SEMIDUALIZING MODULES AND THE DIVISOR CLASS GROUP

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ABSTRACT. Among the finitely generated modules over a Noetherian ring R, the semidualizing modules have been singled out due to their particularly nice duality properties. When R is a normal domain, we exhibit a natural inclusion of the set of isomorphism classes of semidualizing modules $\mathfrak{S}_0(R)$ into the divisor class group $\mathrm{Cl}(R)$. After a description of the basic properties of this inclusion, it is employed to investigate the structure of $\mathfrak{S}_0(R)$. In particular, the question of the finiteness of this set is answered in the affirmative for some interesting classes of Cohen-Macaulay rings.

1. Introduction

Semidualizing modules arise naturally in the investigations of various duality theories in commutative algebra. One instance of this is Grothendieck and Hartshorne's local duality wherein a canonical module, or more generally a dualizing complex, is employed to study local cohomology [23, 24]. Another instance is Auslander and Bridger's methodical study of duality properties with respect to a rank 1 free module that gives rise to the Gorenstein dimension [1, 2]. A free module of rank 1 and a canonical module are both examples of semidualizing modules.

Let R be a Noetherian ring. A finite R-module C is *semidualizing* if the natural homothety map $R \to \mathrm{Hom}_R(C, C)$ is an isomorphism, and $\mathrm{Ext}^i_R(C, C) = 0$ for each integer $i > 0$. The study of such modules in the abstract was initiated by Foxby [15] and Golod [22] where they were called "suitable" modules, and has been continued recently by others; see for example [8, 17, 18, 20, 21, 25].

The semidualizing modules and more generally the semidualizing *complexes* are useful, for example, in identifying local homomorphisms of finite Gorenstein dimension with particularly nice properties as in [4, 16, 26]. This utility along with our desire to expand upon it motivate our investigation of the basic properties of such modules and of the structure of the set of isomorphism classes of semidualizing R-modules, which we denote $\mathfrak{S}_0(R)$.

Surprisingly little is known about his set, though there has been recent progress. For instance, when R is local, there is a nontrivial metric on $\mathfrak{S}_0(R)$; see [18]. However, one of the simplest questions to state is still open in general:

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Question 1.1. When R is local, must $\mathfrak{S}_0(R)$ be finite?

We say that the ring R is $\mathfrak{S}_0$-*finite* if the set $\mathfrak{S}_0(R)$ is finite. The class of rings that are known to be $\mathfrak{S}_0$-finite is rather small. Gorenstein local rings are trivially $\mathfrak{S}_0$-finite, and it is not difficult to show that Cohen-Macaulay rings of minimal multiplicity are so as well. A discussion of the "local" hypothesis follows Corollary 3.5.

The primary goal of this paper is to verify the $\mathfrak{S}_0$-finiteness of certain classes of rings, and moreover to say as much as possible about the total structure of $\mathfrak{S}_0(R)$ for these rings. While we are mainly interested in the local case, a number of our methods and results are nonlocal in nature.

The main tool employed in this investigation is the divisor class group, as motivated by the work of Bruns [5]. When R is a normal domain, we exhibit a natural inclusion $\mathfrak{S}_0(R) \hookrightarrow \mathrm{Cl}(R)$ that behaves well with respect to certain standard operations. This is the content of Section 3. This inclusion allows us to exploit the known behavior of $\mathrm{Cl}(R)$ to gain insight into the structure of $\mathfrak{S}_0(R)$. For instance, it is immediately clear that, if $\mathrm{Cl}(R)$ is finite, then R is $\mathfrak{S}_0$-finite.

Section 4 includes analyses of several particular cases where the divisor class group can be used to give a complete description of $\mathfrak{S}_0(R)$, in particular, addressing the $\mathfrak{S}_0$-finiteness question for these rings. We recount here three such situations, focusing our attention on the size of $\mathfrak{S}_0(R)$.

First, taking our cues from [5], we describe the semidualizing modules over a determinantal ring in Theorem 4.3.

Theorem A. *Let A be a Cohen-Macaulay normal domain and m, n, r nonnegative integers such that $r < \min\{m, n\}$. With $X = X_{ij}$ an $m \times n$ matrix of indeterminants, set $R = A[X]/I_{r+1}(X)$ where $I_{r+1}(X)$ is the ideal generated by the minors of X of size $r+1$. The ring R is $\mathfrak{S}_0$-finite if and only if A is so. More specifically, we have the following cases.*

(a) *If $r = 0$ or $m = n$, then there is a bijection $\mathfrak{S}_0(A) \to \mathfrak{S}_0(R)$.*
(b) *If $r > 0$ and $m \neq n$, then there is a bijection $\mathfrak{S}(A) \times \{0, 1\} \to \mathfrak{S}(R)$.*

When A is the a graded Cohen-Macaulay (super-)normal domain with A_0 local (and complete), this result extends to the localization (and completion) of $A[X]/I_{r+1}(X)$ at its graded maximal ideal; see Corollaries 4.4 and 4.5.

The next result demonstrates how this technique yields information about rings that are themselves not normal domains. For instance, a complete description of $\mathfrak{S}_0(R)$ is given when R is a quotient of a sufficiently nice normal domain by a sum of powers of ideals generated by parts of a regular sequence. Here is a special case; the general statement is Corollary 4.7.

Theorem B. *Let $A = \coprod_{i \geq 0} A_i$ be a graded super-normal Cohen-Macaulay domain with A_0 local and complete. Let $\mathfrak{n}$ be the graded maximal ideal of A and $\widehat{A}$ the $\mathfrak{n}$-adic completion. Let $y = y_1, \ldots, y_q \in \mathfrak{m}$ be an $A_{\mathfrak{n}}$-sequence and fix an integer $m > 1$. There are bijections*

$$\mathfrak{S}_0(A_{\mathfrak{n}}) \to \mathfrak{S}_0(A_{\mathfrak{n}}/(y)) \qquad \mathfrak{S}_0(A_{\mathfrak{n}}) \times \{0,1\} \to \mathfrak{S}_0(A_{\mathfrak{n}}/(y)^m)$$
$$\mathfrak{S}_0(\widehat{A}) \to \mathfrak{S}_0(\widehat{A}/(y)) \qquad \mathfrak{S}_0(\widehat{A}) \times \{0,1\} \to \mathfrak{S}_0(\widehat{A}/(y)^m)$$

The third application of our method yields Corollary 4.8 wherein $\mathfrak{S}_0(R)$ is described for a ring R obtained from a sufficiently nice normal domain by iterated trivial extensions.

Theorem C. *Let $(A, \mathfrak{n})$ be a complete local Cohen-Macaulay super-normal domain and t a positive integer. For $l = 1, \dots, t$ fix a positive integer q_l and set*

$$R = (A \ltimes A^{q_1}) \otimes_A \cdots \otimes_A (A \ltimes A^{q_t}).$$

If s is the number of indices l with $q_l > 1$, then and there is a bijection

$$\mathfrak{S}_0(A) \times \{0, 1, \dots, 2^s - 1\} \xrightarrow{\cong} \mathfrak{S}_0(R).$$

The statements of our main results are module-theoretic in nature. However, we often employ tools from the derived category. We include a summary of these tools in Section 2 along with basic facts about semidualizing modules and the divisor class group.

2. Background

All rings in this paper are commutative Noetherian with unity, and all modules are unital.

Let R be a ring. An *R-complex* is a sequence of R-module homomorphisms

$$X = \cdots \xrightarrow{\partial^X_{i+1}} X_i \xrightarrow{\partial^X_i} X_{i-1} \xrightarrow{\partial^X_{i-1}} \cdots$$

with $\partial^X_i \partial^X_{i+1} = 0$ for each i. We work occasionally in the derived category $\mathcal{D}(R)$ whose objects are the R-complexes; excellent references on the subject include [19, 23, 27, 28, 29]. The category of R-modules Mod-R is naturally identified with the full subcategory of $\mathcal{D}(R)$ whose objects are the complexes concentrated in degree 0. For R-complexes X and Y the left derived tensor product complex is denoted $X \otimes^{\mathbf{L}}_R Y$ and the right derived homomorphism complex is $\mathbf{R}\mathrm{Hom}_R(X, Y)$. For an integer n, the nth *shift* or *suspension* of X is denoted $\mathsf{\Sigma}^n X$ where $(\mathsf{\Sigma}^n X)_i = X_{i-n}$ and $\partial^{\mathsf{\Sigma}^n X}_i = (-1)^n \partial^X_{i-n}$. The symbol "$\simeq$" indicates an isomorphism in $\mathcal{D}(R)$, and "$\sim$" indicates an isomorphism up to shift.

A complex X is *homologically finite*, respectively *homologically degreewise finite*, if its total homology module $\mathrm{H}(X)$, respectively each individual homology module $\mathrm{H}_i(X)$, is a finite R-module. The *infimum*, *supremum*, and *amplitude* of X are

$$\begin{aligned} \inf(X) &= \inf\{i \in \mathbb{Z} \mid \mathrm{H}_i(X) \neq 0\} \\ \sup(X) &= \sup\{i \in \mathbb{Z} \mid \mathrm{H}_i(X) \neq 0\} \\ \mathrm{amp}(X) &= \sup(X) - \inf(X) \end{aligned}$$

respectively, with the conventions $\inf \emptyset = \infty$ and $\sup \emptyset = -\infty$. When R is local with residue field k, the Bass series of a homologically finite complex X is the formal Laurent series $I^X_R(t) = \sum_i \mu^i_R(X) t^i$ where

$$\mu^i_R(X) = \mathrm{rank}_k \mathrm{H}_{-i}(\mathbf{R}\mathrm{Hom}_R(k, X))$$

for each integer i. From [14, (13.11)] one has $\mathrm{id}_R X$ finite if and only if $I^X_R(t)$ is a Laurent polynomial.

Associated to a complex K is a natural homothety morphism

$$\chi^R_K \colon R \to \mathbf{R}\mathrm{Hom}_R(K, K).$$

When K is homologically finite, it is *semidualizing* if χ^R_K is an isomorphism. A complex D is *dualizing* if it is semidualizing and has finite injective dimension. The set of shift-isomorphism classes of semidualizing R-complexes is denoted $\mathfrak{S}(R)$, and the class of a semidualizing complex K in $\mathfrak{S}(R)$ is denoted $[K]_R$ or simply $[K]$ when there is no danger of confusion. The ring R is $\mathfrak{S}$-*finite* if $\mathfrak{S}(R)$ is a finite set.

For a finite R-module C, this definition is equivalent to that given in the introduction: C is *semidualizing* if the natural homothety map $R \to \operatorname{Hom}_R(C,C)$ is an isomorphism and $\operatorname{Ext}^i_R(C,C) = 0$ for each integer $i > 0$. The module R is semidualizing. When R is Cohen-Macaulay, a *canonical module* is a semidualizing module of finite injective dimension. The set of isomorphism classes of semidualizing R-modules is denoted $\mathfrak{S}_0(R)$. The identification of Mod-R with a subcategory of $\mathcal{D}(R)$ provides a natural inclusion $\mathfrak{S}_0(R) \hookrightarrow \mathfrak{S}(R)$, and we shall usually identify $\mathfrak{S}_0(R)$ with its image in $\mathfrak{S}(R)$. In particular, the class of a semidualizing module C in $\mathfrak{S}_0(R)$ is denoted $[C]_R$ or $[C]$. The ring R is $\mathfrak{S}_0$-*finite* if $\mathfrak{S}_0(R)$ is a finite set.

Some of our favorite ring theoretic properties have characterizations in terms of semidualizing objects. If R is Cohen-Macaulay local, then $\mathfrak{S}(R) = \mathfrak{S}_0(R)$. If R is Gorenstein local, then $\mathfrak{S}(R) = \{[R]\}$. The converses of these statements hold when R admits a dualizing complex; see [8, (3.7),(8.6)].

Let K be a semidualizing complex. A homologically finite complex X is K-*reflexive* when $\mathbf{R}\operatorname{Hom}_R(X,K)$ is homologically bounded and the natural biduality morphism

$$\delta^K_X \colon X \to \mathbf{R}\operatorname{Hom}_R(\mathbf{R}\operatorname{Hom}_R(X,K),K)$$

is an isomorphism. For instance, the complexes R and K are both K-reflexive, and K is dualizing if and only if every homologically finite R-complex is K-reflexive. The G_K-*dimension* of X is

$$\mathrm{G}_K\text{-}\dim_R X = \begin{cases} \inf K - \inf \mathbf{R}\operatorname{Hom}_R(X,K) & \text{when } X \text{ is } K\text{-reflexive} \\ \infty & \text{otherwise.} \end{cases}$$

When C is a semidualizing module, the G_K-dimension of a finite R-module M can be described in terms of resolutions. A finite R-module G is *totally K-reflexive* if the natural biduality map $G \to \operatorname{Hom}_R(\operatorname{Hom}_R(G,C),C)$ is bijective, and $\operatorname{Ext}^i_R(G,C) = 0 = \operatorname{Ext}^i_R(\operatorname{Hom}_R(G,C),C)$ for each $i > 0$. A finite R-module M then has finite G_K-dimension if and only if it admits a resolution

$$0 \to G_g \to \cdots \to G_0 \to M \to 0$$

with each G_i totally K-reflexive; the G_K-dimension of M is then the minimum integer g admitting such a resolution. When R is local and M has finite G_K-dimension, the AB formula [8, (3.14)] reads

$$\mathrm{G}_K\text{-}\dim_R M = \operatorname{depth} R - \operatorname{depth}_R M.$$

Here is a group of results that are standard in the local case. The first includes global versions of theorems of Gerko [20, (3.1),(3.4)].

Lemma 2.1. *Let R be a ring and K, K' semidualizing complexes.*

(a) *The complex $\mathbf{R}\operatorname{Hom}_R(K',K)$ is semidualizing if and only if K' is K-reflexive.*
(b) *If K' is K-reflexive, then the evaluation map $\mathbf{R}\operatorname{Hom}_R(K',K) \otimes^{\mathbf{L}}_R K' \to K$ is an isomorphism.*

(c) *If the complex* $K \otimes^{\mathbf{L}}_R K'$ *is semidualizing, then* K, K' *are both* $K \otimes^{\mathbf{L}}_R K'$*-reflexive and there are isomorphisms*

$$\mathbf{R}\mathrm{Hom}_R(K, K \otimes^{\mathbf{L}}_R K') \simeq K' \qquad \mathbf{R}\mathrm{Hom}_R(K', K \otimes^{\mathbf{L}}_R K') \simeq K.$$

Proof. (a) There is a commutative diagram

$$\begin{array}{ccc} R & \xrightarrow{(1)} & \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(K', K), \mathbf{R}\mathrm{Hom}_R(K', K)) \\ {\scriptstyle \chi^R_{K'}}\downarrow{\scriptstyle \simeq} & & {\scriptstyle (2)}\downarrow{\scriptstyle \simeq} \\ \mathbf{R}\mathrm{Hom}_R(K', K') & \xrightarrow{(3)} & \mathbf{R}\mathrm{Hom}_R(K', \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(K', K), K)) \end{array}$$

where (1) is $\chi^R_{\mathbf{R}\mathrm{Hom}_R(K',K)}$, (2) is from adjunction and the commutativity of tensor product [8, (1.5)], and (3) is $\mathbf{R}\mathrm{Hom}_R(K', \delta^K_{K'})$. It suffices to assume that $\mathbf{R}\mathrm{Hom}_R(K', K)$ is homologically finite, and prove that $\chi^R_{\mathbf{R}\mathrm{Hom}_R(K',K)}$ and $\delta^K_{K'}$ are isomorphisms simultaneously. From the diagram, the morphisms $\chi^R_{\mathbf{R}\mathrm{Hom}_R(K',K)}$ and $\mathbf{R}\mathrm{Hom}_R(K', \delta^K_{K'})$ are isomorphisms simultaneously. The proof of [20, (3.1)], applied locally, shows that $\mathbf{R}\mathrm{Hom}_R(K', \delta^K_{K'})$ and $\delta^K_{K'}$ are locally isomorphisms simultaneously, and are therefore isomorphisms simultaneously.

(b) The map is locally an isomorphism by [20, (3.1)] and is thus an isomorphism.

(c) The natural maps

$$\mathbf{R}\mathrm{Hom}_R(K, K \otimes^{\mathbf{L}}_R K') \to K' \qquad \mathbf{R}\mathrm{Hom}_R(K', K \otimes^{\mathbf{L}}_R K') \to K$$

are isomorphisms locally by [20, (3.4)], and are therefore isomorphisms. Similarly for the biduality morphisms. □

We now interpret Lemma 2.1 for modules.

Lemma 2.2. *Let* R *be a ring and* C, C' *semidualizing* R*-modules.*

(a) *The following conditions are equivalent:*
   (i) *The module* C' *is totally* C*-reflexive;*
   (ii) *The module* C' *has finite* G_C*-dimension;*
   (iii) *The module* $\mathrm{Hom}_R(C', C)$ *is semidualizing and* $\mathrm{Ext}^i_R(C', C) = 0$ *for each* $i > 0$.

When these conditions hold, the module $\mathrm{Hom}_R(C', C)$ *is totally* C*-reflexive*

(b) *If* C' *is totally* C*-reflexive, then the evaluation map* $\mathrm{Hom}_R(C', C) \otimes_R C' \to C$ *is an isomorphism and* $\mathrm{Tor}^R_i(\mathrm{Hom}_R(C', C), C') = 0$ *for each* $i > 0$.

(c) *If the module* $C \otimes_R C'$ *is semidualizing and* $\mathrm{Tor}^R_i(C, C') = 0$ *for each* $i > 0$, *then* C, C' *are both totally* $C \otimes_R C'$*-reflexive and there are isomorphisms*

$$\mathrm{Hom}_R(C, C \otimes^{\mathbf{L}}_R C') \cong C' \qquad \mathrm{Hom}_R(C', C \otimes^{\mathbf{L}}_R C') \simeq C.$$

Proof. (a) The implication (i) $\Longrightarrow$ (iii) follows from Lemma 2.1(a) and the definition of totally C-reflexive. The implication (iii) $\Longrightarrow$ (ii) is handled similarly since condition (iii) implies that the complex $\mathbf{R}\mathrm{Hom}_R(C', C)$ is semidualizing.

For the implication (ii) $\Longrightarrow$ (i), it suffices to show that $\mathrm{G}_C\text{-}\dim_R(C') = 0$, which is a local computation: for each prime $\mathfrak{p} \subset R$, the $R_\mathfrak{p}$-module $C'_\mathfrak{p}$ is semidualizing with finite $\mathrm{G}_{C_\mathfrak{p}}$-dimension, so the AB-formula gives $\mathrm{G}_{C_\mathfrak{p}}\text{-}\dim_{R_\mathfrak{p}}(C'_\mathfrak{p}) = 0$ since $\mathrm{depth}_{R_\mathfrak{p}} C'_\mathfrak{p} = \mathrm{depth} R_\mathfrak{p}$ by [8, (3.2)].

When the conditions (i)–(iii) hold, the fact that $\mathrm{G}_C\text{-}\dim_R(C')$ is finite implies that $\mathrm{G}_C\text{-}\dim_R(\mathbf{R}\mathrm{Hom}_R(C', C))$ is finite as well, that is, $\mathrm{G}_C\text{-}\dim_R(\mathrm{Hom}_R(C', C)) <$

∞ by (iii). Since $\mathrm{Hom}_R(C,C')$ is semidualizing, the implication (ii) $\implies$ (i) shows that $\mathrm{Hom}_R(C',C)$ is totally C-reflexive.

For part (b) take homology in Lemma 2.1 (b) and use part (a). For (c), use Lemma 2.1(c) with part (a). $\square$

The next result showcases a standard method of generating new semidualizing complexes from existing ones.

Lemma 2.3. *Let $\sigma\colon V \to W$ be a homomorphism of finite flat dimension.*

(a) *If K is a semidualizing V-complex, then the W-complex $K \otimes^{\mathbf{L}}_V W$ is semidualizing.*
(b) *If C is a semidualizing V-module and σ is either flat, local, or surjective, then the W-module $C \otimes_V W$ is semidualizing and $\mathrm{Tor}_i^V(C,W)=0$ for each $i>0$.*

Proof. Part (a) is in [8, (5.2)]. For part (b), it suffices to prove the vanishing statement, as once this is known one has an isomorphism $C \otimes^{\mathbf{L}}_V W \simeq C \otimes_V W$ and the result follows from part (a). The flat case is easy, and the local case is [8, (5.8)], so assume that σ is surjective. To prove that $\mathrm{Tor}_i^V(C,W)=0$ for each $i>0$ it suffices to do so locally, so fix a prime ideal $\mathfrak{q} \subset W$. With $\mathfrak{p}=\sigma^{-1}(\mathfrak{q})$, the localized homomorphism $\sigma_{\mathfrak{q}}\colon V_{\mathfrak{p}} \to W_{\mathfrak{q}}$ is surjective and local with finite flat dimension. For $i>0$, the isomorphism in the following sequence is standard

$$\mathrm{Tor}_i^V(C,W)_{\mathfrak{q}} \cong \mathrm{Tor}_i^{V_{\mathfrak{p}}}(C_{\mathfrak{p}},W_{\mathfrak{q}})=0$$

while the equality is by the local case. $\square$

Here is another method of building semidualizing complexes along with some useful properties.

Lemma 2.4. *Let $\sigma\colon V \to W$ be a module-finite homomorphism of finite projective dimension and K a semidualizing complex on V.*

(a) *The W-complex $\mathbf{R}\mathrm{Hom}_V(W,K)$ is semidualizing. In particular, the W-complex $\mathbf{R}\mathrm{Hom}_V(W,V)$ is semidualizing.*
(b) *There is an isomorphism $\mathbf{R}\mathrm{Hom}_V(W,V) \otimes^{\mathbf{L}}_W (K \otimes^{\mathbf{L}}_V W) \simeq \mathbf{R}\mathrm{Hom}_V(W,K)$.*
(c) *The complexes $\mathbf{R}\mathrm{Hom}_V(W,V)$ and $K \otimes^{\mathbf{L}}_V W$ are $\mathbf{R}\mathrm{Hom}_V(W,K)$-reflexive with*

$$\mathbf{R}\mathrm{Hom}_W(K \otimes^{\mathbf{L}}_V W, \mathbf{R}\mathrm{Hom}_V(W,K)) \simeq \mathbf{R}\mathrm{Hom}_V(W,V)$$
$$\mathbf{R}\mathrm{Hom}_W(\mathbf{R}\mathrm{Hom}_V(W,V), \mathbf{R}\mathrm{Hom}_V(W,K)) \simeq K \otimes^{\mathbf{L}}_V W.$$

(d) *Assume σ is surjective. For each prime ideal $\mathfrak{q} \subset W$, one has*

$$\inf \mathbf{R}\mathrm{Hom}_V(W,V)_{\mathfrak{q}} = \mathrm{depth} W_{\mathfrak{q}} - \mathrm{depth} V_{\mathfrak{p}}.$$

where $\mathfrak{p}=\sigma^{-1}(\mathfrak{q})$.

Proof. For part (a), see [8, (6.2)], noting that, even though this result is only stated for the local case, its proof yields the desired conclusion.

Part (b) is given by the following sequence of isomorphisms

$$\begin{aligned}\mathbf{R}\mathrm{Hom}_V(W,V) \otimes^{\mathbf{L}}_W (K \otimes^{\mathbf{L}}_V W) &\simeq \mathbf{R}\mathrm{Hom}_V(W,V) \otimes^{\mathbf{L}}_V K\\ &\simeq \mathbf{R}\mathrm{Hom}_V(W,V \otimes^{\mathbf{L}}_V K)\\ &\simeq \mathbf{R}\mathrm{Hom}_V(W,K)\end{aligned}$$

where the first and third isomorphisms are standard and the second is tensor-evaluation [8, (1.5.4)]. Part (c) follows from (b) using Lemma 2.1(c).

For part (d) consider the isomorphism

$$\mathbf{R}\mathrm{Hom}_V(W,V)_{\mathfrak{q}} \simeq \mathbf{R}\mathrm{Hom}_{V_{\mathfrak{p}}}(W_{\mathfrak{q}},V_{\mathfrak{p}}).$$

Since σ is surjective and $\mathrm{pd}_V W$ is finite, the same is true of $\mathrm{pd}_{V_{\mathfrak{p}}} W_{\mathfrak{q}}$, and the desired equality can be found in the proof of [18, (A.10)]. □

The next result of this ilk is a slight generalization of the fact mentioned above that, when R is Cohen-Macaulay local, the inclusion of $\mathfrak{S}_0(R)$ in $\mathfrak{S}(R)$ is a bijection.

Lemma 2.5. *Let R be a Cohen-Macaulay ring and K a semidualizing R-complex.*

(a) *The function $i_K\colon \mathrm{Spec}(R) \to \mathbb{Z}$ given by $\mathfrak{p} \mapsto \inf(K_{\mathfrak{p}})$ is locally constant on* $\mathrm{Spec}(R)$.

(b) *If $\mathrm{Spec}(R)$ is connected, then $\mathrm{H}_{\inf(K)}(K)$ is a semidualizing module and $\mathrm{H}_i(K) = 0$ for each $i \neq \inf(K)$.*

Proof. Note that, since R is Cohen-Macaulay and K is semidualizing, $\mathrm{H}(K_{\mathfrak{p}}) \neq 0$ and $\inf(K_{\mathfrak{p}}) = \sup(K_{\mathfrak{p}})$ for each prime ideal $\mathfrak{p}$ by [8, (3.7)].

For (a), fix $\mathfrak{p} \in \mathrm{Spec}(R)$ and set $i = \inf(K_{\mathfrak{p}})$. It suffices to show that there exists $s \in R \smallsetminus \mathfrak{p}$ such that, for each prime ideal $\mathfrak{q}$ with $s \notin \mathfrak{q}$, one has $\inf(K_{\mathfrak{q}}) = i$. This is straightforward by the above observation as K is homologically finite and $i = \inf(K_{\mathfrak{p}}) = \sup(K_{\mathfrak{p}})$. Part (b) now follows easily. □

The connectedness assumption in part (b) of this lemma is essential. Indeed, if R_1, R_2 are local Cohen-Macaulay rings with canonical modules ω_1, ω_2 respectively, then the complex $K = \omega_1 \times \Sigma\omega_2$ is semidualizing for $R = R_1 \times R_2$ with $\mathrm{amp}(K) = 1$, even though R is Cohen-Macaulay.

The last result of this section is a translation of Lemma 2.4 to the Cohen-Macaulay situation where each complex is guaranteed to be shift-isomorphic to a module. First, recall that an ideal I of a local ring R is *Gorenstein* if it is perfect (i.e., $\mathrm{pd}_R R/I = \mathrm{grade}\, I$) and $\mathrm{Ext}^g_R(R/I,R)$ is cyclic for $g = \mathrm{grade}\, I$. A surjective homomorphism $\sigma\colon V \to W$ of finite projective dimension is *Gorenstein* (*locally Gorenstein* in the language of [3]) if, for each maximal ideal $\mathfrak{m}$ of V containing $\ker(\sigma)$, the localized ideal $\ker(\sigma)_{\mathfrak{m}}$ of $V_{\mathfrak{m}}$ is Gorenstein.

Lemma 2.6. *Let $\sigma\colon V \to W$ be a surjective homomorphism of finite projective dimension where W is Cohen-Macaulay with $\mathrm{Spec}(W)$ connected. Let C be a semidualizing module on V.*

(a) *For prime ideals $\mathfrak{q}_1, \mathfrak{q}_2 \subset W$, let $\mathfrak{p}_i = \sigma^{-1}(\mathfrak{q}_i)$. There is an equality*

$$\mathrm{depth} V_{\mathfrak{p}_1} - \mathrm{depth} W_{\mathfrak{q}_1} = \mathrm{depth} V_{\mathfrak{p}_2} - \mathrm{depth} W_{\mathfrak{q}_2}.$$

Set $d = \mathrm{depth} V_{\mathfrak{p}_1} - \mathrm{depth} W_{\mathfrak{q}_1}$.

(b) *The W-module $\mathrm{Ext}^d_V(W,C)$ is semidualizing and $\mathrm{Ext}^i_V(W,C) = 0$ for each $i \neq d$. In particular, the W-module $\mathrm{Ext}^d_V(W,V)$ is semidualizing.*

(c) *There is an isomorphism $\mathrm{Ext}^d_V(W,V) \otimes_W (C \otimes_V W) \cong \mathrm{Ext}^d_V(W,C)$, and for each $i > 0$ one has $\mathrm{Tor}_W(\mathrm{Ext}^d_V(W,V), C \otimes_V W) = 0$.*

(d) *The modules $C \otimes_V W$ and $\mathrm{Ext}^d_V(W,V)$ are totally $\mathrm{Ext}^d_V(W,C)$-reflexive. with*

$$\mathrm{Hom}_W(C \otimes_V W, \mathrm{Ext}^d_V(W,C)) \cong \mathrm{Ext}^d_V(W,V)$$

$$\mathrm{Hom}_W(\mathrm{Ext}^d_V(W,V), \mathrm{Ext}^d_V(W,C)) \cong C \otimes_V W$$

(e) *If σ is Gorenstein, then the W-module $\operatorname{Ext}^d_V(W,V)$ is locally free of rank 1.*

Proof. (a) The W-complex $\mathbf{R}\mathrm{Hom}_V(W,V)$ is semidualizing by Lemma 2.4(a), so Lemma 2.5 implies that it is homologically concentrated in a single degree d and that $\mathrm{H}_d(\mathbf{R}\mathrm{Hom}_V(W,V))$ is a semidualizing module. In particular, there is an equality $\mathrm{Supp}_R(\mathrm{H}_d(\mathbf{R}\mathrm{Hom}_V(W,V))) = \mathrm{Spec}(R)$, and this yields the second and third equalities in the following sequence

$$\begin{aligned}
\operatorname{depth} W_{\mathfrak{q}_1} - \operatorname{depth} V_{\mathfrak{p}_1} &= \inf \mathbf{R}\mathrm{Hom}_V(W,V)_{\mathfrak{q}_1} \\
&= \inf \mathbf{R}\mathrm{Hom}_V(W,V) \\
&= \inf \mathbf{R}\mathrm{Hom}_V(W,V)_{\mathfrak{q}_2} \\
&= \operatorname{depth} W_{\mathfrak{q}_2} - \operatorname{depth} V_{\mathfrak{p}_2}
\end{aligned}$$

where the first and fourth equalities are from Lemma 2.4(d).

(b) Since the W-complexes $\mathbf{R}\mathrm{Hom}_V(W,C)$ and $\mathbf{R}\mathrm{Hom}_V(W,V)$ are semidualizing, each one is homologically concentrated in a single degree by Lemma 2.5, and it remains to show that each one has infimum $-d$. That this is true for $\mathbf{R}\mathrm{Hom}_V(W,V)$ is in the proof of part (a). For $\mathbf{R}\mathrm{Hom}_V(W,C)$, suspend the isomorphism in Lemma 2.4(b) to obtain

$$(\dagger) \qquad \Sigma^d \mathbf{R}\mathrm{Hom}_V(W,V) \otimes^{\mathbf{L}}_W (C \otimes^{\mathbf{L}}_V W) \simeq \Sigma^d \mathbf{R}\mathrm{Hom}_V(W,C).$$

Since each of the complexes on the left of this isomorphism is homologically concentrated in degree 0, using Nakayama's lemma locally with the equalities

$$\mathrm{Supp}_R(\mathrm{H}_0(\Sigma^d \mathbf{R}\mathrm{Hom}_V(W,V))) = \mathrm{Spec}(R) = \mathrm{Supp}_R(\mathrm{H}_0(C \otimes^{\mathbf{L}}_V W))$$

one has $\inf(\Sigma^d \mathbf{R}\mathrm{Hom}_V(W,C)) = 0$ which gives the desired formula.

Part (d) is an application of Lemmas 2.4(c) and 2.2(a). Part (c) follows from (d) using Lemma 2.2(b). The statement of (e) is local, so assume that σ is local and Gorenstein. One deduces from the above arguments and the proof of [18, (A.10)] that $\operatorname{Ext}^d_V(W,V) \cong W$. □

This section concludes with the definition of the divisor class group of a normal domain R. Let $(-)^\star$ denote the functor $\mathrm{Hom}_R(-,R)$. The *divisor class group* of R, denoted $\mathrm{Cl}(R)$, is the set of isomorphism classes of reflexive R-modules of rank 1, i.e., the set of isomorphism classes of nonzero reflexive ideals of R. The isomorphism class of a reflexive module M is denoted $[M]_R$ or $[M]$ when there is no risk of confusion[1]. The set $\mathrm{Cl}(R)$ admits an Abelian group structure: when M,N are rank 1 reflexive modules

$$[M] + [N] = [(M \otimes_R N)^{\star\star}] \qquad [M] - [N] = [\mathrm{Hom}_R(N,M)].$$

If $\mathfrak{a},\mathfrak{b}$ are ideals with $\mathfrak{a} \cong M$ and $\mathfrak{b} \cong N$, then $[M]+[N] = [\mathfrak{a}] + [\mathfrak{b}] = [(\mathfrak{a}\mathfrak{b})^{\star\star}]$.

3. Semidualizing modules as divisor classes

The following proposition compares directly to the "classical" result for the canonical module which is the prime motivation for our techniques; see, e.g., [6, (3.3.18)]. Recall that a finite R-module M has *rank* (respectively, *rank r*) if $M_\mathfrak{p}$

[1] Our use of the same notation $[-]$ to describe elements of $\mathfrak{S}_0(R)$ and $\mathrm{Cl}(R)$ may seem confusing. However, this notation is perfectly natural when one thinks of these sets as contained in the set of isomorphism classes of all R-modules.

is free (respectively, free of rank r) over $R_{\mathfrak{p}}$ for each associated prime $\mathfrak{p}$ of R. A finitely generated module has rank if and only if it has finite rank.

Proposition 3.1. *Let R be a ring and C a semidualizing R-module, and consider the following conditions.*

(i) *For each $\mathfrak{p} \in \mathrm{Ass}(R)$, the localization $R_{\mathfrak{p}}$ is Gorenstein.*
(ii) *C has rank 1;*
(iii) *C has rank;*
(iv) *C is isomorphic to an ideal $\mathfrak{a}$ of R such that the quotient $R/\mathfrak{a}$ is torsion.*

The implications (i) $\Longrightarrow$ (ii) $\Longleftrightarrow$ (iii) $\Longleftrightarrow$ (iv) *hold.*

Of course, condition (i) is satisfied if R is a domain. Also, using $C = R$ one sees that the implication (ii) $\Longrightarrow$ (i) fails in general; see Proposition 3.2(c).

Proof. (i) $\Longrightarrow$ (ii). For each associated prime $\mathfrak{p}$, the ring $R_{\mathfrak{p}}$ is Gorenstein and therefore the semidualizing $R_{\mathfrak{p}}$-module $C_{\mathfrak{p}}$ is isomorphic to $R_{\mathfrak{p}}$ by [8, (8.6)].

(ii) $\Longrightarrow$ (iii) is trivial. For the converse, since $C_{\mathfrak{p}}$ is semidualizing for $R_{\mathfrak{p}}$, it is straightforward to check that, if $C_{\mathfrak{p}}$ is free over $R_{\mathfrak{p}}$, then it is free of rank 1.

(ii) $\Longleftrightarrow$ (iv). It is straightforward to show that the semidualizing module C is torsion-free; in fact, $\mathrm{Ass}(R) = \mathrm{Ass}_R(C)$. The desired biimplication now follows from a standard exercise; see for instance [6, (1.4.18)]. □

An ideal $\mathfrak{a}$ of R is a *semidualizing ideal* if it is semidualizing as an A-module and has rank. One consequence of the proposition is that, when R is a domain, every semidualizing module is isomorphic to a semidualizing ideal. The next result provides basic properties of such ideals; it compares directly to [6, (3.3.18)]. We restrict our attention to proper ideals as the case $\mathfrak{a} = R$ is tedious.

Proposition 3.2. *Let R be a Cohen-Macaulay ring of dimension d and $\mathfrak{a}$ a proper semidualizing ideal of R.*

(a) $\mathrm{ht}(\mathfrak{a}) = 1$ *and $R/\mathfrak{a}$ is Cohen-Macaulay of dimension $d-1$.*
(b) *The quotient $R/\mathfrak{a}$ is $\mathfrak{a}$-reflexive and* $\mathbf{R}\mathrm{Hom}_R(R/\mathfrak{a}, \mathfrak{a}) \simeq \Sigma^{-1} R/\mathfrak{a}$.
(c) *Consider the following conditions:*
  (i) *The quotient $R/\mathfrak{a}$ is a Gorenstein ring;*
  (ii) *The ideal $\mathfrak{a}$ is dualizing for R;*
  (iii) *R is generically Gorenstein.*
  The implications (i) $\Longleftrightarrow$ (ii) $\Longrightarrow$ (iii) hold.

A principal ideal generated by a non-zerodivisor is semidualizing, but is dualizing if and only if R is Gorenstein. Thus, the implication (iii) $\Longrightarrow$ (ii) fails in general.

Proof. The proof of (a) is nearly identical to that of [6, (3.3.18.b)], so we omit it here. For part (b), use the exact sequence

$$0 \to \mathfrak{a} \to R \to R/\mathfrak{a} \to 0 \tag{$\dagger$}$$

with the fact that $\mathfrak{a}$ and R are both $\mathfrak{a}$-reflexive to conclude that $R/\mathfrak{a}$ is $\mathfrak{a}$-reflexive. To check the isomorphism $\mathbf{R}\mathrm{Hom}_R(R/\mathfrak{a}, \mathfrak{a}) \simeq \Sigma^{-1} R/\mathfrak{a}$, it suffices to verify that $\mathrm{Ext}^i_R(R/\mathfrak{a}, \mathfrak{a}) = 0$ for $i \neq 1$. Indeed, it will then follow that applying $\mathrm{Hom}_R(-, \mathfrak{a})$ to ($\dagger$) yields the exact sequence

$$0 \to \underbrace{\mathrm{Hom}_R(R, \mathfrak{a})}_{\mathfrak{a}} \to \underbrace{\mathrm{Hom}_R(\mathfrak{a}, \mathfrak{a})}_{R} \to \mathrm{Ext}^1_R(R/\mathfrak{a}, \mathfrak{a}) \to 0$$

which supplies the isomorphism $\operatorname{Ext}^1_R(R/\mathfrak{a},\mathfrak{a}) \cong R/\mathfrak{a}$ and yields the desired isomorphism in the derived category.

To check the desired vanishing of $\operatorname{Ext}^i_R(R/\mathfrak{a},\mathfrak{a})$ we may assume that R is local. Since $\operatorname{depth}_R(R/\mathfrak{a}) = \operatorname{depth} R - 1$, we have $\operatorname{G}_{\mathfrak{a}}\text{-}\dim(R/\mathfrak{a}) = 1$, so that $\operatorname{Ext}^i_R(R/\mathfrak{a},\mathfrak{a}) = 0$ for $i > 1$. Furthermore, since $\mathfrak{a}$ has rank, it contains an element that is both R-regular and $\mathfrak{a}$-regular. It then follows that $\operatorname{Hom}_R(R/\mathfrak{a},\mathfrak{a}) = 0$.

For part (c) we assume again that R is local. In the following sequence of formal equalities of Bass series, the first is by [8, (1.6.7)] and the third is standard

$$I^{\mathfrak{a}}_R(t) = I^{\mathbf{R}\operatorname{Hom}_R(R/\mathfrak{a},\mathfrak{a})}_{R/\mathfrak{a}}(t) = I^{\Sigma^{-1}R/\mathfrak{a}}_{R/\mathfrak{a}}(t) = t \cdot I^{R/\mathfrak{a}}_{R/\mathfrak{a}}(t)$$

while the second is a consequence of part (b). It follows that $\operatorname{id}_R(\mathfrak{a})$ and $\operatorname{id}_{R/\mathfrak{a}}(R/\mathfrak{a})$ are simultaneously finite. This gives the equivalence of (i) and (ii), and the implication (ii) $\Longrightarrow$ (iii) is part of [6, (3.3.18)]. $\square$

The final basic property will be applied to the situation in Lemma 2.2 when C' and $\operatorname{Hom}_R(C',C)$ are semidualizing ideals.

Proposition 3.3. *Let R be a ring and $\mathfrak{a},\mathfrak{b}$ semidualizing ideals such that $\mathfrak{a}\otimes_R\mathfrak{b}$ is semidualizing. The natural multiplication map $\mathfrak{a}\otimes_R\mathfrak{b}\to\mathfrak{a}\mathfrak{b}$ is an isomorphism.*

Proof. The map $\mathfrak{a}\otimes_R\mathfrak{b}\to\mathfrak{a}\mathfrak{b}$ is always surjective, so it remains to verify injectivity. Let S denote the compliment in R of the union of the associated primes of R. Since $\mathfrak{a}$ and $\mathfrak{b}$ have rank, the same is true of $\mathfrak{a}\otimes_R\mathfrak{b}$. Furthermore, the fact that $\mathfrak{a}\otimes_R\mathfrak{b}$ is semidualizing implies that $\mathfrak{a}\otimes_R\mathfrak{b}$ is torsion-free. This yields the injectivity of the localization map $\mathfrak{a}\otimes_R\mathfrak{b}\to S^{-1}(\mathfrak{a}\otimes_R\mathfrak{b})$ in the commuting diagram

$$\begin{array}{ccccccc}
\mathfrak{a}\otimes_R\mathfrak{b} & \xrightarrow{(1)} & \mathfrak{a}\mathfrak{b} & \hookrightarrow & R & \hookrightarrow & S^{-1}R \\
\downarrow & & & & & & \downarrow{=} \\
S^{-1}(\mathfrak{a}\otimes_R\mathfrak{b}) & \xrightarrow{\cong} & S^{-1}\mathfrak{a}\otimes_{S^{-1}R}S^{-1}\mathfrak{b} & \xhookrightarrow{(2)} & (S^{-1}\mathfrak{a})(S^{-1}\mathfrak{b}) & \hookrightarrow & S^{-1}R
\end{array}$$

where the maps (1) and (2) are given by the appropriate multiplication and the others are the natural ones. The map (2) is injective, since $S^{-1}\mathfrak{a}$ and $S^{-1}\mathfrak{b}$ are $S^{-1}R$-free of rank 1. It follows that the map (1) must be injective, as desired. $\square$

A specialization to the case of a normal domain supplies the main tool for this investigation. As was discussed at the end of Section 2 we think of the sets $\mathfrak{S}_0(R), \operatorname{Cl}(R)$ as being contained in the set of isomorphism classes of all R-modules.

Theorem 3.4. *Let R be a normal domain. Each semidualizing R-module C is a rank 1 reflexive module. Thus, there is a natural inclusion $\mathfrak{S}_0(R)\subseteq\operatorname{Cl}(R)$.*

Proof. It suffices to verify the first statement. Proposition 3.1 shows that C has rank 1. For each prime ideal $\mathfrak{p}$ of height 1, the ring $R_{\mathfrak{p}}$ is regular since R is (R_1), and therefore $C_{\mathfrak{p}}\cong R_{\mathfrak{p}}$. Using the fact that R is (S_2) and that $\operatorname{depth}_{R_{\mathfrak{p}}}(C_{\mathfrak{p}}) = \operatorname{depth}(R_{\mathfrak{p}})$ for every prime ideal $\mathfrak{p}$, the reflexivity of C follows from [6, (1.4.1)]. $\square$

We record an immediate corollary.

Corollary 3.5. *A normal domain with finite divisor class group is $\mathfrak{S}_0$-finite. A Cohen-Macaulay normal domain with finite divisor class group is $\mathfrak{S}$-finite.*

Since a Cohen-Macaulay normal domain R with $\mathrm{Cl}(R) = 0$ is Gorenstein, it is worth noting that there are non-Gorenstein rings that satisfy the hypotheses of the corollary. For instance, if k is a field and X a symmetric $n \times n$ matrix of indeterminants and r an integer such that $0 < r < n$, then the ring $R = k[X]/I_{r+1}(X)$ is a Cohen-Macaulay normal domain with $\mathrm{Cl}(R) \cong \mathbb{Z}/(2)$ and is non-Gorenstein if and only if $r \equiv n \pmod 2$. Here $I_{r+1}(X)$ is the ideal generated by the minors of X of size $r + 1$; see [6, (7.3.7.c)]. Determinantal rings will be of particular interest in Section 4.

Theorem 3.4 points toward a plethora of examples of nonlocal rings that are neither $\mathfrak{S}$-finite nor $\mathfrak{S}_0$-finite. First, recall that the *Picard group* of a normal domain R, denoted $\mathrm{Pic}(R)$, is the set of isomorphism classes of finitely generated locally free (i.e., projective) R-modules of rank 1. A result of Claborn [13, (14.10)] states that any Abelian group G can be realized as the divisor class group of a Dedekind domain. In [17] it is shown that there are natural inclusions $\mathrm{Pic}(R) \subseteq \mathfrak{S}_0(R) \subseteq \mathrm{Cl}(R)$ for any normal domain R, and each of these is an equality when R is a Dedekind domain by [13, (18.5)]. In particular, for any Abelian group G, regardless of the cardinality, there is a Dedekind domain R such that $\mathfrak{S}(R) = \mathfrak{S}_0(R) \cong G$.

The inclusion $\mathfrak{S}_0(R) \subseteq \mathrm{Cl}(R)$ is not a group homomorphism, as the set $\mathfrak{S}_0(R)$ does not possess a group structure in general. However, the following gives a limited compatibility between addition and subtraction in $\mathrm{Cl}(R)$ and the operations $\otimes$ and Hom in $\mathfrak{S}_0(R)$. We observe that Lemma 2.2 implies that the hypothesis of (b) is satisfied when C is C''-reflexive and $C' = \mathrm{Hom}_R(C, C'')$.

Proposition 3.6. *Let R be a normal domain and C, C' semidualizing modules.*

(a) *If C' is C-reflexive, then $[\mathrm{Hom}_R(C', C)] = [C] - [C']$ in $\mathrm{Cl}(R)$.*
(b) *If $C' \otimes_R C$ is a semidualizing module, then $[C' \otimes_R C] = [C'] + [C]$ in $\mathrm{Cl}(R)$.*

Proof. Part (a) is simply the definition of subtraction in $\mathrm{Cl}(R)$; see Section 2. Part (b) follows similarly since, when $C' \otimes_R C$ is semidualizing, it is reflexive, so that $(C' \otimes_R C)^{\star\star} \cong C' \otimes_R C$. □

The inclusion $\mathfrak{S}_0(R) \subseteq \mathrm{Cl}(R)$ is well-behaved with respect to certain operations that are defined on both sets. The remainder of this section is devoted to describing some of this behavior.

Let $\varphi\colon R \to S$ be a homomorphism of finite flat dimension.

(a) When K is a semidualizing complex, the S-complex $K \otimes^{\mathbf{L}}_R S$ is semidualizing by Lemma 2.3(a), and the assignment $K \mapsto K \otimes^{\mathbf{L}}_R S$ gives rise to a well-defined map of sets $\mathfrak{S}(\varphi)\colon \mathfrak{S}(R) \to \mathfrak{S}(S)$.
(b) Assume that φ is either flat, local, or surjective. When C is a semidualizing R-module, the S-module $C \otimes_R S$ is semidualizing by Lemma 2.3(b), and the assignment $C \mapsto C \otimes_R S$ induces to a well-defined map of sets $\mathfrak{S}_0(\varphi)\colon \mathfrak{S}_0(R) \to \mathfrak{S}_0(S)$.
(c) If R and S are both normal domains and φ is flat, then the assignment $M \mapsto M \otimes^{\mathbf{L}}_R S$ yields a well-defined group homomorphism $\mathrm{Cl}(\varphi)\colon\ \mathrm{Cl}(R) \to \mathrm{Cl}(S)$.

It is clear that these operations are functorial on the appropriate category of rings.

Proposition 3.7. *Let $\varphi\colon R \to S$ be a ring homomorphism of finite flat dimension.*

(a) *If φ is either flat, local, or surjective, then the following diagram commutes.*

$$\begin{array}{ccc} \mathfrak{S}_0(R) & \hookrightarrow & \mathfrak{S}(R) \\ {\scriptstyle \mathfrak{S}_0(\varphi)}\downarrow & & \downarrow{\scriptstyle \mathfrak{S}(\varphi)} \\ \mathfrak{S}_0(S) & \hookrightarrow & \mathfrak{S}(S) \end{array}$$

(b) *If R and S are normal domains and φ is flat, then the following diagram commutes.*

$$\begin{array}{ccc} \mathfrak{S}_0(R) & \hookrightarrow & \mathrm{Cl}(R) \\ {\scriptstyle \mathfrak{S}_0(\varphi)}\downarrow & & \downarrow{\scriptstyle \mathrm{Cl}(\varphi)} \\ \mathfrak{S}_0(S) & \hookrightarrow & \mathrm{Cl}(S) \end{array}$$

(c) *If φ faithfully flat, then $\mathfrak{S}_0(\varphi)$ and $\mathfrak{S}(\varphi)$ are injective.*

(d) *If φ is faithfully flat with R, S normal domains and $\mathrm{Cl}(\varphi)$ is surjective, then $\mathfrak{S}_0(\varphi)$ is bijective.*

Proof. That the diagrams in (a) and (b) commute follows readily from the definitions. Assume for the rest of the proof that φ is faithfully flat.

(c) By the commutativity of the diagram in (a), it suffices to show that $\mathfrak{S}(\varphi)$ is injective[2]. When φ is local, it is verified in [18, (3.3)] that $\mathfrak{S}(\varphi)$ is injective. For the general case, let K, K' be semidualizing R-complexes such that $\mathfrak{S}(\varphi)([K]) = \mathfrak{S}(\varphi)([K'])$, i.e., such that $K \otimes^{\mathbf{L}}_R S \sim K' \otimes^{\mathbf{L}}_R S$. Apply an appropriate suspension to assume that $K \otimes^{\mathbf{L}}_R S$ and $K' \otimes^{\mathbf{L}}_R S$ are isomorphic. Using the faithful flatness of φ and the local case, it follows that $K_{\mathfrak{p}} \simeq K'_{\mathfrak{p}}$ for each prime ideal $\mathfrak{p}$ of R. There is a commutative diagram

$$\begin{array}{ccc} (\mathbf{R}\mathrm{Hom}_R(K', K) \otimes^{\mathbf{L}}_R K') \otimes^{\mathbf{L}}_R S & \xrightarrow{\alpha \otimes^{\mathbf{L}}_R S} & K \otimes^{\mathbf{L}}_R S \\ \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle =} \\ \mathbf{R}\mathrm{Hom}_S(K' \otimes^{\mathbf{L}}_R S, K \otimes^{\mathbf{L}}_R S) \otimes^{\mathbf{L}}_S (K' \otimes_R S) & \xrightarrow{\alpha'} & K \otimes^{\mathbf{L}}_R S \end{array}$$

where and α and α' are the natural evaluation maps. The isomorphism between the semidualizing S-complexes $K' \otimes^{\mathbf{L}}_R S$ and $K \otimes^{\mathbf{L}}_R S$ guarantees that the map α' is an isomorphism by Lemma 2.1(b). Thus, $\alpha \otimes^{\mathbf{L}}_R S$ is an isomorphism, so the faithful flatness of φ implies that α is one as well.

The isomorphisms $S \simeq \mathbf{R}\mathrm{Hom}_S(K' \otimes^{\mathbf{L}}_R S, K \otimes^{\mathbf{L}}_R S) \simeq \mathbf{R}\mathrm{Hom}_R(K', K) \otimes^{\mathbf{L}}_R S$ coupled with a straightforward faithful flatness argument provide an isomorphism $\mathbf{R}\mathrm{Hom}_R(K', K) \simeq R$. Thus, the isomorphism α yields

$$K \simeq \mathbf{R}\mathrm{Hom}_R(K', K) \otimes^{\mathbf{L}}_R K' \simeq R \otimes^{\mathbf{L}}_R K' \simeq K'$$

as desired.

(d) Assume that R, S are normal domains and that $\mathrm{Cl}(\varphi)$ is surjective. To verify that $\mathfrak{S}_0(\varphi)$ is bijective, it suffices to check surjectivity. For this, one must show that an R-module M is semidualizing if and only if the S-module $M \otimes_R S$ is semidualizing. In the local case, this is shown in [18, (A.13)], and the general case follows as above. $\square$

[2] If one prefers to work with modules, the proof that $\mathfrak{S}(\varphi)$ is injective can be translated directly to a proof of the injectivity of $\mathfrak{S}_0(\varphi)$.

Here are some examples of faithfully flat ring homomorphisms φ with $\mathfrak{S}_0(\varphi)$ bijective. Others are given in Proposition 3.14 and Corollary 3.11.

Corollary 3.8. *Let R be a normal domain and $X = X_1, \dots, X_n$ indeterminants. For the following flat R-algebras S, the map $\mathfrak{S}_0(R) \to \mathfrak{S}_0(S)$ is bijective:*

(a) *$S = R[X]$;*
(b) *$S = R[X][f_1^{-1}, \dots, f_i^{-1}]$ where $f_1, \dots, f_i$ are prime elements of $R[X]$ such that the ring homomorphism $R \to R[X][f_1^{-1}, \dots, f_i^{-1}]$ is faithfully flat;*
(c) *$S = R[X]_{\mathfrak{m}R[X]}$ when R is local with maximal ideal $\mathfrak{m}$;*
(d) *$S = R[\![X]\!]_{\mathfrak{m}R[\![X]\!]}$ when R is local with maximal ideal $\mathfrak{m}$ and the $\mathfrak{m}$-adic completion of R is normal.*

Proof. By the previous proposition, it suffices to note that the maps $\mathrm{Cl}(R) \to \mathrm{Cl}(S)$ are bijective; see [13, (7.3),(8.1),(8.9),(19.15)]. □

The following is an important case when localization induces a bijection on the set of semidualizing modules.

Proposition 3.9. *Let $R = \coprod_{i \geq 0} R_i$ be a graded normal domain with $(R_0, \mathfrak{m}_0)$ local. Setting $\mathfrak{m} = \mathfrak{m}_0 + \coprod_{i \geq 1} R_i$, the natural map $\mathfrak{S}_0(R) \to \mathfrak{S}_0(R_\mathfrak{m})$ is a bijection.*

Proof. Let $\varphi\colon R \to R_\mathfrak{m}$ be the localization map. Using Lemma 3.10 below, the argument of [13, (10.3)] yields the bijectivity of the homomorphism $\mathrm{Cl}(\varphi)\colon \mathrm{Cl}(R) \to \mathrm{Cl}(R_\mathfrak{m})$. It follows from Proposition 3.7(c) that $\mathfrak{S}_0(\varphi)$ is injective. To show surjectivity, fix a semidualizing $R_\mathfrak{m}$-module L. Use the surjectivity of $\mathrm{Cl}(\varphi)$ and [13, (10.2)] to obtain a homogeneous reflexive ideal $\mathfrak{a}$ of R such that $\mathfrak{a}_\mathfrak{m} \cong L$. To complete the proof, it suffices to show that $\mathfrak{a}$ is semidualizing. To this end, consider the following commutative diagram where the unmarked arrows are localizations.

$$\begin{array}{ccccc}
R & \longrightarrow & R_\mathfrak{m} & \xrightarrow{=} & R_\mathfrak{m} \\
\Big\downarrow{\scriptstyle \chi^R_\mathfrak{a}} & & \Big\downarrow{\scriptstyle (\chi^R_\mathfrak{a})_\mathfrak{m}} & & \Big\downarrow{\scriptstyle \chi^{R_\mathfrak{m}}_{\mathfrak{a}_\mathfrak{m}}} \\
\mathrm{Hom}_R(\mathfrak{a},\mathfrak{a}) & \longrightarrow & \mathrm{Hom}_R(\mathfrak{a},\mathfrak{a})_\mathfrak{m} & \xrightarrow{\cong} & \mathrm{Hom}_{R_\mathfrak{m}}(\mathfrak{a}_\mathfrak{m},\mathfrak{a}_\mathfrak{m})
\end{array}$$

By assumption, the map $\chi^{R_\mathfrak{m}}_{\mathfrak{a}_\mathfrak{m}}$ is bijective, implying that $(\chi^R_\mathfrak{a})_\mathfrak{m}$ is as well. An application of Lemma 3.10 to the kernel and cokernel of $\chi^R_\mathfrak{a}$ shows that $\chi^R_\mathfrak{a}$ is bijective. Finally, for $i > 0$ one has $\mathrm{Ext}^i_R(\mathfrak{a},\mathfrak{a})_\mathfrak{m} \cong \mathrm{Ext}^i_{R_\mathfrak{m}}(\mathfrak{a}_\mathfrak{m},\mathfrak{a}_\mathfrak{m}) = 0$, and another application of Lemma 3.10 implies that $\mathrm{Ext}^i_R(\mathfrak{a},\mathfrak{a}) = 0$, completing the proof. □

Lemma 3.10. *Let $R = \coprod_{i \geq 0} R_i$ be a graded ring where R_0 is local with maximal ideal $\mathfrak{m}_0$. Set $\mathfrak{m} = \mathfrak{m}_0 + \coprod_{i \geq 1} R_i$ and let M be a finitely generated graded R-module. Then $M = 0$ if and only if $M_\mathfrak{m} = 0$.*

Proof. One implication is trivial. For the other implication, assume that $M_\mathfrak{m} = 0$. Letting $\overline{R} = R/\mathfrak{m}_0 R$ and $\overline{m} = \mathfrak{m}/\mathfrak{m}_0 R$, it follows that the finitely generated graded $\overline{R}$-module $M/\mathfrak{m}_0 M$ satisfies $(M/\mathfrak{m}_0 M)_{\overline{\mathfrak{m}}} = 0$. Since $\overline{R}$ is a graded ring with $\overline{R}_0 = R_0/\mathfrak{m}_0$ a field, it follows from [13, (10.4)] that $M/\mathfrak{m}_0 M = 0$, that is, $M_i/\mathfrak{m}_0 M_i = 0$ for each integer i. An application of Nakayama's lemma provides the equality $M_i = 0$ for each i, so that $M = 0$ as desired. □

If R is a local ring with completion map $\varphi\colon R \to \widehat{R}$, then the map $\mathfrak{S}_0(\varphi)$ is not usually surjective. Indeed, there exist a Cohen-Macaulay local ring R that does not admit a canonical module; the complete local ring $\widehat{R}$ does admit a canonical module ω, and it is straightforward to show that $[\omega] \in \mathfrak{S}_0(\widehat{R})$ cannot be in the image of $\mathfrak{S}_0(\varphi)$. However, a result of Flenner [12, (1.4)] can be applied in certain cases to provide bijectivity; see Corollary 3.12 for a slight generalization. Recall that a ring is *super-normal* if it satisfies Serre's conditions (S_3) and (R_2).

Corollary 3.11. *Let* $R = \coprod_{i\geq 0} R_i$ *be a graded super-normal domain with* $(R_0, \mathfrak{m}_0)$ *local and complete. Setting* $\mathfrak{m} = \mathfrak{m}_0 + \coprod_{i\geq 1} R_i$ *and* $\widehat{R} = \prod_{i\geq 0} R_i$*, the induced maps* $\mathfrak{S}_0(R) \to \mathfrak{S}_0(\widehat{R})$ *and* $\mathfrak{S}_0(R_\mathfrak{m}) \to \mathfrak{S}_0(\widehat{R})$ *are both bijective.*

Proof. The ring $\widehat{R}$ is the $\mathfrak{m}$-adic completion of $R_\mathfrak{m}$, and since R is excellent and super-normal, the same is true of $R_\mathfrak{m}$ and $\widehat{R}$. Let $\varphi\colon R \to R_\mathfrak{m}$ be the localization map and $\psi\colon R_\mathfrak{m} \to \widehat{R}$ the completion map. By Proposition 3.9, the map $\mathfrak{S}_0(\varphi)$ is bijective, so the equality $\mathfrak{S}_0(\psi\varphi) = \mathfrak{S}_0(\psi)\mathfrak{S}_0(\varphi)$ shows that we need only verify that $\mathfrak{S}_0(\psi)$ is bijective. Since ψ is faithfully flat, Proposition 3.7(c) supplies the injectivity of $\mathfrak{S}_0(\psi)$. The surjectivity is also a consequence of this proposition, as the aforementioned result of Flenner [12, (1.4)] guarantees that $\mathrm{Cl}(\psi\varphi) = \mathrm{Cl}(\psi)\,\mathrm{Cl}(\varphi)$ is surjective, and it follows that $\mathrm{Cl}(\psi)$ is surjective. □

Here is the first indication that our methods have applications outside the normal domain arena. See Corollary 4.7 for a more general statement.

Corollary 3.12. *With* $R, \mathfrak{m}, \widehat{R}$ *as in Corollary 3.11, fix a sequence* $y = y_1, \ldots, y_q$ *in the maximal ideal* $\mathfrak{m}$*.*

(a) *If* y *is* $R_\mathfrak{m}$*-regular, then the natural maps induce bijections*

$$\mathfrak{S}_0(R_\mathfrak{m}/(y)) \cong \mathfrak{S}_0(R_\mathfrak{m}) \cong \mathfrak{S}_0(\widehat{R}) \cong \mathfrak{S}_0(\widehat{R}/(y)).$$

(b) *If* y *is* R*-regular, then the composition*

$$\mathfrak{S}_0(R) \to \mathfrak{S}_0(R/(y)) \to \mathfrak{S}_0(R_\mathfrak{m}/(y))$$

is bijective; thus, the first map is injective and the second is surjective.

Proof. (a) The rings under consideration fit into a commutative diagram

$$\begin{array}{ccc} R_\mathfrak{m} & \longrightarrow & \widehat{R} \\ \downarrow & & \downarrow \\ R_\mathfrak{m}/(y) & \longrightarrow & \widehat{R}/(y) \end{array}$$

where the horizontal maps are faithfully flat and the vertical ones are the natural surjections. Applying $\mathfrak{S}_0(-)$ gives rise to a second commutative diagram

$$\begin{array}{ccc} \mathfrak{S}_0(R_\mathfrak{m}) & \xrightarrow[(1)]{\cong} & \mathfrak{S}_0(\widehat{R}) \\ \Big\downarrow {\scriptstyle (2)} & & {\scriptstyle \cong} \Big\downarrow {\scriptstyle (3)} \\ \mathfrak{S}_0(R_\mathfrak{m}/(y)) & \xrightarrow[(4)]{} & \mathfrak{S}_0(\widehat{R}/(y)) \end{array}$$

where the bijectivity of (1) follows from Corollary 3.11; (3) is bijective can be found in [18, (3.5)]; see also [21, (3)]. From the diagram, it follows that (4) is surjective, and hence bijective by Proposition 3.7(c). Thus, (2) is bijective, as well.

(b) When y is R-regular, there is another commutative diagram

$$\begin{array}{ccc} \mathfrak{S}_0(R) & \xrightarrow{\cong} & \mathfrak{S}_0(R_{\mathfrak{m}}) \\ \downarrow & & \cong\downarrow \\ \mathfrak{S}_0(R/(y)) & \longrightarrow & \mathfrak{S}_0(R_{\mathfrak{m}}/(y)) \end{array}$$

and the desired results follow immediately. □

The surjectivity of the natural map $\mathfrak{S}_0(R_{\mathfrak{m}}) \to \mathfrak{S}_0(R_{\mathfrak{m}}/(y))$ in the previous corollary is somewhat surprising, as it does not hold for more general local rings. Indeed, let $(A, \mathfrak{n})$ be a local Cohen-Macaulay ring that does not admit a canonical module. The natural map into the $\mathfrak{n}$-adic completion $\varphi\colon A \to \widehat{A}$ factors through a power series extension $A \xrightarrow{\dot{\varphi}} A[\![X_1, \ldots, X_e]\!] \xrightarrow{\varphi'} \widehat{A}$ where φ' is surjective with kernel generated by a regular sequence. As mentioned above, the map $\mathfrak{S}_0(\varphi)$ is injective but not surjective. Proposition 3.14(a) below guarantees that $\mathfrak{S}_0(\dot{\varphi})$ is bijective, and it follows from the equality $\mathfrak{S}_0(\varphi) = \mathfrak{S}_0(\varphi')\mathfrak{S}_0(\dot{\varphi})$ that $\mathfrak{S}_0(\varphi')$ is not surjective.

Maps of the type considered in the previous paragraph are always injective, as the next lemma shows. When R is local, this is [18, (3.3)]. The proof of the general case is similar to that of Proposition 3.7(c), so we omit it here.

Lemma 3.13. *Let R be a ring and $y = y_1, \ldots, y_q$ an R-sequence in the Jacobson radical of R. The natural surjection $R \to R/(y)$ induces injective maps*

$$\mathfrak{S}_0(R) \hookrightarrow \mathfrak{S}_0(R/(y)) \qquad \mathfrak{S}(R) \hookrightarrow \mathfrak{S}(R/(y)).$$ □

Here are more examples of ring homomorphisms that induce bijections between the sets of semidualizing modules. Note that, except in part (c), the rings involved are not assumed to be normal domains, so one cannot use the divisor class group directly. However, the method of proof is taken directly from the corresponding results for divisor class groups. It is worth noting that the property for $\mathrm{Cl}(-)$ analogous to part (a) does not hold in general; see, for instance, Danilov [9, 10, 11].

Proposition 3.14. *Let R be a ring and $X = X_1, \ldots, X_n$ indeterminants.*

(a) *The natural maps $R \xrightarrow{\varphi} R[\![X]\!]$ and $R[\![X]\!] \xrightarrow{\psi} R$ induce bijections*

$$\mathfrak{S}_0(R) \xrightarrow{\cong} \mathfrak{S}_0(R[\![X]\!]) \xrightarrow{\cong} \mathfrak{S}_0(R)$$
$$\mathfrak{S}(R) \xrightarrow{\cong} \mathfrak{S}(R[\![X]\!]) \xrightarrow{\cong} \mathfrak{S}(R).$$

(b) *If $(R, \mathfrak{m})$ is local and $a_1, \ldots, a_n \in R$, set $\mathfrak{n} = (\mathfrak{m}, X_1 - a_1, \ldots, X_n - a_n)R[X]$. The natural maps $R \to R[X]_{\mathfrak{n}}$ and $R[X]_{\mathfrak{n}} \to R$ induce bijections*

$$\mathfrak{S}_0(R) \xrightarrow{\cong} \mathfrak{S}_0(R[X]_{\mathfrak{n}}) \xrightarrow{\cong} \mathfrak{S}_0(R)$$
$$\mathfrak{S}(R) \xrightarrow{\cong} \mathfrak{S}(R[X]_{\mathfrak{n}}) \xrightarrow{\cong} \mathfrak{S}(R)$$

(c) *The natural map $R[X] \to R$ induces surjections*

$$\mathfrak{S}_0(R[X]) \twoheadrightarrow \mathfrak{S}_0(R) \qquad \mathfrak{S}(R[X]) \twoheadrightarrow \mathfrak{S}(R).$$

When R is a normal domain, the map $\mathfrak{S}_0(R[X]) \to \mathfrak{S}_0(R)$ is bijective.

Proof. (a) Since the composition $\psi\varphi\colon R \to R$ is the identity, the same is true of the composition $\mathfrak{S}_0(\psi)\mathfrak{S}_0(\varphi)$. In particular, $\mathfrak{S}_0(\psi)$ is surjective. Since X is in the Jacobson radical of $R[\![X]\!]$, Lemma 3.13 guarantees that this map is also injective, and so it is bijective. Since the composition $\mathfrak{S}_0(\psi)\mathfrak{S}_0(\varphi)$ is also bijective, the same is true of $\mathfrak{S}_0(\varphi)$. The same argument works for $\mathfrak{S}(\psi)$ and $\mathfrak{S}(\varphi)$.

The proof of part (b) is nearly identical to that of (a), as is the surjectivity statement in part (c). For the bijectivity in (c), use Corollary 4.6(a). □

4. Analysis of special cases

We begin with some notation that will be used throughout the section. Let A be a Noetherian local ring and m, n, r nonnegative integers with $r < \min\{m, n\}$ and $X = X_{ij}$ an $m \times n$ matrix of indeterminants. Set

$$R_{r+1}(A; m, n) = A[X]/I_{r+1}(X)$$

where $I_{r+1}(X)$ denotes the ideal generated by the minors of X of size $r+1$. For the theory of determinantal rings, consult Bruns and Vetter [7]. Recall that, if A is a normal domain (respectively, is Cohen-Macaulay or is Cohen-Macaulay with canonical module or is (S_3) or is (R_2)) then so is $R_{r+1}(A; m, n)$. Also, R is Gorenstein if and only if A is Gorenstein and either $m = n$ or $r = 0$.

Theorem 4.1. *Let k be a field and m, n, r nonnegative integers such that $r < \min\{m, n\}$. The ring $R = R_{r+1}(k; m, n)$ satisfies $\mathfrak{S}_0(R) = \{[R], [\omega]\}$ where ω is a canonical module for R. In particular the cardinality of $\mathfrak{S}_0(R)$ is*

$$\operatorname{card}\mathfrak{S}_0(R) = \begin{cases} 1 & \text{when } m = n \text{ or } r = 0 \\ 2 & \text{when } m \neq n \text{ and } r \neq 0. \end{cases}$$

Proof. When $r = 0$ or $m = n$, then R is Gorenstein and the result is trivial. Assume for the remainder of the proof that $r > 0$ and $m \neq n$. We may also assume that $n \leq m$, as replacing X with its transpose yields an isomorphism $R_{r+1}(k; m, n) \cong R_{r+1}(k; n, m)$. Let $x_{ij} \in R$ denote the residue of X_{ij}.

Let $\mathfrak{p}$ be the ideal of R generated by the r-minors of the first r rows of the residue matrix x. Consult [7] for the following facts. The canonical module ω of R is unique up to isomorphism; it is isomorphic to the symbolic power $\mathfrak{p}^{(m-n)}$. The Abelian group $\operatorname{Cl}(R)$ is free of rank 1 with generator $[\mathfrak{p}]$, and one has $(m-n)[\mathfrak{p}] = [\mathfrak{p}^{(m-n)}] = [\omega]$.

Following [6, (7.3.6)], we reduce to the case $r = 1$. Suppose that $r > 1$. Let $Y = Y_{pq}$ be an $(m-1)\times(n-1)$ matrix of indeterminants and set $R' = R_r(Y; m-1, n-1)$. Let $\mathfrak{p}'$ be the ideal of R' generated by the $(r-1)$-minors of the first $r-1$ rows of the residue matrix y. By [6, (7.3.3)] there is an isomorphism

$$R[x_{11}^{-1}] \cong R'[X_{11}, \ldots, X_{m1}, X_{12}, \ldots, X_{1n}, X_{11}^{-1}].$$

The natural map $\varphi\colon R \to R[x_{11}^{-1}]$ is flat and induces an isomorphism on the level of divisor class groups. The map $\psi\colon R' \to R'[X_{11}, \ldots, X_{m1}, X_{12}, \ldots, X_{1n}, X_{11}^{-1}]$ is faithfully flat and also induces an isomorphism on the level of divisor class groups. In particular, $\mathfrak{S}_0(\psi)$ is a bijection. Furthermore, $\operatorname{Cl}(\varphi)([\mathfrak{p}]) = \operatorname{Cl}(\psi)([\mathfrak{p}'])$. Let C be a semidualizing module on R and let c be the unique integer with $[C] = c[\mathfrak{p}]$ in $\operatorname{Cl}(R)$. If we can show that $c = 0$ or $c = m - n$, then we are done since then either $C \cong \mathfrak{p}^{(0)} = R$ or $C \cong \mathfrak{p}^{(m-n)} \cong \omega$. The $R[x_{11}^{-1}]$-module $C \otimes_R R[x_{11}^{-1}]$ is semidualizing, and its class in $\operatorname{Cl}(R[x_{11}^{-1}])$ is $c \cdot \operatorname{Cl}(\psi)([\mathfrak{p}'])$. Since $\mathfrak{S}_0(\psi)$ is bijective, it follows that $(\mathfrak{p}')^{(c)}$ is R'-semidualizing. By induction, we conclude that $c = 0$ or $c = m - n$, giving the desired reduction.

Assume that $r = 1$. Let C be a semidualizing module on R and c the unique integer with $[C] = c[\mathfrak{p}]$ in $\mathrm{Cl}(R)$. As above, it suffices to show that $c = 0$ or $c = m - n$. Again, consult [7] for the following facts. The ring R is a standard graded ring over a field and $\mathfrak{p}$ is a homogeneous prime ideal. For each $v > 0$ the symbolic power $\mathfrak{p}^{(v)}$ is homogeneous, and so we may speak of its minimal number of generators, denoted $\beta_0(\mathfrak{p}^{(v)})$. As is noted in [7, (9.20)], there is an equality $\mathfrak{p}^{(v)} = \mathfrak{p}^v$ and the homogeneous minimal generators of $\mathfrak{p}^{(v)}$ are in bijection with the monomials of degree v in the ring $k[Z_1, \dots, Z_n]$. In particular, for $u > 0$ one has $\mathfrak{p}^{(u)}\mathfrak{p}^{(v)} = \mathfrak{p}^{(u+v)}$. Since $n \geq 2$, an easy argument shows that

$$(\dagger) \qquad \beta_0(\mathfrak{p}^{(u)})\beta_0(\mathfrak{p}^{(v)}) > \beta_0(\mathfrak{p}^{(u+v)}).$$

Also, when $u \leq v$, the homothety map $\mathfrak{p}^{(v-u)} \to \mathrm{Hom}_R(\mathfrak{p}^{(u)}, \mathfrak{p}^{(v)})$ is an isomorphism, so we identify $\mathrm{Hom}_R(\mathfrak{p}^{(u)}, \mathfrak{p}^{(v)})$ with $\mathfrak{p}^{(v-u)}$.

First, suppose that $0 < c < m - n$. Then $C \cong \mathfrak{p}^{(c)}$. Since $\mathfrak{p}^{(m-n)}$ is a canonical module for R, it follows from Lemma 2.2(a) that $\mathrm{Hom}_R(\mathfrak{p}^{(c)}, \mathfrak{p}^{(m-n)}) \cong \mathfrak{p}^{(m-n-c)}$ is semidualizing. Furthermore, Proposition 3.3 yields an isomorphism

$$\mathfrak{p}^{(c)} \otimes_R \mathfrak{p}^{(m-n-c)} \xrightarrow{\cong} \mathfrak{p}^{(m-n)}$$

and thus the equality $\beta_0(\mathfrak{p}^{(m-n)}) = \beta_0(\mathfrak{p}^{(c)})\beta_0(\mathfrak{p}^{(m-n-c)})$, contradicting $(\dagger)$.

Next, suppose that $c > m - n$. As above, we have

$$\beta_0(\mathfrak{p}^{(c)}) > \beta_0(\mathfrak{p}^{(m-n)}) = \beta_0(\mathfrak{p}^{(c)})\beta_0(\mathrm{Hom}_R(\mathfrak{p}^{(c)}, \mathfrak{p}^{(m-n)})) > \beta_0(\mathfrak{p}^{(c)})$$

yielding a contradiction.

Finally, suppose that $c < 0$. Then $\mathrm{Hom}_R(C, \omega) \cong \mathfrak{p}^{(m-n-c)}$ is semidualizing. However, $c < 0$ implies that $m - n - c > m - n$ which gives a contradiction to the previous case. This concludes the proof. □

Of course, our focus is on the local situation.

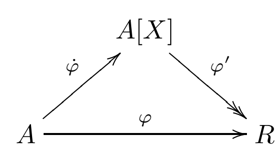
Corollary 4.2. *With notation as in Theorem 4.1, let $\mathfrak{m}$ denote the maximal ideal of R generated by the residues of the variables X_{ij}. Let R' denote either the localization $R_{\mathfrak{m}}$ or its $\mathfrak{m}$-adic completion $\widehat{R}$. Then $\mathfrak{S}_0(R') = \{[R'], [\omega']\}$ where ω' is a canonical module for R'. In particular the cardinality of $\mathfrak{S}_0(R')$ is*

$$\mathrm{card}\,\mathfrak{S}_0(R') = \begin{cases} 1 & \text{when } m = n \text{ or } r = 0 \\ 2 & \text{when } m \neq n \text{ and } r \neq 0. \end{cases}$$

Proof. The ring R satisfies the hypotheses of Corollary 3.11 as it is Cohen-Macaulay and (R_2) by [7, (6.12)]. □

The next result is a generalization of Theorem 4.1 where the hypotheses on the ring of coefficients are relaxed. For this we need some notation. Let A be a Cohen-Macaulay normal domain and m, n, r nonnegative integers such that $r < \min\{m, n\}$. Set $R = R_{r+1}(A; m, n)$ and consider the commutative diagram of natural ring homomorphisms

$(*)$



noting that both φ and $\dot{\varphi}$ are faithfully flat and that φ' is surjective with finite projective dimension. By Lemma 2.6(a), the integer $d = \operatorname{depth}(A[X]_{\mathfrak{p}}) - \operatorname{depth}(R_{\mathfrak{q}})$ is independent of the prime ideal $\mathfrak{q} \subset R$, where $\mathfrak{p}$ is the contraction of $\mathfrak{q}$ to $A[X]$. In particular, letting $\mathfrak{q} = (0)R$, we have

$$d = \operatorname{depth}(A[X]_{I_{r+1}(X)}) = \operatorname{ht}(I_{r+1}(X)) = \dim A[X] - \dim R = mn - r(m+n-r).$$

For a semidualizing A-module C, Lemma 2.3(b) guarantees that the $A[X]$-module $C \otimes_A A[X]$ is semidualizing, as are the R-modules

$$C \otimes_A R \qquad \text{and} \qquad C(\varphi) = \operatorname{Ext}^d_{A[X]}(R, C \otimes_A A[X]).$$

The essence of our next result is that the semidualizing R-modules obtained in this way are the only ones. This encompasses Theorem A from the introduction.

Theorem 4.3. *Let A be a Cohen-Macaulay normal domain and m, n, r nonnegative integers such that $r < \min\{m, n\}$. The ring $R = R_{r+1}(A; m, n)$ is $\mathfrak{S}_0$-finite if and only if A is so. More specifically, we have the following cases.*

(a) *If $r = 0$ or $m = n$, then the map $\mathfrak{S}_0(\varphi)$ is a bijection*

$$\mathfrak{S}_0(\varphi) \colon \mathfrak{S}_0(A) \xrightarrow{\cong} \mathfrak{S}_0(R).$$

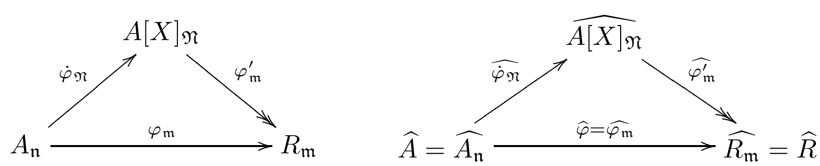
(b) *If $r > 0$ and $m \neq n$, then the assignment*

$$([C]_A, 0) \mapsto [C \otimes_A R]_R \qquad\qquad ([C]_A, 1) \mapsto [C(\varphi)]_R$$

describes a bijection

$$h \colon \mathfrak{S}_0(A) \times \{0, 1\} \xrightarrow{\cong} \mathfrak{S}_0(R).$$

The proof of this result is rather long, so it is included at the end of the section; see 4.10. For now we focus on some consequences of the theorem. First, we discuss the local cases. Continue with the notation preceding Theorem 4.3. Let $\mathfrak{n}$ be a prime ideal of A and consider the prime ideals $\mathfrak{N} = (\mathfrak{n}, X)A[X]$ and $\mathfrak{m} = (\mathfrak{n}, x)R$. Localizing and completing the diagram $(*)$ yield similar commutative diagrams



For semidualizing $A_{\mathfrak{n}}$- and $\widehat{A}$-modules C_0 and C_1, respectively, we set

$$C_0(\varphi_{\mathfrak{m}}) = \operatorname{Ext}^d_{A[X]_{\mathfrak{N}}}(R_{\mathfrak{m}}, C_0 \otimes_{A_{\mathfrak{n}}} A[X]_{\mathfrak{N}}) \qquad \text{(semidualizing for } R_{\mathfrak{m}})$$

$$C_1(\widehat{\varphi}) = \operatorname{Ext}^d_{\widehat{A[X]_{\mathfrak{N}}}}(\widehat{R}, C_1 \otimes_{\widehat{A}} \widehat{A[X]_{\mathfrak{N}}}) \qquad \text{(semidualizing for } \widehat{R})$$

This situation is discussed somewhat extensively in [18].

Corollary 4.4. *Let $A = \coprod_{i \geq 0} A_i$ be a graded normal Cohen-Macaulay domain with $(A_0, \mathfrak{n}_0)$ local, and set $\mathfrak{n} = \mathfrak{n}_0 + \coprod_{i \geq 1} A_i$.*

(a) *If $r = 0$ or $m = n$, then the map $\mathfrak{S}_0(\varphi_{\mathfrak{m}})$ is a bijection*

$$\mathfrak{S}_0(\varphi_{\mathfrak{m}}) \colon \mathfrak{S}_0(A_{\mathfrak{n}}) \xrightarrow{\cong} \mathfrak{S}_0(R_{\mathfrak{m}})$$

(b) *If $r > 0$ and $m \neq n$, then the assignment*

$$([C]_{A_\mathfrak{n}}, 0) \mapsto [C \otimes_{A_\mathfrak{n}} R_\mathfrak{m}]_{R_\mathfrak{m}} \qquad ([C]_{A_\mathfrak{n}}, 1) \mapsto [C(\varphi_\mathfrak{n})]_{R_\mathfrak{m}}$$

describes a bijection

$$h_\mathfrak{m} \colon \mathfrak{S}_0(A_\mathfrak{n}) \times \{0,1\} \xrightarrow{\cong} \mathfrak{S}_0(R_\mathfrak{m})$$

Proof. The following diagrams (one for each of our cases) commute.

$$\begin{array}{ccc} \mathfrak{S}_0(A) & \xrightarrow{\cong} & \mathfrak{S}_0(A_\mathfrak{n}) \\ \cong \big\downarrow \mathfrak{S}_0(\varphi) & & \big\downarrow \mathfrak{S}_0(\varphi_\mathfrak{m}) \\ \mathfrak{S}_0(R) & \xrightarrow{\cong} & \mathfrak{S}_0(R_\mathfrak{m}) \end{array} \qquad \begin{array}{ccc} \mathfrak{S}_0(A) \times \{0,1\} & \xrightarrow{\cong} & \mathfrak{S}_0(A_\mathfrak{n}) \times \{0,1\} \\ \cong \big\downarrow h & & \big\downarrow h_\mathfrak{m} \\ \mathfrak{S}_0(R) & \xrightarrow{\cong} & \mathfrak{S}_0(R_\mathfrak{m}) \end{array}$$

The four horizontal maps are bijective by Proposition 3.9, and two of the vertical ones are bijective by Theorem 4.3. Thus, the two remaining maps are bijective. □

Next, we give the completed case.

Corollary 4.5. *Let $A = \coprod_{i \geq 0} A_i$ be a graded super-normal Cohen-Macaulay domain with $(A_0, \mathfrak{n}_0)$ local and complete. Set $\mathfrak{n} = \mathfrak{n}_0 + \coprod_{i \geq 1} A_i$, and let $\mathfrak{m}, \widehat{A}, \widehat{R}$ be as in the paragraph preceding Corollary 4.4.*

(a) *If $r = 0$ or $m = n$, then the map $\mathfrak{S}_0(\widehat{\varphi})$ is a bijection*

$$\mathfrak{S}_0(\widehat{\varphi}) \colon \mathfrak{S}_0(\widehat{A}) \xrightarrow{\cong} \mathfrak{S}_0(\widehat{R}).$$

(b) *If $r > 0$ and $m \neq n$, then the assignment*

$$([C]_{\widehat{A}}, 0) \mapsto [C \otimes_{\widehat{A}} \widehat{R}]_{\widehat{R}} \qquad ([C]_{\widehat{A}}, 1) \mapsto [C(\widehat{\varphi})]_{\widehat{R}}$$

describes a bijection

$$\widehat{h} \colon \mathfrak{S}_0(\widehat{A}) \times \{0,1\} \xrightarrow{\cong} \mathfrak{S}_0(\widehat{R}).$$

Proof. The proof is almost identical to the previous one, using Corollary 3.11 in place of Proposition 3.9. It suffices to note that, since A is Cohen-Macaulay and super-normal, the same is true of R by [7, (5.17),(6.12)]. □

The next step is to iterate this process.

Corollary 4.6. *Let A be a Cohen-Macaulay normal domain and t a positive integer. For $l = 1, \dots, t$ fix integers r_l, m_l, n_l such that $0 \leq r_l < \min\{m_l, n_l\}$ and let $X_{l**} = X_{lij}$ be an $m_l \times n_l$ matrix of indeterminants. Let X denote the entire list of variables $X_{111}, \dots, X_{lm_ln_l}$ and set*

$$R = A[X] / \sum_{l=1}^{t} I_{r_l+1}(X_{l**})$$

with x the image in R of the sequence X. Let s be the number of indices l such that $r_l > 0$ and $m_l \neq n_l$.

(a) *There is a bijection*

$$\mathfrak{S}_0(A) \times \{0, 1, \dots, 2^s - 1\} \xrightarrow{\cong} \mathfrak{S}_0(R).$$

In particular, if A is Gorenstein with trivial Picard group, then $\mathfrak{S}_0(R)$ has cardinality 2^s.

(b) *With $A, \mathfrak{n}$ as in Corollary 4.4 and $\mathfrak{m} = (\mathfrak{n}, x)R$, there is a bijection*

$$\mathfrak{S}_0(A_\mathfrak{n}) \times \{0, 1, \dots, 2^s - 1\} \xrightarrow{\cong} \mathfrak{S}_0(R_\mathfrak{m}).$$

In particular, if $A_\mathfrak{n}$ is Gorenstein, then $\mathfrak{S}_0(R_\mathfrak{m})$ has cardinality 2^s.

(c) *With $A, \mathfrak{n}$ as in Corollary 4.5 and $\mathfrak{m} = (\mathfrak{n}, x)R$, let $\widehat{A}$ and $\widehat{R}$ denote the $\mathfrak{n}$-adic and $\mathfrak{m}$-adic completions of A and R, respectively. There is a bijection*

$$\mathfrak{S}_0(\widehat{A}) \times \{0, 1, \dots, 2^s - 1\} \xrightarrow{\cong} \mathfrak{S}_0(\widehat{R}).$$

In particular, if $\widehat{A}$ is Gorenstein, then $\mathfrak{S}_0(\widehat{R})$ has cardinality 2^s.

We note here some conditions guaranteeing that $\mathrm{Pic}(A) = 0$. If A is local, then $\mathrm{Pic}(A) = 0$. If A is graded as in part (b) or (c) of the corollary, then one sees from [13, (10.2)–(10.4)] that $\mathrm{Pic}(A) = 0$. (The group $\mathrm{Pic}(A)$ should not be confused with $\mathrm{Pic}(\mathrm{Proj}(A))$.) We consider $\mathrm{Pic}(A)$ in these statements because, as is discussed in [6, (3.3.17)], when A has a canonical module, the set of canonical modules of A is in bijection with $\mathrm{Pic}(A)$. In particular, when A is Gorenstein with trivial Picard group, one has $\mathfrak{S}_0(A) = \{[A]\}$. See [17] for more details on the interaction between $\mathrm{Pic}(A)$ and $\mathfrak{S}_0(A)$.

Proof. Write $R_0 = A$ and for $l = 1, \dots t$ set $R_l = R_{r_l+1}(R_{l-1}; m_l, n_l)$. Then $R_t \cong R$ and part (a) is proved by induction on t using Theorem 4.3. Parts (b) and (c) now follow from Proposition 3.9 and Corollary 3.11, respectively. □

We now apply the results of this section to obtain results for some Cohen-Macaulay rings that are not domains. The first result generalizes Corollary 3.12 and contains Theorem B from the introduction.

Corollary 4.7. *With $A, \mathfrak{n}$ as in Corollary 4.5, let t be a positive integer and for $l = 1, \dots, t$ fix positive integers m_l, q_l and a sequence $y_{l*} = y_{l1}, \dots, y_{lq_l} \in \mathfrak{n}$. Let y denote the full list $y = y_{11}, \dots, y_{tq_t}$ and set*

$$I = \sum_{l=1}^{t} (y_{l*})^{m_l} A.$$

Let s be the number of indices l such that $m_l > 1$.

(a) *If y is $A_\mathfrak{n}$-regular, then there are bijections*

$$\mathfrak{S}_0(A_\mathfrak{n}) \times \{0, 1, \dots, 2^s - 1\} \xrightarrow{\cong} \mathfrak{S}_0(A_\mathfrak{n}/IA_\mathfrak{n})$$
$$\mathfrak{S}_0(\widehat{A}) \times \{0, 1, \dots, 2^s - 1\} \xrightarrow{\cong} \mathfrak{S}_0(\widehat{A}/I\widehat{A}).$$

(b) *If y is A-regular, then there is an injection*

$$\mathfrak{S}_0(A) \times \{0, 1, \dots, 2^s - 1\} \hookrightarrow \mathfrak{S}_0(A/I).$$

Proof. For $l = 1, \dots, t$ set $r_l = m_l - 1$ and $n_l = m_l + q_l - 1$, and let $R, \mathfrak{m}$ be as in Corollary 4.6. Arguing as in [7, discussion after (2.14)], there is a surjection $\tau \colon R \to A/I$. When y is $A_\mathfrak{n}$-regular, the kernel of the induced surjections $R_\mathfrak{m} \to A_\mathfrak{n}/IA_\mathfrak{n}$ and $\widehat{R} \to \widehat{A}/I\widehat{A}$ are generated by a sequence in $\mathfrak{m}R_\mathfrak{m}$ that is $R_\mathfrak{m}$- and $\widehat{R}$-regular. When y is A-regular, the kernel of τ is generated by an R-sequence in $\mathfrak{m}$. The result now follows from Corollaries 3.12 and 4.6. □

We now consider tensor products of trivial extensions. Let A be a ring and t a positive integer. For $l = 1, \dots, t$ fix a positive integer q_l and indeterminants $Y_{l*} = Y_{l1}, \dots, Y_{lq_l}$. We consider the ring

$$R = A[Y_{1*}]/(Y_{1*})^2 \otimes_A \cdots \otimes_A A[Y_{t*}]/(Y_{t*})^2$$

which can be thought of in several different ways. First off, each ring $A[Y_{l*}]/(Y_{l*})^2$ is isomorphic to the trivial extension $A \ltimes A^{q_l}$, so there is an isomorphism

$$R \cong (A \ltimes A^{q_1}) \otimes_A \cdots \otimes_A (A \ltimes A^{q_t})$$

Next, set $R_0 = A$ and take successive trivial extensions $R_l = R_{l-1} \ltimes (R_{l-1})^{q_l}$. From the previous description, it is clear that there is an isomorphism $R \cong R_t$. Finally, let Y denote the full list of variables $Y = Y_{11}, \dots, Y_{tq_t}$. From the definition of R, one obtains the isomorphism

$$R \cong A[Y]/\sum_{l=1}^{t}(Y_{l*})^2 A[Y].$$

With this notation, let y denote the image in R of the sequence Y. For a maximal ideal $\mathfrak{n}$ of A set $\mathfrak{m} = (\mathfrak{n}, y)R$, and let $\widehat{A}$ and $\widehat{R}$ denote the $\mathfrak{n}$-adic and $\mathfrak{m}$-adic completions, respectively. The final result of this paper contains Theorem C from the introduction.

Corollary 4.8. *With $A, \mathfrak{n}$ as in Corollary 4.5, let t be a positive integer. For $l = 1, \dots, t$ fix a positive integer q_l, and let s denote the number of indices l such that $q_l > 1$. With*

$$R = (A \ltimes A^{q_1}) \otimes_A \cdots \otimes_A (A \ltimes A^{q_t})$$

and $\mathfrak{m}, \widehat{A}, \widehat{R}$ as above, there is an injection

$$\mathfrak{S}_0(A) \times \{0, 1, \dots, 2^s - 1\} \hookrightarrow \mathfrak{S}_0(R)$$

and there are bijections

$$\mathfrak{S}_0(A_\mathfrak{n}) \times \{0, 1, \dots, 2^s - 1\} \xrightarrow{\cong} \mathfrak{S}_0(R_\mathfrak{m})$$
$$\mathfrak{S}_0(\widehat{A}) \times \{0, 1, \dots, 2^s - 1\} \xrightarrow{\cong} \mathfrak{S}_0(\widehat{R}).$$

Proof. In light of the final description of R in the preceding paragraph, the result follows immediately from Corollary 4.7. □

To keep things tangible, we give an explicit description of the injection

$$\mathfrak{S}_0(A) \times \{0, 1, \dots, 2^s - 1\} \hookrightarrow \mathfrak{S}_0(R)$$

from the previous corollary. (The two bijections are described analogously.) Set $R_0 = A$ and take successive trivial extensions $R_l = R_{l-1} \ltimes (R_{l-1})^{q_l}$ so that $R \cong R_t$. Each faithfully flat local homomorphism $\varphi_{l-1} \colon R_{l-1} \to R_l$ induces an injective map:

(a) If $q_l = 1$, then set $f_{l-1} = \mathfrak{S}_0(\varphi_{l-1}) \colon \mathfrak{S}_0(R_{l-1}) \to \mathfrak{S}_0(R_l)$;
(b) If $q_l > 1$, then let $f_{l-1} \colon \mathfrak{S}_0(R_{l-1}) \times \{0, 1\} \to \mathfrak{S}_0(R_l)$ be given by

$$([C]_{R_{l-1}}, 0) \mapsto [C \otimes_{R_{l-1}} R_l]_{R_l}$$
$$([C]_{R_{l-1}}, 1) \mapsto [\operatorname{Hom}_{R_{l-1}}(R_l, C)]_{R_l}.$$

The desired inclusion is exactly the composition $f_{t-1} \cdots f_0$, as one sees from [18, (A.10)] and the explicit descriptions of Corollary 3.12 and Theorem 4.3.

The calculations of this section motivate the following refinement of Question 1.1.

Question 4.9. If R is a local ring, must the sets $\mathfrak{S}_0(R)$ and $\mathfrak{S}(R)$ have cardinality equal to a power of 2?

The discussion following Corollary 3.5 explains the need for the "local" hypothesis. Beyond the results of this section, evidence justifying this question can be found in [18, (5.6)]: If R is a non-Gorenstein ring admitting a dualizing complex and $\mathfrak{S}(R)$ is a finite set, then $\mathfrak{S}(R)$ has even cardinality.

We conclude this section with the proof of Theorem 4.3.

4.10. *(Proof of Theorem 4.3.)* It suffices to prove (a) and (b). Let x_{ij} denote the residue of X_{ij} in R and set $d = \dim R - \dim A = (m+n-r)r$ and

$$\Delta = \det \begin{pmatrix} X_{11} & \dots & X_{1r} \\ \vdots & & \vdots \\ X_{r1} & \dots & X_{rr} \end{pmatrix} \in A[X] \qquad \delta = \varphi'(\Delta) = \det \begin{pmatrix} x_{11} & \dots & x_{1r} \\ \vdots & & \vdots \\ x_{r1} & \dots & x_{rr} \end{pmatrix} \in R.$$

By [7, (6.4)] there is a prime element $\zeta \in A[T_1, \dots, T_d]$ and isomorphisms

$$R \otimes_{A[X]} A[X][\Delta^{-1}] \xrightarrow[\tau]{\cong} R[\delta^{-1}] \xleftarrow[\epsilon]{\cong} A[T_1, \dots, T_d][\zeta^{-1}]$$

Furthermore, the natural map $\alpha\colon A \to A[T_1, \dots, T_d][\zeta^{-1}]$ is faithfully flat so $\mathfrak{S}_0(\alpha)$ is bijective by Corollary 3.8(b).

Let $U = A \smallsetminus (0)$ and set $F = U^{-1}A$. Using Proposition 3.7(b), the natural maps $\beta\colon R \to R[\delta^{-1}]$ and $\gamma\colon R \to U^{-1}R$ along with $\epsilon\alpha$ yield a commutative diagram

$$(\dagger) \qquad \begin{array}{ccccc} \mathfrak{S}_0(R) & \xrightarrow{f} & \mathfrak{S}_0(R[\delta^{-1}]) \times \mathfrak{S}_0(U^{-1}R) & \xleftarrow{g} & \mathfrak{S}_0(A) \times \mathfrak{S}_0(U^{-1}R) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Cl}(R) & \xrightarrow{\cong} & \mathrm{Cl}(R[\delta^{-1}]) \times \mathrm{Cl}(U^{-1}R) & \xleftarrow{\cong} & \mathrm{Cl}(A) \times \mathrm{Cl}(U^{-1}R) \end{array}$$

where the horizontal maps are given by

$$[C] \longmapsto ([C \otimes_R R[\delta^{-1}]\,], [C \otimes_R U^{-1}R])$$

$$([C' \otimes_A R[\delta^{-1}]\,], [C'']) \longleftarrow ([C'], [C''])$$

and the vertical arrows are induced by the respective inclusions. The lower horizontal arrows are bijective by [7, (8.3)] and [13, (7.3),(8.1)]. In particular, the maps f, g are injective, and g is bijective by Proposition 3.7(c).

(a) Assuming that $r = 0$ or $m = n$, Theorem 4.1 implies that $\mathfrak{S}_0(R_{r+1}(F; m, n))$ is trivial, and hence so is $\mathfrak{S}_0(U^{-1}R)$ since $U^{-1}R \cong R_{r+1}(F; m, n)$. Thus, the top row of (†) reduces to

$$\mathfrak{S}_0(R) \xhookrightarrow{\mathfrak{S}_0(\beta)} \mathfrak{S}_0(R[\delta^{-1}]) \xleftarrow[\cong]{\mathfrak{S}_0(\epsilon\alpha)} \mathfrak{S}_0(A).$$

The functoriality of $\mathfrak{S}_0(-)$ with the commuting diagram of ring homomorphisms

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & A[T][\zeta^{-1}] \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \epsilon} \\ R & \xrightarrow{\beta} & R[\delta^{-1}] \end{array}$$

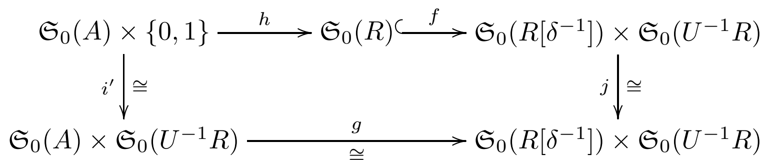
yields the equality $\mathfrak{S}_0(\beta)\mathfrak{S}_0(\varphi) = \mathfrak{S}_0(\epsilon\alpha)$. Since $\mathfrak{S}_0(\epsilon\alpha)$ is bijective, it follows that $\mathfrak{S}_0(\beta)$ is surjective. The map β is faithfully flat, and therefore $\mathfrak{S}_0(\beta)$ is also injective, so it follows that $\mathfrak{S}_0(\varphi) = \mathfrak{S}_0(\beta)^{-1}\mathfrak{S}_0(\epsilon\alpha)$ is bijective as desired.

(b) Assume now that $r > 0$ and $m \neq n$. The isomorphism $U^{-1}R \cong R_{r+1}(F; m, n)$ in conjunction with Theorem 4.1 yields a bijection $i\colon \{0,1\} \xrightarrow{\cong} \mathfrak{S}_0(U^{-1}R)$ given by $i(0) = [U^{-1}R]$ and $i(1) = [\omega_{U^{-1}R}]$ where $\omega_{U^{-1}R}$ is a canonical module for $U^{-1}R$. Let $i'\colon \mathfrak{S}_0(A) \times \{0,1\} \to \mathfrak{S}_0(A) \times \mathfrak{S}_0(U^{-1}R)$ be the induced bijection.

Below we construct a bijection

$$j\colon \mathfrak{S}_0(R[\delta^{-1}]) \times \mathfrak{S}_0(U^{-1}R) \to \mathfrak{S}_0(R[\delta^{-1}]) \times \mathfrak{S}_0(U^{-1}R)$$

such that $gi' = jfh$, that is, such that the following diagram commutes.



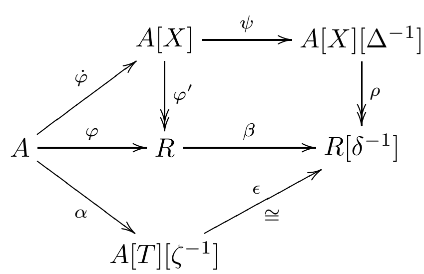
Once this is done, the proof will be complete by a simple diagram chase.

Localize the surjection $\varphi'\colon A[X] \to R$ by inverting Δ to obtain a surjection $\rho\colon A[X][\Delta^{-1}] \to R[\delta^{-1}]$. Since φ' has finite projective dimension, so does ρ. Furthermore, for each prime prime ideal $\mathfrak{q} \subset R[\delta^{-1}]$ one verifies easily that

$$\operatorname{depth}(A[X][\Delta^{-1}]_{\mathfrak{p}}) - \operatorname{depth}(R[\delta^{-1}]_{\mathfrak{q}}) = d$$

where $\mathfrak{p} = \rho^{-1}(\mathfrak{q})$.

We claim that ρ is Gorenstein. To see this, consider the commutative diagram of ring homomorphisms.



The homomorphism α is faithfully flat. Furthermore, for each prime ideal $\mathfrak{p}$ of A, the fibre $\kappa(\mathfrak{p}) \otimes_A A[T][\zeta^{-1}] \cong \kappa(\mathfrak{p})[T][\zeta^{-1}]$ is Gorenstein. The desired conclusion now follows from [3, (6.2),(6.3)].

Set $\omega_\rho = \operatorname{Ext}^d_{A[X][\Delta^{-1}]}(R[\delta^{-1}, A[X][\Delta^{-1}])$, which is $R[\delta^{-1}]$-semidualizing by Lemma 2.6(b). Moreover, it is locally free of rank 1. Setting

$$\omega_\rho^{-1} = \operatorname{Hom}_{R[\delta^{-1}]}(\omega_\rho, R[\delta^{-1}])$$

thus yields an isomorphism

$$\omega_\rho \otimes_{R[\delta^{-1}]} \omega_\rho^{-1} \cong R[\delta^{-1}]. \qquad (\ddagger)$$

We now define the aforementioned map j and demonstrate that it has the desired properties. For each semidualizing $R[\delta^{-1}]$-module C, set

$$\begin{aligned} j([C],[U^{-1}R]) &= ([C],[U^{-1}R]) \\ j([C],[\omega_{U^{-1}R}]) &= ([\omega_\rho^{-1} \otimes_{R[\delta^{-1}]} C],[\omega_{U^{-1}R}]). \end{aligned}$$

It follows from the isomorphism (‡) that the assignment

$$\begin{aligned} ([C],[U^{-1}R]) &\mapsto ([C],[U^{-1}R]) \\ ([C],[\omega_{U^{-1}R}]) &\mapsto ([\omega_\rho \otimes_{R[\delta^{-1}]} C],[\omega_{U^{-1}R}]) \end{aligned}$$

describes an inverse of j, so that j is bijective. It remains only to show that $gi' = jfh$, so fix a semidualizing A-module C. First, there are equalities

$$\begin{aligned} jfh([C],0) &= jf([C \otimes_A R]) \\ &= j([C \otimes_A R \otimes_R R[\delta^{-1}]])],[C \otimes_A R \otimes_R U^{-1}R]) \\ &\overset{(1)}{=} j([C \otimes_A R[\delta^{-1}]],[(C \otimes_A U^{-1}A) \otimes_{U^{-1}A} U^{-1}R]) \\ &\overset{(2)}{=} j([C \otimes_A R[\delta^{-1}]],[U^{-1}A \otimes_{U^{-1}A} U^{-1}R]) \\ &= j([C \otimes_A R[\delta^{-1}]],[U^{-1}R]) \\ &= g([C],[U^{-1}R]) \\ &= gi'([C],0) \end{aligned}$$

where each of the unmarked equalities follows either from a definition or by a standard isomorphism. Equality (1) uses the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & R \\ \downarrow & & \downarrow \\ U^{-1}A & \xrightarrow{U^{-1}\varphi} & U^{-1}R \end{array}$$

where the vertical maps are the natural ones. Equality (2) follows because $U^{-1}A$ is a field, and therefore has only the trivial semidualizing module.

To compute $jfh([C]_A,1)$, we first describe some isomorphisms:

$$\begin{aligned} &C(\varphi) \otimes_R R[\delta^{-1}] \\ &\qquad \cong \operatorname{Ext}^d_{A[X]}(R, C \otimes_A A[X]) \otimes_R R[\delta^{-1}] \\ &\qquad \overset{(3)}{\cong} \operatorname{Ext}^d_{A[X]}(R, C \otimes_A A[X]) \otimes_R (R \otimes_{A[X]} A[X][\Delta^{-1}]) \\ &\qquad \cong \operatorname{Ext}^d_{A[X]}(R, C \otimes_A A[X]) \otimes_{A[X]} A[X][\Delta^{-1}] \\ &\qquad \overset{(4)}{\cong} \operatorname{Ext}^d_{A[X][\Delta^{-1}]}(R \otimes_{A[X]} A[X][\Delta^{-1}], C \otimes_A A[X] \otimes_{A[X]} A[X][\Delta^{-1}]) \\ &\qquad \overset{(5)}{\cong} \operatorname{Ext}^d_{A[X][\Delta^{-1}]}(R[\delta^{-1}], C \otimes_A A[X][\Delta^{-1}]) \\ &\qquad \overset{(6)}{\cong} \omega_\rho \otimes_{R[\delta^{-1}]} (C \otimes_A A[X][\Delta^{-1}] \otimes_{A[X][\Delta^{-1}]} R[\delta^{-1}]) \\ &\qquad \cong \omega_\rho \otimes_{R[\delta^{-1}]} (C \otimes_A R[\delta^{-1}]) \end{aligned}$$

Each of the unmarked isomorphisms is either by definition or standard. Isomorphisms (3) and (5) are via the isomorphism τ, whereas (4) is from the flatness of

ψ, and (6) is by Lemma 2.6(c) and the definition of ω_ρ. Similar explanations yield the unmarked isomorphisms in the following sequence

$$\begin{aligned}
C(\varphi)\otimes_R U^{-1}R &\cong \operatorname{Ext}^d_{A[X]}(R, C\otimes_A A[X])\otimes_R U^{-1}R\\
&\cong \operatorname{Ext}^d_{A[X]}(R, C\otimes_A A[X])\otimes_R (R\otimes_{A[X]} U^{-1}A[X])\\
&\cong \operatorname{Ext}^d_{A[X]}(R, C\otimes_A A[X])\otimes_{A[X]} U^{-1}A[X]\\
&\cong \operatorname{Ext}^d_{U^{-1}A[X]}(R\otimes_{A[X]} U^{-1}A[X], C\otimes_A A[X]\otimes_{A[X]} U^{-1}A[X])\\
&\cong \operatorname{Ext}^d_{U^{-1}A[X]}(U^{-1}R, C\otimes_A U^{-1}A[X])\\
&\cong \operatorname{Ext}^d_{U^{-1}A[X]}(U^{-1}R, (C\otimes_A U^{-1}A)\otimes_{U^{-1}A} U^{-1}A[X])\\
&\cong \operatorname{Ext}^d_{U^{-1}A[X]}(U^{-1}R, U^{-1}A\otimes_{U^{-1}A} U^{-1}A[X])\\
&\cong \operatorname{Ext}^d_{U^{-1}A[X]}(U^{-1}R, U^{-1}A[X])\\
&\overset{(7)}{\cong} \omega_{U^{-1}R}
\end{aligned}$$

while isomorphism (7) is standard. Indeed, the ring $U^{-1}A[X]$ is regular and surjects onto $U^{-1}R$ so that $\operatorname{Ext}^d_{U^{-1}A[X]}(U^{-1}R, U^{-1}A[X])$ is a canonical module for $U^{-1}R$, and is therefore isomorphic to $\omega_{U^{-1}R}$ since the canonical module of $U^{-1}R$ is unique up to isomorphism.

These isomorphisms yield equality (8) in the final computation

$$\begin{aligned}
jfh([C],1) &= jf([C(\varphi)])\\
&= j([C(\varphi)\otimes_R R[\delta^{-1}]], [C(\varphi)\otimes_R U^{-1}R])\\
&\overset{(8)}{=} j([\omega_\rho\otimes_{R[\delta^{-1}]} (C\otimes_A R[\delta^{-1}])], [\omega_{U^{-1}R}])\\
&= ([\omega_\rho^{-1}\otimes_{R[\delta^{-1}]} \omega_\rho\otimes_{R[\delta^{-1}]} (C\otimes_A R[\delta^{-1}])], [\omega_{U^{-1}R}])\\
&\overset{(9)}{=} ([C\otimes_A R[\delta^{-1}]], [\omega_{U^{-1}R}])\\
&= g([C], [\omega_{U^{-1}R}])\\
&= gi'([C],1).
\end{aligned}$$

while (9) is by (‡), and the others are by definition. This completes the proof. □

In closing we note that the maps $\mathfrak{S}_0(\varphi)$ and h from Theorem 4.3 can be defined using the methods of [18] without the hypothesis "Cohen-Macaulay normal domain"; furthermore, these maps are always injective. At this time, though, it is only by using the divisor class group that we are able to prove surjectivity.

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