On the associated primes of generalized local cohomology modules

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1. Introduction

Throughout this paper, R is a commutative Noetherian ring with identity. For an ideal $\mathfrak{a}$ of R and $i \geq 0$, the i-th local cohomology module of M is defined as:

$$H^i_{\mathfrak{a}}(M) = \varinjlim_n \operatorname{Ext}^i_R(R/\mathfrak{a}^n, M).$$

In [**8**], Huneke conjectured that if M is a finitely generated R-module, then the set of associated primes of $H^i_{\mathfrak{a}}(M)$ is finite. Singh [**15**] provides a counter example for this conjecture. However, it is known that the conjecture is true in many situations. For example, in [**11**] it is shown that if R is local and $\dim R/\mathfrak{a} = 1$, then for a finitely generated R-module M, the set $\operatorname{Ass}_R(H^i_{\mathfrak{a}}(M))$ is finite for all $i \geq 0$.

Also, Brodmann and Lashgari [**2**] showed that the first non-finitely generated local cohomology module of a finitely generated R-module has only finitely many associated primes. Also, see [**10**] and [**4**] for a far reaching generalizations of this result.

The following generalization of local cohomology theory is due to Herzog [**7**] (see also [**17**]). The generalized local cohomology functor $H^i_{\mathfrak{a}}(.,.)$ is defined by

$$H^i_{\mathfrak{a}}(M, N) = \varinjlim_n \operatorname{Ext}^i_R(M/\mathfrak{a}^n M, N)$$

for all R-modules M and N. Clearly, this is a generalization of the usual local cohomology functor. Recently, there are some new interest in generalized local cohomology (see e.g. [**1**], [**5**], [**6**] and [**18**]). Our main aim in this paper is to establish the following.

Theorem 1.1. *Let $\mathfrak{a}$ be an ideal of R and let M and N be two finitely generated R-modules. Then the following statements hold.*

(i) For any positive integer t,

$$\mathrm{Ass}_R(H^t_{\mathfrak{a}}(M,N)) \subseteq \bigcup_{i=0}^{t} \mathrm{Ass}_R(\mathrm{Ext}^i_R(M, H^{t-i}_{\mathfrak{a}}(N))).$$

(ii) If $d = pd(M)$ *and* $n = \dim N$ *are finite, then* $H^{n+d}_{\mathfrak{a}}(M,N)$ *is Artinian. In particular* $\mathrm{Ass}_R(H^{n+d}_{\mathfrak{a}}(M,N))$ *consists of finitely many maximal ideals.*
(iii) Suppose that $(R,\mathfrak{m})$ *is local with dimension* n *and that* $d = pd(M)$ *is finite. Then* $\mathrm{Supp}_R(H^{n+d-1}_{\mathfrak{a}}(M,N))$ *is finite.*

Clearly (i) extends the main results of [**10**, Theorem B], [**2**, Theorem 2.2] and [**4**, Corollary 2.7], (ii) extends [**12**, Theorem 2.2], and (iii) is an improvement of [**11**, Corollary 2.4].

2. The results

First, we recall the definition of a weakly Laskerian module. An R-module M is said to be Laskerian if any submodule of M is an intersection of a finite number of primary submodules. Obviously, any Noetherian module is Laskerian. In [**4**], as a generalization of this notion, we introduced the following definition.

Definition 2.1. An R-module M is said to be *weakly Laskerian* if the set of associated primes of any quotient module of M is finite.

Example 2.2. (i) Every Laskerian module is weakly Laskerian.
(ii) Any module with finite support is weakly Laskerian. In particular, any Artinian R-module is weakly Laskerian.

Theorem 2.3. *Let* $\mathfrak{a}$ *be an ideal of* R *and* M *be a finitely generated* R*-module. Let* N *be an* R*-module and* t *a positive integer. Then*

$$\mathrm{Ass}_R(H^t_{\mathfrak{a}}(M,N)) \subseteq \bigcup_{i=0}^{t} \mathrm{Ass}_R(\mathrm{Ext}^i_R(M, H^{t-i}_{\mathfrak{a}}(N))).$$

Proof. By [**14**, Theorem 11.38], there is a Grothendieck spectral sequence

$$E^{p,q}_2 := \mathrm{Ext}^p_R(M, H^q_{\mathfrak{a}}(N)) \underset{p}{\Longrightarrow} H^{p+q}_{\mathfrak{a}}(M,N).$$

For all $i \geq 2$, we consider the exact sequence

$$0 \longrightarrow \ker d^{0,t}_i \longrightarrow E^{0,t}_i \xrightarrow{d^{0,t}_i} E^{i,t-i+1}_i . (1)$$

Since $E_i^{0,t} = \ker d_{i-1}^{0,t} / \operatorname{im} d_{i-1}^{1-i,t+i-2}$ and $E_i^{i,j} = 0$ for all $j < 0$, we may use (1) to obtain $\ker d_{t+2}^{i,t-i} \cong E_{t+2}^{i,t-i} \cong \cdots = E_\infty^{i,t-i}$ for all $0 \leq i \leq t$. There exists a finite filtration

$$0 = \phi^{t+1} H^t \subseteq \phi^t H^t \subseteq \cdots \subseteq \phi^1 H^t \subseteq \phi^0 H^t = H_{\mathfrak{a}}^t(M, N)$$

such that

$$E_\infty^{i,t-i} = \phi^i H^t / \phi^{i+1} H^t$$

for all $0 \leq i \leq t$.

Now, the exact sequences $0 \longrightarrow \phi^{i+1} H^t \longrightarrow \phi^i H^t \longrightarrow E_\infty^{i,t-i} \longrightarrow 0$ $(0 \leq i \leq t)$ in conjunction with

$$E_\infty^{i,t-i} \cong \ker d_{t+2}^{i,t-i} \subseteq \ker d_2^{i,t-i} \subseteq E_2^{i,t-i}$$

yields

$$\operatorname{Ass}_R(H_{\mathfrak{a}}^t(M, N)) \subseteq \bigcup_{i=0}^{t} \operatorname{Ass}_R(\operatorname{Ext}_R^i(M, H_{\mathfrak{a}}^{t-i}(N))).\square$$

Next, we obtain an extension of [**2**, Theorem 2.2], [**10**, Theorem B], and [**4**, Corollary 2.7].

Corollary 2.4. *Let $\mathfrak{a}$ be an ideal of R, M a finitely generated R-module, and N a weakly Laskerian R-module. If $H_{\mathfrak{a}}^i(N)$ is weakly Laskerian module for all $i < t$, then* $\operatorname{Ass}_R(H_{\mathfrak{a}}^t(M, N))$ *is finite.*

Proof. This is immediate by 2.3 and [**4**, Lemma 2.3 and Corollary 2.7]. $\square$

Corollary 2.5. *Let $(R, \mathfrak{m})$ be a local ring and let* $\dim R \leq 2$. *Let M be a finitely generated R- module and N a weakly Laskerian R-module. Then* $\operatorname{Ass}_R(H_{\mathfrak{a}}^t(M, N)$ *is finite for all $t \geq 0$.*

Proof. By [**11**, Corollaries 2.3, 2.4], [**3**, Theorem 6.1.2] and 2.2(ii), $H_{\mathfrak{a}}^t(N)$ is weakly Laskerian for all $t \geq 1$. Also, $\Gamma_{\mathfrak{a}}(N)$ is weakly Laskerian by 2.1. So, by 2.4, $\operatorname{Ass}_R(H_{\mathfrak{a}}^t(M, N))$ is finite for all $t \geq 0$. $\square$

Corollary 2.6. *Let $\mathfrak{a}$ be an ideal of a local ring $(R, \mathfrak{m})$ and* $\dim R = n$. *Let M be a finitely generated R-module and N be an R-module such that $H_{\mathfrak{a}}^i(N) = 0$ for all $i \neq n-1, n$. Then* $\operatorname{Ass}_R(H_{\mathfrak{a}}^t(M, N))$ *is finite for all $t \geq 0$.*

Proof. By [**11**, Corollaries 2.3, 2.4] and hypothesis, $\operatorname{Supp}_R(H_{\mathfrak{a}}^t(N))$ is finite for all $t \geq 0$; so that, by 2.4, $\operatorname{Ass}_R(H_{\mathfrak{a}}^t(M, N))$ is finite for all $t \geq 0$. $\square$

Corollary 2.7. *Let $\mathfrak{a}$ be an ideal of a local ring $(R, \mathfrak{m})$ with $\dim R/\mathfrak{a} = 1$. Let M and N be two R-modules. Then $\mathrm{Ass}_R(H^t_{\mathfrak{a}}(M,N))$ is finite for all $t \geq 0$.*

Proof. It is clear that $\mathrm{Supp}_R(H^t_{\mathfrak{a}}(N))$ is finite for all $t \geq 0$; hence by 2.4, $\mathrm{Ass}_R(H^t_{\mathfrak{a}}(M,N))$ is finite for all $t \geq 0$. □

Following [**9**], a sequence $x_1, \ldots, x_n$ of elements of $\mathfrak{a}$ is said to be an $\mathfrak{a}$-filter regular sequence on N, if

$$\mathrm{Supp}_R((x_1, \ldots, x_{i-1})N :_N x_i/(x_1, \ldots, x_{i-1})N) \subseteq V(\mathfrak{a})$$

for all $i = 1, \ldots, n$, where $V(\mathfrak{a})$ denotes the set of all prime ideals of R containing $\mathfrak{a}$. The concept of an $\mathfrak{a}$-filter regular sequence is a generalization of the one of a filter regular sequence which has been studied in [**16**, Appendix 2(ii)] and has led to some interesting results. It is easy to see that the analogue of [**16**, Appendix 2(ii)] holds true whenever R is Noetherian, N is a finitely generated R-module and $\mathfrak{m}$ replaced by $\mathfrak{a}$; so that, if $x_1, \ldots . x_n$ be an $\mathfrak{a}$-filter regular sequence on N, then there is an element $y \in \mathfrak{a}$ such that $x_1, \ldots, x_n, y$ is an $\mathfrak{a}$-filter regular sequence on N. Thus for a positive integer n, there exists an $\mathfrak{a}$-filter regular sequence on N of length n.

The following Lemma, which needs the concept of a filter regular sequence, is a generalization of [**13**, Lemma 3.4].

Lemma 2.8. *Let $\mathfrak{a}$ be an ideal of R and M be a finitely generated R-module such that $d = pd(M)$ is finite. Let N be an R-module and assume that $n \in \mathbb{N}$ and $x_1, \ldots, x_n$ is an $\mathfrak{a}$-filter regular sequence on N.Then $H^{i+n}_{\mathfrak{a}}(M,N) \cong H^i_{\mathfrak{a}}(M, H^n_{(x_1,\ldots,x_n)}(N))$ for all $i \geq d$.*

Proof. Consider the spectral sequence

$$E_2^{p,q} := H^p_{\mathfrak{a}}(M, H^q_{(x_1,\ldots,x_n)}(N)) \underset{p}{\Longrightarrow} H^{p+q}_{\mathfrak{a}}(M,N).$$

We have $E_2^{p,q} = 0$ for $q > n$ (by Theorem 3.3.1 of [**3**]) and for $q = n$, $p > d$ (by Proposition 2.5 of [**13**] and Lemma 1.1 of [**18**]). It therefore follows $E_2^{i,n} \cong E_\infty^{i,n}$ and $E_\infty^{i,n} \cong H^{i+n}_{\mathfrak{a}}(M,N)$. This proves the result. □

The following result is a generalization of [**12**, Theorem 2.2] and [**6**, Theorem 1.2].

Theorem 2.9. *Let $\mathfrak{a}$ be an ideal of R and let M and N be two finitely generated R-modules. Assume that $d = \mathrm{pd}(M)$ and $n = \dim N$ are finite. Then $H^{n+d}_{\mathfrak{a}}(M, N)$ is an Artinian R-module. In particular, $\mathrm{Ass}_R(H^{n+d}_{\mathfrak{a}}(M, N)$ is a finite set consisting of maximal ideals.*

Proof. Let $x_1, \ldots, x_n$ be an $\mathfrak{a}$-filter regular sequence on N. Then, by 2.8,

$$H^{n+d}_{\mathfrak{a}}(M, N) \cong H^{d}_{\mathfrak{a}}(M, H^{n}_{(x_1,\ldots,x_n)}(N))$$

and, by [**3**, Exercise 7.1.7], $H^{n}_{(x_1,\ldots,x_n)}(N)$ is Artinian. Put $S = H^{n}_{(x_1,\ldots,x_n)}(N)$. Then $H^{d}_{\mathfrak{a}}(M, S) \cong H^{d}(\mathrm{Hom}(M, \Gamma_{\mathfrak{a}}(\dot{E})))$ by [**6**, Lemma 2.1], where $\dot{E}$ is an injective resolution of S such that its terms are all Artinian modules. Therefore $H^{n+d}_{\mathfrak{a}}(M, N)$ is Artinian and $\mathrm{Ass}_R(H^{n+d}_{\mathfrak{a}}(M, N))$ is a finite set consisting of maximal ideals. □

The following theorem is an improvement of [**11**, Corollary 2.4].

Theorem 2.10. *Let $(R, \mathfrak{m})$ be a local ring of dimension n, N an R-module, M a finitely generated R-module, and $d = pd(M)$ is finite. Then $\mathrm{Supp}_R(H^{n+d-1}_{\mathfrak{a}}(M, N))$ is finite.*

Proof. Consider the Grothendieck spectral sequence

$$E_2^{p,q} := \mathrm{Ext}^p_R(M, H^q_{\mathfrak{a}}(N)) \underset{p}{\Longrightarrow} H^{p+q}_{\mathfrak{a}}(M, N).$$

So, we have a finite filtration

$$0 = \phi^{d+n} H^{d+n-1} \subseteq \phi^{d+n-1} H^{d+n-1} \subseteq \cdots \subseteq \phi^{1} H^{d+n-1} \subseteq \phi^{0} H^{d+n-1} = H^{d+n-1}_{\mathfrak{a}}(M, N)$$

and the equalities $E_\infty^{i,d+n-i-1} = \phi^i H^{n+d-1}/\phi^{i+1} H^{n+d-1}$ for all $o \leq i \leq n + d - 1$.
Since $E_2^{i,n+d-i-1} = 0$ for all $i \neq d-1, d$ and $E_\infty^{i,n+d-i-1}$ is a subquotient $E_2^{i,n+d-i-1}$, it follows that

$$\phi^{d+1} H^{n+d-1} = \phi^{d+2} H^{n+d-1} = \ldots = \phi^{d+n} H^{n+d-1} = 0$$

and that

$$\phi^{d-1} H^{n+d-1} = \phi^{d-2} H^{n+d-1} = \ldots = \phi^{0} H^{n+d-1} = H^{n+d-1}_{\mathfrak{a}}(M, N).$$

Now, using the above consequences in conjunction with [**11**, Corollaries 2.3, 2.4], it is easy to see that $\mathrm{Supp}_R(E_\infty^{d,n-1})$ and $\mathrm{Supp}_R(E_\infty^{d-1,n})$ are finite sets.
Next, consider the exact sequence

$$0 \longrightarrow E_\infty^{d,n-1} \longrightarrow H^{n+d-1}_{\mathfrak{a}}(M, N) \longrightarrow E_\infty^{d-1,n} \longrightarrow 0,$$

to deduce that $\mathrm{Supp}_R(H^{n+d-1}_{\mathfrak{a}}(M,N))$ is a finite. □

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