Deformations of Coxeter hyperplane arrangements and their characteristic polynomials

Christos A. Athanasiadis

Abstract.

Let $\mathcal{A}$ be a Coxeter hyperplane arrangement, that is the arrangement of reflecting hyperplanes of an irreducible finite Coxeter group. A deformation of $\mathcal{A}$ is an affine arrangement each of whose hyperplanes is parallel to some hyperplane of $\mathcal{A}$. We survey some of the interesting combinatorics of classes of such arrangements, reflected in their characteristic polynomials.

§1. Introduction

Much of the motivation for the study of arrangements of hyperplanes comes from Coxeter arrangements. Because of their importance in algebra, Coxeter arrangements have been studied a great deal in the context of representation theory of semisimple Lie algebras (where they arose), invariant theory of reflection groups, combinatorics of root systems and Coxeter groups, combinatorics of convex polytopes and oriented matroids and within the general theory of hyperplane arrangements [42]. From a geometric, combinatorial and algebraic point of view, they are fairly well understood in terms of their classification, facial structure, intersection posets, characteristic polynomials and freeness; see [17, §2.3] and [42, Chapter 6].

A *deformation* of a Coxeter arrangement $\mathcal{A}$ is an affine arrangement each of whose hyperplanes is parallel to some hyperplane of $\mathcal{A}$. Interesting examples of such arrangements first arose in the study of affine Weyl groups by Shi [53, 54] and have appeared since then in various mathematical contexts. Their combinatorics was first investigated systematically by Stanley [59] and relates to objects studied classically in enumeration such as trees, set partitions and partially ordered sets. A major role in this study has been played by the characteristic polynomial.

---

[1]The present article was written while the author was a Hans Rademacher Instructor at the University of Pennsylvania.

In the present exposition we describe some of the lively work on deformations of Coxeter arrangements that has been carried out in the recent past. We emphasize the combinatorial and algebraic properties related to their characteristic polynomials, a topic which we find rich and interesting enough to stand on its own. We discuss some of the relevant motivation and include a number of open questions which are often suggested naturally by the results.

Acknowledgement. The author has benefited from the work of Alexander Postnikov and Richard Stanley [44, Chapter 1][45, 59], some of which was carried out in parallel with his own thesis work [3][4, Part II], as well as from various discussions with Anders Björner, Victor Reiner, Gian-Carlo Rota, Bruce Sagan, Hiroaki Terao and Günter Ziegler. He is indebted to Victor Reiner and Richard Stanley for their valuable comments.

§2. Background

The characteristic polynomial. Let $\mathbb{K}$ be a field. A hyperplane arrangement $\mathcal{A}$ in $\mathbb{K}^\ell$ is a finite collection of affine hyperplanes in $\mathbb{K}^\ell$, i.e. affine subspaces of $\mathbb{K}^\ell$ of codimension one. We will mostly be interested in arrangements over the reals, so that $\mathbb{K} = \mathbb{R}$. We call $\mathcal{A}$ *central* if all hyperplanes in $\mathcal{A}$ are linear. The *characteristic polynomial* of $\mathcal{A}$ is defined as

$$(1) \qquad \chi(\mathcal{A}, q) = \sum_{x \in L_\mathcal{A}} \mu(\hat{0}, x)\, q^{\dim x},$$

where $L_\mathcal{A} = \{\cap \mathcal{F} : \mathcal{F} \subseteq \mathcal{A}\}$ (partially ordered by reverse inclusion) is the *intersection poset* of $\mathcal{A}$, $\hat{0} = \mathbb{K}^\ell$ is the unique minimal element of $L_\mathcal{A}$ (which corresponds to $\mathcal{F}$ being empty) and μ stands for its Möbius function [60, §3.7].

The characteristic polynomial $\chi(\mathcal{A}, q)$ is a fundamental combinatorial and topological invariant of $\mathcal{A}$ and plays a significant role throughout the theory of hyperplane arrangements [42]. If $\mathbb{K} = \mathbb{R}$ then $\chi(\mathcal{A}, q)$ gives valuable enumerative information about the cell decomposition of the space $\mathbb{R}^\ell$, induced by $\mathcal{A}$ [17, §2.1]. The cells in this decomposition are the *faces* of $\mathcal{A}$. The faces of dimension ℓ are simply the connected components of the space obtained from $\mathbb{R}^\ell$ by removing the hyperplanes of $\mathcal{A}$ and are called the *regions* or *chambers* of $\mathcal{A}$.

2.1. Theorem (Zaslavsky [69]). *The number of regions into which $\mathcal{A}$ dissects $\mathbb{R}^\ell$ is equal to $(-1)^\ell \chi(\mathcal{A}, -1)$.*

In particular, for any k, the number of faces of $\mathcal{A}$ of dimension k depends only on $L_\mathcal{A}$ and ℓ.

On the other hand, if $\mathbb{K} = \mathbb{C}$ then $\chi(\mathcal{A}, q)$ gives topological information about the complement $M_\mathcal{A} = \mathbb{C}^\ell - \bigcup_{H \in \mathcal{A}} H$. The following result was proved by Orlik and Solomon in the context of their fundamental work [41] on the cohomology algebra $H^*(M_\mathcal{A}, \mathbb{Z})$ of $M_\mathcal{A}$.

2.2. Theorem (Orlik–Solomon [41]). *If $\mathcal{A}$ is a central arrangement in $\mathbb{C}^\ell$ then*

$$\sum_{i \geq 0} \operatorname{rank} H^i(M_\mathcal{A}, \mathbb{Z})\, q^i = (-q)^\ell \chi(\mathcal{A}, -1/q).$$

For the cohomological significance of $\chi(\mathcal{A}, q)$ when $\mathcal{A}$ is a *subspace arrangement* we refer to Björner [15, §7] and Björner and Ekedahl [16]. The following corollary of Theorem 2.2 continues to hold when $\mathcal{A}$ is a subspace arrangement, see [15, §8.3].

2.3. Corollary. *If $\mathcal{A}$ is an arrangement in $\mathbb{R}^\ell$ then*

$$\sum_{i \geq 0} \operatorname{rank} H^i(M_\mathcal{A}, \mathbb{Z}) = \sum_{i \geq 0} \operatorname{rank} H^i(M_{\mathcal{A}^\mathbb{C}}, \mathbb{Z}),$$

where $M_\mathcal{A}$ is the complement of $\mathcal{A}$ in $\mathbb{R}^\ell$ and $M_{\mathcal{A}^\mathbb{C}}$ is the complement of its complexification $\mathcal{A}^\mathbb{C}$ in $\mathbb{C}^\ell$.

Freeness. Let $\mathcal{A}$ be central and $\mathbb{S} := \mathbb{K}[x_1, x_2, \ldots, x_\ell]$ be the polynomial ring over $\mathbb{K}$ in ℓ variables. Let Q be the product of the linear forms in $\mathbb{S}$ defining the hyperplanes of $\mathcal{A}$, so that Q is unique up to multiplication by an element of $\mathbb{K}^*$, and let $Q\mathbb{S}$ be the principal ideal in $\mathbb{S}$ generated by Q. The *module of derivations* $D(\mathcal{A})$ of $\mathcal{A}$ is the set of all derivations $\theta : \mathbb{S} \to \mathbb{S}$ such that $\theta(Q) \in Q\mathbb{S}$. $D(\mathcal{A})$ is actually a module over $\mathbb{S}$. The arrangement $\mathcal{A}$ is called *free* [63] if $D(\mathcal{A})$ is a free $\mathbb{S}$-module. One can associate to $\mathcal{A}$ a multiset of ℓ nonnegative integers, called the *exponents* of $\mathcal{A}$. They are the degrees of the elements in any basis of the free $\mathbb{S}$-module $D(\mathcal{A})$.

2.4. Theorem (Terao [65][41, Theorem 4.137]). *If $\mathcal{A}$ is free with exponents $e_1, e_2, \ldots, e_\ell$ then*

$$\chi(\mathcal{A}, q) = \prod_{i=1}^{\ell} (q - e_i).$$

Theorem 2.4 is one of a number of results which explain factorization phenomena for $\chi(\mathcal{A}, q)$. Other approaches include supersolvability [58] and its generalizations [14, 19], inductive freeness [63], recursive freeness [75], factorization of rooted complexes [18], factorization [26, 67] and inductive factorization [34]. For background we refer to these sources, [42, Chapter 4] and the survey article [50]. A purely algebraic–combinatorial proof of Theorem 2.4 was given in Solomon and Terao [56]; see also [42, Chapter 4].

Coxeter arrangements. Let Φ be an irreducible root system in $\mathbb{R}^\ell$ [33, §1.2], equipped with the standard inner product. We rely on [33] for basic background and terminology on root systems. The *Coxeter arrangement* $\mathcal{A}_\Phi$ corresponding to Φ is the arrangement of the linear hyperplanes

$$(\alpha, x) = 0$$

orthogonal to the roots $\alpha \in \Phi$, i.e. the reflecting hyperplanes of the associated finite Coxeter group W. See [42, Chapter 6] and [17, §2.3] for expositions of Coxeter arrangements from algebraic-topological and geometric-combinatorial points of view, respectively. The following result will be of interest here.

2.5.   Theorem   (Arnol'd [1, 2], Saito [51, 52]). *The Coxeter arrangement $\mathcal{A}_\Phi$ is free with exponents the exponents of the root system Φ.*

In fact, explicit bases for the modules of derivations were constructed in terms of the *basic invariants* [33, §3.5] of the algebra of W-invariant polynomials by Saito [51] and Terao [64]. The analogue of Theorem 2.5 for complex reflection groups and a generalization to all reflection arrangements appear in Terao [64, 66].

2.6. Corollary.   *If $e_1, e_2, \ldots, e_\ell$ are the exponents of Φ then*

$$\chi(\mathcal{A}_\Phi, q) = \prod_{i=1}^{\ell} (q - e_i).$$

§3.   Deformations of Coxeter arrangements

We now assume that Φ is crystallographic [33, §2.9], so that W is a Weyl group. We let Φ^+ be a choice of positive roots. When we give equations for the hyperplanes of deformations of $\mathcal{A}_\Phi$ we will choose Φ and

Φ^+ as in [33, §2.10] and denote the dimension of the ambient space by n. The *braid arrangement* $\mathcal{A}_n$, for instance, consists of the hyperplanes in $\mathbb{R}^n$ of the form $x_i - x_j = 0$. In this notation we prefer to consider $\mathcal{A}_n$ as an arrangement in $\mathbb{R}^n$ (so that its characteristic polynomial has q as a factor), even though it corresponds to the Coxeter arrangement $\mathcal{A}_\Phi$ for $\Phi = A_{n-1}$, which is an arrangement in $\mathbb{R}^{n-1}$ (so that $\ell = n - 1$). We extend this convention to deformations of $\mathcal{A}_n$ as well.

We begin with three motivating classes of examples to which we will come back in the next section.
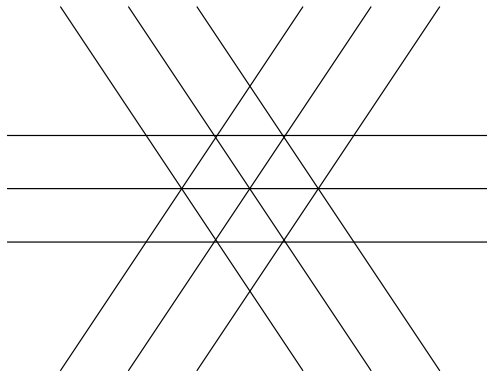

FIG 1. The Catalan arrangement for A_2.

The Catalan arrangement. The *Catalan arrangement* Cat_Φ consists of the hyperplanes

$$\begin{aligned}
(\alpha, x) &= -1, \\
(\alpha, x) &= 0, \qquad \alpha \in \Phi^+ \\
(\alpha, x) &= 1,
\end{aligned}$$

in $\mathbb{R}^\ell$. It is invariant under the action of the Weyl group W and is shown in Figure 1 for $\Phi = A_2$. For $\Phi = A_{n-1}$ the hyperplanes are

$$x_i - x_j = -1, 0, 1 \ \text{ for } \ 1 \le i < j \le n.$$

We denote this arrangement in $\mathbb{R}^n$ by Cat_n. The terminology "Catalan arrangement" comes from the fact that the number of regions of Cat_n, divided by $n!$, is equal to the nth Catalan number. It was observed by Stanley [59, §2] that the regions of Cat_n within the fundamental Weyl chamber of $\mathcal{A}_n$ are in bijection with *unit interval orders* with n elements, i.e. partial orders which come from unit intervals $I_1, I_2, \ldots, I_n$ on the

real line by letting $I_i < I_j$ if I_i lies entirely to the left of I_j. To see this, it suffices to let the ith interval be $[x_i, x_i+1]$, where $x_1 < x_2 < \cdots < x_n$, and observe that the partial order defined by these n intervals depends only on the region of Cat_n in which the point $(x_1, x_2, \ldots, x_n)$ lies. For a treatment of the theory of interval orders see [27].

In another direction, it was observed by Postnikov (see Remark 2 in [47, §6] and [7]) that the regions of Cat_Φ within the fundamental Weyl chamber of $\mathcal{A}_\Phi$ are in bijection with *nonnesting partitions* on Φ, i.e. antichains in the *root order* of Φ, defined on Φ^+ by $\alpha \le \beta$ if $\beta - \alpha$ is a linear combination of positive roots with nonnegative coefficients. The following theorem is a special case of Theorem 4.6 for the classical root systems and has also been verified for $\Phi = G_2, F_4$ and E_6 (see [24, §3]).

3.1. Theorem ([3, 4]). *Let Φ be of type A, B, C or D. We have $\chi(\mathrm{Cat}_\Phi, q) = \chi(\mathcal{A}_\Phi, q - h)$, where h is the Coxeter number of Φ. In particular, the number of regions of Cat_Φ is equal to*

$$\prod_{i=1}^{\ell} (e_i + h + 1)$$

and the number of nonnesting partitions on Φ is equal to

$$\prod_{i=1}^{\ell} \frac{e_i + h + 1}{e_i + 1},$$

*where $e_1, e_2, \ldots, e_\ell$ are the exponents of Φ.*

The Shi arrangement. The *Shi arrangement* $\mathcal{S}_\Phi$ consists of the hyperplanes

$$\begin{aligned} (\alpha, x) &= 0, \\ (\alpha, x) &= 1, \end{aligned} \quad \alpha \in \Phi^+$$

in $\mathbb{R}^\ell$. This is shown in Figure 2 for $\Phi = A_2$. For $\Phi = A_{n-1}$ the hyperplanes are

$$\begin{aligned} x_i - x_j &= 0 \ \text{ for } \ 1 \le i < j \le n, \\ x_i - x_j &= 1 \ \text{ for } \ 1 \le i < j \le n. \end{aligned}$$

We denote this arrangement in $\mathbb{R}^n$ by $\mathcal{S}_n$. The arrangement $\mathcal{S}_\Phi$ was first considered by Shi in his investigation of the affine Weyl group $\widetilde{A}_{n-1}$ of type A_{n-1} [53, §7]. The regions of $\mathcal{S}_n$ correspond to certain equivalence classes of elements of $\widetilde{A}_{n-1}$, called "admissible sign types", which were shown by Shi to play a significant role in the Kazhdan–Lusztig theory of cells [36] for this group.
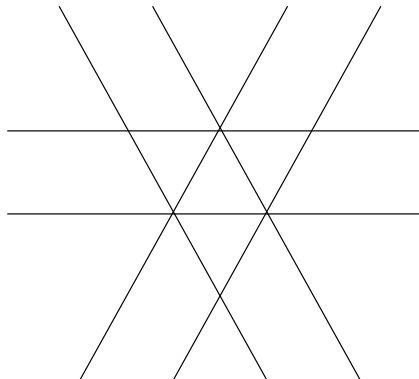
FIG 2. The Shi arrangement for A_2.

Since Shi's work, the arrangement $\mathcal{S}_\Phi$ has continued to appear as related to affine Weyl groups in Headley [30, 31, 32], invariant theory of finite Coxeter groups in Solomon and Terao [57], and representations of affine Hecke algebras in Ram [46], as an object of independent interest in enumerative combinatorics in the type A case [5, 10, 59, 61], and as a particularly nice example where techniques from the theory of hyperplane arrangements apply [3, 4, 6, 32, 44, 45]. Much of the interest initially attracted by the Shi arrangement is due to the following surprising result.

3.2. Theorem (Shi [54], [53, Corollary 7.3.10] for $\Phi = A_{n-1}$). *The number of regions of $\mathcal{S}_\Phi$ is $(h+1)^\ell$, where h is the Coxeter number of Φ. In particular, the number of regions of $\mathcal{S}_n$ is $(n+1)^{n-1}$.*

Shi gave a constructive proof of this fact for $\mathcal{S}_n$ [53] by considering the elements of the affine Weyl group of type A_{n-1} which correspond to the regions and a uniform but lengthy proof in the general case [54] using his notion of "sign type" for affine Weyl groups. More direct combinatorial proofs in the type A case can be found in Headley [31], Stanley [59] and Athanasiadis and Linusson [10, §2]. The proof in [59] yields an interesting refinement of the enumeration of the regions by a certain distance statistic; see Theorem 6.13.

The following stronger result, via Theorem 2.1, on the characteristic polynomial of $\mathcal{S}_\Phi$ was proved by Headley, whose argument relied on Theorem 3.2 and induction.

3.3. Theorem (Headley [30, 31, 32]). *We have* $\chi(\mathcal{S}_\Phi, q) = (q-h)^\ell$, *where h is the Coxeter number of Φ. In particular, we have $\chi(\mathcal{S}_n, q) = q(q-n)^{n-1}$.*
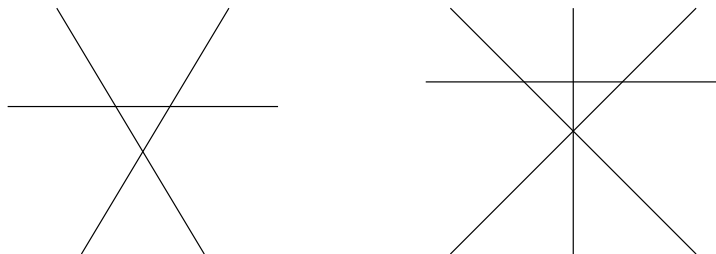

FIG 3. The Linial arrangements for A_2 and B_2.

The Linial arrangement. In the rest of the paper we allow Φ to be the non-reduced system BC_n, which is the union of B_n and C_n in the standard choice of [33, §2.10].

The *Linial arrangement* $\mathcal{L}_\Phi$ consists of the hyperplanes

$$(\alpha, x) = 1, \quad \alpha \in \Phi^+$$

in $\mathbb{R}^\ell$. It is shown in Figure 3 for $\Phi = A_2$ and B_2. For $\Phi = A_{n-1}$ the hyperplanes are

$$x_i - x_j = 1 \ \text{ for } \ 1 \le i < j \le n.$$

We denote this arrangement in $\mathbb{R}^n$ by $\mathcal{L}_n$. Interest in the arrangement $\mathcal{L}_n$ came from a surprising conjecture of Linial, Ravid and Stanley (see [59, §4]) stating that the number of regions of $\mathcal{L}_n$ is equal to the number f_n of *alternating trees* on $n+1$ vertices, i.e. trees on the vertex set $\{1, 2, \ldots, n+1\}$ such that no $i < j < k$ are consecutive vertices of a path in the tree, in the order i, j, k. Alternating trees first appeared in [28]. The explicit formula

$$(2) \qquad\qquad f_n = \frac{1}{2^n} \sum_{k=0}^n \binom{n}{k} (k+1)^{n-1}$$

was found by Postnikov [43, Theorem 1], who later proved the conjecture about $\mathcal{L}_n$ as follows.

3.4. Theorem (Postnikov [44, Theorem 1.4.5][45, Theorem 8.1]). *The number of regions of the Linial arrangement $\mathcal{L}_n$ is equal to f_n.*

There is no bijective proof of the Linial–Ravid–Stanley conjecture at present. Postnikov's theorem naturally suggests the problem of finding directly an explicit formula for the characteristic polynomial of $\mathcal{L}_n$. Such a formula was first given in [3, 4]; see also [44, §1.5][45, §9] and [59, Corollary 4.2]. The proof in [3, 4] was simplified in [9, §3]; see also Section 4.

3.5. Theorem ([3, Theorem 4.2][4, Theorem 6.4.2]). *The Linial arrangement $\mathcal{L}_n$ has characteristic polynomial*

$$\chi(\mathcal{L}_n, q) = \frac{q}{2^n} \sum_{k=0}^{n} \binom{n}{k} (q - k)^{n-1}.$$

This expression implies Theorem 3.4, via Theorem 2.1. For results on the asymptotic behaviour of $\chi(\mathcal{L}_n, q)$ for large n, see Postnikov [44, §1.6.3]. The analogous problem to compute $\chi(\mathcal{L}_\Phi, q)$ in general is also suggested by a conjecture of Postnikov and Stanley [45, §9] which, in a special case, states that all roots of $\chi(\mathcal{L}_\Phi, q)$ have the same real part; see Conjecture 3.6.

The affine Weyl arrangement. As is apparent from the previous examples, interesting deformations of $\mathcal{A}_\Phi$ often occur as subarrangements of the *affine Weyl arrangement* $\widetilde{\mathcal{A}}_\Phi$

$$(\alpha, x) = k, \quad \alpha \in \Phi^+, \quad k \in \mathbb{Z},$$

the arrangement of reflecting hyperplanes of the *affine Coxeter group* $\widetilde{W}$. For integers $a \le b$ we denote by $\mathcal{A}_\Phi^{[a,b]}$ the subarrangement of hyperplanes

$$(\alpha, x) = k, \quad \alpha \in \Phi^+, \quad k = a, a+1, \ldots, b.$$

These include $\mathcal{A}_\Phi$, Cat_Φ, $\mathcal{S}_\Phi$ and $\mathcal{L}_\Phi$ and, more generally, the *extended Catalan arrangements* $\mathcal{A}_\Phi^{[-a,a]}$

$$(\alpha, x) = -a, -a+1, \ldots, a, \quad \alpha \in \Phi^+,$$

the *extended Shi arrangements* $\mathcal{A}_\Phi^{[-a+1,a]}$

$$(\alpha, x) = -a+1, -a+2, \ldots, a, \quad \alpha \in \Phi^+$$

and the *extended Linial arrangements* $\mathcal{A}_\Phi^{[1,b]}$

$$(\alpha, x) = 1, 2, \ldots, b, \quad \alpha \in \Phi^+.$$

These extended analogues have similar properties with those of Cat_Φ, $\mathcal{S}_\Phi$ and $\mathcal{L}_\Phi$, respectively; see Section 4. The connection between interval orders and deformations of $\mathcal{A}_n$, for instance, was extended in [59, §2] by considering labeled *marked intervals* with arbitrary prescribed lengths. As an example, suppose that the ith interval $I_i = [x_i, x_i + \lambda_i - 1]$ has integral length $\lambda_i - 1$ and is marked at all its points $x_i + k$ which are an integral distance k from the endpoint x_i. The number of inequivalent orders for placing these marked intervals on a line such that no two marks coincide is equal to the number of regions of the deformation of $\mathcal{A}_n$ with hyperplanes

$$(3) \quad x_i - x_j = -\lambda_i + 1, \ldots, -1, 0, 1, \ldots, \lambda_j - 1 \ \text{ for } \ 1 \leq i < j \leq n,$$

since comparing the marks $x_i + k$ and $x_j + l$ amounts to choosing one of the halfspaces determined by the hyperplane $x_i + k = x_j + l$. These placements correspond to nonnesting set partitions [7] whose blocks are labeled and have sizes $\lambda_1, \lambda_2, \ldots, \lambda_n$ (a set partition π of $[m] := \{1, 2, \ldots, m\}$ is *nonnesting* if whenever $a < b < c < d$ and a, d are *consecutive* elements of a block B of π, then b, c are not both contained in a block B' of π). They have also appeared in a geometric context related to monotone paths on polytopes [8]. The characteristic polynomials of the arrangements (3), which include the extended Catalan arrangements of type A, and those of root system analogues of (3) have turned out to be useful for the enumeration of nonnesting partitions by block size; see Proposition 4.7 and [7].

The family of arrangements in the following conjecture includes the extended Shi and Linial arrangements.

3.6. Conjecture (Postnikov–Stanley [45, §9]). *If a, b are non-negative integers, not both zero, satisfying $a \leq b$ then all roots of the polynomial $\chi(\mathcal{A}_\Phi^{[-a+1, b]}, q)$ have the same real part.*

For a semi-generic deformation of $\mathcal{A}_n$, see [59, §3][45, §6]. Other deformations of Coxeter arrangements appear in [59, §2], [4, Chapters 6–7].

§4. The characteristic polynomial

The examples in the previous section make it clear that tools to compute the characteristic polynomial explicitly are desirable. Such tools have traditionally included the following.

Definition: the defining equation (1) [32] or the equivalent expression, given in [42, Lemma 2.55],

$$(4) \qquad \chi(\mathcal{A}, q) = \sum_{\mathcal{F} \subseteq \mathcal{A}} (-1)^{\#\mathcal{F}} \ q^{\dim \mathcal{F}},$$

where the sum is over all subarrangements $\mathcal{F}$ of $\mathcal{A}$ and $\dim \mathcal{F}$ is the dimension of the intersection of the hyperplanes in $\mathcal{F}$; see, e.g., [44, 45].

Deletion-Restriction: this powerful technique in the theory of arrangements yields the formula ([42, Theorem 2.56])

$$\chi(\mathcal{A}, q) = \chi(\mathcal{A}', q) - \chi(\mathcal{A}'', q),$$

where $\mathcal{A}'$ and $\mathcal{A}''$ are obtained from $\mathcal{A}$ by deleting or restricting on a hyperplane $H \in \mathcal{A}$ [42, p. 14]; see, e.g., [23, 24, 6].

Chromatic Polynomials: the "signed chromatic polynomial" interpretation of Zaslavsky [70] (if $\mathcal{A}$ consists of some of the reflecting hyperplanes of Coxeter type B) and its generalization to "gain graph coloring" [73, §4]; see, e.g., [70, 71, 20], [42, §2.4] and [29, 73, 74], respectively.

Factorizations: the theory of supersolvable [58], inductively free [63] or, more generally, free arrangements [63] [42, Chapter 4], when $\chi(\mathcal{A}, q)$ factors; see, e.g., [42, §4.3] and [23, 24, 6, 12, 35].

For a discussion from a matroid theoretic point of view we refer to Kung [38, §5]; see also Zaslavsky [72]. Here we mention that Crapo's identity [21] [38, p. 49], which, in the language of arrangements, expresses the characteristic polynomial of a subarrangement of $\mathcal{A}$ in terms of those of its restrictions, has been of use in this context; see, e.g., [39].

Useful tools have resulted recently by interpreting the right hand sides of (1) or (4) using Möbius inversion or inclusion-exclusion, respectively. This is easily done when $\mathcal{A}$ is defined over a finite field $\mathbb{F}_q$, since then $q^{\dim x}$ is the cardinality of x. The following theorem appeared, in a dual formulation, as early as 1970 in the work of Crapo and Rota [22, §16] (see the discussion in [3, §1]) and was stated in the language of arrangements by Terao [66, Proposition 4.10]; see also [42, Theorem 2.69]. The proof is an easy application of Möbius inversion [49] [60, §3.7]. We denote by $V_{\mathcal{A}}$ the union of the hyperplanes of $\mathcal{A}$.

4.1. Theorem (Crapo–Rota [22], Terao [66]). *If $\mathcal{A}$ is an arrangement in $\mathbb{F}_q^n$ then the cardinality of $\mathbb{F}_q^n - V_{\mathcal{A}}$ is equal to the value $\chi(\mathcal{A}, q)$ of the characteristic polynomial of $\mathcal{A}$ at q.*

Let $\mathcal{A}$ be an arrangement in $\mathbb{K}^n$ where $\mathbb{K}$ has characteristic zero, say $\mathbb{K} = \mathbb{R}$ for simplicity, and let $\mathbb{Z}_q$ denote the abelian group of integers modulo q. We call $\mathcal{A}$ a $\mathbb{Z}$-*arrangement* if its hyperplanes are given by equations with integer coefficients. Such equations define subsets of the finite set $\mathbb{Z}_q^n$ if we reduce their coefficients modulo q. We still denote by $V_{\mathcal{A}}$ the union of these subsets, supressing q in the notation. If q avoids a finite set of prime factors, which depends on $\mathcal{A}$, then the intersection poset of the reduced arrangement in $\mathbb{Z}_q^n$ is isomorphic to that of $\mathcal{A}$ and Theorem 4.1 gives a combinatorial interpretation to the value $\chi(\mathcal{A}, q)$. This idea was first used for the purpose of computing the characteristic polynomial in [3][4, Part II] and allows for a variety of techniques from enumerative combinatorics to be employed.

The next theorem, stated as in [9, Theorem 2.1], generalizes easily to subspace arrangements [3, Theorem 2.2][4, Theorem 5.2.1] [16]. It was given independently by Björner and Ekedahl in their recent work [16] on the cohomology of subspace arrangements over finite fields; see Proposition 3.2 and Lemma 5.1 in [16].

4.2. Theorem (Athanasiadis [3, 4, 9], Björner–Ekedahl [16]). *Let $\mathcal{A}$ be a $\mathbb{Z}$-arrangement in $\mathbb{R}^n$. There exist positive integers m, k which depend only on $\mathcal{A}$, such that for all q relatively prime to m with $q > k$,*

$$\chi(\mathcal{A}, q) = \# \left(\mathbb{Z}_q^n - V_{\mathcal{A}} \right).$$

For subarrangements of the Coxeter arrangement of type B, Theorem 4.2 specializes to Zaslavsky's chromatic polynomial interpretation [70] or its generalization to subspace arrangements by Blass and Sagan [20, Theorem 2.1]. For a different generalization of Theorem 4.1 in the context of the Tutte polynomial see Reiner [48]. Finally, an interesting point of view and interpretation to (1) and (4) in terms of *valuations* appears in Ehrenborg and Readdy [25], who give several applications to classes of complex arrangements.

Theorem 4.2 has been quite useful for classes of deformations of Coxeter arrangements [3, 4, 7, 8, 9] [68, §4]. In the remainder of this section we give applications related to the examples in Section 3. For an illustration, we give a proof of Theorem 3.3 in the case $\Phi = A_{n-1}$, taken from [3, 4].

Proof of Theorem 3.3 for $\Phi = A_{n-1}$. Theorem 4.2 implies that, for large primes q, $\chi(\mathcal{S}_n, q)$ counts the number of n-tuples $x = (x_1, x_2, \ldots, x_n) \in \mathbb{F}_q^n$ which satisfy

$$x_i - x_j \neq 0, 1$$

in $\mathbb{F}_q$ for all $1 \leq i < j \leq n$. Since x satisfies these conditions if and only if $x+m := (x_1+m, \ldots, x_n+m)$ does so, this number is the number of such x with $x_n = 0$, multiplied by q. These n-tuples x are in bijection with linear orderings of the integers $1, 2, \ldots, n$ and $q - n$ indistinguishable objects such that n is first in the ordering and no two integers $i < j$ occur consecutively in the order j, i. Indeed, let i be in position $k + 1$ if $x_i = k \bmod q$, to get such an ordering.

To construct these orderings, one can place the $q-n$ objects along a line, place n first from the left and then insert $1, \ldots, n-1$ in $(q-n)^{n-1}$ ways, so that between any two consecutive objects or to the right of the rightmost one, the integers are ordered in increasing order. This shows that $\chi(\mathcal{S}_n, q) = q(q-n)^{n-1}$ for infinitely many values of q and proves the result. $\qquad\square$

The next few results can be proved by variations of the argument in the previous proof. The proofs of Theorems 4.3 and 4.6 are case by case. The next result was also obtained by Postnikov and Stanley [44, p. 39] [45, §9.2] for $\Phi = A_{n-1}$ (see also [9, Proposition 5.3]) and generalizes Theorem 3.3 for the classical root systems.

4.3. Theorem ([4, §7.1–7.2])**.** *Let Φ be of type A, B, C, D or BC. For the extended Shi arrangement $\mathcal{A} = \mathcal{A}_\Phi^{[-a+1,a]}$ we have*

$$\chi(\mathcal{A}, q) = (q - ah)^\ell,$$

where for $\Phi = BC_n$ the Coxeter number is defined as $h = 2n + 1$. In particular, the number of regions of $\mathcal{A}_\Phi^{[-a+1,a]}$ is $(ah + 1)^\ell$.

An application in the spirit of [23, 35] comes from considering arrangements between the braid and Catalan arrangement $\mathcal{A}_n$ and Cat_n. For $G \subseteq \mathcal{E}_n := \{(i, j) \in [n] \times [n] : i \neq j\}$, let $\mathcal{A}_{n,G}$ be the arrangement of hyperplanes

(5)
$$\begin{aligned} x_i - x_j &= 0 \ \text{ for } 1 \leq i < j \leq n, \\ x_i - x_j &= 1 \ \text{ for } (j, i) \in G. \end{aligned}$$

Note that if G is empty, $G = \{(j, i) \in \mathcal{E}_n : i < j\}$, or $G = \mathcal{E}_n$, then $\mathcal{A}_{n,G}$ specializes to $\mathcal{A}_n$, $\mathcal{S}_n$, or Cat_n, respectively, to which the next proposition applies. For a generalization and analogous results for other root systems see [4, Theorem 6.2.10 and §6.3][3, §3].

4.4. Proposition ([3, Theorem 3.9][4, Theorem 6.2.7])**.** *Suppose that the set $G \subseteq \mathcal{E}_n$ has the following properties:*

(i) *If $i, j < k$, $i \neq j$ and $(i, j) \in G$, then $(i, k) \in G$ or $(k, j) \in G$.*

(ii) *If $i, j < k$, $i \neq j$ and $(i, k) \in G$, $(k, j) \in G$, then $(i, j) \in G$.*
Then

$$\chi(\mathcal{A}_{n,G}, q) = q \prod_{1 < j \leq n} (q - c_j),$$

where $c_j = n + a_j - j + 1$ and a_j is the number of (j, i) or (i, j) in G with $i < j$.

The conditions in Proposition 4.4 become simpler if $\mathcal{A}_{n,G}$ contains hyperplanes of the form $x_i - x_j = 1$ only for $i < j$, i.e. if it lies between $\mathcal{A}_n$ and $\mathcal{S}_n$. We state this special case for later reference.

4.5. Corollary ([3, Theorem 3.4][4, Theorem 6.2.2]). *Suppose that the set $G \subseteq \{(j, i) \in \mathcal{E}_n : i < j\}$ has the following property: if $1 \leq i < j < k \leq n$ and $(j, i) \in G$ then $(k, i) \in G$. Then*

$$\chi(\mathcal{A}_{n,G}, q) = q \prod_{1 < j \leq n} (q - c_j),$$

where $c_j = n - \#\{i < j : (j, i) \notin G\}$.

Recall from Theorem 2.6 that the characteristic polynomial of $\mathcal{A}_\Phi$ factors with roots the exponents of Φ. The following result was also obtained in [44, Proposition 1.5.8][45, Theorem 9.8] for $\Phi = A_{n-1}$; see also [3, Theorem 5.5] and [9, Proposition 5.3].

4.6. Theorem ([4, Corollary 7.2.3 and Theorem 7.2.6]). *Let Φ be of type A, B, C, D or BC. For the extended Catalan arrangement $\mathcal{A} = \mathcal{A}_\Phi^{[-a,a]}$ we have*

$$\chi(\mathcal{A}, q) = \chi(\mathcal{A}_\Phi, q - ah),$$

if Φ has type A, B, C or D and

$$\chi(\mathcal{A}, q) = \begin{cases} \chi(\mathcal{A}_\Phi, q - (2n+1)a), & \text{if } a \text{ is even,} \\ \chi(\mathcal{A}_\Phi, q - (2n+1)a - 1), & \text{if } a \text{ is odd,} \end{cases}$$

if Φ has type BC.

The arrangement (3) reduces to the extended Catalan arrangement of type A for $\lambda_1 = \lambda_2 = \cdots = \lambda_n = a + 1$. Its characteristic polynomial can be computed by an easy application of Theorem 4.2.

4.7. Proposition ([8, §4]). *If $\mathcal{A}$ is the arrangement (3) and m is the sum of the positive integers λ_i for $1 \leq i \leq n$, then*

$$\chi(\mathcal{A}, q) = q \prod_{j=m-n+1}^{m-1} (q - j).$$

In particular, the number of nonnesting partitions of $[m]$ with block sizes $\lambda_1, \lambda_2, \ldots, \lambda_n$ is equal to

$$\frac{m!}{r_\lambda \, (m - n + 1)!},$$

where $r_\lambda = r_1! r_2! \cdots$ and r_j is the number of indices i with $\lambda_i = j$.

Root system analogues of Proposition 4.7 for $\Phi = B_n$ and C_n appear in [7].

The expression for $\chi(\mathcal{L}_n, q)$ in Theorem 3.5 was obtained in [3, 4] by a similar but less straightforward argument, based on Theorem 4.2. It generalizes easily to the extended Linial arrangements. Let S be the *shift operator*, acting on polynomials in y by

$$Sf(y) := f(y - 1).$$

We state the next result in the elegant form given in [44, 45]. For a relatively short proof based on Theorem 4.2 see [9, §3].

4.8. Theorem ([3, §4][4, §6.4] [45, Theorem 9.7]). *For $\Phi = A_{n-1}$ and $a \geq 1$, the extended Linial arrangement has characteristic polynomial*

$$\chi(\mathcal{A}_\Phi^{[1,a]}, q) = \frac{1}{(a + 1)^n} \, (1 + S + S^2 + \cdots + S^a)^n \, q^{n-1}.$$

Theorem 4.8 implies the fact that all roots of $\chi(\mathcal{A}_\Phi^{[1,a]}, q)$ have the same real part. Indeed, if the polynomial f has this property then so does $(S+\zeta)f$, if $\zeta \in \mathbb{C}$ satisfies $|\zeta| = 1$; see [44, Lemma 1.5.12][45, Lemma 9.12] for an elegant, short proof. Using this reasoning, Postnikov and Stanley settled their Conjecture 3.6 in the type A case.

4.9. Theorem ([44, Theorem 1.5.11] [45, Theorem 9.11]). *Conjecture 3.6 is true for $\Phi = A_{n-1}$.*

Explicit formulae for the characteristic polynomials of the arrangements in Conjecture 3.6 were obtained in [9, §4–5] for the other classical root systems. The proofs follow the ones for the type A case in [9, §3]

but are more involved. We give the formulae for the extended Linial arrangements.

4.10. Theorem ([9, §4]). *For the extended Linial arrangement* $\mathcal{A} = \mathcal{A}_{\Phi}^{[1,a]}$ *and for* a *even or odd, respectively,* $\chi(\mathcal{A}, q)$ *is equal to*

$$
\begin{cases}
\frac{1}{(a+1)^{n+1}} \, (1 + S^2 + S^4 + \cdots + S^{2a})^{n-1} \, (1 + S + S^2 + \cdots + S^a) \, q^n, \\
\frac{4S}{(a+1)^{n+1}} \, (1 + S^2 + S^4 + \cdots + S^{2a})^{n-1} \, (1 + S^2 + S^4 + \cdots + S^{a-1}) \, q^n
\end{cases}
$$

if $\Phi = B_n$ *or* C_n,

$$
\begin{cases}
\frac{1}{(a+1)^{n+1}} \, (1 + S^2 + S^4 + \cdots + S^{2a})^{n-2} \, (1 + S + S^2 + \cdots + S^a)^4 \, q^n, \\
\frac{8S}{(a+1)^{n+1}} \, (1 + S^2) \, (1 + S^2 + \cdots + S^{2a})^{n-1} \, (1 + S^2 + \cdots + S^{a-1})^4 \, q^n
\end{cases}
$$

if $\Phi = D_n$ *and*

$$
\begin{cases}
\frac{1}{(a+1)^{n+1}} \, (1 + S^2 + S^4 + \cdots + S^{2a})^{n} \, (1 + S + S^2 + \cdots + S^a) \, q^n, \\
\frac{2S}{(a+1)^{n+1}} \, (1 + S^2 + S^4 + \cdots + S^{2a})^{n} \, (1 + S^2 + S^4 + \cdots + S^{a-1}) \, q^n
\end{cases}
$$

if $\Phi = BC_n$.

The next result follows as in the type A case; see [9].

4.11. Theorem ([9, Theorem 1.2]). *Conjecture 3.6 is true for all root systems of type* A, B, C, D *or* BC.

§5. Freeness

Recall from Theorem 2.4 that the characteristic polynomial of a free arrangement factors completely over the nonnegative integers and from Theorem 2.5 that the Coxeter arrangement $\mathcal{A}_{\Phi}$ is free with exponents the exponents $e_1, e_2, \ldots, e_\ell$ of Φ. In view of the numerous instances in Sections 3 and 4 in which $\chi(\mathcal{A}, q)$ factors, it is natural to ask whether various deformations of $\mathcal{A}_{\Phi}$ are free, when homogenized to central arrangements by the cone operation [42, Definition 1.15].

Freeness of the cones of the extended Catalan and Shi arrangements was conjectured in [24] and remains unsettled, except for the type A case [24, §3] [6, §3]. We continue to denote by h the Coxeter number of Φ.

5.1. Conjecture (Edelman–Reiner [24, Conjecture 3.3]). *The cone of the extended Catalan arrangement* $\mathcal{A}_{\Phi}^{[-a,a]}$ *is free with exponents* $1, e_1 + ah, e_2 + ah, \ldots, e_\ell + ah$.

5.2. Conjecture (Edelman–Reiner [24, Conjecture 3.3]). *The cone of the extended Shi arrangement $\mathcal{A}_\Phi^{[-a+1,a]}$ is free with exponents 1 with multiplicity one, and ah with multiplicity ℓ.*

Edelman and Reiner have stated these conjectures for an irreducible crystallographic root system Φ. In view of Theorems 4.3 and 4.6, it is natural to include the non-reduced system BC_n. The conjectures are not true in general in the non-crystallographic case; see the comments after Conjecture 3.3 in [24].

Except for Theorems 4.3 and 4.6, evidence in support of the conjectures is provided by the fact that they have been verified in the case of type A; see the proof of [24, Theorem 3.2] and [6, Corollary 3.4], respectively. Moreover, in the case of Conjecture 5.2, additional evidence is provided by work of Solomon and Terao [57] on the double Coxeter arrangement, which we will briefly describe.

Suppose $\mathcal{A}$ is central in $\mathbb{K}^\ell$ and that α_H is the linear form which defines $H \in \mathcal{A}$, so that $H = \ker(\alpha_H)$. Let $\mathbb{S} = \mathbb{K}[x_1, x_2, \ldots, x_\ell]$ be the polynomial ring, as in Section 2, and $\mathrm{Der}_\mathbb{S}$ be its module of derivations. In his theory of *free multiarrangements* [76], Ziegler has defined the $\mathbb{S}$-module

$$E(\mathcal{A}) = \{\theta \in \mathrm{Der}_\mathbb{S} : \theta(\alpha_H) \in \mathbb{S}\,\alpha_H^2 \ \text{ for } \ H \in \mathcal{A}\},$$

which is a submodule of $D(\mathcal{A})$. Note that the restriction of the cone of $\mathcal{S}_\Phi$ to the hyperplane at infinity $x_0 = 0$ is the *double Coxeter arrangement*, i.e. $\mathcal{A}_\Phi$ with each hyperplane having multiplicity two. Thus by Ziegler's [76, Theorem 11], the $a = 1$ case of Conjecture 5.2 implies that the double Coxeter arrangement is free, in the sense of the following theorem.

5.3. Theorem (Solomon–Terao [57]). *Let Φ be any irreducible root system. The module $E(\mathcal{A}_\Phi)$ is free with all degrees of the elements in a basis equal to the Coxeter number h.*

Moreover, Solomon and Terao [57, Theorem 1.4] construct an explicit basis of $E(\mathcal{A}_\Phi)$ in terms of the invariant theory of the Coxeter group W. This raises naturally the following question.

5.4. Question.   *Is there a basis of the module of derivations of the cone of $\mathcal{S}_\Phi$ which can be described explicitly in terms of the invariant theory of the Weyl group W?*

Beginning with work of Stanley [58] on subarrangements of the braid arrangement $\mathcal{A}_n$, called *graphical arrangements*, classes of subarrangements of Coxeter arrangements have been studied [35] and characterized

[23, 13] from the point of view of freeness; see also [24], [11, §7][12]. It was shown by Stanley [58] that the supersolvable – or free – graphical arrangements correspond to *chordal* graphs and by Edelman and Reiner [23] that the free arrangements between $\mathcal{A}_n$ and the Coxeter arrangement of type B_n correspond to *threshold* graphs. For interesting classes of free or non-free subarrangements, in particular for non-free graphical arrangements whose characteristic polynomials factor completely over the integers, see Kung [38].

Various deformations of $\mathcal{A}_n$ were studied in this sense in [6]. We mention a complete characterization for the family of arrangements $\mathcal{A}_{n,G}$, defined in (5), which lie between $\mathcal{A}_n$ and $\mathcal{S}_n$. The class of arrangements in this family with free cones turns out to be, essentially, the class which appears in Corollary 4.5. The condition in Corollary 4.5 has also appeared in the characterization of freeness in a different family; see Bailey [11, Theorem 7.3][12].

5.5. Theorem ([6, Theorem 4.1]). *Let $G \subseteq E_n := \{(j,i) \in [n] \times [n] : i < j\}$. The following are equivalent:*

(i) *$\mathcal{A}_{n,G}$ is inductively free.*
(ii) *The cone of $\mathcal{A}_{n,G}$ is free.*
(iii) *There is a permutation $w = w_1 w_2 \cdots w_n$ of $[n]$ such that*

$$w^{-1} \cdot G = \{(j,i) : (w_j, w_i) \in G\}$$

is contained in E_n and satisfies the condition in Corollary 4.5.

A similar characterization for the family of arrangements between $\mathcal{S}_n$ and Cat_n is given in [6, Theorem 4.3]. Specifically, if $E_n \subseteq G \subseteq \mathcal{E}_n$ and $\overline{G} = \{(j,i) : (i,j) \in \mathcal{E}_n - G\} \subseteq E_n$, then $\mathcal{A}_{n,G}$ has free cone if and only if so does $\mathcal{A}_{n,\overline{G}}$. In contrast with the situation in [23], most of the free arrangements of Theorem 5.5 are not supersolvable; see [6, Theorem 4.2]. For characterizations of supersolvability for deformations of $\mathcal{A}_n$, see Zaslavsky [74, §3].

§6.   Remarks and open questions

In this section we include a number of questions other than Conjecture 3.6 (which is still open for the exceptional root systems), Conjectures 5.1 and 5.2 and Question 5.4. Our main objective is to point out that from many perspectives, the classes of deformations of Coxeter arrangements we have discussed are still not well understood.

All known proofs of Theorem 3.3 proceed with a case by case verification. A positive answer to Question 5.4 would give a uniform proof,

via Terao's factorization theorem. The fact that Theorem 3.3 is stated in an elementary, uniform way suggests the following question.

6.1. Question. *Is there an elementary, case-free proof of Theorem 3.3?*

Similar questions can be asked about Theorems 4.3 and 4.6 and the curious property of $\chi(\mathcal{A}, q)$ in Conjecture 3.6, which Postnikov and Stanley refer to as the "Riemann hypothesis" for $\mathcal{A}$ [44, 45]. In particular, it is natural to ask whether Theorems 4.3, 4.6 and 4.11 extend to the exceptional root systems.

6.2. Question. *Are there case-free proofs of Theorems 4.3 and 4.6? In particular, are these theorems valid for the exceptional crystallographic root systems?*

6.3. Question. *Is there a case-free, conceptual proof of Conjecture 3.6?*

It would also be desirable to find simpler derivations of the formulae in Theorem 4.10 than those of [9], which may not give the best insight possible, especially in the case of the root system D_n. In particular, there is no conceptual explanation to the fact that these formulae coincide for the root systems B_n and C_n.

The Riemann hypothesis for $\mathcal{A}$ does not apply exclusively to the arrangements of Conjecture 3.6, as the following example shows.

6.4. Example ([9, Proposition 6.1]). *The arrangement with hyperplanes*

$$2x_i = 0, 1, 2, \ldots, 2a \quad for \quad 1 \leq i \leq n,$$
$$x_i - x_j = 0, 1, \ldots, a \quad for \quad 1 \leq i < j \leq n,$$
$$x_i + x_j = 0, 1, \ldots, a \quad for \quad 1 \leq i < j \leq n$$

has characteristic polynomial

$$\frac{1}{a^{n+1}} \, S^{2n+1} \, (1 + S^2 + S^4 + \cdots + S^{2a-2})^{n+1} \, q^n$$

and hence satisfies Conjecture 3.6.

By analogy with the numerous theories built to explain the phenomenon of complete factorization of $\chi(\mathcal{A}, q)$ over the integers (see Section 2), we ask the following.

6.5. Question. *Is there a natural algebraic condition on $\mathcal{A}$ which implies the Riemann hypothesis of Conjecture 3.6 for $\chi(\mathcal{A}, q)$?*

In various characterizations of freeness, such as those in [23, 24, 6, 11], the families of arrangements under consideration are indexed by undirected graphs on n vertices. Classes of arrangements which correspond to pairs of graphs seem to be more challenging to analyze from the point of view of freeness. It is not known, for instance, which the free subarrangements of the Coxeter arrangement of type B_n are; see [23]. Proposition 4.4 suggests an explicit characterization of the arrangements between $\mathcal{A}_n$ and Cat_n with free cones.

6.6. Conjecture. *For $G \subseteq \mathcal{E}_n$, the cone of $\mathcal{A}_{n,G}$ is free if and only if G satisfies the two conditions in Proposition 4.4.*

Motivated by the fact that Coxeter arrangements are $K(\pi, 1)$ [42, Chapter 6] we ask the following about the topology of the complexifications of $\mathcal{S}_\Phi$ and Cat_Φ.

6.7. Question. *Is the Shi arrangement $\mathcal{S}_\Phi$ a $K(\pi, 1)$ arrangement? Is the Catalan arrangement Cat_Φ a $K(\pi, 1)$ arrangement?*

Finally, we collect some questions and facts about the combinatorics of the face structure of the arrangements in Section 3.

Direct bijective proofs of Theorem 3.2 for the type A case can be found in [61, §2] [10, §2]; see also [31] and Remark 1 in [10, §4] for a proof by deletion-restriction. The bijections in [61, 10] generalize to the extended Shi arrangements. The one in [10] generalizes also to the family of arrangements between $\mathcal{A}_n$ and $\mathcal{S}_n$ [10, Theorem 1.2].

6.8. Question. *Are there simple bijective proofs of Theorem 3.2 for cases other than that of type A?*

For the braid arrangement $\mathcal{A}_n$, it is well known that faces of a fixed dimension k correspond to ordered partitions of the set $[n]$ with k blocks. In the case of type A, Shi's formula for the number of regions of $\mathcal{S}_n$ was generalized to k-dimensional faces in [3, Theorem 6.5][4, Corollary 8.2.2] as follows.

6.9. Theorem ([3, 4])**.** *For $0 \le k \le n$, the number of faces of $\mathcal{S}_n$ of dimension k is given by*

$$f_k(\mathcal{S}_n) = \binom{n}{k} \sum_{i=0}^{n-k} (-1)^i \binom{n-k}{i} (n-i+1)^{n-1}.$$

Equivalently,

$$(6) \qquad f_k(\mathcal{S}_n) = \binom{n}{k} \# \{f : [n-1] \to [n+1] \mid [n-k] \subseteq \mathrm{Im} f\},$$

where $\mathrm{Im} f$ *is the image of the map* f.

6.10. Question. *Is there a simple bijective proof of Theorem 6.9? Can the poset of faces of* $\mathcal{S}_n$, *partially ordered by inclusion of their closures, be described in terms of the maps in (6)?*

It is plausible that such a bijection will specialize to the one between regions of $\mathcal{S}_n$ and parking functions given in [10, §2] for $k = n$. Theorem 6.9 generalizes to the extended Shi arrangements [4, Theorem 8.2.1].

The "coincidence" of the formulae for the number f_n of regions of the Linial arrangement $\mathcal{L}_n$ and alternating trees on $n + 1$ vertices suggests the following question.

6.11. Question. *Is there a bijective proof of Theorem 3.4?*

We refer to [59, §4] for a number of combinatorial interpretations and expressions for f_n. In particular, Postnikov [43, §4][44, Theorem 1.4.3] has given a bijection between alternating trees on $n + 1$ vertices and *local binary search trees* on n vertices. Here we remark that f_n is also equal to the number of n-tuples $x = (x_1, x_2, \ldots, x_n) \in \mathbb{Z}_{n+1}^n$ which satisfy $x_i - x_j \neq 1$ in $\mathbb{Z}_{n+1}$ for $1 \leq i < j \leq n$ and $x_1 = 0$ or, in other words, to the number of ways to distribute $1, 2, \ldots, n$ in $n + 1$ boxes arranged cyclically, with repetitions allowed, such that no j is immediately followed clockwise by an $i < j$. This follows from the proof of Theorem 3.5 in [3, 4, 9] by letting $q = n + 1$.

The regions of Cat_n, $\mathcal{S}_n$ and $\mathcal{L}_n$ are in bijection with certain classes of posets that can be characterized in terms of forbidden induced sub-posets, see [44, §1.3.1][45, §7], [5] and [44, §1.4.6][45, §8.2], respectively. It would be interesting to find other instances of this phenomenon.

The enumeration of regions by the "distance statistic" has been of interest in the context of deformations of Coxeter arrangements. The *distance* $\rho_{R_0}(R)$ of a region R of $\mathcal{A}$ from a fixed base region R_0 is the number of hyperplanes of $\mathcal{A}$ which separate R from R_0. The following result for Coxeter arrangements is classical.

6.12. Theorem (Solomon [55]). *For any irreducible root system* Φ *we have*

$$\sum_R q^{\rho_{R_0}(R)} = \prod_{i=1}^{\ell}(1 + q + q^2 + \cdots + q^{e_i}),$$

where R *runs through all regions of* $\mathcal{A}_\Phi$, R_0 *is any fixed region and* $e_1, e_2, \ldots, e_\ell$ *are the exponents of* Φ.

For the Shi arrangement $\mathcal{S}_n$ the distance enumerator, for a suitably chosen base region R_0, turns out to be the *inversion enumerator* for trees [40]. Indeed, let R_0 be the region defined by the inequalities $x_1 > x_2 > \cdots > x_n$ and $x_1 - x_n < 1$. An *inversion* of a tree T on the vertex set $\{0, 1, \ldots, n\}$ is a pair (i, j) with $1 \leq i < j \leq n$ such that vertex j lies on the path in T from 0 to i. The bijection described in [59, §5] and one due to Kreweras [37] yield the following result. A proof and generalization to the extended Shi arrangements is given in [61].

6.13. Theorem (Pak–Stanley [59, Theorem 5.1][61]). *For each* $m = 0, 1, \ldots, \binom{n}{2}$, *the number of regions* R *of* $\mathcal{S}_n$ *with distance* m *from* R_0 *is equal to the number of trees on* $\{0, 1, \ldots, n\}$ *with* $\binom{n}{2} - m$ *inversions.*

It would be interesting to find a simpler and more direct proof of this theorem. See the notes in [61, §3] for related open questions.

6.14. Question (Stanley [62]). *Are there analogues of Theorem 6.13 for root systems other than those of type A? Is there an analogue for the Linial arrangement* $\mathcal{L}_n$?

It was observed by Stanley [62] that the distance enumerator for the Catalan arrangement Cat_n is

$$\sum_R q^{\rho_{R_0}(R)} = C_n(q) \prod_{i=1}^{n-1}(1 + q + q^2 + \cdots + q^i),$$

where R_0 is as in the case of $\mathcal{S}_n$ and

$$C_n(q) = \sum_\lambda q^{|\lambda|},$$

with $\lambda = (\lambda_1, \lambda_2, \ldots)$ running over all partitions with $\lambda_i \leq n - i$.

References

[1] V.I. Arnol'd, Wavefront evolution and equivariant Morse Lemma, *Comm. Pure Appl. Math.* **26** (1976), 557–582.

[2] ———, Indices of singular points of 1-forms on a manifold with boundary, convolution of invariants of reflection groups and singular projections of smooth surfaces, *Russian Math. Surveys* **34** (1979), 1–42.

[3] C.A. Athanasiadis, Characteristic polynomials of subspace arrangements and finite fields, *Adv. Math.* **122** (1996), 193–233.

[4] ———, *Algebraic combinatorics of graph spectra, subspace arrangements and Tutte polynomials*, Ph.D. thesis, MIT, 1996.

[5] ———, A class of labeled posets and the Shi arrangement of hyperplanes, *J. Combin. Theory Ser. A* **80** (1997), 158–162.

[6] ———, On free deformations of the braid arrangement, *European J. Combin.* **19** (1998), 7–18.

[7] ———, On noncrossing and nonnesting partitions for classical reflection groups, *Electronic J. Combin.* **5** (1998), R42, 16 pp.

[8] ———, Piles of cubes, monotone path polytopes and hyperplane arrangements, *Discrete Comput. Geom.* **21** (1999), 117–130.

[9] ———, Extended Linial hyperplane arrangements for root systems and a conjecture of Postnikov and Stanley, *J. Alg. Combin.*, to appear.

[10] C.A. Athanasiadis and S. Linusson, A simple bijection for the regions of the Shi arrangement of hyperplanes, in *Gould Anniversary Volume* (I. Gessel et al. eds.), *Discrete Math.*, in press.

[11] G.D. Bailey, *Tilings of zonotopes: Discriminantal arrangements, oriented matroids, and enumeration*, Ph.D. thesis, University of Minnesota, 1997.

[12] ———, Coherence and enumeration of tilings of 3-zonotopes, *Discrete Comput. Geom.*, to appear.

[13] ———, Inductively factored signed-graphic arrangements of hyperplanes, *Discrete Math.*, to appear.

[14] C. Bennett and B.E. Sagan, A generalization of semimodular supersolvable lattices, *J. Combin. Theory Ser. A.* **72** (1995), 209–231.

[15] A. Björner, Subspace arrangements, in *Proc. of the first European Congress of Mathematics, Paris 1992* (A. Joseph et al. eds.), Progress in Math. **119**, Birkhäuser 1994, pp. 321–370.

[16] A. Björner and T. Ekedahl, Subspace arrangements over finite fields: cohomological and enumerative aspects, *Adv. Math.* **129** (1997), 159–187.

[17] A. Björner, M. Las Vergnas, B. Sturmfels, N. White and G.M. Ziegler, *Oriented Matroids*, Encyclopedia of Mathematics and Its Applications **46**, Cambridge University Press, Cambridge, 1993.

[18] A. Björner and G.M. Ziegler, Broken circuit complexes: Factorizations and generalizations, *J. Combin. Theory Ser. B* **51** (1991), 96–126.

[19] A. Blass and B.E. Sagan, Möbius functions of lattices, *Adv. Math.* **127** (1997), 94–123.

[20] _____, Characteristic and Ehrhart polynomials, *J. Alg. Combin.* **7** (1998), 115–126.

[21] H.H. CRAPO, Möbius inversion in lattices, *Arch. Math.* **19** (1968), 595–607.

[22] H.H. CRAPO AND G.-C. ROTA, *On the Foundations of Combinatorial Theory: Combinatorial Geometries*, preliminary edition, M.I.T. press, Cambridge, MA, 1970.

[23] P.H. EDELMAN AND V. REINER, Free hyperplane arrangements between A_{n-1} and B_n, *Math. Z.* **215** (1994), 347–365.

[24] _____, Free arrangements and rhombic tilings, *Discrete Comput. Geom.* **15** (1996), 307–340.

[25] R. EHRENBORG AND M. READDY, On valuations, the characteristic polynomial and complex subspace arrangements, *Adv. Math.* **134** (1998), 32–42.

[26] M. FALK AND M. JAMBU, Factorizations of combinatorial geometries, Manuscript, 1989.

[27] P.C. FISHBURN, *Interval Orders and Interval Graphs*, John Wiley & Sons, New York, 1985.

[28] I.M. GELFAND, M.I. GRAEV AND A. POSTNIKOV, Combinatorics of hypergeometric functions associated with positive roots, in *Arnol'd-Gelfand Mathematical Seminars*, Birkhäuser, Boston, pp. 205–221.

[29] P. HANLON AND T. ZASLAVSKY, Tractable partially ordered sets derived from root systems and biased graphs, *Order* **14** (1997/98), 229-257.

[30] P. HEADLEY, *Reduced expressions in infinite Coxeter groups*, Ph.D. thesis, University of Michigan, 1994.

[31] _____, On reduced words in affine Weyl groups, in *Proc. Formal Power Series and Algebraic Combinatorics (FPSAC) 1994* (L.J. Billera, C. Greene, R. Simion and R.P. Stanley, eds.), DIMACS Series in Discrete Mathematics and Theoretical Computer Science, Amer. Math. Soc. **24** (1996), pp. 225–232.

[32] _____, On a family of hyperplane arrangements related to affine Weyl groups, *J. Alg. Combin.* **6** (1997), 331–338.

[33] J. HUMPHREYS, *Reflection groups and Coxeter groups*, Cambridge Studies in Advanced Mathematics **29**, Cambridge University Press, Cambridge, England, 1990.

[34] M. JAMBU AND L. PARIS, Combinatorics of inductively factored arrangements, *European J. Combin.* **16** (1995), 267–292.

[35] T. JÓZEFIAK AND B.E. SAGAN, Basic derivations for subarrangements of Coxeter arrangements, *J. Alg. Combin.* **2** (1993), 291–320.

[36] D. KAZHDAN AND G. LUSZTIG, Representations of Coxeter groups and Hecke algebras, *Invent. Math.* **53** (1979), 165–184.

[37] G. KREWERAS, Une famille de polynômes ayant plusieurs propriétés énumeratives, *Period. Math. Hungar.* **11** (1980), 309–320.

[38] J.P.S. KUNG, Critical problems, in *Theory of Matroids*, Contemp. Math. **197**, Amer. Math. Soc., Providence, RI, 1996, pp. 1–127.

[39] _____, A geometric condition for a hyperplane arrangement to be free, *Adv. Math.* **135** (1998), 303–329.

[40] C.L. MALLOWS AND J. RIORDAN, The inversion enumerator for labeled trees, *Bull. Amer. Math. Soc.* **74** (1968), 92–94.

[41] P. ORLIK AND L. SOLOMON, Combinatorics and topology of complements of hyperplanes, *Invent. Math.* **56** (1980), 167–189.

[42] P. ORLIK AND H. TERAO, *Arrangements of Hyperplanes*, Grundlehren 300, Springer-Verlag, New York, NY, 1992.

[43] A. POSTNIKOV, Intransitive trees, *J. Combin. Theory Ser. A* **79** (1997), 360–366.

[44] _____, *Enumeration in algebra and geometry*, Ph.D. thesis, MIT, 1997.

[45] A. POSTNIKOV AND R.P STANLEY, Deformations of Coxeter hyperplane arrangements, Manuscript dated April 14, 1997.

[46] A. RAM, Standard Young tableaux for finite root systems, Manuscript, 1998.

[47] V. REINER, Non-crossing partitions for classical reflection groups, *Discrete Math.* **177** (1997), 195–222.

[48] _____, An interpretation for the Tutte polynomial, *European J. Combin.*, to appear.

[49] G.-C. ROTA, On the foundations of combinatorial theory I. Theory of Möbius functions, *Z. Wahrscheinlichkeitstheorie* **2** (1964), 340–368.

[50] B.E. SAGAN, Why the characteristic polynomial factors, *Bull. Amer. Math. Soc.*, to appear.

[51] K. SAITO, On the uniformization of complements of discriminant loci, in *Seminar Notes*, Amer. Math. Soc. Summer Institute, Williamstown, 1975.

[52] _____, Theory of logarithmic differential forms and logarithmic vector fields, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **27** (1980), 265–291.

[53] J.-Y. SHI, *The Kazhdan-Lusztig cells in certain affine Weyl groups*, Lecture Notes in Mathematics, no. 1179, Springer-Verlag, Berlin/ Heidelberg/New York, 1986.

[54] _____, Sign types corresponding to an affine Weyl group, *J. London Math. Soc.* **35** (1987), 56–74.

[55] L. SOLOMON, The orders of the finite Chevalley groups, *J. Algebra* **3** (1966), 376–393.

[56] L. SOLOMON AND H. TERAO, A formula for the characteristic polynomial of an arrangement, *Adv. Math.* **64** (1987), 305–325.

[57] _____, The double Coxeter arrangement, *Comment. Math. Helv.* **73** (1998), 237–258.

[58] R.P. STANLEY, Supersolvable lattices, *Algebra Universalis* **2** (1972), 197–217.

[59] _____, Hyperplane arrangements, interval orders and trees, *Proc. Nat. Acad. Sci.* **93** (1996), 2620–2625.

[60] _____, *Enumerative Combinatorics*, vol. 1, Cambridge University Press, Cambridge, 1997.

[61] _____, Hyperplane arrangements, parking functions and tree inversions, in *Mathematical Essays in Honor of Gian-Carlo Rota* (B.E. Sagan and R.P. Stanley, eds.), Progress in Math. **161** (1998), Birkhäuser, Boston, pp. 359–375.

[62] _____, Personal communication, 1996–98.

[63] H. TERAO, Arrangement of hyperplanes and their freeness I, II, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **27** (1980), 293–320.

[64] _____, Free arrangement of hyperplanes and unitary reflection groups, *Proc. Japan. Acad. Ser. A* **56** (1980), 389–392.

[65] _____, Generalized exponents of a free arrangement of hyperplanes and the Shepherd-Todd-Brieskorn formula, *Invent. Math.* **63** (1981), 159–179.

[66] _____, The Jacobians and the discriminants of finite reflection groups, *Tôhoku Math. J.* **41** (1989), 237–247.

[67] _____, Factorizations of Orlik-Solomon algebras, *Adv. Math.* **91** (1992), 45–53.

[68] V. WELKER, Colored partitions and a generalization of the braid arrangement, *Electronic J. Combin.* **4** (1997), R4, 12 pp.

[69] T. ZASLAVSKY, Facing up to arrangements: face-count formulas for partitions of space by hyperplanes, *Mem. Amer. Math. Soc.* vol. 1, no. 154, (1975).

[70] _____, Signed graph coloring, *Discrete Math.* **39** (1982), 215–228.

[71] _____, Chromatic invariants of signed graphs, *Discrete Math.* **42** (1982), 287–312.

[72] _____, The Möbius function and the characteristic polynomial, Chapter 7 in *Combinatorial Geometries* (N. White, ed.), Cambridge University Press, Cambridge, 1987, pp. 114–138.

[73] _____, Biased graphs. III. Chromatic and dichromatic invariants, *J. Combin. Theory Ser. B* **64** (1995), 17–88.

[74] _____, Supersolvable frame-matroid and graphic-lift lattices, Manuscript, 1996.

[75] G.M. ZIEGLER, *Algebraic combinatorics of hyperplane arrangements*, Ph.D. thesis, MIT, 1987.

[76] _____, Multiarrangements of hyperplanes and their freeness, in *Singularities*, Contemp. Math. **90**, Amer. Math. Soc., Providence, RI, 1989, pp. 345–359.

Christos A. Athanasiadis
Department of Mathematics
University of Pennsylvania
Philadelphia, PA 19104-6395
E-mail address: athana@math.upenn.edu