An Iterative Power Allocation Algorithm in OFDM System Based on Power Relaxation

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***Abstract*— We present an efficient algorithm for power allocation and bit loading in multi-user OFDM system. We starts with the close-form optimum power solution based on ideal channel coding and allowing negative transmission power. The solution is then revised by removing the subcarrier of negative power from the objective function and power constraints. The result of this procedure is proven to be the optimum power allocation in the continuous space. The continues power/rate solution is then mapped to the feasible (non-continuous) rate space. Our algorithm is very efficient in computation (O(N)) and the resultant discrete space power/rate vector is very close to the optimum solution (less than 1% difference in the overall data rate).**

I. INTRODUCTION

Future high speed wireless networks must satisfy the increasing bandwidth demand to support multimedia applications. A promising solution is the OFDM (Orthogonal Frequency Division Multiplexing) technology and link adaptive modulation. For a point-to-point communication link with slow fading, the transmitter can adaptively decide the modulation scheme and power level for individual subcarrier based on the channel condition. Usually a high level modulation (32-QAM) can be used for subcarrier of high channel gain. When the total transmission power is given, the seeking of maximum data rate becomes a joint optimization problem of bit loading and power assignment. This problem has been extensively studied in literature, such as [3], [5], [6]. With perfect CSI, the water-filling algorithm gives the optimum bit/sec/HZ.

In a one-to-many transmission scenario, a scheme is to load data for different destinations into OFDM subcarriers to maximize packet capacity. Jang [8] showed that the greedy subcarrier assignment is indeed optimal. Thus the multi-destination bit-loading problem is reduced to the single user case with a channel response function composed of the maximum of the CSI over all users.

In this paper, we focus on an efficient power allocation algorithm based on iterative refining of an sub-optimal close-form solution. Using Lagrange's method, a close-form expression for the power-relaxed optimization problem is obtained. The close form expression is obtained with only $O(N)$ computation cost, which is very time-saving compared to the other methods [2], [4].

Based on the close-form solution of the power-budget-constrained problem, we proposed an iterative method to produce the optimum solution with non-negative power constraints. The candidate solution is initially obtained with possible negative power values. Then subcarriers assigned with negative power values are removed and result a revised Lagrange problem. The revise objective function and constraints result in a new candidate solution which contains less "negative-power" subcarriers and preserve the optimality with respect to the modified constraints. The procedure is repeated until all power values are positive. We proved that the solution obtained via this procedure is globally optimal and satisfies all power constraints.

The rest of this paper is organized as follows: Section 2 presents the problem setting; Section 3 provide the close-form expression with power budget; Section 4 describes the iterative procedure. In section 5, the practical modulation is derived from the power assignment.

II. PROBLEM DESCRIPTION

The channel condition for different communication link could be significantly different to each other, even for those "physically closed paths". For instance, assuming node A is communicating to both node B and node C. The multipath between Link A→C contains three taps (1,0,0.5,0.2...), where we assume a unit LOS path at zero relative delay, a path of half signal strength delayed by two sampling period, and a third path of 20% signal strength delayed by three sampling period. The link between A→B has a fourth signal path of 30% signal strength at fourth delay tap. With such two channel power-delay-profiles, the channel gain in OFDM subcarriers is readily obtained via FFT transform. The channel responses for link A→B, and A→C are plot in Figure 1.

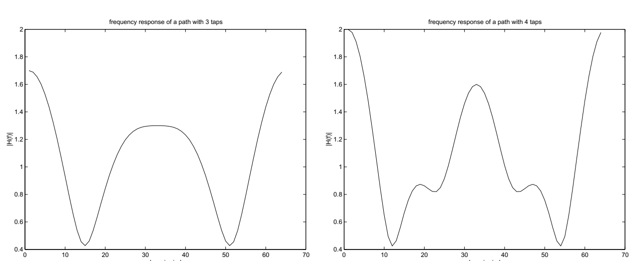

Fig. 1: Frequency Response: (a) node A→ node B, (b) node A → node C

Clearly, link A→C has deep fade in frequency 12, 13, 17, and 20, while link A→B has very good SNR on these subcarriers. Based on this observation, one should load subcarrier 12, 13, 17 and 20 with data from A→C, with high-order modulation scheme, and use data from A→B for the rest of subcarrier. Since high-level modulation is used for subcarrier 12, 13, 17 and 20, the overall data rate will be superior than the traditional bit loading method, where only one of either B or C's data is transmitted any time.

In general, we can consider a virtual channel response $H(i)|i = 1..N$ where $H(i)$ is the largest frequency response of the i^{th} subcarrier among all users. The problem of interest is to decide:

- the power level in each subcarrier $p_j \geq 0, for\ i = 1 \cdots N$,
- and the number of bits to be loaded per OFDM symbol R_j for each subcarrier

such that the summation of the transmitted bits over all subcarriers $R = \sum_{j=1 \cdots N} R_j$ is maximized, subject to the desired symbol error rate

$$SER(p_j, R_j, H(u, v_j, j)) < SER,\ \ forj = 1 \cdots N$$

and the power budget constraints $\sum_{j=1 \cdots N} p_j = P_{budget}$

Here R_j determines the actual modulation scheme to be used for the j^{th} subcarrier. For example, $R_j = 4$ indicate that 16-QAM will be used for the j^{th} subcarrier. In the classical water-filling solution, R_j is calculated via $log(1+SNR)$ once the optimal power allocation is determined. However these data rates are difficult to approach in practical system. The practically allowed modulation schemes are always in discrete space, such as BPSK, QPSK, 8-QAM, 16-QAM, and 32-QAM.

III. Close-form Expression

The classical water-filling algorithm is not computational efficient (in $O(N^2)$). The method in [7] uses table lookup and bisection to reduce the converge time when searching the Lagrange-formulated problem. Compared to those methods, the most important difference of our proposed algorithm is that we avoid the search of optimality from scratch, instead, we find a sub-optimal solution quickly via analytical model. The key to the success of our method are thus: (1) how to quickly identify a good pseudo-solution, and (2) how to convert the pseudo-solution efficiently into a feasible solution.

Our method consists two steps: (1) we obtain an close-form expression of power vector based on a slightly modified problem setting, this will be used as an approximation of the optimum power vector. (2) based on the results in step 1, modify the problem setting so that the power constraints are gradually satisfied. It will be showed later that the computation cost for step (1) only requires $O(N)$ multiplication, and step (2) is also very efficient if the initial power allocation is very close to the optimum point, which accelerates the convergence process.

We consider the following Lagrange problem

$$\begin{aligned} R = & \quad \max_{p_i} \textstyle\sum_{i=1 \ldots N} R_i(p_i) \\ \sum_{i=1 \ldots N} p_i = & \quad P_{budget} \\ where \ \ R_i(p_i) = & \quad \log \tfrac{p_i H_i^2}{\sigma^2} + 1 \end{aligned} \tag{1}$$

Now obtain the Lagrange of (1)

$$L(p, \lambda) = R + \lambda(\sum_{i=1 \ldots N} p_i - P_{budget})$$

It is easy to see the all power-rate function are concave, thus the objective function is also concave. The necessary and sufficient condition for the above convex optimization problem are thus:

$$\frac{\partial L}{\partial p_i} = \frac{\partial R_i}{\partial p_i} = \frac{\frac{H_i^2}{\sigma^2}}{\frac{H_i^2}{\sigma^2} p_i + 1} - \lambda = 0 \tag{2}$$

$$for\ i = 1 \ldots N$$

$$\frac{\partial L}{\partial \lambda} = \sum_{i=1 \ldots N} p_i - P_{budget} = 0 \tag{3}$$

From (3), we have

$$p_i = \frac{\frac{\frac{H_i^2}{\sigma^2}}{\lambda} - 1}{\frac{H_i^2}{\sigma^2}} \tag{4}$$

replace p_i into the original power budget constraint, we obtain

$$\sum_{i=1 \ldots N} \frac{\frac{\frac{H_i^2}{\sigma^2}}{\lambda} - 1}{\frac{H_i^2}{\sigma^2}} = P_{budget}$$

Thus the optimum

$$\lambda = \frac{N}{(P_{budget} + \sum_{i=1..N} \frac{1}{\frac{H_i^2}{\sigma^2}})} \tag{5}$$

Substitute λ back in equation (4), we immediately get the optimum power assignment.

The result here is indeed very similar to that of the the water-filling solution except that we don't impose the non-negative power requirements. We do observe that when the given power budget is too low, the solution obtained above might contains negative power values for some subcarriers, thus is not feasible solution for the original problem. This is illustrated by Figure 2. When the available total power budget is 8 db, 6 of the 64 subcarriers are assigned negative power values. The location of these non-feasible points coincides with the deeply-faded subcarriers. The cause of this phenomena is obvious: in the formulation of the problem, we ignore the non-negative requirement. In next section, we will see how the non-negative power requirement can be enforced in deriving the optimum power assignment.

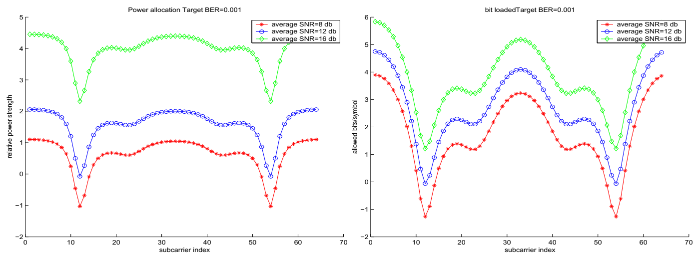

Fig. 2: Optimum power assignment with possible negative power values: (a) power allocation vector, (b) corresponding bit loaded in subcarriers.

IV. Handling Non-negative Power Constraints

When the power constraints $p_i >= 0$ must be satisfied for all subcarriers, we need to use N Lagrange multipliers $\mu_1, ...\mu_N$ for the N additional constraints. The Lagrange becomes

$$L(p, \lambda, \mu_1, ...\mu_N) = R + \lambda(\sum_{i=1...N} p_i - P_{budget}) - \sum_{i=1...N} \mu_i * p_i$$

the Kuhn-Tucker Theorem (KTT) conditions for the optimum solution can be expressed:

$$\frac{\partial L}{\partial p_i} = \frac{\partial R_i}{\partial p_i} - \lambda - \mu_i = \frac{\frac{H_i^2}{\sigma^2}}{\frac{H_i^2}{\sigma^2} p_i + 1} - \lambda(J) - \mu_i = 0 \quad for\ i = 1 \dots N \tag{6}$$

$$\frac{\partial L}{\partial \lambda} = \sum_{i=1...N} p_i - P_{budget} = 0 \tag{7}$$

$$\mu_i * p_i = 0 \quad for\ i = 1 \dots N \tag{8}$$

Denote $J \subset Z_N = \{1, 2, ..., N\}$ and its supplement set $J^C = Z_N - J$. From (8), for any feasible solution $(p_1, ...p_N, \lambda, \mu_1, ...\mu_N)$ satisfying the above KTT conditions, there must exists one such set J such that:

$$\{p_i = 0, u_j = 0 | i \in J, j \in J^C\}$$

For a given J, the close form expression for p_i can be obtained by following steps (we use $\lambda(J)$ to denote the λ value for the particular J):

(1) for $i \in J^C$, we have $\mu_i = 0$. Substitute μ_i into (6) and we have:

$$\frac{\partial L}{\partial p_i} = \frac{\frac{H_i^2}{\sigma^2}}{\frac{H_i^2}{\sigma^2} p_i + 1} - \lambda(J) = 0$$

thus $p_i = \frac{\frac{\frac{H_i^2}{\sigma^2}}{\lambda(J)} - 1}{\frac{H_i^2}{\sigma^2}}$, which is the same case as in the unconstrained case (4).

(2) for $i \in J$, we set $p_i = 0$.

(3)Substitute the results in step 1 and step 2 into equation (7), the corresponding $\lambda(J)$ value is

$$\lambda(J) = \frac{|J^C|}{(P_{budget} + \sum_{i \in J^C} \frac{1}{\frac{H_i^2}{\sigma^2}})} \tag{9}$$

We thus have the following close-form expression for the possible solution

$$p_i = \begin{cases} 0, i \in J \\ \frac{1}{\lambda(J)} - \frac{1}{\frac{H_i^2}{\sigma^2}}, i \in J^C \end{cases} \tag{10}$$

$$u_i = \begin{cases} \frac{H_i^2}{\sigma^2} - \lambda(J), i \in J \\ 0, i \in J^C \end{cases} \tag{11}$$

$$\lambda(J) = \frac{|J^C|}{(P_{budget} + \sum_{i \in J^C} \frac{1}{\frac{H_i^2}{\sigma^2}})} \tag{12}$$

Since there are 2^N different subsets of Z_N, the optimum solution can be found by checking all these cases and select the best one.

V. An Fast Iterative Searching Algorithm

With the non-negative power constraints, the brutal-force method in previous section requires examining 2^N possible candidates. This is not practical when the number of subcarrier is large in actual OFDM system which contains up to 1024 or more subcarriers. Fortunately, with little modification of the solution in section (III), we are able to completely avoid the brutal-force searching and still obtain the optimum solution with non-negative power constraints.

The algorithm is described by an iterative procedure as follows,

1) obtain the initial $p_i(0)$'s by equation (5) and (4), $p_i(0)$ might contains negative value.
2) In the k^{th} iteration, if all $p_i(k) >= 0$, goto step 4 and terminate; otherwise, for all $\{i | p_i < 0\}$, remove corresponding terms of subcarrier i in the optimization problem (1) to obtain a revise problem.
3) Using equation (5) and (4), solve the revised problem and get a new set of $p_i(k+1)$. go back to step 2.
4) The final solution is now obtained by

$$p_i^* = \begin{cases} p_i(k), i^{th}\ subcarrier\ is\ used \\ 0, otherwise \end{cases}$$

In fact, we have the following proposition:

Proposition 1: the solution obtained by above fast algorithm is the optimum solution satisfying all constraints.

Proof: Let $p_i(0)$ be the initial solution which is the global optimum (maximum) point without the non-negative power constraints. Let us also assume that the above procedure terminates at iteration k and results in a power vector $p_i(k)$, with objective function evaluated as $R(k)$. If there exist another power assignment $\hat{\mathbf{P}}$ which also satisfy all power constraints and have a higher objective value $\hat{R}$. There must exists an i such that $p_i(k) = 0$ and $\hat{p}_i > 0$, otherwise the two solutions will be the same. Thus we can found a $0 < \theta < 1$ such that:

$$p_i(k) = \theta p_i(0) + (1 - \theta)\hat{p}_i \quad for\ all\ i$$

That is, we find a small positive value θ to represent the converged power vector $\mathbf{P(k)}$ by the initial power vector $\mathbf{P(0)}$

and the optimum power vector $\hat{\mathbf{P}}$. Due to the concavity of the objective function, we have:

$$\begin{aligned} R(\mathbf{P}(\mathbf{k})) &= R(\theta\mathbf{P}(\mathbf{0}) + (1-\theta)\hat{\mathbf{P}}) \\ &> \theta R(\mathbf{P}(\mathbf{0})) + (1-\theta)R(\hat{\mathbf{p}}_{\mathbf{i}}) \\ &> \theta R(\mathbf{P}(\mathbf{k})) + (1-\theta)R(\mathbf{P}(\mathbf{k})) \\ &> R(p_i(k)) \end{aligned}$$

Thus a conflict. In the above derivation, we use the fact that $R(\mathbf{P}(\mathbf{0})) > R(\mathbf{P}(\mathbf{k}))$. This is because $\mathbf{P}(\mathbf{0})$ is the optimum power vector without non-negative power constraints, thus have a higher $R()$ value than that of $\mathbf{P}(\mathbf{k})$, which is subject to non-negative power constraints.

$R(\hat{\mathbf{P}}) > R(\mathbf{P}(\mathbf{k}))$ since $\hat{\mathbf{P}}$ is the optimum solution in assumption.■

VI. Map into Discrete Space Solution

So far, we already demonstrate an efficient algorithm to obtain an sub-optimum power solution $\mathbf{p}^*$ in the continues space, and thus the corresponding achievable rate $\mathbf{R}^*$, also in continues space. However, the optimum bits/symbol in a practical system can only be selected from a finite set of integer values. Thus we needs to map the continues $(\mathbf{P}^*, \mathbf{R}^*)$ to a feasible $(\mathbf{P}', \mathbf{R}')$ with integer rate.

A potential computation efficient choice now is to apply water-filling based on the solution $\lfloor R_i^* \rfloor$. It is obvious that the overall power usage corresponding to $\lfloor R_i^* \rfloor$ is smaller than the power budget, which give us some residue power. Then we find the subcarrier index j which needs least amount of power to load one more bits, i.e.

$$j = \arg\min p_i(\lfloor R_i \rfloor) - p_i(\lceil R_i \rceil)$$

The bit/symbol for subcarrier j is then increased by one and the transmitting power is adjusted accordingly. If we have more residue power available, the same procedure is repeated until all residue power is used up, or not sufficient to support one more bits for any subcarrier.

The computation cost of this modified water-filling is much smaller than the original algorithm, since in our case, we already fill all subcarrier "close to full" via the continuous space optimum solution before executing the water-filling procedure. The number of iteration at the water-filling stage is thus significantly reduced. Figure 3 shows that power and bit allocation in the continuous space and the discrete space obtained by water-filling post-process.

In Figure 3, the curve with stars is the power/bit allocation obtained from the original water-filling algorithm, which represent the optimum solution in the discrete space. We also modified the original water-filling algorithm which allow bit-allocation in the unit of 0.1 and 0.2 bits. The corresponding power/bit allocations of this modified water-filling are shown in curves with diamonds and circles. The continuous space solution is shown in solid blue line. It is clearly observed that as the bit-step becomes smaller, the water-filling converges to the continuous space solution. Also notice that in Figure 3.b, the allowed bits in each subcarrier is indeed the ceiling or flooring of the corresponding continuous solution. This observation further confirms our earlier claim that the optimum solution in discrete space is indeed a neighboring integer point of the continuous solution.

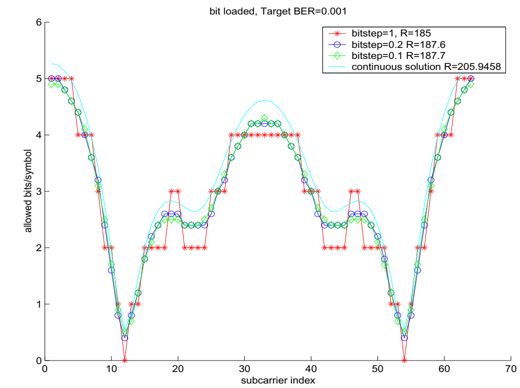

Fig. 3: Comparison of the continues-space solution to the result of water-filling algorithm: corresponding bit loaded in subcarriers. Assumed channel is figure 1.(b), total 64 subcarrier.

We compared the bits/symbol of the continuous space solution, the simple heuristic (as a low bound), pure water-filling algorithm and the improved water-filling algorithm. The overall data rate obtained by our iterative solution is very close to that of water-filling algorithm (high than 99% optimal rate). Thus our algorithm can potential produce good throughput with reasonable computation overhead.

VII. Conclusion

In this paper, we discussed the optimum power/bit allocation for OFDM system with multiple connections. In particular, we present an algorithm with significantly less computation cost yet close-to-optimum performance based on the Lagarange and water-filling methods.

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