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New Topological Aspects of BF Theories

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Abstract

BF theories defined over non trivial line bundles are studied. It is shown that such theories describe a realization of a non trivial higher order bundle. The partition function differs from the usual one -in terms of the Ray Singer Torsion- by a factor that arises from the non triviality of the line bundles.

Topological field theories were introduced in [1] and [2][4]. The partition functions of the abelian BF theories is a topological invariant of the base manifold X, related to the Ray Singer torsion as shown by A. Schwarz in [3]. Other interesting observables may be constructed as correlation functions of Wilson surfaces associated to the A and B fields, these correlation functions determine the linking and intersection numbers of manifolds of several dimensions.

The BRST gauge fixing of the BF action was studied for the abelian case in [4] and completed for the non abelian case in [5] and [6], and several interesting properties of the quantum effective action were analysed.

The interest on the BF theories is not limited to the topological field theory, it happens that the BF lagrangian density appears as an interaction term in several physical theories with local propagating degrees of freedom [7] [8] in particular, as the interaction term relating dual theories of p and $(d-p-2)$-forms in d dimensions [9], consequently the topological contribution comming from BF theories appear even in those physical theories with non trivial physical hamiltonian.

In the analysis of [1][2][4] the connection 1-form A of the BF action is defined over a flat vector bundle, in this note we will consider BF theories over non trivial $U(1)$ bundles. More precisely, we will assume the base X to be a compact, orientable finite dimensional euclidean manifold without boundary, and will consider all the isomorphism classes of $U(1)$ line bundles with connections that can be built over X. We will compare the resulting theory with the standard one, that is, we will compare a BF theory over a trivial line bundle with a BF theory formulated over a non trivial one. We will show that in the latest case the theory describes a realization of a non trivial higher order bundle [9][10][11][12].

The partition function of both formulations have a common factor expressed in terms of the Ray Singer torsion, the new theory however contains an extra topological factor arising solely from the non triviality of the line bundles themselves, which distingushes both theories, this latest factor is introduced by the zero modes of the quantum effective action of the theory. These non trivial contributions to the partition function arising from non trivial line bundles and higher order bundles are also present in any global analysis of duality in quantum field theory, nevertheless, they are usually missing in the literature.

Higher order bundles [9][10][11] constitute geometrical objects that generalize the concept of fiber bundles, from a physical point of view they describe antisymmetric tensor fields with transitions whose effect is detected in generalized Dirac's quantization conditions, and are therefore naturally realized in the Chern Simmons terms in the action for $D = 11$ supergravity and consequently relevant for $\mathcal{M}$ theory. From the mathematical point of view [12] higher order bundles -also called Gerbes- are fiber bundles over a manifold, whose fibers are groupoids.

We begin by reviewing abelian BF theories formulated on trivial bundles. The action for such systems is given by

$$S = \int B \wedge dA \tag{1}$$

under the assumption that A is a $U(1)$ connection over a trivial line bundle it follows that A is a globally defined 1-form over X while B is a globally defined $d-2$-form. The field equations following from (1) are:

$$dA = 0, \qquad dB = 0 \tag{2}$$

that is: A and B are closed 1 and $d-2$ forms. We define the gauge transformations for these fields as:

$$B \to d\theta, \qquad A \to d\Lambda \tag{3}$$

where θ and Λ are a globally defined $d-3$ form and a 0-form respectively. The space of solutions of the field equations coming from the action (1) may then be expressed as:

$$H^{d-2}_{dR}(X, \Re) \otimes H^1_{dR}(X, \Re) \tag{4}$$

where $H^p_{dR}(X, \Re)$ stands for the de Rham Cohomolgy group of degree p. Let us now turn our attention to the quantum theory and consider the BRST invariant effective action associated to (1), to do so we have to introduce the ghosts, antighosts and Lagrange multipliers associated to the gauge symmetries (3) and to the corresponding BRST gauge fixing procedure. We first notice that the gauge symmetries correspond to exact forms and that consequently, the associated ghost fields have no harmonic part. Indeed, since the gauge transformation for the 1-form A is given by

$$A \to A + d\Lambda \tag{5}$$

it follows that the corresponding BRST transformation is [13][14]

$$\hat{\delta} A = dC \tag{6}$$

where the zero form Λ has been replaced by the ghost field C. Since $d\Lambda$ is an exact form dC must also be exact in order to preserve the same degrees of freedom, consequently, dC has no harmonic part. If instead $d\Lambda$ were a closed 1-form, dC should also be closed and hence would contain a harmonic part. We also note that the Lagrange multiplier associated to the gauge fixing introduced by the modified BFV approach [14] does not have any zero mode freedom in agreement with Lagrange's multiplier theorem. Consequently the antighost ($\bar{C}$) has also no harmonic part contribution. This result can be generalized to reducible systems (such as BF theories for $d > 3$) where one may argue that the ghosts and antighosts do not have any zero modes (harmonic parts). In this sense the approach in [14] provides the construction of the BRST invariant effective action without the introduction spurious zero modes on the ghost sector. Going back to the discussion of the quantum theory of the action (1) , we find that the only zero modes are those of the A and B fields which are given by the cohomology classes (4), and which give rise to the nontrivial contributions to the partition function

For the sake of simplicity we consider a straightforward example of the above discussion, namely, the BF theory in $d = 3$ dimensions, the effective action is given by

$$S_{eff} = \int B \wedge dA + \lambda_1 d^* B + \lambda_2 \wedge D^* A + \bar{C}_1 d^* dC_1 + \bar{C}_1 d^* dC_1 \qquad (7)$$

by construction there are no zero modes for the ghosts, antighosts and Lagrange multipliers . Indeed, the potential zero modes for C_1 -for example- arise as solutions of

$$d^* dC_1 \qquad (8)$$

but this condition is simply states that dC_1 is an harmonic 1-form in contradiction with the fact that dC_1 is exact. The only zero modes of the effective action (7) come thus from A and B. The zero modes of the B field satisfy

$$dB = 0, \qquad d^* B = 0 \qquad (9)$$

and similar conditions for the zero modes of A. This set of conditions imply that the zero modes are two copies of the space of harmonic $(d-2) = 1$-forms. That is, the cohomolgy classes defined by $H^1_{dR}(X, \Re) \otimes H^1_{dR}(X, \Re)$.

The evaluation of the partition function of the BF theory in this case -over trivial line bundles- was performed in [2][3] [4], the final result is:

$$\mathcal{Z} = \mathrm{Vol}(\mathcal{ZM})(T(X))^{\alpha} \tag{10}$$

where $\mathrm{Vol}(\mathcal{ZM})$ denotes the volume of the space of zero modes, while $T(X)$ is the Ray Singer torsion, the exponent α being given by ($n = dim(X)$):

$$\alpha = \{ \begin{array}{cc} 2-n+1 & n \text{ even} \\ -1 & n \text{ odd} \end{array} \tag{11}$$

According to the above construction of the effective action, the space of zero modes $\mathcal{ZM}$ is given by

$$\mathcal{ZM} = H^1_{dR}(X, \Re) \otimes H^1_{dR}(X, \Re) \tag{12}$$

Concerning the latest point, we take a different point of view than the one presented in [4] where the gauge symmetry is extended to include harmonic gauge parameters, in that case we agree in that the ghosts sector should include harmonic parts.

We will now turn to our main interest: BF theories defined over non trivial line bundles. Any connection 1-form over a nontrivial line bundle may be decomposed as:

$$A = \hat{A} + a \tag{13}$$

where a is a globally defined 1-form while $\hat{A}$ is a fixed 1-form connection of the same topological bundle as A. A and $\hat{A}$ have the same transitions over X, while a has obviously none. B will be taken as a globally defined $(d-2)$ form.

The BF action for such a system may be written as:

$$S = i\int_X B \wedge F(A) = i\int_X B \wedge F(\hat{A}) + i\int_X B \wedge F(a) \tag{14}$$

Variations of (14) with respect to a, or functional integration with respect to it yields

$$dB = 0 \tag{15}$$

on the other hand, variations of (14) with respect to B or equivalently functional integrations on B yields

$$F(A) \equiv dA = 0 \tag{16}$$

finally, summation over all line bundles gives:

$$\oint_{\Sigma_I} B = 2\pi n^I \tag{17}$$

where $\{\Sigma_I\}$ is a basis of homology of dimension $d-2$, while the numbers n^I are integers associated to each Σ_I. To obtain (17) we recursively integrate by parts using a triangulation of X and proceeding as follows

$$\begin{aligned} i\int_X B \wedge F(\hat{A}) &= (-1)^{d-2} i \int_X d(B \wedge \hat{A}) = (-1)^{d-2} i \sum_{U_i \cap U_j} \int_{U_i \cap U_j} B \wedge (\hat{A}_i - \hat{A}_j) = \\ &= i(-1)^{d-2} \sum_{U_i \cap U_j} \int_{U_i \cap U_j} B \wedge d\Lambda_{ij} = i \sum_{U_i \cap U_j} \int_{U_i \cap U_j} d(B\Lambda_{ij}) = \\ &= i \sum_{U_i \cap U_j \cap U_k} \int_{U_i \cap U_j \cap U_k} B(\Lambda_{ij} + \Lambda_{jk} + \Lambda_{ki}) = \\ &= i \sum_{U_i \cap U_j \cap U_k} \int_{U_i \cap U_j \cap U_k} 2\pi n B = 2\pi m^I \sum_I i \int_{\Sigma^I} B \end{aligned} \tag{18}$$

summation over all m^I yields then (17). In (14) one may identify the last term of the split action, namely: $i \int_X B \wedge F(a)$ as the BF action of a global 1-form a or equivalently of a 1-form connection over a trivial line bundle. The contribution of all non-trivial $U(1)$ line bundles arises from the $i \int_X B \wedge F(\hat{A})$ term of the action.

Let us now analyze the space of solutions of the field equations (15) ,(16), and (17). We first recall that $F(A)$ being the curvature of a connection 1-form A over a line bundle also has integral periods (Dirac quantization conditions) over any basis of homology of dimension 2, but because of (16) the integers are all zero. Condition (16) implies that A is a flat connection 1-form, given A_{flat} one such connection then for any closed 1-form ω over X

$$A_{flat} + \omega \tag{19}$$

is also a flat connection on the same line bundle, moreover these are all the flat connections over the line bundle. The problem then reduces to find all the line bundles which admit a flat connection. There exists generically non-trivial line bundles with flat connections over it. They are line bundles with constant transitions and are classified by the *Čech* cohomology group with values on the constant sheaf $\Re/Z$:

$$\check{H}^1(X, \Re/Z) \tag{20}$$

For any such line bundle the flat connections are in one to one correspondence to the de Rahm cohomology group

$$H^1_{dR}(X, \Re). \tag{21}$$

According to this, the total space of flat connections is determined by the product

$$\check{H}^1(X, \Re/Z) \otimes H^1_{dR}(X, \Re) \tag{22}$$

Let us now analyze the space of solutions to (15) and (17), taking into account the gauge symmetry of the B field

$$B \rightarrow B + d\theta \tag{23}$$

where $d\theta$ is an exact $(d-2)$ form. Conditions $dB = 0$ and $\oint_{\Sigma_I} B = 2\pi n^I$ ensure the existence of both: a higher order $U(1)$ bundle, and a $(d-3)$-form b with non-trivial transitions over X [9] , such that $B = db$. The former result is a generalization of Weil's theorem stating that given a 2-form F satisfying Dirac's quantization conditions there exists a line bundle and a connection A such that F is the curvature of A [12]

In a particular case, B being a 3-form we have, on an open covering $U_l, l \in L$, of X that

$$\begin{gathered} B = db_i \text{ on } U_i, \qquad b_i - b_j = d\eta_{ij} \text{ on } U_i \wedge U_j \neq \emptyset \\ \eta_{ij} + \eta_{jk} + \eta_{ki} = d\Lambda_{ijk} \text{ on } U_i \wedge U_j \wedge U_k \neq \emptyset \\ \sum_{ijkl} \Lambda_{ijkl} = 2\pi n \end{gathered} \tag{24}$$

the cocycle condition being satisfied on any intersection of four open sets. Conversely, given conditions (24), then B satisfies (15) and (17).

The gauge equivalent triplets (b, η, Λ) are defined by

$$\begin{aligned} b_i &\rightarrow b_i + d\eta_i \qquad \text{on } U_i \\ \eta_{ij} &\rightarrow \eta_{ij} + \eta_i - \eta_j + d\Lambda_{ij} \qquad \text{on } U_i \cap U_j \\ \Lambda_{ijk} &\rightarrow \Lambda_{ijk} + \Lambda_{ij} + \Lambda_{jk} + \Lambda_{ki} \qquad \text{on } U_i \cap U_j \cap U_k \end{aligned} \tag{25}$$

It is important to realize that given B satisfying (15) and (17) the triplet (b, η, Λ) in (24) may not be unique in general, indeed, there may exist constant transition $\widetilde{\Lambda}_{ijk}$ on $U_i \cap U_j \cap U_k \cap U_l \neq \emptyset$ satisfying

$$\sum_{ijkl} \widetilde{\Lambda}_{ijk} = 0 \quad \text{on} \quad U_i \cap U_j \cap U_k \cap U_l \neq \emptyset \tag{26}$$

which may be added to any particular triplet (b, η, Λ) satisfying (24) giving rise to a new triplet satisfying the same conditions. The space of all the triplets $(0, 0, \widetilde{\Lambda}_{ijk})$ with constant transitions $\widetilde{\Lambda}_{ijk}$are classified by the *Čech* cohomology group $\check{H}^2(x, \Re/Z)$ with values on the constant sheaf $\Re/Z$. This is the only degeneracy on the triplets satisfying (24) for a given B, a closed 3-form with integral periods. Modulo this degeneracy two triplets (b_1, η_1, Λ_1) and (b_2, η_2, Λ_2) satisfying the cocycle condition with the same set of integers n satisfy

$$(b_2, \eta_2, \Lambda_2) \sim (b_1 + \theta, \eta_1, \Lambda_1) \tag{27}$$

where $\sim$ denotes gauge equivalence (25), and θ is a globally defined 2-form over X. Since this is just the gauge symmetry (23) of the BF action and consequently of its field equations (15) and (17), we conclude that the set of integers n^I associated to a basis of integral homology of dimension 3 determine the space of solutions of (15) and (17). They classify all the higher order bundles of degree 3, that is the *Čech* cohomology group

$$\check{H}^3(X, Z) \tag{28}$$

modulo $\check{H}^2(X, \Re/Z)$ the space of higher order bundles of degree 3 with constant transitions, i.e. two elements of the same equivalence class differ by constant transitions $\widetilde{\Lambda}$.

Finally, from (14) we may determine the partition function of the BF theory on non-trivial line bundles. It has the form $\mathrm{Vol}(\mathcal{ZM})(T(X))^\alpha$ already found in [2][3][4] but now the zero mode space is determined by

$$(\check{H}^1(X,\Re/Z)\otimes H^1_{dR}(X,\Re))\otimes(\check{H}^3(X,Z)/\check{H}^2(X,\Re Z)). \tag{29}$$

In the general case the degree of the $\check{C}ech$ cohomology groups are $d-2$ and $d-3$ respectively and consequently the last formula generalizes to

$$(\check{H}^1(X,\Re/Z)\otimes H^1_{dR}(X,\Re))\otimes(H^{\check{d}-2}(X,Z)/H^{\check{d}-3}(X,\Re Z)). \tag{30}$$

showing that BF theories provide, may be the most elementary realization of higher order bundles in field theory.

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