solved for in the first step, they are computed from the definition as

$$\rho_i(z_i) = \sqrt{\frac{L_{o;x}(z)}{L_{c;x}(z)}}\Bigg|_{z=(0,\ldots,0,\overbrace{z_i}^{i:th},0,\ldots,0)}$$

for $i = 1, \ldots, n_{x_1}$. To obtain the coordinate change Ψ, the same method as above is used. The equation (20) is solved by substituting Ψ with its series expansion (24b), power series expand the right hand side and solve for the unknown coefficients. The singular values $\sigma_i(q_i)$ can then be calculated from (18).

Having the coordinate transformations, it is now possible to express the reduced nonlinear DAE system. Assume that the first j Hankel singular values are significantly larger than the $n-j$ other. Then divide the states q into two subspaces q_a and q_b where the former corresponds to the larger singular values and divide system in a corresponding way,

$$\dot{q}_a = \bar{F}_{1a}(q_a, 0, x_2, u) \quad (27a)$$
$$0 = \bar{F}_2(q_a, 0, x_2, u) \quad (27b)$$
$$y_a = \bar{h}(q_a, 0, x_2) \quad (27c)$$

and

$$\dot{q}_b = \bar{F}_{1b}(0, q_b, x_2, u) \quad (28a)$$
$$0 = \bar{F}_2(0, q_b, x_2, u) \quad (28b)$$
$$y_b = \bar{h}(0, q_b, x_2) \quad (28c)$$

where

$$\bar{F}_1(q_{1a}, q_{1b}, x_2, u) = \left(\tfrac{\partial \Phi \circ \Psi(q)}{\partial q}\right)^{-1} F_1\big(\Phi \circ \Psi(q), x_2, u\big)$$
$$\bar{F}_2(q_{1a}, q_{1b}, x_2, u) = F_2\big(\Phi \circ \Psi(q), x_2, u\big)$$
$$\bar{h}(q_{1a}, q_{1b}, x_2) = h\big(\Phi \circ \Psi(q), x_2\big)$$

Then a j:th order reduced model for the DAE system (1) is given by (27). This system still satisfies Assumption 1.

Let the controllability and observability functions computed for the reduced DAE model be denoted $L_{ca}(q_a)$ and $L_{oa}(q_a)$, and let the corresponding singular values be $\sigma_{ai}(q_{ai})$, $i = (1, \ldots, j)$. Then these will be the same as if $q = (q_a, 0)$ is substituted into $L_c\big(\Phi \circ \Psi(q)\big)$ and $L_o\big(\Phi \circ \Psi(q)\big)$. The same holds symmetrically for the q_b. This results basically comes from (Fujimoto and Tsubakino, 2006), and is formulated as a theorem.

Theorem 10. Consider a DAE system (1), satisfying Assumptions 1 – 7. Then it holds that

$$L_{ca}(q_a) = L_c\big(\Phi \circ \Psi(q_a, 0)\big), \quad L_{oa}(q_a) = L_o\big(\Phi \circ \Psi(q_a, 0)\big)$$
$$L_{cb}(q_b) = L_c\big(\Phi \circ \Psi(0, q_b)\big), \quad L_{ob}(q_b) = L_o\big(\Phi \circ \Psi(0, q_b)\big)$$

and

$$\sigma_{ai}(q_{ai}) = \sigma_i(q_{ai}), \quad i = 1, 2, \ldots, j$$
$$\sigma_{bi}(q_{bi}) = \sigma_i(q_{bi}), \quad i = j+1, j+2, \ldots, n_{x_1}$$

where σ_{ai} and σ_{bi} are the singular values to (27) and (28), respectively.

PROOF. Implicitly the systems (27) and (28) is defined from

$$\dot{q} = \hat{\bar{F}}_1(q, u) = \bar{F}_1\big(q, \bar{\varphi}(q, u), u\big)$$
$$y = \bar{\bar{h}}(q, u) = \bar{h}\big(q, \bar{\varphi}(q, u), u\big)$$

where $\bar{\varphi}(q, u) = \varphi\big(\Phi \circ \Psi(q), u\big)$. Since this is a state-space system it is possible to use the proof in (Fujimoto and Tsubakino, 2006).

Above it has been shown that for quite a large class of DAE's it is possible to find a reduced model computationally. However, it comes with a price. Two of the main drawbacks are that the model is only valid in the vincinity of the origin and the region in which it is valid is not easy to derive analytically. A third drawback is that only an approximate reduced model is obtained, since computationally the power series have to be truncated. In practice, these obstacles can often be handled. The reduced model given by a truncated power series is in many cases a quite good approximation in a reasonably sized region. Furthermore, is often possible to get a quite a good estimate of the region from simulations. A small example of this will be shown in the next section.

4. EXAMPLE

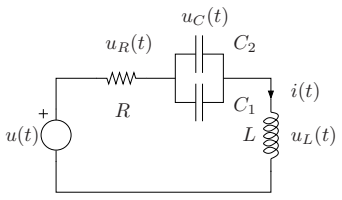

Fig. 1. An electrical circuit.

In this section a small example is shown mostly to demonstrate the method. It describes an electrical circuit, which can be seen in Figure 1. The circuit consists of a voltage source, a resistor, two capacitors and an inductor. The voltage source is assumed to be ideal, *i.e.*, having no internal resistance and giving the voltage u. The inductor is assumed to have a ferromagnetic core resulting in a saturated magnetic flux Φ for large currents i. The capacitors are assumed to

have voltage dependent capacitances C_1 and C_2, and finally, it is assumed that resistor depends both linearly and cubically of the current. Modeling the circuit in an object-oriented fashion, where all components are modeled separately and then connected using Kirchoff's voltage law, gives a model that does not satisfy the assumptions. However, usage of the method in (Kunkel and Mehrmann, 2006) yields a model in the form

$$\dot{u}_{C_1} = \tfrac{1}{2}\tfrac{i_1}{1+10^{-1}u_{C_1}} + \tfrac{1}{2}\tfrac{i_2}{1+10^{-2}u_{C_1}} \tag{29a}$$
$$\dot{\Phi} = u_L \tag{29b}$$
$$0 = \Phi - \arctan(i) \tag{29c}$$
$$0 = u_R - 5i - 10i^3 \tag{29d}$$
$$0 = i - i_1 - i_2 \tag{29e}$$
$$0 = \tfrac{i_1}{1+10^{-1}u_{C_1}} - \tfrac{i_2}{1+10^{-2}u_{C_1}} \tag{29f}$$
$$0 = u - u_R - u_{C_1} - u_L \tag{29g}$$

and

$$h(u_{C_1}, \Phi, i, u_L, u_R) = u_{C_1} \tag{29h}$$

which satisfies all assumptions in Theorems 5 and 6 for $x_1 = (u_{C_1}, \Phi)$ and $x_2 = (i, i_1, i_2, u_L, u_R)$. Then, it is known that there locally exist a controllability function (8) and an observability function (11).

Having the controllability and observability functions, the coordinate transformations Φ and Ψ can be computed using the derived methods. The corresponding second order approximation of the composite transformation becomes

$$\Phi \circ \Psi = \begin{pmatrix} 0.43q_1 - 3.1q_2 - 0.084q_2^2 - 0.0033q_1^2 + 0.073q_1q_2 \\ 0.085q_1 + 32q_2 - 1.8q_2^2 - 6.0\cdot 10^{-4}q_1^2 + 0.0066q_1q_2 \end{pmatrix}$$

Note that here only the second order approximation of the transformation is given to save space, but in the simulations the fourth order approximation has been used. The Hankel singular values $\sigma_i(q_i)$ approximated to second order are

$$\sigma_1(q_1) = 0.51 - 0.00015q_1 - 0.00024q_1^2$$
$$\sigma_2(q_2) = 0.0098 + 0.00054q_2 - 15q_2^2$$

Hence, as can be seen the first state in the balanced realization affects the input-output behavior much more than the other state. Therefore, a one state reduced model can used.

Two comparisons between the output for the reduced model and for the original model can be found in Figure 2 and 3. In both figures, the initial conditions have been $x_1(0) = (0.01, 0.01)$ In Figure 2, the input signal is zero, while in Figure 3 it is chosen as $u(t) = 0.01\sin(t)$. As can be seen, the reduced order model seems to reflect the original system rather well.

5. CONCLUSIONS

In this paper, model reduction of nonlinear DAE systems has been considered. An algorithm for computing the reduced order model has been presented. The challenge computationally was that the implicit function is known to exist, but can seldom be written in closed form. It was shown that rather general nonlinear DAE system can be handled using methods by Kunkel and Mehrmann. The main drawback of the method is that the reduced model is valid only in the vincinity of the origin and the region in which it is valid can often not be estimated analytically. That is, one has to rely on simulations.

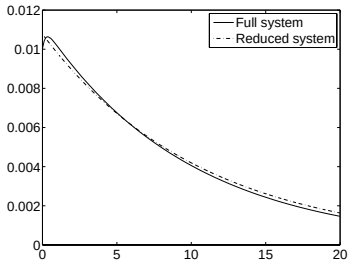

Fig. 2. Comparison between the full and the reduced system, with no input.

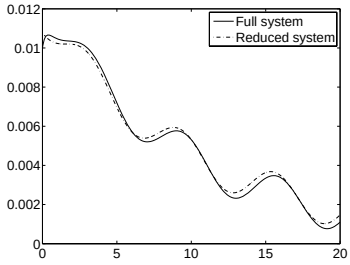

Fig. 3. Comparison between the full and the reduced system, using a sinusoidal as input.

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