J. Algebra **225** (2000), 275–298.

HEREDITARY NOETHERIAN PRIME RINGS 3: INFINITELY GENERATED PROJECTIVE MODULES

LAWRENCE S. LEVY AND J. CHRIS ROBSON

Abstract. We describe the structure of infinitely generated projective modules over hereditary Noetherian prime rings. The isomorphism invariants are uniform dimension and ranks at maximal ideals. Infinitely generated projective modules need not be free. However, every uncountably generated projective module is the direct sum of a finitely generated module and free modules over specific finite overrings of the given ring in its Goldie quotient ring.

1. Introduction

Relatively little seems to be known about infinitely generated (i.e. not finitely generated) projective modules over noncommutative Noetherian rings. A well-known result of Kaplansky states that all infinitely generated projective modules over commutative Dedekind domains are free [9]. Bass extended this by showing that every *uniformly big* projective module P over any (noncommutative) Noetherian ring is free [1]. Here 'uniformly big' means that, for every maximal ideal M of the ring, P/PM requires the same infinite cardinal number of generators as does P.

The present paper adds to these a different type of result. It describes precisely the structure of all infinitely generated projective modules (and their direct sum behaviour) over certain Noetherian rings in a context where these modules can be neither free nor unique direct sums of indecomposables.

Let R denote an HNP ('hereditary Noetherian prime') ring; that is, a prime ring in which every left ideal and every right ideal is a finitely generated projective module. Our recent paper [12] describes the structure of finitely generated projective modules over R. Here we complete the analysis of projective modules over these rings (which, in the commutative case, become the Dedekind domains studied by Kaplansky).

Our results take their simplest form when either the modules are uncountably generated or else R is a *classical hereditary order* (i.e. is module-finite over a central Dedekind domain). So it is a pair of results in these situations which we mention first. Each involves free modules S^α over some overrings S of R in its Goldie quotient ring R_{quo} where α is a cardinal number.

Theorem 1.1. *(See Theorem 4.3) Let R be a classical hereditary order and P a countably generated projective R-module. Then $P \cong H \oplus S^{\aleph_0}$ where H is a finitely*

Date: 5 August, 1998, minor revisions/corrections 4 February, 23 August,1999, 5 January, 2000.

1991 *Mathematics Subject Classification.* 16D40, 16D70,16E60, 16P40.

Key words and phrases. projective, hereditary, HNP, genus.

Research partially supported by NSA grants (US) and an EPSRC visiting fellowship (UK)..

generated projective R-module and S is a ring with $R \subseteq S \subset R_{\text{quo}}$ such that S is a finitely generated right R-module.

Theorem 1.2. *(See Theorem 4.9) Let P be an uncountably generated projective module over any HNP ring R. Then*

$$P \cong H \oplus S_1^{\alpha_1} \oplus S_2^{\alpha_2} \oplus \dots \tag{1.2.1}$$

where H is a finitely generated projective R-module, the α_n are infinite cardinal numbers satisfying $\alpha_1 < \alpha_2 < \dots$, with at least one α_i uncountable, and $S_1 \subset S_2 \subset \dots$ is a finite or countably infinite sequence of rings, each a finitely generated right R-module with $R \subseteq S_n \subset R_{\text{quo}}$.

Furthermore, Theorem 4.10 shows that the sequence of ordered pairs (S_i, α_i) is unique. The contribution of H to the isomorphism class of P is the collection of ranks of H at a certain finite number of maximal ideals of R. Whilst these are determined by P, H itself is not unique.

There remains one case not covered by Theorems 1.1 and 1.2, namely when R is an arbitrary HNP ring and P is countably generated. Here we give a full set of invariants for the isomorphism class of P, but not an easily visualized canonical form for P. See Remark 4.12.

The approach to these results involves the notion of the 'genus' of P which we now describe. First consider a simple Artinian ring, A say. Every right A-module H is the direct sum of some unique number (finite or infinite) of copies of the unique simple A-module. We call this number the *A-rank* of H.

Since our HNP ring R is prime and Noetherian, R_{quo} is a simple Artinian ring. Therefore, for every projective right R-module P, we can define the *uniform dimension* $\text{udim}(P)$ to be the R_{quo}-rank of $P \otimes_R R_{\text{quo}}$. Let W be an unfaithful simple R-module, with annihilator M. Then R/M is a simple Artinian ring [11, Lemma 2.5]. We define $\rho(P, W) = \rho(P, M)$, the *rank* of P at W — equivalently, at M — to be the R/M-rank of P/PM. Thus $\rho(P, W)$ is the largest cardinal number α such that P can be mapped onto the direct sum W^α of α copies of W.

Let $\mathcal{W}$ be a set of representatives of the isomorphism classes of *unfaithful* simple right R-modules. We define the *genus* $\Psi(P)$ of a projective right R-module P to be the function, defined on $\{0\} \bigcup \mathcal{W}$, such that $\Psi(P)_0 = \text{udim}(P)$ and $\Psi(P)_W = \rho(P, W)$ $(W \in \mathcal{W})$. We usually think of $\Psi(P)$ as a family of cardinal numbers — namely $\Psi(P)_0$ and $\Psi(P)_W$ — indexed by $\{0\} \bigcup \mathcal{W}$. Equivalently, the genus of P is the class of all projective right R-modules Q such that $\Psi(Q) = \Psi(P)$. When the HNP ring R is a finitely generated module over a central Dedekind domain, thinking of the genus of P as a class of modules, in this way, agrees with the classical notion of the genus of P [13, 5.1].

The main theorem about genus states simply:

Theorem 1.3. *(See Theorem 4.11) Let P, P' be infinitely generated projective R-modules. Then $P \cong P'$ if and only if $\Psi(P) = \Psi(P')$.*

As with Kaplansky's result, this displays a simplification when compared with the corresponding result for finitely generated projective modules [12, Theorem 4.4] where the Steinitz class of P and P' must also match. One consequence is the determination, in terms of genera, of when one projective R-module P is isomorphic to a proper direct summand of a given infinitely generated projective R-module Q.

When $\operatorname{udim}(Q)$ is uncountable or R is a classical hereditary order, the condition is simply that $\Psi(P) \leq \Psi(Q)$.

The paper is organized as follows: Section 2 reviews some frequently needed results about the finitely generated case; Section 3 establishes precisely when a family of cardinal numbers, indexed by $\{0\} \cup \mathcal{W}$, is the genus of some infinitely generated projective R-module; Section 4 gives the proofs of Theorems 1.1, 1.2 and 1.3; and Section 5 demonstrates when one projective R-module P is isomorphic to a proper direct summand of another.

Notation 1.4. Throughout the paper R denotes a hereditary Noetherian prime ring — an HNP ring, for short. To avoid trivialities, we always assume that R is not simple Artinian; i.e. $R \neq R_{\text{quo}}$. We usually use the word 'module' to indicate a right module. In particular, a *finite overring* of R is a ring S such that S_R is finitely generated and $R \subseteq S \subset R_{\text{quo}}$.

The phrase 'infinitely generated' will be taken to mean 'not finitely generated'; and 'uncountably generated' means 'not countably generated'. Finally, $\mathcal{W}$ denotes a set of representatives of the isomorphism classes of *unfaithful* simple right modules.

2. Results about the Finitely Generated Case

There is a relatively complete structure theory for finitely generated projective R-modules which appears in [12]. That, in turn, relies upon results from [11] about simple R-modules and their extensions. We collect together here the basic ideas and facts needed in this paper.

A *cycle tower* is [11, Definition 3.3] a finite sequence $W_1, W_2, \ldots, W_n$ of nonisomorphic unfaithful simple right R-modules such that $\operatorname{Ext}^1_R(W_i, W_{i+1}) \neq 0$ for $i \neq n$ and $\operatorname{Ext}^1_R(W_n, W_1) \neq 0$. As suggested by the name, this is considered to be the same cycle tower as $W_2, W_3, \ldots, W_n, W_1$. For any cycle tower $\mathcal{C}$ we define $\rho(R, \mathcal{C}) = \sum_{W \in \mathcal{C}} \rho(R, W)$.

A *faithful tower* is a finite sequence $W_0, W_1, \ldots, W_n$ of nonisomorphic simple right R-modules such that each $\operatorname{Ext}^1_R(W_i, W_{i+1}) \neq 0$, W_0 is faithful, each other W_i is unfaithful, and the sequence cannot be extended to the right. In fact, every simple right R-module W belongs to a unique cycle tower or faithful tower (but never both) [11, Theorem 3.4]. A tower is *trivial* if it consists of a single simple module.

Theorem 2.1. [12, Theorem 2.16] *Let Φ be a family of nonnegative integers, indexed by $\{0\} \cup \mathcal{W}$, such that $\Phi_0 \neq 0$. Then Φ is the genus of some nonzero finitely generated projective R-module if and only if the following two conditions hold.*

(i) *Φ has* almost standard rank; *that is, $\Phi_W = \rho(R, W) \cdot \Phi_0 / \operatorname{udim}(R)$ for almost all (i.e. all but finitely many) $W \in \mathcal{W}$; and*

(ii) *Φ has* cycle-standard rank; *that is,*

$$\sum_{W \in \mathcal{C}} \Phi_W = \rho(R, \mathcal{C}) \cdot \Phi_0 / \operatorname{udim}(R)$$

for every cycle tower $\mathcal{C}$.

The special case of this that we use most often in the present paper involves an *essential* right ideal H — that is, a right ideal with $\operatorname{udim}(H) = \operatorname{udim}(R)$. In this situation the theorem becomes:

Corollary 2.2. *Let Φ be a family of nonnegative integers, indexed by $\{0\} \cup \mathcal{W}$, such that $\Phi_0 = \operatorname{udim}(R)$. Then $\Phi = \Psi(H)$ for some essential right ideal H if and only if*

(i) (Almost standard rank) $\Phi_W = \rho(R, W)$ *for almost all* $W \in \mathcal{W}$*;*
(ii) (Cycle-standard rank) $\sum_{W \in \mathcal{C}} \Phi_W = \rho(R, \mathcal{C})$ *for every cycle tower* $\mathcal{C}$*.*

We use the name *essential genus* for any indexed family Φ satisfying the conditions in this corollary; i.e. for the genus of some essential right ideal.

Corollary 2.3. *Let $\mathcal{F}$ be a finite subset of $\mathcal{W}$ containing no entire cycle tower, and for each $W \in \mathcal{F}$, let $r(W)$ be a nonnegative integer. Then there is a finitely generated projective R-module H such that $\rho(H, W) = r(W)$ for all $W \in \mathcal{F}$.*

Proof. By Theorem 2.1 a family Φ of nonnegative integers indexed by $\{0\} \bigcup \mathcal{W}$ is the genus of some nonzero finitely generated projective R-module H if and only if Φ has almost standard rank and cycle-standard rank.

We construct Φ as follows. Let $\Phi_0 = \operatorname{udim} R \cdot \sup\{r(W) \mid W \in \mathcal{F}\}$ and let $\Phi_W = r(W)$ for each $W \in \mathcal{F}$. For each W not so far considered and either belonging to a faithful tower, or belonging to a cycle tower which does not include any member of $\mathcal{F}$, we set Φ_W to be standard; i.e. $\Phi_W = \rho(R, W) \cdot \Phi_0 / \operatorname{udim}(R)$.

Finally, consider the simple modules belonging to some cycle tower $\mathcal{C}$ which does include a member of $\mathcal{F}$. Those in $\mathcal{F}$ are already dealt with. By hypothesis, some W still remain. For all but one, we make Φ_W standard. Note that the choice of Φ_0 ensures that all ranks so far chosen are either standard or less than standard. For the remaining W in $\mathcal{C}$ we choose Φ_W so as to ensure cycle-standard rank for $\mathcal{C}$. We repeat this for each of the finitely many cycle towers thus involved.

We can now apply Theorem 2.1 to give the desired module H. □

Given two genera Ψ and Φ, we write $\Psi \leq \Phi$ to mean that $\Psi_0 \leq \Phi_0$ and $\Psi_W \leq \Phi_W$ for all $W \in \mathcal{W}$. The next result is proved in [19, Corollary 7.2] and in [12, Theorem 5.1].

Theorem 2.4. *Let P and Q be finitely generated projective R-modules such that $\Psi(P) \leq \Psi(Q)$ and $\operatorname{udim}(P) < \operatorname{udim}(Q)$. Then $Q \cong P \oplus X$ for some $X \neq 0$.*

This result is closely connected with the following cancellation theorem [12, Theorem 3.13].

Theorem 2.5. *Suppose that $P \oplus X \cong Q \oplus X$ for finitely generated projective R-modules P, Q, X such that $\operatorname{udim}(P) \geq 2$. Then $P \cong Q$.*

We end this section with a part of [11, Theorem 7.17]. Recall that a *finite overring* S of R is a ring such that S_R is finitely generated and $R \subseteq S \subseteq R_{\text{quo}}$. When dealing with R together with other subrings of R_{quo}, we write ρ_R for rank as an R-module.

Theorem 2.6. *Let $\mathcal{F}$ be any finite subset of $\mathcal{W}$ containing no entire cycle tower. Then there exists a unique finite overring $S(\mathcal{F})$ of R such that, for every $W \in \mathcal{W}$,*

$$\rho_R(S(\mathcal{F}), W) = 0 \iff W \in \mathcal{F}. \tag{2.6.1}$$

Moreover:

(i) $S(\mathcal{F}) \subseteq S(\mathcal{G}) \iff \mathcal{F} \subseteq \mathcal{G}$.
(ii) *Every finite overring of R equals $S(\mathcal{F})$ for some unique $\mathcal{F}$.*

(iii) *Let P be a finitely generated projective R-module. Then P is an $S(\mathcal{F})$-module if and only if $\rho_R(P,W) = 0$ for all $W \in \mathcal{F}$.*

3. Pregenus and genus

Throughout this section P denotes an infinitely generated projective R-module. We claim that P has a decomposition:

$$P = \oplus_{i \in I} P_i \quad \text{with each } P_i \text{ isomorphic to an essential right ideal of } R. \tag{3.0.1}$$

Since R is hereditary, every projective R-module is isomorphic to a direct sum of right ideals of R [2, Theorem 5.3]. Since R is Noetherian, each of these right ideals is finitely generated. Therefore the preceding decomposition can be refined to a decomposition of P into a direct sum of modules isomorphic to *uniform* right ideals [11, Lemma 2.1]. Finally, since this last decomposition contains infinitely many terms, we can group the terms together — $\operatorname{udim}(R)$ at a time — getting the desired decomposition (3.0.1).

Next we note, by tensoring (3.0.1) with R_{quo} over R and remembering that each $\operatorname{udim}(P_i)$ is finite, that

$$\operatorname{udim}(P) = |I| \tag{3.0.2}$$

where $|I|$ denotes the cardinality of I. This is the way we always view $\operatorname{udim}(P)$. Note that P is countably or uncountably generated precisely when $|I|$ is countable or uncountable, respectively.

We fix decomposition (3.0.1) throughout this section.

Definitions 3.1. Let Φ be a family of cardinals indexed by $\{0\} \cup \mathcal{W}$ (for example, the genus of an infinitely generated projective module). We call Φ a *pregenus* if it satisfies the following three conditions.

(i) Φ_0 is infinite, and $\Phi_0 \geq \Phi_W$ for all $W \in \mathcal{W}$.

(ii) (a) For every cardinal number $\alpha < \Phi_0$, there are only finitely many $W \in \mathcal{W}$ such that $\Phi_W \leq \alpha$.

(b) For every positive integer n there are only finitely many $W \in \mathcal{W}$ such that $\Phi_W \leq n \cdot \rho(R,W)$.

(iii) In each cycle tower $\mathcal{C}$ there is at least one member W such that $\Phi_W = \Phi_0$.

Our aim in the present section is to show that a pregenus is the same thing as the genus of an infinitely generated projective module. We do the easier half of this now.

Theorem 3.2. *Let P be an infinitely generated projective R-module. Then $\Psi = \Psi(P)$ is a pregenus.*

Proof. We check the three conditions in the definition of 'pregenus'.

(i) (3.0.2) shows that $\Psi_0 = \operatorname{udim}(P) = |I|$, which is infinite by hypothesis. Now consider any $W \in \mathcal{W}$. Since each P_i in decomposition (3.0.1) is finitely generated, we have $\rho(P_i, W) < \aleph_0$ and therefore

$$\rho(P,W) = \sum_{i \in I} \rho(P_i, W) \leq |I| \cdot \aleph_0 = |I| = \Psi_0.$$

(ii)(a), when α is infinite. Suppose, to the contrary, that infinitely many such W exist, and choose a countably infinite subset $\{W_1, W_2, \dots\}$ of $\mathcal{W}$ such that every $\rho(P, W_j) \leq \alpha$. Let $I(W_j) = \{i \in I \mid \rho(P_i, W_j) \neq 0\}$. It follows that

$$|I(W_j)| \leq \sum_{i \in I(W_j)} \rho(P_i, W_j) = \rho(P, W_j) \leq \alpha.$$

Let $I' = \bigcup I(W_j)$. Since the number of sets $I(W_j)$ is $\aleph_0$ we have $|I'| \leq \aleph_0 {\cdot} \alpha = \alpha < |I|$. Hence there exists $i \in I - I'$; and then $\rho(P_i, W_j) = 0$ for all W_j, contradicting almost standard rank of the essential right ideal P_i [Corollary 2.2].

(ii)(a) when $\alpha = n$ is finite, and (ii)(b). Choose any $n+1$ of the summands P_i, calling them $P_1, \dots, P_{n+1}$. For almost all $W \in \mathcal{W}$, each of the essential right ideals $P_1, \dots, P_{n+1}$ has the standard W-rank, $\rho(R, W)$. For any such W we have

$$\rho(P, W) \geq \rho(P_1 \oplus \ldots \oplus P_{n+1}, W) = (n+1) \cdot \rho(R, W) > n \cdot \rho(R, W)$$

which proves (ii)(b) and completes the proof of (ii)(a).

(iii) This holds by cycle-standard rank [Corollary 2.2], since $\operatorname{udim}(P_i) = \operatorname{udim}(R) < \infty$ for all $i \in I$. □

Definitions 3.3. Given a pregenus Φ we define $\mathcal{D}(\Phi)$ to be the set of distinct cardinals in Φ, that is, of the form Φ_W or Φ_0. Note that $\mathcal{D}(\Phi)$, like any set of cardinals, is well-ordered by the ordering of the cardinal numbers.

Before establishing the converse to Theorem 3.2, we need some facts about $\mathcal{D}(\Phi)$. We use the notation ω to denote the order type of the natural numbers.

Lemma 3.4. *Let Φ be a pregenus.*

(i) *The well-ordered set $\mathcal{D}(\Phi)$ is either finite or has order type $\omega + 1$.*
(ii) *If the order type is $\omega + 1$ — that is, $\mathcal{D}(\Phi) = \{\alpha_1 < \alpha_2 < \cdots < \alpha_\omega \ (= \Phi_0)\}$ — we have $\sup\{\alpha_i \mid i < \omega\} = \Phi_0$.*
(iii) *If Φ_0 is uncountable, then only finitely many elements of $\mathcal{D}(\Phi)$ are finite (and each of these finite ranks is attained at only finitely many $W \in \mathcal{W}$).*

Proof. (i) These are the well-ordered sets having a greatest element and such that all but the greatest element have only finitely many predecessors.

(ii) Let $\sup\{\alpha_i \mid i < \omega\} = \gamma$. Since Φ_0 is the largest element in $\mathcal{D}(\Phi)$, then $\gamma \leq \Phi_0$. If $\gamma < \Phi_0$ then (by the definition of pregenus) γ can have only finitely many predecessors in $\mathcal{D}(\Phi)$, yielding the contradiction that $\mathcal{D}(\Phi)$ is a finite set.

(iii) If Φ_0 is uncountable then $\aleph_0 < \Phi_0$. Now use condition (ii) in Definitions 3.1. □

The restriction that $\mathcal{D}(\Psi(P))$ contains only finitely many finite elements when $\Psi(P)_0 = \operatorname{udim}(P)$ is uncountable [Lemma 3.4(iii)] can fail if P is countably generated, as we show in the next example. However, this cannot occur for classical hereditary orders – see Theorem 4.3.

Example 3.5. Let R be some HNP ring which has infinitely many nontrivial towers (as provided by [17] or [18]; or see [11, remarks above Lemma 3.8]). Choose a countably infinite subset $\mathcal{F} = \{W_1, W_2, \dots\}$ of $\mathcal{W}$ which does not include any complete cycle tower. Let $\mathcal{F}_n$ denote the subset $\{W_1, W_2, \dots, W_{n-1}\}$. By Theorem 2.6 there is a finite overring $S_n = S(\mathcal{F}_n)$ of R such that $\rho_R(S_n, W_i) = 0$ if and only if $i < n$. Let $P = \oplus S_n$. Then $n \leq \rho(P, W_n) < \aleph_0$ for each n, and $\rho(P, W) = \aleph_0$

for all unfaithful simple modules $W \notin \mathcal{F}$. Thus $\mathcal{D}(\Psi(P))$ contains infinitely many positive integers.

Note also that, in this example, $\mathcal{D}(\Psi(P))$ has order type $\omega + 1$. Moreover, $\Psi(P)_W = \rho(P, W)$ is finite for all $W \in \mathcal{W}$ if and only if $\mathcal{F} = \mathcal{W}$. (This can happen if R has only countably many nontrivial towers, all faithful.)

We can now prove the second half of our main result.

Theorem 3.6. *Let Φ be a pregenus. Then Φ is a genus; that is, $\Phi = \Psi(P)$ for some infinitely generated P.*

Proof. By Lemma 3.4, $\mathcal{D}(\Phi)$ is either a finite sequence or an infinite sequence $\alpha_1 < \alpha_2 < \dots$, followed by $\alpha_\omega = \Psi_0$. We consider two cases separately.

Case 1: Suppose that only finitely many α_i are finite. Note that each finite α_i has only finite multiplicity in Φ, by condition (ii)(a) in the definition of pregenus. Hence the set $\mathcal{F}$ of all $W \in \mathcal{W}$ such that Φ_W is finite is again a finite set. Note that $\mathcal{F}$ contains no entire cycle tower, by condition (iii) in the definition of pregenus. Therefore, by Corollary 2.3, there is a finitely generated H such that $\rho(H, W) = \alpha_i$ for all W such that $\Phi_W = \alpha_i$ is finite.

Next, let $\beta_1 < \beta_2 < \dots$ be the infinite elements of the sequence $\alpha_1 < \alpha_2 < \dots$. (Thus we are excluding α_ω, if it exists.) Consider some β_i. Since $\beta_i < \Phi_0$, which is infinite, the definition of pregenus implies that the set $\mathcal{G}_i = \{W \in \mathcal{W} \mid \Phi_W < \beta_i\}$ is finite. Moreover, $\mathcal{G}_i$ contains no entire cycle tower by (iii) in the definition of pregenus. Let $S_i = S(\mathcal{G}_i)$, the finite overring of R given by Theorem 2.6. Then, for $W \in \mathcal{W}$, $\rho(S_i, W) = 0$ if and only if $\Phi_W < \beta_i$. We set

$$P = H \oplus S_1^{\beta_1} \oplus S_2^{\beta_2} \oplus \dots,$$

noting that the β_i can form either a finite or infinite sequence. It is easily checked that $\Psi(P) = \Phi$, which completes the proof in this case.

Case 2: Now consider the alternative case, when infinitely many α_i are finite. Lemma 3.4 (i) and (ii) imply that α_ω exists and that $\alpha_\omega = \Phi_0 = \aleph_0$; so the P we want is countably generated. Also α_i is finite whenever $i \neq \omega$.

It suffices to describe a collection of essential genera Φ_n such that $\sum_{n=1}^{\infty} \Phi_n = \Phi$. For then, since each Φ_n is an essential genus, there is an essential right ideal P_n such that $\Phi_n = \Psi(P_n)$. Then $P = \oplus P_n$ is as required.

Fix a value of n. Since we want Φ_n to be essential, we set $(\Phi_n)_0 = \operatorname{udim}(R)$. We still need to choose $(\Phi_n)_W$ for each $W \in \mathcal{W}$, bearing in mind the two requirements of Corollary 2.2 that Φ_n satisfy almost standard rank and cycle-standard rank. First we define some subsets of $\mathcal{W}$, namely

$$\mathcal{F}_n = \{W \in \mathcal{W} \mid \Phi_W \leq n \cdot \rho(R, W)\},$$

with the convention that $\mathcal{F}_0$ is the empty set. Note that $\mathcal{F}_{n-1} \subseteq \mathcal{F}_n$ and, by (ii)(b) in the definition of a pregenus, $\mathcal{F}_n$ is a finite set. The basic formula for $(\Phi_n)_W$ which comes next will apply to most $W \in \mathcal{W}$, as specified later.

(3.6.1)

$$(\Phi_n)_W = \begin{cases} \rho(R, W) & \text{if } W \notin \mathcal{F}_n; \\ \Phi_W - (n-1)\rho(R, W) & \text{if } W \in \mathcal{F}_n - \mathcal{F}_{n-1}; \\ 0 & \text{if } W \in \mathcal{F}_{n-1}. \end{cases}$$

For the second part of (3.6.1) to be a legitimate definition we need to know that $(\Phi_n)_W \geq 0$. In fact, the definition of $\mathcal{F}_n$ shows that

$$0 < (\Phi_n)_W \leq \rho(R, W) \quad \text{if } W \notin \mathcal{F}_{n-1}. \tag{3.6.2}$$

Now we need to specify to which W this applies. Every W belongs to some unique tower $\mathcal{C}$. We subdivide the definition of $(\Phi_n)_W$ into three cases according to the nature of $\mathcal{C}$.

Case (a): If $\mathcal{C}$ is a faithful tower then we simply apply (3.6.1).

Case (b): If $\mathcal{C}$ is a cycle tower that does not meet $\mathcal{F}_n$, we use (the first part of) (3.6.1).

Case (c): Suppose $\mathcal{C}$ is a cycle tower that meets $\mathcal{F}_n$. By condition (iii) in the definition of a pregenus, Φ_W cannot be finite for all $W \in \mathcal{C}$. Fix an element $W'(\mathcal{C}) \in \mathcal{C}$ such that $\Phi_{W'(\mathcal{C})} = \aleph_0$. For $W \in \mathcal{C} - \{W'(\mathcal{C})\}$, use (3.6.1); and, having done this, set:

$$(\Phi_n)_{W'(\mathcal{C})} = \rho(R, \mathcal{C}) - \sum \{ (\Phi_n)_W \mid W \in \mathcal{C} - \{W'(\mathcal{C})\} \}. \tag{3.6.3}$$

Again we must verify that this is positive. In fact, we prove:

$$(\Phi_n)_{W'(\mathcal{C})} \geq \rho(R, W'(\mathcal{C})). \tag{3.6.4}$$

First note that, by definition of $\rho(R, \mathcal{C})$, equality holds when $(\Phi_n)_W$ is standard for all $W \in \mathcal{C} - \{W'(\mathcal{C})\}$. Therefore it suffices to verify that any nonstandard rank that we assigned in (3.6.1) is less than the standard rank $\rho(R, W)$. Only the second of the three situations in (3.6.1) is nontrivial, and this is done in (3.6.2).

We now verify that Φ_n is an essential pregenus.

Almost standard rank. Only finitely many cycle towers meet the finite sets $\mathcal{F}_n$ and $\mathcal{F}_{n-1}$. Therefore, in verifying almost standard rank, we can ignore any W in cases (a)–(c) that belongs to either of these sets; and we can ignore case (c) completely. For all remaining W, $(\Phi_n)_W$ equals the standard rank $\rho(R, W)$.

Cycle-standard rank. We may ignore case (a); and cycle-standard rank is obviously satisfied in case (b). It holds in case (c) by (3.6.3).

Finally, having established that each Φ_n is an essential pregenus, we need to check that $\sum_n (\Phi_n)_W = \Phi_W$ for every W. Suppose first that Φ_W is infinite, and hence $W \notin \bigcup_n \mathcal{F}_n$. Here it suffices to prove that $(\Phi_n)_W > 0$ for all n. In cases (a) and (b) this is given in the first part of (3.6.1). In case (c) it is true by the first part of (3.6.1) except if $W = W'(\mathcal{C})$, where it is given by (3.6.4).

This leaves the case that Φ_W is finite, in which case there is a smallest n, which we fix, such that $W \in \mathcal{F}_n$; and then $W \in \mathcal{F}_i$ for all $i \geq n$. In case (a) (3.6.1) states that $(\Phi_i)_W$ is standard for the first $n-1$ values of i and is zero for $i \geq n+1$. Adding all this to the second part of (3.6.1) gives the desired sum. Case (b) does not concern us here. In case (c) we can ignore $W'(\mathcal{C})$ since $\Phi_{W'(\mathcal{C})}$ is infinite. Therefore the proof for case (a) works here, too. □

Case 2 of the above proof is the first instance of a recurring theme in this paper: countably generated projective modules can be more complicated than uncountably generated projectives because they can have infinitely many finite ranks. (As mentioned above, this does not happen with classical hereditary orders.)

As we shall see in the next section, the module P in the above theorem is determined up to isomorphism by Φ.

4. Structure Theorems

Throughout this section P denotes an infinitely generated projective R-module which therefore has a decomposition $P = \oplus\{P_i \mid i \in I\}$ with $|I|$ infinite and each P_i isomorphic to an essential right ideal [see(3.0.1)]. The first main result establishes Theorem 1.3 when P is countably generated.

Theorem 4.1. *If P is countably generated and if $\Psi(P) = \Psi(Q)$ then $P \cong Q$.*

Proof. Write both P and Q as countable direct sums of essential right ideals, as at the beginning of this section: $P = P_1 \oplus P_2 \oplus \dots$ and $Q = Q_1 \oplus Q_2 \oplus \dots$. We start by comparing certain initial segments of these sums.

Choose n. Since the rank of each P_i and each Q_i is almost standard (i.e. equals $\rho(R, W)$ for almost all W), we have $\rho(P_1 \oplus P_2 \oplus \dots \oplus P_n, W) = \rho(Q_1 \oplus Q_2 \oplus \dots \oplus Q_n, W)$ for almost all W. Suppose that W is one of the finite number of exceptions to this, and recall that $\rho(P, W) = \rho(Q, W)$ for all W since $\Psi(P) = \Psi(Q)$. Therefore we can choose an $m > n$ so that $\sum_{i=1}^{m} \rho(P_i, W) \geq \sum_{i=1}^{n} \rho(Q_i, W)$. Since the number of exceptions is finite, we can deduce the first statement in (4.1.0) — and the second statement follows by symmetry.

(4.1.0)
$$(\forall n)(\exists m > n) \quad \Psi(P_1 \oplus P_2 \oplus \dots \oplus P_m) \geq \Psi(Q_1 \oplus Q_2 \oplus \dots \oplus Q_n).$$
$$(\forall m)(\exists n > m) \quad \Psi(P_1 \oplus P_2 \oplus \dots \oplus P_m) \leq \Psi(Q_1 \oplus Q_2 \oplus \dots \oplus Q_n).$$

Now choose any positive integer $m(1)$, and let $S_1 = P_1 \oplus \dots \oplus P_{m(1)}$. Then there exists $n(1) > m(1)$ such that $\Psi(S_1) \leq \Psi(T_1)$ where $T_1 = Q_1 \oplus \dots \oplus Q_{n(1)}$. Therefore, by Theorem 2.4, S_1 is isomorphic to a direct summand of T_1. This yields $X_1 \neq 0$ such that the first isomorphism in (4.1.1) below holds. In the same way, there exists $m(2) > n(1)$ such that the initial segment $S_2 = P_1 \oplus \dots \oplus P_{m(2)}$ satisfies the second isomorphism in (4.1.1) for some $Y_1 \neq 0$.

(4.1.1)
$$S_1 \oplus X_1 \cong T_1, \qquad T_1 \oplus Y_1 \cong S_2.$$

Continuing in this fashion, we obtain an infinite sequence of initial segments $S_1, T_1, S_2, T_2, \dots$ each containing strictly more terms than the previous one. Then, as in (4.1.1), we have another pair of relations

(4.1.2)
$$S_2 \oplus X_2 \cong T_2, \qquad T_2 \oplus Y_2 \cong S_3;$$

and so on.

Now, for $j > i$ (and by slight abuse of notation) define $S_j - S_i = \oplus_{m(i) < k \leq m(j)} P_k$, the direct sum of all P_k that appear in S_j but not in S_i; and define $T_j - T_i$ analogously. The two isomorphisms in (4.1.1) yield:

$$S_1 \oplus (S_2 - S_1) \cong S_2 \cong S_1 \oplus X_1 \oplus Y_1.$$

We have $\mathrm{udim}(X_1 \oplus Y_1) \geq 2$ since both X_1 and Y_1 are nonzero. Therefore we can cancel S_1 [Theorem 2.5] getting the first isomorphism in (4.1.3) below. To obtain the second isomorphism in (4.1.3), use the first isomorphism in (4.1.2) together with the second isomorphism in (4.1.1).

(4.1.3)
$$S_2 - S_1 \cong X_1 \oplus Y_1, \qquad T_2 - T_1 \cong Y_1 \oplus X_2.$$

After obtaining an infinite sequence of pairs of isomorphisms analogous to (4.1.3) we show that $P \cong Q$ as follows:

$$\begin{aligned} P &\cong S_1 \oplus (S_2 - S_1) \oplus (S_3 - S_2) \oplus \dots \\ &\cong S_1 \oplus (X_1 \oplus Y_1) \oplus (X_2 \oplus Y_2) \oplus \dots \\ &\cong (S_1 \oplus X_1) \oplus (Y_1 \oplus X_2) \oplus (Y_2 \oplus X_3) \oplus \dots \\ &\cong T_1 \oplus (T_2 - T_1) \oplus (T_3 - T_2) \oplus \dots \quad \cong Q. \end{aligned}$$

□

This enables us to provide a simple description of all countably generated projective modules over a classical hereditary order (that is, an HNP which is a finitely generated module over a central Dedekind domain). Our proof requires the following lemma , which uses the notions of "Dedekind closure" and "merging" from [11, 7.12 and 6.1].

Lemma 4.2. *Suppose that R is a classical hereditary order. Then:*

(i) *The centre C of R is a Dedekind domain and R is a finitely generated C-module.*
(ii) *R has only finitely many nontrivial cycle towers and no faithful towers.*
(iii) *Every Dedekind closure S of R is a finitely generated C-module (i.e. S is a classical maximal C-order containing R).*

Proof. (i) Since R is module-finite over a central commutative ring, R satisfies a polynomial identity. In view of this, (i) is proved in [14, 13.9.16].

(ii) By Kaplansky's Theorem on rings satisfying a polynomial identity, each primitive factor ring of R is simple Artinian. Therefore each simple R-module is unfaithful and so R has no faithful towers. By [14, 13.9.13], R has only finitely many idempotent prime ideals. Therefore, by [11, Lemma 3.7], R has only finitely many nontrivial cycle towers.

(iii) [11, Theorem 7.13] shows that each such S is obtained by merging all nontrivial R-towers into simple S-modules. Since these towers are finite in number, [11, Corollary 6.5] asserts that S is a finite overring of R and so is finitely generated over C. □

Theorem 4.3. *Let R be a classical hereditary order. Then every countably generated projective right R-module has the form $Q = H \oplus S^{\aleph_0}$ for some finite overring S of R and some finitely generated projective right R-module H.*

Proof. Let $P = \oplus\{P_i \mid i \in I\}$ be given, with $\text{udim}(P) = \aleph_0$. Lemma 4.2 shows that R has only finitely many nontrivial cycle towers and no faithful towers. Let $\mathcal{F} = \{W_1, \dots, W_u\}$ be the finite set of elements in these towers at which P has finite rank. Note that $\mathcal{F}$ contains no entire cycle tower, since every cycle tower contains at least one element at which P has infinite rank [Theorem 3.2 and Definition 3.1]. Let H be the direct sum of the finite set of all P_i such that $\rho(P_i, W)$ is nonzero for some $W \in \mathcal{F}$; and let $S(\mathcal{F})$ be the finite overring of R such that $\rho(S(\mathcal{F}), W) = 0$ if and only if $W \in \mathcal{F}$ [Theorem 2.6]. It is easy to check from this that P and Q have the same ranks at all elements of $\mathcal{W}$ and hence, by Theorem 4.1, that $P \cong Q$. □

Exactly the same proof establishes:

Corollary 4.4. *Let P be a countably generated projective module over any HNP ring R. Then the following are equivalent:*

(i) *P has only finitely many distinct finite ranks at elements $W \in \mathcal{W}$;*
(ii) *$P \cong H \oplus S^{\aleph_0}$ for some finite overring S of R and some finitely generated projective right R-module H.*

Example 3.5 shows that countably generated modules over HNP rings can have infinitely many distinct finite ranks. Such modules are discussed in Remark 4.12.

We now turn towards the uncountably generated case; so we fix $P = \oplus_{i \in I} P_i$ with $|I|$ uncountable and each P_i isomorphic to an essential right ideal. The proof of Theorem 1.3, in this case, involves an intricate grouping of the summands in this decomposition, and our next few results prepare for that.

Recall that $\mathcal{D}(\Psi(P))$ denotes the set of distinct cardinal numbers in the genus $\Psi(P)$ [Definitions 3.1]. Let $\mathcal{D}'(\Psi(P))$ be the set of distinct nonzero cardinal numbers in $\Psi(P)$ of the form $\Psi(P)_W$ with $W \in \mathcal{W}$. Notice that it may or may not be the case that $\operatorname{udim}(P) \in \mathcal{D}'(\Psi(P))$.

We first deal with an easy special case.

Lemma 4.5. *If $\mathcal{D}'(\Psi(P))$ is empty, then:*
(i) *$\mathcal{W}$ is a finite set and R has no cycle towers;*
(ii) *$P \cong S^{|I|}$ where $S = S(\mathcal{W})$ is the finite overring of R [given by Theorem 2.6] such that $\rho_R(S, W) = 0$ for all $W \in \mathcal{W}$.*

Proof. (i) Consider any of the summands P_i. By hypothesis, $\rho(P_i, W) = 0$ for all $W \in \mathcal{W}$. This contradicts P_i having almost standard rank unless $\mathcal{W}$ is finite; and it contradicts P_i having cycle-standard rank unless there are no cycle towers.

(ii) It suffices to prove this assertion when $|I| = \aleph_0$ since I is a disjoint union of countably infinite sets. Since both P and $S^{|I|}$ are now countably generated, it therefore suffices to check that $\Psi(P) = \Psi(S^{|I|})$ [Theorem 4.1]. But, by hypothesis, $\operatorname{udim}(P) = \operatorname{udim}(S^{|I|}) = \aleph_0$ and $\rho(P, W) = \rho(S^{|I|}, W) = 0$ for all $W \in \mathcal{W}$. □

The crux of our argument involves the following temporary hypothesis.

Hypothesis 4.6. $\mathcal{D}'(\Psi(P))$ is nonempty and all of its elements are infinite.

Lemma 4.7. *Assume that Hypothesis 4.6 holds. Let α be the smallest element of $\mathcal{D}'(\Psi(P))$. Then there is a decomposition $P = P(\alpha) \oplus P'$ in which $P(\alpha)$ and P' are direct sums of complementary subsets of the set of summands P_i of P, and for each $W \in \mathcal{W}$:*
(i) *$\rho(P(\alpha), W) = \alpha$ if $\rho(P, W) \geq \alpha$ and $\rho(P(\alpha), W) = 0$ otherwise;*
(ii) *$P(\alpha) = \oplus_c Q(\alpha, c)$ where the index of summation c ranges through a set of cardinality α, each $Q(\alpha, c)$ is non-zero and is the direct sum of a finite number of the summands P_i, and*

(4.7.1)
$$\rho(Q(\alpha, c), W) \neq 0 \iff \rho(P, W) \geq \alpha \quad (\iff \rho(P, W) \neq 0);$$

(iii) *$\rho(P', W) = 0$ for every W such that $\rho(P, W) = \alpha$ and $\rho(P', W) = \rho(P, W)$ for every W such that $\rho(P, W) > \alpha$;*
(iv) *$\mathcal{D}'(\Psi(P')) = \mathcal{D}'(\Psi(P)) - \{\alpha\}$.*

Proof. For $W \in \mathcal{W}$, let $I(W) = \{i \in I \mid \rho(P_i, W) \neq 0\}$. Similarly, for any cardinal number α, let $\mathcal{W}(\alpha) = \{W \in \mathcal{W} \mid \rho(P, W) = \alpha\}$. Note that if $W \in \mathcal{W}(\alpha)$ then, since each $\rho(P_i, W)$ is finite, Hypothesis 4.6 insists that $|I(W)|$ is infinite and hence $|I(W)| = \alpha$.

Now let $J = \bigcup\{I(W) \mid W \in \mathcal{W}(\alpha)\}$, the set of $j \in I$ such that $\rho(P_j, W) \neq 0$ for some $W \in \mathcal{W}(\alpha)$. We claim that $|J| = \alpha$. To see this, first consider the case that $\alpha = \text{udim}(P)$. Choose any $W \in \mathcal{W}(\alpha)$. Then we have

$$\text{udim}(P) = \alpha = |I(W)| \leq |J| \leq |I| = \text{udim}(P)$$

and therefore $|J| = \alpha$. Suppose, on the other hand, that $\alpha < \text{udim}(P)$. Since every genus is a pregenus, Definitions 3.1(ii) shows that $\mathcal{W}(\alpha)$ is a finite set. Therefore J is the union of a finite number of sets each having (infinite) cardinality α. Thus we again have $|J| = \alpha$ as claimed.

We now begin the construction. We will define the $Q(\alpha, c)$ by transfinite induction, with their indices c running through a subset of J. Well-order J, making its order type the smallest ordinal, necessarily a limit ordinal, of cardinality $|J|$. Then every $j \in J$ has fewer than $|J|$ predecessors in J.

Choose $c \in J$. Suppose that, for every $j < c$, we have either: (a) decided not to define $Q(\alpha, j)$; or (b) chosen a module $Q(\alpha, j)$ which has the property described in (4.7.1) and which is the direct sum of P_j and finitely many summands P_i with $i > j$. Now we describe whether and (if so) how to construct $Q(\alpha, c)$.

If P_c is a summand of $Q(\alpha, j)$ for some $j < c$ then we do not construct $Q(\alpha, c)$. Suppose, on the other hand, that P_c is not a summand of any previously defined $Q(\alpha, j)$. Then we make P_c a summand of a new module $Q(\alpha, c)$. By almost standard rank [Corollary 2.2] we have $\rho(P_c, W) \neq 0$ for almost all W. Consider now the finite set of W such that $\rho(P_c, W) = 0$ but $\rho(P, W) \neq 0$. If the set is empty, let $Q(\alpha, c) = P_c$. Otherwise, for each W in that set, we will include an additional direct summand P_i in $Q(\alpha, c)$ chosen such that $\rho(P_i, W) \neq 0$ and P_i is not contained in any $Q(\alpha, j)$ with $j < c$. We need to show that there is such a P_i.

By our choice of well-ordering of J, the number of predecessors of c is less than $|J| = \alpha$. Since each $Q(\alpha, j)$ that is already defined contains only finitely many summands P_i, and only finitely many additional summands can already have been chosen for $Q(\alpha, c)$ then the total number of P_i that already belong to some $Q(\alpha, j)$ is less than α. On the other hand, $|I(W)| = \alpha$, as shown at the beginning of this proof. This proves that at least one such P_i is available for inclusion in $Q(\alpha, c)$. Thus we have found a P_i for each W. We then let $Q(\alpha, c)$ be the direct sum of them and P_c.

This completes the construction of $Q(\alpha, c)$ satisfying (4.7.1). Then transfinite induction completes the construction of all the $Q(\alpha, c)$.

To see that the number of $Q(\alpha, c)$ that we have defined is α, first note that since c runs through a subset of J, the number of these modules is no more than $|J| = \alpha$. The opposite inequality follows from the fact that every P_i with $i \in J$ is contained in some $Q(\alpha, c)$, and each $Q(\alpha, c)$ contains only finitely many P_i.

The remaining properties are easily verified. □

Lemma 4.8. *Assume that Hypothesis 4.6 holds and let the elements of $\mathcal{D}'(\Psi(P))$ be $\alpha_1 < \alpha_2 < \dots$, a finite sequence, or an infinite sequence possibly followed by α_ω. Then there is a decomposition*

$$P = \left[\oplus_{n \neq \omega} P(\alpha_n)\right] \oplus P(0) \tag{4.8.1}$$

in which each term $P(\alpha_n)$ and $P(0)$ is a direct sum of a subset of the original summands P_i of P, and for $W \in \mathcal{W}$ and for all n:

(i) $\rho(P(\alpha_n), W) = \alpha_n$ *if* $\rho(P, W) \geq \alpha_n$ *and* $\rho(P(\alpha_n), W) = 0$ *otherwise.*

(ii) $P(\alpha_n) = \oplus_c Q(\alpha_n, c)$ *where the index of summation c ranges through a set of cardinality α_n, each $Q(\alpha_n, c)$ is non-zero and is the direct sum of a finite number of the summands P_i, and*

$$\rho(Q(\alpha_n, c), W) \neq 0 \iff \rho(P, W) \geq \alpha_n. \tag{4.8.2}$$

(iii) $\rho(P(0), W) = 0$ *for all $W \in \mathcal{W}$ and, if $P(0) \neq 0$, then* $\operatorname{udim}(P(0)) = \operatorname{udim}(P) > \alpha_n$ *for all n, and only finitely many α_n occur.*

Proof. Lemma 4.7 yields a decomposition $P = P(\alpha_1) \oplus P'$ where $P(\alpha_1)$ is as desired and $\mathcal{D}'(\Psi(P')) = \mathcal{D}'(\Psi(P)) - \{\alpha_1\}$.

If $\mathcal{D}'(\Psi(P'))$ is nonempty, apply the previous procedure to P' in place of P, getting $P = P(\alpha_1) \oplus P(\alpha_2) \oplus P''$ with $\mathcal{D}'(\Psi(P'')) = \mathcal{D}'(\Psi(P)) - \{\alpha_1, \alpha_2\}$, and so on. Let the result of the n^{th} iteration of this procedure be $P = S_n \oplus T_n$, where $S_n = \oplus_{k=1}^n P(\alpha_k)$ and $\rho(T_n, W) = 0$ whenever $\rho(P, W) \in \{\alpha_1, \ldots, \alpha_n\}$. We distinguish two cases.

Case 1: $\mathcal{D}'(\Psi(P))$ is an infinite set. Then we have a decomposition $P = S_n \oplus T_n$ for every positive integer n. Letting $S_\infty = \bigcup_n S_n$ and $T_\infty = \bigcap_n T_n$ yields the decomposition $P = S_\infty \oplus T_\infty$. Note that no subtleties are involved here: we are only rearranging the terms of the original decomposition of P. To complete the proof of the lemma in this case, it suffices to show that $T_\infty = 0$. If $T_\infty \neq 0$, it contains some summand P_i. On the one hand, P_i has almost standard rank [Corollary 2.2], and hence $\rho(P_i, W) = 0$ for only finitely many W. On the other hand, $P_i \subseteq T_\infty \subseteq T_n$ for all n and therefore $\rho(P_i, W) = 0$ for infinitely many W (at least one for each α_n). This contradiction completes the proof of Case 1.

Case 2: $\mathcal{D}'(\Psi(P))$ is a finite set. Let α_s be the largest cardinal in $\mathcal{D}'(\Psi(P))$. Then $\mathcal{D}'(T_s)$ is empty, and we have $P = S_s \oplus T_s$; that is:

$$P = P(\alpha_1) \oplus \ldots \oplus P(\alpha_s) \oplus T_s \tag{4.8.3}$$

If $T_s = 0$ we are done: take $P(0) = 0$. So suppose that $T_s \neq 0$.

We split the rest of the proof into two cases.

Case 2a: $\operatorname{udim}(T_s) \leq \alpha_s$. We complete the proof here by showing that we can modify the summands $P(\alpha_s)$ and T_s in (4.8.3) so that $T_s = 0$, and then we again set $P(0) = 0$. The number of summands $Q(\alpha_s, c) \subseteq P(\alpha_s)$ is α_s. Since the number of direct summands P_i of T_s is α_s or less, we can modify a subset of the summands $Q(\alpha_s, c)$ of $P(\alpha_s)$ by including one of the summands P_i of T_s in each of them. This does not change any ranks of $P(\alpha_s)$ at elements of $\mathcal{W}$, and does not change the fact that each $Q(\alpha_s, c)$ is the direct sum of finitely many summands P_i of P. Moreover, (4.8.3) still holds, with this new $P(\alpha_s)$ and with $T_s = 0$.

Case 2b: $\operatorname{udim}(T_s) > \alpha_s$. From (4.8.3) we have $\operatorname{udim}(P) = \alpha_1 + \cdots + \alpha_s + \operatorname{udim}(T_s)$. Since every $\alpha_n < \operatorname{udim}(T_s)$ this shows that $\operatorname{udim}(T_s) = \operatorname{udim}(P)$. Set $P(0) = T_s$. □

Our next objective is to show that every summand listed in (4.8.1) is a free module over an appropriate finite overring of R, thus establishing the second main result of this section.

Theorem 4.9. *For every uncountably generated projective right R-module P, we have*

$$P \cong H \oplus S_1^{\alpha_1} \oplus S_2^{\alpha_2} \oplus \ldots \tag{4.9.1}$$

where H is a finitely generated projective R-module, the α_n are infinite cardinal numbers satisfying $\alpha_1 < \alpha_2 < \dots$ with at least one being uncountable, and $S_1 \subset S_2 \subset \dots$ is a finite or a countably infinite sequence of finite overrings of R.

Proof. Assume first that Hypothesis 4.6 holds. We will obtain decomposition (4.9.1), with $H = 0$, in this situation.

Start with the decomposition in (4.8.1), and fix n. We claim that

$$P(\alpha_n) \cong S_n^{\alpha_n} \tag{4.9.2}$$

for some finite overring S_n of R which we need to define. First we define $\mathcal{F}_n$ by the first equality in (4.9.3) below. Then note that, for each of the direct summands $Q(\alpha_n, c)$ of $P(\alpha_n)$, the second equality in (4.9.3) holds by (4.8.2).

(4.9.3)
$$\mathcal{F}_n = \{W \in \mathcal{W} \mid \rho(P, W) < \alpha_n\} \;=\; \{W \in \mathcal{W} \mid \rho(Q(\alpha_n, c), W) = 0\}.$$

Since $Q(\alpha_n, c)$ is finitely generated, it has almost standard rank [Theorem 2.1] and therefore $\mathcal{F}_n$ is a finite set. Since $\rho(Q(\alpha_n, c), W) = 0$ for every $W \in \mathcal{F}_n$, cycle-standard rank [Theorem 2.1], shows that $\mathcal{F}_n$ contains no entire cycle tower. Therefore, by Theorem 2.6, there is a finite overring $S_n = S(\mathcal{F}_n)$ of R such that for $W \in \mathcal{W}$ we have:

$$\rho_R(S_n, W) = 0 \iff W \in \mathcal{F}_n. \tag{4.9.4}$$

The first equality in (4.9.3) shows that $\mathcal{F}_n \subset \mathcal{F}_{n+1}$ if α_{n+1} exists and so $S_n \subset S_{n+1}$ by Theorem 2.6.

We now turn to isomorphism (4.9.2). The R-module $P(\alpha_n)$ is the direct sum of α_n modules $Q(\alpha_n, c)$. Since α_n is an infinite cardinal number, every set of cardinality α_n is the disjoint union of α_n countably infinite sets. Consequently, $P(\alpha_n)$ can be written as the direct sum of modules $Q'(\alpha_n, d)$, where the index of summation d runs through a set of cardinality α_n and each $Q'(\alpha_n, d)$ is the direct sum of $\aleph_0$ of the modules $Q(\alpha_n, c)$. Similarly, $S_n^{\alpha_n}$ is the direct sum of α_n copies of $S_n^{\aleph_0}$. Therefore, in order to prove (4.9.2) it suffices to prove that, for each d,

$$Q'(\alpha_n, d) \cong S_n^{\aleph_0} \qquad \text{(as } R\text{-modules).} \tag{4.9.5}$$

Since S_n and each $Q(\alpha_n, c)$ is a nonzero finitely generated R-module, both sides of (4.9.5) are countably generated. It therefore suffices, by Theorem 4.1, to prove that both sides of (4.9.5) are in the same genus; and we already know that each has uniform dimension $\aleph_0$. Take any $W \in \mathcal{W}$. By (4.9.3) and (4.9.4) both $Q(\alpha_n, c)$ and S_n have W-rank 0 if $W \in \mathcal{F}_n$ and have finite nonzero W-rank otherwise. It follows that both sides of (4.9.5) have W-rank 0 if $W \in \mathcal{F}_n$ and W-rank $\aleph_0$ otherwise, proving (4.9.5), and therefore proving (4.9.2).

In view of (4.9.2), decomposition (4.8.1) can be rewritten

$$P \cong \left[\oplus_n S^{\alpha_n}\right] \oplus P(0). \tag{4.9.6}$$

Thus we have proved (4.9.1) when $P(0) = 0$, and can now assume that $P(0) \neq 0$, in which case Lemma 4.8 shows that only finitely many α_n occur. Let α_s be the largest. Lemma 4.8 further shows that $\operatorname{udim}(P(0)) = \operatorname{udim}(P) > \alpha_s$. We set $\operatorname{udim}(P(0)) = \alpha_{s+1}$ and note, by Lemma 4.5 that $P(0) \cong S_{s+1}^{\alpha_{s+1}}$ where $S_{s+1} = S(\mathcal{W})$.

This proves the theorem when Hypothesis 4.6 is satisfied. Now drop Hypothesis 4.6. It suffices to prove that P is the direct sum of a finitely generated module H and a module that satisfies Hypothesis 4.6. Since P is uncountably generated, there are only finitely many $W \in \mathcal{W}$ such that $\rho(P, W)$ is finite [Lemma 3.4 and Theorem 3.2]; and if $\rho(P, W)$ is finite and nonzero, there are only finitely many terms P_i such that $\rho(P_i, W) \neq 0$. Let H be the direct sum of all P_i such that, for some W, $\rho(P_i, W)$ is finite and nonzero. Then H is finitely generated, and $P = H \oplus P'$ where P' is the direct sum of all P_i not contained in H. Then $\mathrm{udim}(P') = \mathrm{udim}(P)$ and either P' satisfies Hypothesis 4.6 or else P' satisfies the hypotheses of Lemma 4.5. Thus the proof is complete. □

We next demonstrate what level of uniqueness there is in the description of P in the preceding result and in Theorem 4.3 and Corollary 4.4.

Theorem 4.10. *Let Q be a right R-module of the form*

$$Q = H \oplus S_1^{\alpha_1} \oplus S_2^{\alpha_2} \oplus \dots \tag{4.10.1}$$

where H is a finitely generated projective R-module and $S_1 \subset S_2 \subset \dots$ is a nonempty, strictly increasing, finite or countably infinite sequence of finite overrings of R, and $\aleph_0 \leq \alpha_1 < \alpha_2 < \dots$. Then the sequence of ordered pairs $(S_1, \alpha_1),\ (S_2, \alpha_2),\ \dots$ together with the ranks $\rho(H, W)$ for those $W \in \mathcal{W}$ such that $\rho(S_1, W) = 0$

(i) *is determined by $\Psi(Q)$, and*
(ii) *determines the isomorphism class of Q.*

Proof. (i) Let Q be as in (4.10.1). By Theorem 2.6 we know that each $S_n = S(\mathcal{F}_n)$ for some unique finite set $\mathcal{F}_n$ of elements of $\mathcal{W}$ containing no entire cycle tower, and $\mathcal{F}_n \subset \mathcal{F}_{n+1}$ whenever both sets are defined. The nontrivial part of this proof is to establish the following statement:

(4.10.2) The sequence $(S_1, \alpha_1),\ (S_2, \alpha_2),\ \dots$ is determined by $\Psi(Q)$.

To establish this, it suffices to prove that the sequence $(\mathcal{F}_1, \alpha_1),\ (\mathcal{F}_2, \alpha_2),\ \dots$ is determined by $\Psi(Q)$.

First we identify $\mathcal{F}_1$ as the set of all $W \in \mathcal{W}$ such that $\rho(Q, W)$ is finite. To see this, note first that, when $W \in \mathcal{F}_1$, every $\rho(S(\mathcal{F}_n), W) = 0$. Therefore $\rho(Q, W) = \rho(H, W)$ which is finite since H is finitely generated. On the other hand, for any other W, $\rho(Q, W) \geq \rho(S(\mathcal{F}_1)^{\alpha_1}, W) = \alpha_1$ which is infinite. Notice that this establishes the other part of what is claimed in (i); namely that $\Psi(Q)$ determines the ranks $\rho(H, W)$ for those $W \in \mathcal{W}$ such that $\rho(S_1, W) = 0$. It also shows that α_1 is identified as the smallest infinite cardinal α such that, for some W, $\rho(Q, W) = \alpha$.

Next we show that the set of W such that $\rho(Q, W) = \alpha_1$ is $\mathcal{F}_2 - \mathcal{F}_1$; thus $\mathcal{F}_2$ is identified. The critical observation establishing this is that, for $W \in \mathcal{F}_2 - \mathcal{F}_1$ we have $\rho(Q, W) = \rho(H, W) + \rho(S(\mathcal{F}_1)^{\alpha_1}, W) = \rho(H, W) + \alpha_1 = \alpha_1$ (the last equality since H is finitely generated). Now α_2 is identified as the smallest cardinal $\alpha > \alpha_1$ such that, for some W, $\rho(Q, W) = \alpha$.

If there are infinitely many $\mathcal{F}_n$, then we continue in this way, eventually proving that every α_n and $\mathcal{F}_n$ is determined by $\Psi(Q)$.

Suppose, therefore, that there are only finitely many $\mathcal{F}_n$, and call the last one $\mathcal{F}_u$. We can continue the foregoing reasoning until we have proved that α_{u-1} and $\mathcal{F}_u - \mathcal{F}_{u-1}$ (hence $\mathcal{F}_u$) are determined by $\Psi(Q)$. We still need to prove that α_u is so determined. We must use different reasoning here because, in the case that $\mathcal{W}$

is a finite set and $\mathcal{F}_u = \mathcal{W}$, there are no further $W \in \mathcal{W}$ to use. Instead, we note that

$$\mathrm{udim}(Q) = \mathrm{udim}(H) + \alpha_1 + \alpha_2 + \cdots + \alpha_u = \alpha_u$$

with the latter equality because $\mathrm{udim}(H)$ is finite and the terms α_n form an increasing sequence of infinite cardinal numbers. This completes the proof of (i).

(ii) By hypothesis, all terms in (4.10.1) other than H are given. It remains to consider H (which, as we shall see, is *not* uniquely determined).

We can replace the module H in (4.10.1) by $X = H \oplus S(\mathcal{F}_1)^{\aleph_0}$ without altering the isomorphism class of Q because $S(\mathcal{F}_1)^{\aleph_0} \oplus S(\mathcal{F}_1)^{\alpha_1} \cong S(\mathcal{F}_1)^{\alpha_1}$. It is therefore enough to prove that the module X is determined up to isomorphism by the given invariants. Since X is countably generated, its isomorphism class is determined by its ranks at elements of $\mathcal{W}$ [Theorem 4.1]. If $W \in \mathcal{F}_1$, we have $\rho(X, W) = \rho(H, W)$ which is given. Otherwise $\rho(X, W) = \rho(H \oplus S(\mathcal{F}_1)^{\aleph_0}, W) = \aleph_0$. This completes the proof of the theorem. □

We can now complete the proof of Theorem 1.3

Theorem 4.11. *Let P, P' be infinitely generated projective R-modules. Then $P \cong P'$ if and only if $\Psi(P) = \Psi(P')$.*

Proof. This has already been established in Theorem 4.1 in the case of countably generated projective modules. So the case remains when P is uncountably generated. We know, by Theorem 4.9, that P is isomorphic to a module of the form Q in (4.10.1). However, Theorem 4.10 shows that the isomorphism class of Q is determined by its genus; i.e. by $\Psi(P)$. □

Remark 4.12. Unfortunately, we have no canonical form for those countably generated projective R-modules P such that $\mathcal{D}(\Psi(P))$ is an infinite set. However, the complete structural details are located in various parts of this paper, and we give a brief directory to them, here.

Let Φ be any 'countable' pregenus; that is, a function from $\{0\} \bigcup \mathcal{W}$ to the set of finite and countably infinite cardinals, such that Φ satisfies the conditions of Definitions 3.1. Since every pregenus is a genus [Theorem 3.6] there is a P such that $\Psi(P) = \Phi$. The proof of this theorem contains a construction of such a P. What that theorem does not state, but is proved in Theorem 4.1, is that *this determines P up to isomorphism.*

Remark 4.13. The canonical form (4.9.1) of an uncountably generated projective R-module is not well-suited to determining the canonical form of a direct sum of such modules. For example, let S be a finite overring of R such that $S \neq R$, and let $\alpha \geq \beta$ be uncountable cardinals. Then the uncountably generated projective R-module $P = R^\alpha \oplus S^\beta$ does not appear to have the form displayed in (4.9.1). However, one can readily check that $\Psi(P) = \Psi(R^\alpha)$ and so Theorem 4.11 shows that $P \cong R^\alpha$.

As this example illustrates, whether we are discussing countably or uncountably generated projective modules, we can check the isomorphism class of a direct sum simply by adding the genera and then using Theorem 4.11; and if the resulting genus is of an appropriate type, we can readily obtain its canonical form.

5. Direct Summand Theorem

Let P, Q be projective R-modules, with Q infinitely generated. When is P isomorphic to a *proper* direct summand of Q? i.e. $Q \cong P \oplus X$ for some $X \neq 0$? An obviously necessary condition is $\Psi(P) \leq \Psi(Q)$. In the finitely generated case we get a necessary and sufficient condition by adding the condition $\operatorname{udim}(P) < \operatorname{udim}(Q)$ [Theorem 2.4]. This last condition is not necessary in the infinitely generated case. For example, every infinitely generated free module is obviously isomorphic to a proper direct summand of itself. The answer in brief, when Q is infinitely generated, is that the condition $\Psi(P) \leq \Psi(Q)$ is necessary and sufficient if Q is uncountably generated or R is a classical hereditary order. But additional complications arise if Q is countably generated, as we show at the end of the section.

Theorem 5.1. *Let P, Q be projective R-modules such that Q is infinitely generated and $\Psi(P) \leq \Psi(Q)$. Then there is an infinitely generated projective R-module X such that $P \oplus X \cong Q$ if and only if:*

(5.1.1) For every nonnegative integer n, there exist only finitely many $W \in \mathcal{W}$ such that $\rho(Q, W)$ [hence $\rho(P, W)$] is finite and $\rho(Q, W) - \rho(P, W) \leq n \cdot \rho(R, W)$.

Proof. Suppose that $P \oplus X \cong Q$ — and hence $\Psi(P) + \Psi(X) = \Psi(Q)$ — but condition (5.1.1) fails. Then there exist a nonnegative integer n and infinitely many W such that $\rho(X, W) \leq n \cdot \rho(R, W)$. Therefore $\Psi(X)$ does not satisfy the finite multiplicity condition (ii)(b) in the definition of a pregenus, and hence X cannot be an infinitely generated projective R-module [Theorem 3.2].

Conversely, suppose that P and Q are given and condition (5.1.1) holds. Define the function Φ by $\Phi_W = \rho(Q, W) - \rho(P, W)$ whenever $W \in \mathcal{W}$ and $\rho(Q, W)$ is finite, $\Phi_W = \rho(Q, W)$ when the $\rho(Q, W)$ is infinite, and $\Phi_0 = \operatorname{udim}(Q)$. We claim that Φ is a pregenus.

Condition (ii)(b) in the definition of a pregenus [Definitions 3.1] is satisfied by hypothesis (5.1.1). The remaining conditions in the definition of a pregenus are satisfied because $\Psi(Q)$ is the genus of an infinitely generated projective module.

Since the claim holds, Φ is the genus of an infinitely generated projective module [Theorem 3.6], say $\Phi = \Psi(X)$. It is then easily verified that $\Psi(P) + \Psi(X) = \Psi(Q)$, and therefore $P \oplus X \cong Q$ [Theorem 4.11]. □

Corollary 5.2. *Let P, Q be nonzero projective R-modules such that $\Psi(P) \leq \Psi(Q)$ and such that $\rho(Q, W)$ is infinite for almost all $W \in \mathcal{W}$. Then there exists an infinitely generated projective R-module X such that $P \oplus X \cong Q$.*

Proof. Clear from the theorem. □

Corollary 5.3. *Let P, Q be nonzero projective R-modules such that $\Psi(P) \leq \Psi(Q)$. Suppose that either:*

(i) *Q is uncountably generated, or*
(ii) *R is a classical hereditary order.*

Then $\rho(Q, W)$ is infinite for almost all $W \in \mathcal{W}$ and (therefore) P is isomorphic to a proper direct summand of Q.

Proof. By Corollary 5.2 it is enough to establish the first half of the claim. In situation (i) this is done in Lemma 3.4. So consider situation (ii). By Lemma 4.2 R has only finitely many nontrivial cycle towers and no faithful towers. Moreover every cycle tower contains at least one W such that $\rho(Q, W)$ is infinite [Theorem 3.2 and Definitions 3.1(iii)]; and therefore $\rho(Q, W)$ is infinite for all $W \in \mathcal{W}$ except some of the finitely many belonging to nontrivial cycle towers. □

Corollary 5.4. *Let P be an infinitely generated projective R-module. Then the following are equivalent:*

(i) *$P \oplus X \cong P$ for some nonzero R-module X;*
(ii) *$P \oplus X \cong P$ for some infinitely generated R-module X;*
(iii) *$\rho(P, W)$ is infinite for almost all $W \in \mathcal{W}$.*

Proof. It is a triviality that (ii) implies (i); and it is immediate from Corollary 5.2 that (iii) implies (ii). Finally, suppose that (i) holds but (iii) fails. Then $\rho(X, W) = 0$ for the infinitely many $W \in \mathcal{W}$ for which $\rho(P, W)$ is finite. If X were infinitely generated, this would contradict condition (ii)(a) in the definition of 'pregenus'; and if X were finitely generated and nonzero this would violate almost standard rank [Theorem 2.1]. This contradiction completes the proof. □

Corollary 5.5. (i) *Every uncountably generated projective R-module is isomorphic to a proper direct summand of itself.*
(ii) *Every infinitely generated projective module over a classical hereditary order is isomorphic to a proper direct summand of itself.*

Proof. This follows directly from Corollaries 5.3 and 5.4. □

We conclude this section with some anomalies that occur in the countably generated case, beginning with an extreme illustration of the fact that $\Psi(P) \leq \Psi(Q)$ does not imply that P is isomorphic to a proper direct summand of Q.

Example 5.6. There exists a countably generated projective R-module which is not isomorphic to a proper direct summand of itself. This follows from Corollary 5.4 if we take any projective R-module P (necessarily countably generated) with infinitely many finite ranks at elements of $\mathcal{W}$ — for example, as in Example 3.5.

Example 5.7. There exist infinitely generated projective R-modules P, Q (necessarily with countable udim) such that $P \oplus X \cong Q$ for some nonzero finitely generated X but for no infinitely generated X. To see this, let P be as in Example 5.6 and let $Q = P \oplus R$. Then there are infinitely many W such that $\rho(Q, W)$ is finite and $\rho(Q, W) - \rho(P, W) = \rho(R, W)$. Therefore there is no infinitely generated X such that $Q \cong P \oplus X$ [Lemma 5.1].

A particularly interesting case of this example occurs if $\mathcal{W}$ is infinite, and all ranks $\rho(P, W)$ are finite. Then one can show that $P \oplus X \cong P \oplus R$ if and only if $\Psi(X) = \Psi(R)$.

References

[1] H. Bass, "Big projective modules are free", *Israel J. Math.* **7** (1963), pp. 24–31.
[2] H. Cartan and S. Eilenberg, *Homological Algebra* (Princeton, 1956).
[3] D. Eisenbud and J. C. Robson, "Modules over Dedekind prime rings", *J. Algebra* **16** (1970), pp. 67–85.
[4] D. Eisenbud and J. C. Robson, "Hereditary Noetherian prime rings", *J. Algebra* **16** (1970), pp. 86–104.
[5] D. Eisenbud and P. Griffith, "Serial rings", *J. Algebra* **17** (1971), pp. 389–400.
[6] K. R. Goodearl, "Localization and splitting in hereditary Noetherian prime rings", *Pac. J. Math.* **53** (1974), pp. 137–151.

[7] K. R. Goodearl and R. B. Warfield, Jr., "Simple modules over hereditary Noetherian prime rings", *J. Algebra* **57** (1979), pp. 82–100.
[8] T. J. Hodges, "K-theory and right ideal class groups for HNP rings", *Trans. Amer. Math. Soc.* **302** (1987), pp. 751–767.
[9] I. Kaplansky, "Modules over Dedekind rings and valuation rings", *Trans. Amer. Math. Soc.* **72** (1952), pp. 327–340.
[10] I. Kaplansky, "Projective modules", *Ann. of Math.* (2), **68** (1958), pp. 372–377.
[11] L. S. Levy and J. C. Robson,"Hereditary Noetherian prime rings 1: Integrality and simple modules" *J. Algebra* **218** (1999), 307–337.
[12] L. S. Levy and J. C. Robson,"Hereditary Noetherian prime rings 2: Finitely generated projective modules" *J. Algebra* **218** (1999), 338–372.
[13] L. S. Levy, J. C. Robson, and J. T. Stafford, "Hidden matrices", *Proc. London Math. Soc.* (3) **69** (1994), pp. 277–308.
[14] J. C. McConnell and J.C. Robson, *Noncommutative Noetherian Rings*, Wiley-Interscience, Chichester, 1987.
[15] J. C. Robson, "Idealizers and hereditary Noetherian prime rings", *J. Algebra* **22** (1972), pp. 45–81.
[16] L. Silver, "Non-commutative localizations and applications", *J. Algebra* **7** (1967) pp. 44–76.
[17] J. T. Stafford and R. B. Warfield, Jr., "Hereditary orders with infinitely many idempotent ideals", *J. Pure and Applied Algebra* **31** (1984), pp. 217–225.
[18] J. T. Stafford and R. B. Warfield, Jr., "Constructions of hereditary Noetherian rings and simple rings", *Proc. London Math. Soc.* **51** (1985), pp. 1-20.
[19] R.B. Warfield, Jr., "The number of generators of a module over a fully bounded ring", *J. Algebra* **66** (1980), pp. 425–447.

L. S. Levy, Mathematics Department, University of Wisconsin, Madison WI 53706-1388, USA
E-mail address: levy@math.wisc.edu

J. C. Robson, School of Mathematics, University of Leeds, Leeds LS2 9JT, England
E-mail address: J.C.Robson@leeds.ac.uk