

Phase Space Bounds for Quantum Mechanics on a Compact Lie Group

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Abstract: Let K be a compact, connected Lie group and $K_{\mathbb{C}}$ its complexification. I consider the Hilbert space $\mathcal{H}L^2(K_{\mathbb{C}}, \nu_t)$ of holomorphic functions introduced in [H1], where the parameter t is to be interpreted as Planck's constant. In light of [L-S], the complex group $K_{\mathbb{C}}$ may be identified canonically with the cotangent bundle of K . Using this identification I associate to each $F \in \mathcal{H}L^2(K_{\mathbb{C}}, \nu_t)$ a "phase space probability density." The main result of this paper is Theorem 1, which provides an upper bound on this density which holds uniformly over all F and all points in phase space. Specifically, the phase space probability density is at most $a_t (2\pi t)^{-n}$, where $n = \dim K$ and a_t is a constant which tends to one exponentially fast as t tends to zero. At least for small t , this bound cannot be significantly improved.

With t regarded as Planck's constant, the quantity $(2\pi t)^{-n}$ is precisely what is expected on physical grounds. Theorem 1 should be interpreted as a form of the Heisenberg uncertainty principle for K , that is, a limit on the concentration of states in phase space. The theorem supports the interpretation of the Hilbert space $\mathcal{H}L^2(K_{\mathbb{C}}, \nu_t)$ as the phase space representation of quantum mechanics for a particle with configuration space K .

The phase space bound is deduced from very sharp pointwise bounds on functions in $\mathcal{H}L^2(K_{\mathbb{C}}, \nu_t)$ (Theorem 2). The proofs rely on precise calculations involving the heat kernel on K and the heat kernel on $K_{\mathbb{C}}/K$.

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1. Introduction

The classical Segal-Bargmann space [B, Se1-3] is the space of holomorphic functions F on \mathbb{C}^n satisfying

$$\|F\|_t^2 \equiv \int_{\mathbb{C}^n} |F(z)|^2 \nu_t(z) dz < \infty,$$

where

$$\nu_t(z) = (\pi t)^{-n/2} e^{-(\text{Im}z)^2/t}.$$

Here t is a positive parameter. (This is the “invariant” form of the Segal-Bargmann space in which the measure is constant in the real directions. See the appendix in [H1] for its relationship to other forms.) We will denote this space $\mathcal{HL}^2(\mathbb{C}^n, \nu_t)$, where \mathcal{H} indicates holomorphic.

I wish to interpret $\mathcal{HL}^2(\mathbb{C}^n, \nu_t)$ as the “phase space Hilbert space” for quantum mechanics of a particle moving in \mathbb{R}^n . In this case, t is to be interpreted as Planck’s constant (\hbar). There is a natural unitary map, called the Segal-Bargmann transform, which connects this phase space Hilbert space to the customary “configuration space Hilbert space” $L^2(\mathbb{R}^n, dx)$. However, the transform is not directly relevant to the present paper. A phase space Hilbert space is a natural and useful setting for semiclassical analysis [V, P-U, G-P, T-W, C].

If we normalize $F \in \mathcal{HL}^2(\mathbb{C}^n, \nu_t)$ so that $\|F\|_t = 1$, then

$$\int_{\mathbb{C}^n} |F(z)|^2 \nu_t(z) dz = 1.$$

The quantity $|F(z)|^2 \nu_t(z)$ is to be interpreted as a sort of “phase space probability density.” Although other definitions of the phase density are possible, this one is natural in many respects. (See [H3].) The results of Bargmann [B, (1.7)], adapted to our normalization, show that

$$|F(z)|^2 \nu_t(z) \leq (2\pi t)^{-n} \tag{1}$$

for all $F \in \mathcal{HL}^2(\mathbb{C}^n, \nu_t)$ with $\|F\|_t = 1$ and for all $z \in \mathbb{C}^n$. The quantity $(2\pi t)^n = (2\pi\hbar)^n$ is the volume of a semiclassical cell in phase space. Thus (1) tells us that if E is a region of phase space whose volume is p times the volume of a cell, then the particle has probability at most p of being in E . This is a form of the Heisenberg uncertainty principle. The fact that the right side of (1) is independent of z reflects the fact that the group of translations of \mathbb{C}^n acts in a projective unitary fashion on $\mathcal{HL}^2(\mathbb{C}^n, \nu_t)$. (See [B, (3.5)].)

The purpose of this paper is to prove a similar result for a particle whose configuration space is an arbitrary connected compact Lie group K . In [H1] I construct an analog on K of the Segal-Bargmann transform. (See also [H2, D, D-G, G-M, A, Hi1-2].) Let $K_{\mathbb{C}}$ denote the complexification of K (Sect. 2). The range of the generalized Segal-Bargmann transform is $\mathcal{HL}^2(K_{\mathbb{C}}, \nu_t)$, that is, the space of holomorphic functions F on $K_{\mathbb{C}}$ for which

$$\|F\|_t^2 \equiv \int_{K_{\mathbb{C}}} |F(g)|^2 \nu_t(g) dg < \infty.$$

Here dg is Haar measure on $K_{\mathbb{C}}$ and ν_t is (Sect. 2) the heat kernel on $K_{\mathbb{C}}/K$, viewed as a K -invariant function on $K_{\mathbb{C}}$. (More precisely, this space is the image of the K -invariant form C_t of the generalized Segal-Bargmann transform [H1, Thm. 2].) I wish to interpret

$\mathcal{H}L^2(K_{\mathbb{C}}, \nu_t)$ as the phase space Hilbert space for a quantum particle with configuration space K .

The usual phase space for a particle with configuration space K is the cotangent bundle of K , $T^*(K)$. In Sect. 3, we will discover a canonical diffeomorphism Φ between $T^*(K)$ and the complex group $K_{\mathbb{C}}$, obtained by means of the results of Lempert and Szöke [L-S, Sz1-2] or the largely equivalent results of Guillemin and Stenzel [G-S1-2]. For each $F \in \mathcal{H}L^2(K_{\mathbb{C}}, \nu_t)$ with $\|F\|_t = 1$, the associated phase space probability density is

$$|F(g)|^2 \nu_t(g) \sigma(g),$$

where σ is the “Jacobian” of Φ . Let $n = \dim K$. The main result of this paper (Theorem 1) is that for any F in $\mathcal{H}L^2(K_{\mathbb{C}}, \nu_t)$ with $\|F\|_t = 1$, the phase space probability density satisfies

$$|F(g)|^2 \nu_t(g) \sigma(g) \leq a_t (2\pi t)^{-n}, \quad (2)$$

where a_t is a constant that tends to one exponentially fast as t tends to zero. In particular, for each fixed t there is a bound on the phase space probability density that holds uniformly over all F and all points in phase space. I prove that, at least for small t , the bound (2) cannot be substantially improved.

The optimal bound for the left side of (2) is a non-constant function of g , given in (6) below. This non-constancy reflects the fact that $T^*(K)$ is less symmetric than \mathbb{C}^n . Unless K is commutative there is no obvious transitive group of canonical transformations of $T^*(K)$; in particular, the symplectic structure on $K_{\mathbb{C}}$ obtained via Φ is neither left- nor right-invariant. Nevertheless, the right side of (6) is *nearly* constant. According to Theorem 1, it is bounded above by a constant for all t and bounded below by a constant for small t , and the ratio of the upper and lower bounds tends to one exponentially fast as t tends to zero.

Theorem 1 supports the view that $\mathcal{H}L^2(K_{\mathbb{C}}, \nu_t)$ is the “right” phase space Hilbert space for a quantum particle with configuration space K . This view is also supported by the inversion formula in [H2], which says (roughly) that the configuration space wave function can be obtained from the phase space wave function by integrating over the momentum variables.

As explained in Sect. 4, the phase space density is bounded by the product of three quantities—the function ν_t , the “Jacobian” of the map Φ , and a certain analytic continuation of the heat kernel on K . Gangolli [G] gives an exact formula for ν_t , the Jacobian of Φ can be computed exactly, and the analytic continuation of the heat kernel on K can be estimated by analyzing the Poisson summation formula of Urakawa [U]. When we multiply a miracle occurs: everything cancels except for the physically expected quantity $(2\pi t)^{-n}$, times a function which tends to one uniformly as t tends to zero. The miraculous nature of these cancellations suggests that some more general principle is at work.

The results of [H1] and [L-S] carry over to the case of compact symmetric spaces. (See [H1, Sect. 11] and [Sz1, Thm. 2.5].) However, the present paper relies on heat kernel formulas which hold only in the group case. I conjecture that some analog of Theorem 1 holds for general compact symmetric spaces.

I thank Ping Feng for helping me to understand the map Φ and Chris Herald for inspiring me to use the Fourier transform in the proof of Proposition 3.

2. Preliminaries

The setup is as follows. We let K be an arbitrary compact connected Lie group with Lie algebra \mathfrak{k} . We fix an inner product $\langle \cdot, \cdot \rangle$ on \mathfrak{k} which is invariant under the adjoint action of K . This inner product determines a bi-invariant Riemannian metric on K . We will let dx denote Haar measure on K normalized to coincide with Riemannian volume measure. With this normalization the volume of K need not equal one.

Let Δ denote the Laplace-Beltrami operator associated with this Riemannian metric. The heat kernel ρ_t at the identity on K is defined by the conditions that ρ_t satisfy the heat equation

$$\frac{d\rho}{dt} = \frac{1}{2}\Delta\rho_t$$

and that

$$\lim_{t \rightarrow 0} \int_K f(x) \rho_t(x) dx = f(e)$$

for all continuous functions f on K . For each $t > 0$, the heat kernel is a C^∞ , strictly positive function on K which satisfies $\int_K \rho_t(x) dx = 1$.

Let $K_{\mathbb{C}}$ be the complexification of K . (See [H1, Sect. 3] for the definition.) Then $K_{\mathbb{C}}$ is a connected complex Lie group whose Lie algebra $\mathfrak{k}_{\mathbb{C}}$ is the complexification of \mathfrak{k} , and which contains K as a subgroup. For example, if $K = \mathbf{SU}(n)$, then $K_{\mathbb{C}} = \mathbf{SL}(n; \mathbb{C})$. The inner product on \mathfrak{k} extends to a real-valued inner product on $\mathfrak{k}_{\mathbb{C}}$ satisfying

$$\langle X_1 + iY_1, X_2 + iY_2 \rangle = \langle X_1, X_2 \rangle + \langle Y_1, Y_2 \rangle$$

for $X_k, Y_k \in \mathfrak{k}$. This inner product determines a left-invariant Riemannian metric on $K_{\mathbb{C}}$. We will let dg denote Haar measure on $K_{\mathbb{C}}$ normalized to coincide with Riemannian volume measure.

As proved in [H1, Sect. 4], the heat kernel ρ_t has a unique analytic continuation from K to $K_{\mathbb{C}}$. The “reproducing kernel” described in Sect. 4 is expressed in terms of the analytic continuation of ρ_t .

The quotient space $K_{\mathbb{C}}/K$ is a manifold with a transitive left action of $K_{\mathbb{C}}$. The tangent space to $K_{\mathbb{C}}/K$ at the identity coset can be thought of as $i\mathfrak{k} \subset \mathfrak{k}_{\mathbb{C}}$. There exists a unique $K_{\mathbb{C}}$ -invariant Riemannian structure on $K_{\mathbb{C}}/K$ which at the identity agrees with our inner product on $i\mathfrak{k} \subset \mathfrak{k}_{\mathbb{C}}$. We will let ν_t be the solution to the equation

$$\frac{d\nu}{dt} = \frac{1}{4}\Delta\nu_t$$

subject to the condition that

$$\lim_{t \rightarrow 0} \int_{K_{\mathbb{C}}/K} f(m) \nu_t(m) dm = f([e])$$

for all continuous functions f of compact support. Here dm denotes Riemannian volume measure and Δ the Laplace-Beltrami operator on $K_{\mathbb{C}}/K$. The function ν_t is positive and C^∞ and satisfies $\int \nu_t(m) dm = 1$.

We will think of ν_t as a right- K -invariant function on $K_{\mathbb{C}}$, one which turns out to be left- K -invariant as well. The normalization of ν_t as a function on $K_{\mathbb{C}}$ is

$$\int_{K_{\mathbb{C}}} \nu_t(g) dg = \text{Vol}(K).$$

This is proved in Lemma 6 in Sect. 4. This normalization guarantees that ν_t as defined in this paper coincides with ν_t as defined in [H1, Thm. 2], since the function μ_t in [H1] integrates to one. An explicit formula for $\nu_t(g)$, due to Gangolli, is given in (11) below.

The space $\mathcal{H}L^2(K_{\mathbb{C}}, \nu_t)$ will denote the space of holomorphic functions F on $K_{\mathbb{C}}$ satisfying

$$\int_{K_{\mathbb{C}}} |F(g)|^2 \nu_t(g) dg < \infty.$$

The norm in this space will be denoted $\|F\|_t$. An explicit formula for the measure $\nu_t(g) dg$ in natural coordinates is given in Lemma 5.

We will use the standard machinery for compact Lie groups. (See [B-D].) Let T be a maximal torus in K , and \mathfrak{t} its Lie algebra. Using the inner product on \mathfrak{k} (restricted to \mathfrak{t}) we will identify \mathfrak{t}^* with \mathfrak{t} . Let $R \subset \mathfrak{t}$ be the real roots, that is, the non-zero α in \mathfrak{t} for which there exists a non-zero $X \in \mathfrak{k}_{\mathbb{C}}$ with

$$[H, X] = i \langle \alpha, H \rangle X$$

for all $H \in \mathfrak{t}$. Let R^+ be a set of positive roots and let ρ be half the sum of the positive roots. Let W be the Weyl group. Let $\Gamma \subset \mathfrak{t}$ be the kernel of the exponential mapping for \mathfrak{t} . Let π denote the polynomial on \mathfrak{t} given by

$$\pi(H) = \prod_{\alpha \in R^+} \langle \alpha, H \rangle.$$

In light of [B, V.(4.10)], π is alternating with respect to the action of the Weyl group.

We will use the polar decomposition for $K_{\mathbb{C}}$, which states that every $g \in K_{\mathbb{C}}$ can be written uniquely in the form $g = xe^{iY}$, with $x \in K$ and $Y \in \mathfrak{k}$. In fact, the map $\Phi : K \times \mathfrak{k} \rightarrow K_{\mathbb{C}}$ given by $\Phi(x, Y) = xe^{iY}$ is a diffeomorphism. (See the proof of Lemma 12 in [H1, Sect. 11].) Since every $Y \in \mathfrak{k}$ can be moved into \mathfrak{t} by the adjoint action of K , every $g \in K_{\mathbb{C}}$ can be written as $g = xe^{iH}y$, with $x, y \in K$ and $H \in \mathfrak{t}$. While this decomposition is not unique, H is unique up to the action of the Weyl group.

3. The Complex Structure on Phase Space

A phase space probability density should be a positive function on phase space (that is on the cotangent bundle $T^*(K)$) which integrates to one with respect to the natural phase volume measure. In Sect. 4 we will associate such a probability density to each $F \in \mathcal{H}L^2(K_{\mathbb{C}}, \nu_t)$ with $\|F\|_t = 1$. The probability density depends on an identification of $K_{\mathbb{C}}$ with $T^*(K)$. In this section we will discover the “right” such identification.

We may identify the cotangent bundle $T^*(K)$ with $K \times \mathfrak{k}^*$ by means of left-translation, and then with $K \times \mathfrak{k}$ by means of the inner product on \mathfrak{k} . (Under this identification, the phase volume measure is simply Haar measure on K times Lebesgue measure on \mathfrak{k} . See Lemma 4.) We then use the diffeomorphism $\Phi : K \times \mathfrak{k} \rightarrow K_{\mathbb{C}}$ of Sect. 2 given by

$$\Phi(x, Y) = xe^{iY}, \quad x \in K, Y \in \mathfrak{k}.$$

Physically, x represents position and Y momentum. Since we are identifying $T^*(K)$ with $K \times \mathfrak{k}$, we will regard Φ as a map from $T^*(K)$ to $K_{\mathbb{C}}$.

The map Φ is natural in several respects. First, it takes the obvious copy of K in $T^*(K)$ to the obvious copy of K in $K_{\mathbb{C}}$, and it intertwines the action of $K \times K$ on $T^*(K)$ with the action of $K \times K$ on $K_{\mathbb{C}}$. Second, if you use Φ to transfer the complex structure

of $K_{\mathbb{C}}$ back to $T^*(K)$, this complex structure fits together with the symplectic structure of $T^*(K)$ to give you a Kähler manifold. (More on this below.) These two conditions already severely constrain what Φ can be. Third, there is a canonical “adapted” complex structure J on $T^*(K)$, and, as explained below, Φ is the unique biholomorphism of $(T^*(K), J)$ with $K_{\mathbb{C}}$ which restricts to the identity map of $K \subset T^*(K)$ onto $K \subset K_{\mathbb{C}}$. Last, the Jacobian of Φ comes out in precisely the right way to give the physically natural bounds on the phase space probability density.

The map Φ is a diffeomorphism of the symplectic manifold $T^*(K)$ with the complex manifold $K_{\mathbb{C}}$. We may use Φ to transfer the complex structure of $K_{\mathbb{C}}$ to a complex structure J on $T^*(K)$. The resulting complex symplectic manifold is in fact a Kähler manifold. This means that $\omega(JX, JY) = \omega(X, Y)$ and that $\omega(X, JX) \geq 0$ for all tangent vectors X and Y , where ω is the canonical 2-form on $T^*(K)$. While the Kählerness of $(T^*(K), J, \omega)$ can be proved directly by differentiating Φ as in Sect. 4, the result also follows from the results of Lempert-Szöke and Guillemin-Stenzel [L-S, Sz1-2, G-S1-2] which I now recap briefly.

Let M be a real-analytic Riemannian manifold and $T(M)$ its tangent bundle. Since M is Riemannian, the tangent and cotangent bundles are identified. A complex structure on $T(M)$ is said to be *adapted* if for each geodesic γ in M the map

$$\tau + i\sigma \rightarrow (\gamma(\tau), \sigma\dot{\gamma}(\tau))$$

is a holomorphic mapping of \mathbb{C} into $T(M)$. If an adapted complex structure exists, then it is unique. Moreover, in this case if we identify $T^*(M)$ and $T(M)$ using the Riemannian structure, then the symplectic structure of $T^*(M)$ and the adapted complex structure of $T(M)$ fit together to give a Kähler manifold. In general, an adapted complex structure may not exist on all of $T(M)$. If M is compact, then an adapted complex structure exists at least on a tube of some radius. If M is a compact Lie group with a bi-invariant metric, then an adapted complex structure exists on all of $T(M)$.

Now, the geodesics in K (with a bi-invariant metric) are precisely the curves of the form $\gamma(\tau) = xe^{\tau Y}$, for $x \in K, Y \in \mathfrak{k}$. If we identify $T(K)$ with $K \times \mathfrak{k}$ via left-translation, then $(\gamma(\tau), \sigma\dot{\gamma}(\tau)) = (xe^{\tau Y}, \sigma Y)$. Thus

$$\Phi(\gamma(\tau), \sigma\dot{\gamma}(\tau)) = xe^{\tau Y} e^{i\sigma Y} = xe^{(\tau+i\sigma)Y}.$$

This last expression clearly depends holomorphically on $z = \tau + i\sigma$. Thus the complex structure on $T(K)$ induced by the map Φ is adapted. Equivalently, if J is the unique adapted complex structure on $T^*(K)$, then Φ is a holomorphism of $(T^*(K), J)$ with $K_{\mathbb{C}}$. So the fact that Φ makes $K_{\mathbb{C}}$ into a Kähler manifold follows from, say, Cor. 5.5 and Thm. 5.6 of [L-S]. (See also [Sz2, Sect. 4].)

4. Phase Space Bounds

I would like to interpret $F \in \mathcal{HL}^2(K_{\mathbb{C}}, \nu_t)$ as the phase space wave function for a quantum particle with configuration space K . Such an interpretation would be impossible if F were an arbitrary element of $L^2(K_{\mathbb{C}}, \nu_t)$, for then F could be supported in an arbitrarily small region of phase space, violating the uncertainty principle. Fortunately, F is required to be holomorphic, which, as we shall see, imposes very precise conditions on how concentrated F can be in phase space.

The natural “reference measure” on $K_{\mathbb{C}}$ is not Haar measure but rather the Liouville phase volume measure, which can be thought of as a measure on $K_{\mathbb{C}}$ by means of

the diffeomorphism Φ between $T^*(K)$ and $K_{\mathbb{C}}$. In terms of the position-momentum coordinates (x, Y) , phase volume measure is simply $dx dY$, that is, Haar measure in x times Lebesgue measure in Y (Lemma 4). Let $\sigma(g)$ denote the density of Haar measure with respect to phase volume measure (Lemma 5). Then for F with $\|F\|_t = 1$, the quantity

$$|F(g)|^2 \nu_t(g) \sigma(g) \tag{3}$$

is the phase space probability density and integrates to one with respect to phase volume measure.

As on any reasonable L^2 -space of holomorphic functions, the pointwise evaluation maps $F \rightarrow F(g)$ are bounded linear functionals on $\mathcal{H}L^2(K_{\mathbb{C}}, \nu_t)$. Estimates on the norms of these functionals will give us bounds on the density (3). Now, as a consequence of [H1, Thm. 6], “evaluation at g ” may be computed as

$$F(g) = \int_{K_{\mathbb{C}}} \overline{\rho_{2t}(hg^*)} F(h) \nu_t(h) dh. \tag{4}$$

Here ρ_{2t} refers to the analytic continuation of ρ_{2t} from K to $K_{\mathbb{C}}$, and the map $g \rightarrow g^*$ is the unique antiholomorphic antiautomorphism of $K_{\mathbb{C}}$ with the property that $g^* = \bar{g}^{-1}$ for $g \in K$. (In the notation of [H1, Sect. 3], $g^* = \bar{g}^{-1}$.) The function $\overline{\rho_{2t}(hg^*)}$ is called the reproducing kernel or Bergman kernel.

For each g , $\rho_{2t}(hg^*)$ is [H1, Thm. 6] holomorphic and square-integrable with respect to h . So (4) tells us that the norm of “evaluation at g ” is equal to the L^2 norm of $\rho_{2t}(hg^*)$. But by (4),

$$\int_{K_{\mathbb{C}}} \overline{\rho_{2t}(hg^*)} \rho_{2t}(hg^*) \nu_t(h) dh = \rho_{2t}(gg^*)$$

because $\rho_{2t}(hg^*)$ is holomorphic and square-integrable with respect to h . So we obtain the bound

$$|F(g)|^2 \leq \rho_{2t}(gg^*) \|F\|_t^2. \tag{5}$$

This bound is sharp in the sense that for each g there is a non-zero F for which equality holds. We will obtain explicit upper bounds (for all t) and lower bounds (for small t) on the function $\rho_{2t}(gg^*)$.

The pointwise bounds (5) lead immediately to sharp bounds on the phase space probability density (3):

$$|F(g)|^2 \nu_t(g) \sigma(g) \leq \rho_{2t}(gg^*) \nu_t(g) \sigma(g) \tag{6}$$

for all g and all F with $\|F\|_t = 1$. The bound in Theorem 1 follows from the estimates for $\rho_{2t}(gg^*)$ in Theorem 2 together with explicit formulas for ν_t and σ .

Theorem 1. *Let $n = \dim K$. For each $t > 0$, there exists a constant a_t such that for all $F \in \mathcal{H}L^2(K_{\mathbb{C}}, \nu_t)$ with $\|F\|_t = 1$ the phase space probability density satisfies*

$$|F(g)|^2 \nu_t(g) \sigma(g) \leq a_t (2\pi t)^{-n}$$

for all $g \in K_{\mathbb{C}}$.

For all sufficiently small $t > 0$, there exists a positive constant b_t such that for each $g \in K_{\mathbb{C}}$ there is $F \in \mathcal{H}L^2(K_{\mathbb{C}}, \nu_t)$ with $\|F\|_t = 1$ such that

$$|F(g)|^2 \nu_t(g) \sigma(g) \geq b_t (2\pi t)^{-n}.$$

The optimal constants a_t and b_t satisfy

$$\lim_{t \rightarrow 0} a_t = \lim_{t \rightarrow 0} b_t = 1,$$

and the convergence is exponentially fast.

Theorem 2. Let $n = \dim K$. For each $g \in K_{\mathbb{C}}$, write g in the form $g = xe^{iH}y$, with $x, y \in K$ and $H \in \mathfrak{t}$. Then for each $t > 0$ there exists a constant a_t such that for all $F \in \mathcal{H}L^2(K_{\mathbb{C}}, \nu_t)$ with $\|F\|_t = 1$

$$|F(g)|^2 \leq \rho_{2t}(gg^*) \leq a_t e^{|\rho|^2 t} (4\pi t)^{-n/2} e^{|H|^2/t} \prod_{\alpha \in R^+} \frac{\langle \alpha, H \rangle}{\sinh \langle \alpha, H \rangle}.$$

Here R^+ is the set of positive roots, and ρ is half the sum of the positive roots.

For all sufficiently small $t > 0$, there exists a positive constant b_t such that for each $g \in K_{\mathbb{C}}$ there is $F \in \mathcal{H}L^2(K_{\mathbb{C}}, \nu_t)$ with $\|F\|_t = 1$ such that

$$|F(g)|^2 = \rho_{2t}(gg^*) \geq b_t e^{|\rho|^2 t} (4\pi t)^{-n/2} e^{|H|^2/t} \prod_{\alpha \in R^+} \frac{\langle \alpha, H \rangle}{\sinh \langle \alpha, H \rangle}.$$

The optimal constants a_t and b_t satisfy

$$\lim_{t \rightarrow 0} a_t = \lim_{t \rightarrow 0} b_t = 1,$$

and the convergence is exponentially fast.

Remarks. 1) If K is commutative or if $K = \text{SU}(2)$, then the constant b_t in the preceding theorems exists not just for small times, but for all times t . I will point out how this is proved after the end of the proof of Theorem 2. It is reasonable to conjecture that this holds for all K .

2) The proof of Theorem 1 relies on a strong similarity between the formula (8) for $\rho_{2t}(gg^*)$ and the formula (11) for $\nu_t(g)$. This similarity is not coincidental. As a consequence of [H2, Thm. 5] $\rho_{2t}(gg^*)$, viewed as a function on $K_{\mathbb{C}}/K$, satisfies the inverse heat equation. In fact it is possible to show that each term in (8) satisfies the inverse heat equation. The $\gamma_0 = 0$ term is (up to a constant) the solution to the inverse heat equation obtained by formally replacing t by $-t$ in the formula (11) for ν_t .

3) The ‘‘averaging lemma’’ [H1, Lem. 11], together with Theorem 2, gives pointwise bounds on functions in the space $\mathcal{H}L^2(K_{\mathbb{C}}, \mu_t)$ of [H1]. The bounds are the same as in Theorem 2, except that a_t and b_t do not tend to one as t tends to zero. These bounds for $\mathcal{H}L^2(K_{\mathbb{C}}, \mu_t)$ are stronger than the bounds of Driver and Gross [D-G, Cor. 3.10], both because $|H| \leq |g|$, and because of the exponentially decaying factors $\langle \alpha, H \rangle / \sinh \langle \alpha, H \rangle$ in Theorem 2. On the other hand, the bounds of Driver and Gross hold in much greater generality.

Proof of Theorem 2. We will use an extension of Urakawa’s [U] Poisson summation formula for the restriction of the heat kernel ρ_t to the maximal torus T . Recall that Γ denotes the kernel of the exponential mapping for \mathfrak{t} , R^+ denotes the set of positive roots, and ρ denotes half the sum of the positive roots. For $\gamma \in \Gamma$, let $\epsilon(\gamma) = \exp i \langle \rho, \gamma \rangle$, so that $\epsilon(\gamma) = \pm 1$. Then

$$\rho_t(e^H) = (2\pi t)^{-n/2} e^{|\rho|^2 t/2} \left(\prod_{\alpha \in R^+} \frac{1}{2 \sin \frac{1}{2} \langle \alpha, H \rangle} \right) \sum_{\gamma \in \Gamma} \epsilon(\gamma) \pi(H - \gamma) e^{-|H - \gamma|^2/2t} \tag{7}$$

for all $H \in \mathfrak{t}$ for which e^H is regular. Here $n = \dim K$ and $\pi(H) = \prod_{\alpha \in R^+} \langle \alpha, H \rangle$.

If K is simply connected then $\epsilon(\gamma) \equiv 1$ and (7) reduces essentially to the formula in [U]. (There is a question about the overall constant, which will be addressed below.) The general result can be reduced to the simply connected case as follows. If K is commutative then R^+ is empty, so $\rho = 0$, $\epsilon(\gamma) \equiv 1$, and $\pi(H) \equiv 1$; thus (7) reduces to the usual Poisson summation formula for the heat kernel on a torus. A general compact connected Lie group is of the form $K = (K_1 \times S)/N$, where K_1 is simply connected, S is a torus, and N is a finite subgroup of the center of $K_1 \times S$. The Lie algebras of K_1 and S are automatically orthogonal with respect to any invariant inner product, and so the heat kernel on $K_1 \times S$ factors, establishing (7) on $K_1 \times S$. To get the heat kernel on K one simply periodizes over the action of N . But it is not hard to see that if γ is in the kernel of the exponential mapping for K then

$$\prod_{\alpha \in R^+} \sin \frac{1}{2} \langle \alpha, H - \gamma \rangle = \epsilon(\gamma) \prod_{\alpha \in R^+} \sin \frac{1}{2} \langle \alpha, H \rangle.$$

From this it is straightforward to see that periodization over N yields (7) for K .

The formula in [U] contains an overall constant which is not computed explicitly. However, because we are normalizing Haar measure on K to coincide with Riemannian volume measure, we are able to pin down this constant. To see that the constant in (7) is correct, note that by Minakshisundaram’s expansion [U, (1.2), (1.7)] and my normalization of the heat equation, ρ_t must satisfy

$$\text{Vol}(K) \rho_t(e) = (2\pi t)^{-n/2} [\text{Vol}(K) + O(t)].$$

So $\rho_t(e) \sim (2\pi t)^{-n/2}$. But as proved in detail below, $\rho_t(e)$ is well approximated for small t by the limit as $H \rightarrow 0$ of just the $\gamma = 0$ term in (7), which goes as $(2\pi t)^{-n/2}$. (Let $H \rightarrow 0$ in (8) using Prop. 3.)

We wish to estimate $\rho_{2t}(gg^*)$. As in Sect. 2, we write $g = xe^{iH}y$, with $x, y \in K$ and $H \in \mathfrak{t}$. Then $gg^* = xe^{iH}yy^{-1}e^{iH}x^{-1} = xe^{2iH}x^{-1}$. Since the analytically continued heat kernel is a class function, this means that $\rho_{2t}(gg^*) = \rho_{2t}(e^{2iH})$. It is not hard to show that (7) can be analytically continued term by term, so that we may simply replace H by $2iH$ (and t by $2t$). The analytic continuation of $|H - \gamma|^2 = \langle H - \gamma, H - \gamma \rangle$ is accomplished by taking a complex bilinear extension of $\langle \cdot, \cdot \rangle$, giving

$$“\langle 2iH - \gamma, 2iH - \gamma \rangle” = -4 \langle H, H \rangle - 4i \langle H, \gamma \rangle + \langle \gamma, \gamma \rangle.$$

Now, every $\gamma \in \Gamma$ is contained in the orbit under W of a unique γ_0 in the closed fundamental Weyl chamber \bar{C} . Letting $W \cdot \gamma_0$ denote the orbit of γ_0 and doing the algebra gives

$$\begin{aligned} \rho_{2t}(e^{2iH}) &= e^{|\rho|^2 t} (4\pi t)^{-n/2} e^{|H|^2/t} \left(\prod_{\alpha \in R^+} \frac{\langle \alpha, H \rangle}{\sinh \langle \alpha, H \rangle} \right) \\ &\times \sum_{\gamma_0 \in \Gamma \cap \bar{C}} \epsilon(\gamma_0) e^{-|\gamma_0|^2/4t} \frac{\sum_{\gamma \in W \cdot \gamma_0} \pi \left(H - \frac{1}{2i} \gamma \right) e^{i \langle H, \gamma \rangle / t}}{\pi(H)}. \end{aligned} \tag{8}$$

We have used the easily verified fact that $\epsilon(w \cdot \gamma_0) = \epsilon(\gamma_0)$ for all $w \in W$, and we have multiplied and divided each term by $\pi(H) = \prod_{\alpha \in R^+} \langle \alpha, H \rangle$.

Strictly speaking this formula is valid only on the complement of the hyperplanes $\langle \alpha, H \rangle = 0$. However, the complement of the hyperplanes is dense, so bounds that apply there continue to hold for all H . We will show directly that the right side of (8) extends to a smooth function on all of \mathfrak{t} .

We now need to estimate the sum

$$\sum_{\gamma_0 \in \Gamma \cap \overline{C}} \epsilon(\gamma_0) e^{-|\gamma_0|^2/4t} \frac{\sum_{\gamma \in W \cdot \gamma_0} \pi \left(H - \frac{1}{2i} \gamma \right) e^{i \langle H, \gamma \rangle / t}}{\pi(H)} \tag{9}$$

in (8). We will show that (9) is a bounded function of H for all t , and that this function tends to one uniformly in H as t tends to zero. Note that the $\gamma_0 = 0$ term is identically equal to one and that all of the other terms are small for small t . So it is easy to see that (9) tends to one for each fixed H not in any hyperplane. But because of the factor of $\pi(H)$ in the denominator, we will have to work much harder to get uniform estimates.

Proposition 3. *There exists a polynomial P , whose degree is equal to twice the number of positive roots, such that*

$$\left| \frac{\sum_{\gamma \in W \cdot \gamma_0} \pi \left(H - \frac{1}{2i} \gamma \right) e^{i \langle H, \gamma \rangle / t}}{\pi(H)} \right| \leq P \left(\frac{|\gamma_0|}{\sqrt{t}} \right) \tag{10}$$

for all H and γ_0 in \mathfrak{t} and all $t > 0$.

This proposition is the key technical result in the proof of Theorem 2. Its proof is deferred to an appendix.

Using Proposition 3 we see easily that the sum (9) is a bounded function of H for each t . If a_t is the supremum over H of this sum, then (8) gives us the first part of Theorem 2. Furthermore, the $\gamma_0 = 0$ term in (9) is one and all the other terms are uniformly small for small t because of Proposition 3 and the factor $\exp(-|\gamma_0|^2/4t)$. It is easy to see, then, that (9) tends to one uniformly in H as $t \rightarrow 0$. Thus the infimum b_t over H will be positive for all sufficiently small t , giving the second part of Theorem 2. The constants a_t and b_t tend to one as t tends to zero, and it is not hard to see that the convergence is exponentially fast, essentially because $\exp(-|\gamma_0|^2/4t)$ tends to zero exponentially fast for each non-zero γ_0 . This gives the last part of the theorem. \square

If K is commutative then R^+ is empty, $\pi(H) \equiv 1$, $\epsilon(\gamma) \equiv 1$, and $\Gamma \cap \overline{C} = \Gamma$. Thus the sum (9) is periodic. But $\rho_{2t}(e^{2iH})$ must be strictly positive, since it is the norm squared of the “evaluation at g ” functional, which is non-zero (e.g., with $F \equiv 1$). So the sum (9) is a strictly positive continuous periodic function, which must therefore be bounded away from zero.

If $K = \mathbf{SU}(2)$ then $\epsilon(\gamma) \equiv 1$, Γ may be identified with the integer lattice in \mathbb{R} , and π is linear. The Weyl group is $\{1, -1\}$ and $\overline{C} = [0, \infty)$. So if y is a suitable linear coordinate on \mathfrak{t} , the sum (9) becomes

$$\left[1 + 2 \sum_{n=1}^{\infty} e^{-n^2/4t} \cos \left(\frac{ny}{t} \right) \right] - \sum_{n=1}^{\infty} e^{-n^2/4t} \frac{n \sin \left(\frac{ny}{t} \right)}{y}.$$

The first term is periodic and is essentially a heat kernel on the circle. It is therefore strictly positive. The second goes to zero as $y \rightarrow \infty$. So $\rho_{2t}(e^{2iH})$ is a strictly positive

continuous function which is the sum of a strictly positive continuous periodic function and a function which goes to zero at infinity. A simple compactness argument then shows that $\rho_{2t}(e^{2iH})$ must be bounded away from zero.

Thus the constant b_t in Theorem 2, and so also in Theorem 1, exists for all t if K is commutative or if $K = \text{SU}(2)$.

Proof of Theorem 1. Recall from Sect. 2 that each $g \in K_{\mathbb{C}}$ can be written in the form $xe^{iH}y$, with $x, y \in K$ and $H \in \mathfrak{k}$ and that ν_t is bi- K -invariant. The formula for ν_t is the following

$$\nu_t(xe^{iH}y) = e^{-|\rho|^2 t} (\pi t)^{-n/2} e^{-|H|^2/t} \prod_{\alpha \in R^+} \frac{\langle \alpha, H \rangle}{\sinh \langle \alpha, H \rangle}. \tag{11}$$

(See also Lemma 5.) If K is semisimple, then this is (up to a constant) a formula of Gangolli [G, Prop. 3.2], where ν_t is $g_{t/4}$ in Gangolli's notation. What Gangolli calls $|\rho_*|$ is $2|\rho|$ in our notation; see the expression for $\rho(H)$ near the top of p.159 in [G].

If K is commutative, then $K_{\mathbb{C}}/K$ is isometric to \mathbb{R}^n , and (11) is the usual Gaussian heat kernel. In general, $K = (K_1 \times S)/N$ with K_1 semisimple, S a torus, and N a finite central subgroup. It follows from the polar decomposition that $K_{\mathbb{C}}/K$ is isometric to $(K_{1,\mathbb{C}}/K_1) \times (S_{\mathbb{C}}/S)$. So (11) holds for K .

As in the compact case, there is a question about the overall constant in (11). The constant can be verified as follows. By Lemma 5 below and our normalization of ν_t ,

$$\text{Vol}(K) = \int_{K_{\mathbb{C}}} \nu_t(g) dg = \text{Vol}(K) \int_{\mathfrak{k}} \nu_t(e^{iY}) \sigma(Y) dY,$$

where σ is given explicitly in the lemma. Cancelling $\text{Vol}(K)$ and letting $t \rightarrow 0$, we see that ν_t should satisfy

$$\lim_{t \rightarrow 0} \int_{\mathfrak{k}} \nu_t(e^{iY}) \sigma(Y) dY = 1. \tag{12}$$

The limit may be computed by making the change of variable $Z = Y/\sqrt{t}$ and moving the limit inside the integral. Since $\sigma(0) = 1$ and $\lim_{H \rightarrow 0} \langle \alpha, H \rangle / \sinh \langle \alpha, H \rangle = 1$, (12) becomes

$$\pi^{-n/2} \int_{\mathfrak{k}} e^{-|Z|^2} dZ = 1,$$

which is true. So the constant in (11) must be correct.

In the next three lemmas we will give an explicit formula for phase volume measure, compute the Jacobian factor σ , and verify that the normalization of ν_t in this paper is consistent with that in [H1]. This last point is necessary because we are using the formula from [H1] for the reproducing kernel. Then to prove Theorem 1 we will simply put everything together.

Lemma 4. *Identify $T^*(K)$ with $K \times \mathfrak{k}$ via left-translation and the inner product on \mathfrak{k} . Then the integral of a function f with respect to phase volume measure is given by*

$$\int_K \int_{\mathfrak{k}} f(x, Y) dx dY,$$

where dx is Haar measure on K normalized to coincide with Riemannian volume measure and dY is Lebesgue measure on \mathfrak{k} normalized by means of the inner product.

Lemma 5. *If f is a continuous function of compact support, then*

$$\int_{K_{\mathbb{C}}} f(g) dg = \int_K \int_{\mathfrak{k}} f(xe^{iY}) dx \sigma(Y) dY,$$

where σ is an Ad - K -invariant function on \mathfrak{k} which satisfies

$$\sigma(H) = \prod_{\alpha \in R^+} \left(\frac{\sinh \langle \alpha, H \rangle}{\langle \alpha, H \rangle} \right)^2$$

for $H \in \mathfrak{k}$.

The measure $\nu_t(g) dg$ is given in (x, Y) coordinates by

$$\nu_t(g) dg = e^{-|\rho|^2 t} (\pi t)^{-n/2} e^{-|Y|^2/t} \eta(Y) dx dY, \tag{13}$$

where $\eta(Y)$ is the Ad - K -invariant function given by

$$\eta(H) = \prod_{\alpha \in R^+} \frac{\sinh \langle \alpha, H \rangle}{\langle \alpha, H \rangle}$$

for $H \in \mathfrak{k}$.

Lemma 6. *Normalizing things as in Sect. 2 we have*

$$\int_{K_{\mathbb{C}}} \nu_t(g) dg = \text{Vol}(K).$$

Proof of Lemma 4. If \mathcal{M} is any Riemannian manifold, the phase volume on $T^*(\mathcal{M})$ may be computed by integrating over the cotangent spaces with respect to Lebesgue measure (normalized by the inner product) and then integrating over \mathcal{M} with respect to Riemannian volume measure. To see this note that the phase volume measure is given by integrating the Liouville $2n$ -form

$$dq^1 \wedge \cdots \wedge dq^n \wedge dp_1 \wedge \cdots \wedge dp_n,$$

where the q 's are local coordinates on \mathcal{M} and the p 's are the associated coordinates on the cotangent spaces. But this is equal to

$$(\sqrt{g} dq^1 \wedge \cdots \wedge dq^n) \wedge \left(\frac{1}{\sqrt{g}} dp_1 \wedge \cdots \wedge dp_n \right)$$

which corresponds to volume measure on \mathcal{M} times normalized Lebesgue measure on the cotangent spaces.

If we use the metric to identify $T^*(\mathcal{M})$ and $T(\mathcal{M})$, we get a similar statement on $T(\mathcal{M})$. The lemma is then just a special case of this general result, in which all the tangent spaces to K are identified isometrically with \mathfrak{k} . \square

Proof of Lemma 5. We have to compute the ‘‘Jacobian’’ of the map $\Phi : K \times \mathfrak{k} \rightarrow K_{\mathbb{C}}$ given by $\Phi(x, Y) = xe^{iY}$. Now

$$\begin{aligned}
\left. \frac{d}{ds} \right|_{s=0} \Phi(xe^{sX}, Y) &= \left. \frac{d}{ds} \right|_{s=0} xe^{iY} e^{-iY} e^{sX} e^{iY} \\
&= (L_{xe^{iY}})_* e^{-i\text{ad}Y}(X) \\
&= (L_{xe^{iY}})_* (\cos \text{ad}Y(X) - i \sin \text{ad}Y(X)).
\end{aligned}$$

Using the formula for the differential of the exponential mapping [He, Thm. II.1.7]

$$\begin{aligned}
\left. \frac{d}{ds} \right|_{s=0} \Phi(x, Y + sX) &= (L_{xe^{iY}})_* \frac{1 - e^{-i\text{ad}Y}}{i\text{ad}Y}(iX) \\
&= (L_{xe^{iY}})_* \left(\frac{1 - \cos \text{ad}Y}{\text{ad}Y}(X) + i \frac{\sin \text{ad}Y}{\text{ad}Y}(X) \right).
\end{aligned}$$

Using left-translation on K we think of the tangent space at each point of $K \times \mathfrak{k}$ as $\mathfrak{k} \oplus \mathfrak{k}$. Using left-translation on $K_{\mathbb{C}}$, we think of the tangent space at each point of $K_{\mathbb{C}}$ as $\mathfrak{k}_{\mathbb{C}} = \mathfrak{k} \oplus \mathfrak{k}$. Thus the differential of Φ at the point (x, Y) is represented by the block matrix

$$\Phi_*(x, Y) = \begin{pmatrix} \cos \text{ad}Y & \frac{1 - \cos \text{ad}Y}{\text{ad}Y} \\ -\sin \text{ad}Y & \frac{\sin \text{ad}Y}{\text{ad}Y} \end{pmatrix}. \quad (14)$$

The cotangent space at each point to $K \times \mathfrak{k}$ is $\mathfrak{k}^* \oplus \mathfrak{k}^*$, which we identify with $\mathfrak{k} \oplus \mathfrak{k}$ via the inner product. Let $\{e_j\}$ be an orthonormal basis for the first copy of \mathfrak{k} and $\{f_j\}$ an orthonormal basis for the second copy of \mathfrak{k} . By Lemma 4, the Liouville form on $K \times \mathfrak{k}$ is

$$e_1 \wedge \cdots \wedge e_n \wedge f_1 \wedge \cdots \wedge f_n. \quad (15)$$

The cotangent space at each point of $K_{\mathbb{C}}$ is similarly identified with $\mathfrak{k} \oplus \mathfrak{k}$, and the $2n$ -form that gives Haar measure on $K_{\mathbb{C}}$ is also given by (15). Thus the density σ of Haar measure with respect to phase volume measure will be given by the determinant of the matrix in (14), which is evidently a function of Y only. Since the blocks of (14) commute, its determinant as a $2n \times 2n$ matrix may be computed by first taking the blockwise ‘‘determinant,’’ which comes out to be $\sin \text{ad}Y/\text{ad}Y$, and then taking the determinant of the result as an $n \times n$ matrix. So

$$\sigma(Y) = \det \left(\frac{\sin \text{ad}Y}{\text{ad}Y} \right).$$

It is clear from this expression that $\sigma(Y)$ is $\text{Ad-}K$ -invariant, so it suffices to compute σ for $Y = H \in \mathfrak{t}$. Now, $\sin \theta/\theta = 1$ when $\theta = 0$, so only the non-zero eigenvalues of $\text{ad}H$ contribute to the determinant. But the non-zero eigenvalues of $\text{ad}H$ are of the form $i\langle \alpha, H \rangle$, with $\alpha \in R$. Since $\sin i\theta/i\theta = \sinh \theta/\theta$ we have

$$\sigma(H) = \prod_{\alpha \in R} \frac{\sinh \langle \alpha, H \rangle}{\langle \alpha, H \rangle} = \prod_{\alpha \in R^+} \left(\frac{\sinh \langle \alpha, H \rangle}{\langle \alpha, H \rangle} \right)^2.$$

This is the formula we want.

Meanwhile, to get the formula (13) for the measure $\nu_t(g) dg$, we take the formula for the function $\nu_t(g)$ and multiply by $\sigma(g)$. Note that the exponentially growing factor $\sinh \langle \alpha, H \rangle$ is in the numerator in the formula for the measure $\nu_t(g) dg$. \square

Proof of Lemma 6. The Riemannian volume measure on $K_{\mathbb{C}}/K$ is invariant under the action of $K_{\mathbb{C}}$. Haar measure on $K_{\mathbb{C}}$ pushed forward under the quotient map to $K_{\mathbb{C}}/K$ is also invariant under the action of $K_{\mathbb{C}}$. It follows automatically that pushed-forward Haar measure equals a constant times Riemannian volume measure. To establish the lemma, we need to show that this constant is $\text{Vol}(K)$.

Now, the quotient map takes the set $P = \exp i\mathfrak{k}$ diffeomorphically onto $K_{\mathbb{C}}/K$; we may thus identify $K_{\mathbb{C}}/K$ with P . Lemma 5 works just as well with xe^{iY} replaced by $e^{iY}x$, and so integration with respect to pushed-forward Haar measure amounts to

$$\text{Vol}(K) \int_{\mathfrak{k}} f(e^{iY}) \sigma(Y) dY.$$

Meanwhile, under the identification of $K_{\mathbb{C}}/K$ with P , the map $Y \rightarrow e^{iY}$ is the geometric exponential mapping for $K_{\mathbb{C}}/K$, which is a diffeomorphism in this case. It follows that integration with respect to Riemannian volume measure is given by

$$\int_{\mathfrak{k}} f(e^{iY}) \phi(Y) dY,$$

where ϕ is a positive density equal to one at the origin. The functions ϕ and σ must differ at most by a multiplicative constant; since $\phi(0) = \sigma(0) = 1$, the constant is one. \square

We are now ready to put everything together. We use formula (11) for ν_t , the formula in Lemma 5 for σ , and the pointwise estimates in Theorem 2. In the resulting bounds on $|F(g)|^2 \nu_t(g) \sigma(g)$, everything miraculously cancels, except for the constant, a factor of $(4\pi t)^{-n/2}$ from Theorem 2, and a factor of $(\pi t)^{-n/2}$ from ν_t . These combine to give you a constant $(a_t \text{ or } b_t)$ times $(2\pi t)^{-n}$, which is Theorem 1. \square

5. Appendix

Proof of Proposition 3. We may write \mathfrak{k} as $\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{a}$, where \mathfrak{k}_1 is semisimple and \mathfrak{a} is abelian, in which case $\mathfrak{t} = \mathfrak{t}_1 \oplus \mathfrak{a}$, where \mathfrak{t}_1 is a maximal abelian subalgebra of \mathfrak{t} . If we identify \mathfrak{t}^* with \mathfrak{t} , then all the roots lie in \mathfrak{t}_1 . Furthermore, we may write γ_0 as $\gamma_1 + \gamma_2$, with $\gamma_1 \in \mathfrak{t}_1$ and $\gamma_2 \in \mathfrak{a}$. Then the contribution of γ_2 to the expression in the proposition is just a multiplicative factor of absolute value one. Since $|\gamma_1| \leq |\gamma_0|$, there is no harm in assuming \mathfrak{k} is semisimple.

We will proceed by computing the Fourier transform, in the sense of tempered distributions, of the fraction in the proposition. The Fourier transform of the numerator is easily computed as a linear combination of derivatives of δ -functions. To compute the Fourier transform of the fraction we will compute the Fourier transform of the numerator and then integrate, in a sense to be described below. The key result will be that the Fourier transform of the fraction has compact support. (See Lemma 9.)

Let a **cone over \mathbf{R}^+** denote a set of the form

$$\{x_0 + a_1\alpha_1 + \dots + a_k\alpha_k \mid a_j \geq 0\} \tag{16}$$

with $x_0 \in \mathfrak{t}$, where $R^+ = \{\alpha_1, \dots, \alpha_k\}$ is the set of positive roots. Analogously define a **cone over \mathbf{R}^-** to be a set of the same form but with $a_j \leq 0$. The set (16) is the

same as $\{x_0 + a_1\alpha_1 + \cdots + a_m\alpha_m \mid a_j \geq 0\}$, where $\alpha_1, \dots, \alpha_m$ are the positive simple roots, which (since we assume \mathfrak{k} is semisimple) form a basis for \mathfrak{t} . Every compact set is contained in a cone over R^+ and in a cone over R^- . The intersection of a cone over R^+ and a cone over R^- is compact.

Definition 7. Suppose $f \in C^\infty(\mathfrak{t})$ and f is supported in some cone over R^+ . Then for $\alpha \in R^+$ define

$$I_\alpha f(x) = \int_0^\infty f(x - t\alpha) dt.$$

The condition on f guarantees that the integral exists, since for all sufficiently large t , $x - t\alpha$ will be outside the cone supporting f . Note also that if x is not in the cone supporting f , then neither is $x - t\alpha$ ($t > 0$). Thus $I_\alpha f$ will again be supported in a cone over R^+ . It is easy to verify that $I_\alpha f$ is C^∞ and that $D_\alpha I_\alpha f = f$, where D_α denotes the directional derivative in the α direction. It is also true that $I_\alpha D_\alpha f = f$, since $I_\alpha D_\alpha f - f$ must be constant along each line of the form $\{x - t\alpha\}$, and is zero when t is large. If f is supported in a cone over R^+ , then for $\alpha, \beta \in R^+$, $I_\alpha I_\beta f$ and $I_\beta I_\alpha f$ both make sense, and must be equal because I_α and I_β are two-sided inverses of D_α and D_β , which commute.

Of course, by reversing signs we can define $I_{-\alpha} f$ for f supported on a cone over R^- . Integration by parts shows that if f is supported on a cone over R^+ and g is supported on a cone over R^- , then

$$\int_{\mathfrak{t}} I_\alpha f(x) g(x) dx = \int_{\mathfrak{t}} f(x) I_{-\alpha} g(x) dx. \quad (17)$$

The integrals make sense because in both cases the integrand is supported on the intersection of a cone over R^+ and a cone over R^- .

Definition 8. Let T be a distribution supported on a cone over R^+ . Then for $\alpha \in R^+$, define a distribution $I_\alpha T$ by

$$(I_\alpha T, f) = (T, I_{-\alpha} f)$$

for all $f \in C_c^\infty(\mathfrak{t})$.

Note that T is supported on a cone over R^+ and $I_{-\alpha} f$ is supported on a cone over R^- . The expression $(T, I_{-\alpha} f)$ really means $(T, \phi I_{-\alpha} f)$, where ϕ is any C^∞ function of compact support which is equal to one in a neighborhood of $\text{supp}(T) \cap \text{supp}(I_{-\alpha} f)$. If f is supported outside a cone over R^+ , then so is $I_{-\alpha} f$. Thus the distribution $I_\alpha T$ will again be supported on a cone over R^+ . If T is a C^∞ function, then by (17) $I_\alpha T$ defined as a distribution coincides with $I_\alpha T$ defined as a function. The results $I_\alpha D_\alpha T = D_\alpha I_\alpha T = T$ and $I_\alpha I_\beta T = I_\beta I_\alpha T$ follow from the corresponding results for functions.

Lemma 9. Let T be a compactly supported distribution which is alternating with respect to the action of the Weyl group. Let $R^+ = \{\alpha_1, \dots, \alpha_k\}$ be the set of positive roots. Then

$$S = I_{\alpha_1} I_{\alpha_2} \cdots I_{\alpha_k} T$$

has compact support, and the convex hull of the support of S is contained in the convex hull of the support of T .

Proof. The distribution T can be approximated, in the sense of distribution, by alternating C^∞ functions T_ϵ such that every point in the support of T_ϵ is within ϵ of a point in the support of T . It suffices, then, to prove the lemma under the assumption that T is an alternating C^∞ function of compact support.

Let E denote the convex hull of the support of T . If f is any C^∞ function supported in E , then it is easy to see that $I_\alpha f$ will be supported in E if and only if

$$\int_{-\infty}^{\infty} f(x + t\alpha) dt = 0 \tag{18}$$

for all x . Let α and β be distinct elements of R^+ , and suppose f , $I_\alpha f$, and $I_\beta f$ are all supported in E . Then

$$\begin{aligned} \int_{-\infty}^{\infty} I_\beta f(x + t\alpha) dt &= \int_{-\infty}^{\infty} \int_0^{\infty} f(x + t\alpha - s\beta) ds dt \\ &= \int_0^{\infty} \int_{-\infty}^{\infty} f(x + t\alpha - s\beta) dt ds \\ &= 0. \end{aligned}$$

Here Fubini applies because α and β are distinct (hence non-parallel) elements of R^+ , so that $f(x + t\alpha - s\beta)$ is zero for all sufficiently large s and t . Thus we see that $I_\alpha I_\beta f$ also is supported in E . Applying this argument repeatedly we see that if f is supported in E , and $I_\alpha f$ is supported in E for each $\alpha \in R^+$, then $I_{\alpha_1} \cdots I_{\alpha_k} f$ is supported in E .

Since T is alternating, $T(s_\alpha x) = -T(x)$, where s_α is the reflection about the hyperplane perpendicular to α . It follows that for any $\alpha \in R^+$, condition (18) holds, so $I_\alpha T$ will be supported in E . But then by the preceding paragraph, $I_{\alpha_1} \cdots I_{\alpha_k} T$ will be supported in E . \square

Now, since π is homogeneous, the expression on the left side of Proposition 3 may be written as

$$\frac{\sum_{\gamma \in W \cdot \gamma_0} \pi \left(\frac{H}{\sqrt{t}} - \frac{1}{2i} \frac{\gamma}{\sqrt{t}} \right) \exp i \left\langle \frac{H}{\sqrt{t}}, \frac{\gamma}{\sqrt{t}} \right\rangle}{\pi \left(\frac{H}{\sqrt{t}} \right)}.$$

Thus the supremum over H of this expression will be a function of γ_0/\sqrt{t} . So it suffices to prove the proposition with $t = 1$.

Since π is alternating, the inner product is Weyl invariant, and γ ranges over a Weyl invariant set, we see that the numerator in the proposition,

$$\sum_{\gamma \in W \cdot \gamma_0} \pi \left(H - \frac{1}{2i} \gamma \right) \exp i \langle H, \gamma \rangle, \tag{19}$$

is alternating. Let T denote the Fourier transform of (19), in the sense of tempered distributions. Then T is also alternating.

Now (19) can be expanded as a linear combination of at most $2^k |W|$ terms of the form

$$\langle \alpha_{i_1}, \gamma \rangle \cdots \langle \alpha_{i_l}, \gamma \rangle \langle \alpha_{i_{l+1}}, H \rangle \cdots \langle \alpha_{i_k}, H \rangle e^{i \langle H, \gamma \rangle},$$

with coefficients independent of γ and H . (Here k is the number of positive roots.) Taking the Fourier transform of this gives an irrelevant constant times

$$\langle \alpha_{i_1}, \gamma \rangle \cdots \langle \alpha_{i_l}, \gamma \rangle D_{\alpha_{i_{l+1}}} \cdots D_{\alpha_{i_k}} \delta_\gamma,$$

where δ_γ denotes a δ -function at γ . Thus $S = I_{\alpha_1} \cdots I_{\alpha_k} T$ is a linear combination of terms of the form

$$\langle \alpha_{i_1}, \gamma \rangle \cdots \langle \alpha_{i_l}, \gamma \rangle I_{\alpha_{i_1}} \cdots I_{\alpha_{i_l}} \delta_\gamma.$$

Now, $I_{\alpha_{i_1}} \cdots I_{\alpha_{i_l}} \delta_\gamma$ is a positive measure, so S is a complex measure. By Lemma 9, S is supported on E , where E is the convex hull of the support of T —that is, E is the convex hull of $W \cdot \gamma_0$. Let C_1 be the smallest cone over R^+ containing E , C_2 the smallest cone over R^- containing E , and $P = C_1 \cap C_2$, so that P is a parallelepiped. There exists a constant c , independent of γ_0 , so that $\text{diam}(P) \leq c \text{diam}(E) \leq 2c|\gamma_0|$. It is a straightforward calculation to see that the measure of the set E with respect to the measure $I_{\alpha_{i_1}} \cdots I_{\alpha_{i_l}} \delta_\gamma$ is at most $\text{diam}(P)^l \leq (2c|\gamma_0|)^l$. Taking into account the factors $\langle \alpha_{i_1}, \gamma \rangle \cdots \langle \alpha_{i_l}, \gamma \rangle$ and the fact that $l \leq k$, we see that the total variation norm of S will be bounded by

$$\text{const.} \left(1 + |\gamma_0|^{2k}\right). \quad (20)$$

But if \mathcal{F} denotes the Fourier transform, then

$$\pi(H) \mathcal{F}^{-1}(S) = \text{const.} \mathcal{F}^{-1}(T) = \text{const.} \sum_{\gamma \in W \cdot \gamma_0} \pi\left(H - \frac{1}{2i}\gamma\right) \exp i \langle H, \gamma \rangle.$$

But both $\mathcal{F}^{-1}(S)$ and $\mathcal{F}^{-1}(T)$ are C^∞ functions, so

$$\frac{\sum_{\gamma \in W \cdot \gamma_0} \pi\left(H - \frac{1}{2i}\gamma\right) \exp i \langle H, \gamma \rangle}{\pi(H)} = \text{const.} \mathcal{F}^{-1}(S), \quad (21)$$

where all the constants are independent of γ_0 . While the left side of (21) is defined only when $\pi(H) \neq 0$, we see that it extends to a C^∞ function on all of \mathfrak{t} .

The expression (21) together with the bound (20) on the total variation of S gives the desired estimate. \square

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