THE UNIFORM CLOSURE OF RATIONAL MODULES

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ABSTRACT

For a smooth function g, the module $\mathscr{R}(X)+g\mathscr{R}(X)$ is dense in $C(X)$ if and only if $\mathscr{R}(Z)$ is dense in $C(Z)$, where X is compact and nowhere dense and Z is the subset of X on which $\bar{\partial}g$ vanishes. The "localness" of the module $\mathscr{R}(X)+g\mathscr{R}(X)$ is also investigated.

Let X be a compact subset of the complex plane $\mathbb{C}$ and let g be a continuous function on X. We denote by $\mathscr{R}(X, g)$ the rational module

$$\mathscr{R}(X)+\mathscr{R}(X)g = \{r_0(z)+r_1(z)g(z)\},$$

where each r_i denotes a rational function with poles off X. The purpose of this paper is to study the uniform closure of such modules on X.

In the case that $g(z) = \bar{z}$, the closure of $\mathscr{R}(X, \bar{z})$ in various topologies was considered in [3], [4] and was applied to rational approximation problems in lip α. Further results were obtained in [6], [7]. A question which arose from these investigations concerned the characterization of the uniform closure of $\mathscr{R}(X, \bar{z})$ in $C(X)$ when X has empty interior. This was settled in [5] by showing that $\mathscr{R}(X, \bar{z})$ is uniformly dense in $C(X)$ whenever X has empty interior.

I. For the first section we assume that the interior of X is empty. We are interested in determining functions g for which $\mathscr{R}(X, g)$ is (uniformly) dense in $C(X)$. It is well known that if X is a "Swiss cheese" then $\mathscr{R}(X)$ fails to be dense in $C(X)$ (see [2]). Thus if $\mathscr{R}(X, g)$ is to be dense in $C(X)$, we are led to make a natural assumption concerning the subset of X on which $\bar{\partial}g$ is zero. Here $\bar{\partial}$ denotes the operator $\frac{1}{2}(\partial_x+i\partial_y)$. We denote the uniform closures on X of $\mathscr{R}(X, g)$ and $\mathscr{R}(X)$ by $R(X, g)$ and $R(X)$ respectively.

Assume that g is continuously differentiable in a neighborhood of X. Denote by Z the subset of X on which $\bar{\partial}g$ vanishes. Let $\mathbb{C}$ denote the complex plane. We have the following theorem:

THEOREM 1. *Let $X \subset \mathbb{C}$ be a compact set with empty interior. Then $R(X, g) = C(X)$ if and only if $R(Z) = C(Z)$.*

Before giving the proof we state a corollary, followed by several lemmas needed to establish the theorem.

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COROLLARY. *Let $X \subset \mathbb{C}$ be compact with empty interior and let p be a nonconstant polynomial in $\bar{z}$ with coefficients in $R(X)$. Then*

$$R(X, p) = C(X).$$

Proof. If Z is the subset of X in which $\bar{\partial} p$ vanishes then $m(Z) = 0$, where m denotes area Lebesgue measure on $\mathbb{C}$. Appealing to a theorem of Hartogs and Rosenthal ([2], p. 47), $\mathscr{R}(Z)$ is dense in $C(Z)$. Thus Theorem 1 applies.

For the next three lemmas, g will denote a continuously differentiable function on $\mathbb{C}$. Let μ be a finite Borel measure on X. For λ in $\mathbb{C}$ we define the functions $\tilde{\mu}$ and $\hat{\mu}$ by

$$\tilde{\mu}(\lambda) = \int_X \frac{g(z)-g(\lambda)}{z-\lambda}\, d\mu(z) \quad \text{and} \quad \hat{\mu}(\lambda) = \int_X \frac{1}{z-\lambda}\, d\mu(z).$$

(Here $\tilde{\mu}$ depends on the appropriate g considered in context; and $\hat{\mu}$ is the Cauchy transform of μ (see [1], p. 37).)

LEMMA 1. *If $\mu \perp R(X, g)$, then $\tilde{\mu} \equiv 0$ off X.*

Proof. We have $\tilde{\mu}(\lambda) = [g d\mu]\hat{}\,(\lambda) - g(\lambda)\hat{\mu}(\lambda)$. If $\mu \perp R(X, g)$, then $\mu \perp R(X)$ and $g d\mu \perp R(X)$. Thus by Theorem 8.1 of [2], $\hat{\mu}$ and $[g d\mu]\hat{}$ vanish off X.

LEMMA 2. *Except for at most a countable subset of X, $\tilde{\mu}$ is continuous on $\mathbb{C}$.*

Proof. Since μ has finite total variation, it is sufficient to show that $\tilde{\mu}$ is continuous on the set of those λ in $\mathbb{C}$ with $\mu(\{\lambda\}) = 0$. Because g is C^1, $\dfrac{g(z)-g(\lambda)}{z-\lambda}$ is bounded on X for $z \neq \lambda$. Let $\lambda_n \to \lambda_0$ with $\mu(\{\lambda_n\}) = 0$ for $n = 0, 1, \ldots$. Clearly

$$\frac{g(z)-g(\lambda_n)}{z-\lambda_n} \text{ converges pointwise to } \frac{g(z)-g(\lambda_0)}{z-\lambda_0}$$

for $z \neq \lambda_0$. Apply the dominated convergence theorem to complete the proof.

The key needed in the proof of Theorem 1 is the following representation result.

LEMMA 3. *Let Z_1 denote the subset of $\mathbb{C}$ on which $\bar{\partial} g$ vanishes. Let $h \in C_c^1(\mathbb{C})$ with $h \equiv 0$ in a neighborhood of Z_1. There exists a function $f \in C_c^1(C)$ and an entire function q with*

$$\frac{1}{\pi} \int_{\mathbf{C}} \bar{\partial} f(z) \frac{g(z)-g(\lambda)}{z-\lambda}\, dm(z) = h(\lambda) + q(\lambda)$$

for $\lambda \in \mathbb{C}$. (The left hand side of the equality is the Vitushkin operator evaluated at g (see [2], p. 64).)

Proof. Let $\phi \in C_c^1(\mathbb{C})$. Then

$$(*) = \frac{1}{\pi}\int_{\mathbb{C}} \bar\partial\phi(z)\frac{g(z)-g(\lambda)}{z-\lambda}\,dm(z)$$

$$= \left[\bar\partial\phi g\frac{dm}{\pi}\right]^{\wedge}(\lambda) - g(\lambda)\left[\bar\partial\phi\frac{dm}{\pi}\right]^{\wedge}(\lambda)$$

$$= \left[\bar\partial\phi g\frac{dm}{\pi}\right]^{\wedge}(\lambda) + g(\lambda)\phi(\lambda).$$

The last equality can be found in ([2], p. 26). Since

$$\bar\partial\left(\left[\bar\partial\phi g\frac{dm}{\pi}\right]^{\wedge}(\lambda)\right) = -\bar\partial\phi(\lambda)g(\lambda),$$

applying $\bar\partial$ to (*) at λ, we get

$$\bar\partial(*) = \bar\partial g(\lambda)\phi(\lambda).$$

Define

$$f(\lambda) = \begin{cases} 0 & \text{if } \lambda \in Z_1 \\ \dfrac{\bar\partial h(\lambda)}{\bar\partial g(\lambda)} & \text{if } \lambda \notin Z_1\,. \end{cases}$$

The assumption on h ensures that $f \in C_c^1(\mathbb{C})$. The previous computation applied to f gives

$$\bar\partial\left[\frac{1}{\pi}\int_{\mathbb{C}} \bar\partial f(z)\frac{g(z)-g(\lambda)}{z-\lambda}\,dm(z)\right] = \bar\partial h(\lambda).$$

Since the solutions of $\bar\partial F \equiv 0$ in $\mathbb{C}$ are entire functions, the lemma is proved.

We use a duality argument to prove Theorem 1. Note that $g \in R(Z)$ by ([9], p. 160) so one direction is trivial.

Proof of Theorem 1. Let g be continuously differentiable in $\mathbb{C}$. Suppose that μ is a finite Borel measure on X with $\mu \perp R(X, g)$. By Lemma 1, $\tilde\mu \equiv 0$ off X. Since X has empty interior, Lemma 2 says that $\tilde\mu = 0$, except for at most a countable subset of X. Choose any h in $C_c^1(\mathbb{C})$ which vanishes in a neighborhood of Z_1. Then there exists an $f \in C_c^1(\mathbb{C})$ and an entire function q satisfying Lemma 3.

$$\int_X h\,d\mu = \int_X h+q\,d\mu = \int_{\mathbb{C}} \bar\partial f\,\tilde\mu\frac{dm}{\pi} = 0\,.$$

The second equality follows from Fubini's theorem and the third since $\tilde\mu = 0$ almost everywhere with respect to the measure m. Hence $\mu|_{X-Z}$ is the zero measure and the support of μ is contained in Z. Since X has no interior and $\mu|_{X-Z}$ is the zero measure, continuity shows that $\mu \perp R(Z, g)$ and thus $\mu \perp R(Z)$. If we assume that $\mathscr{R}(Z)$ is dense in $C(Z)$, then we must have $\mu|_Z \equiv 0$ also. Hence $\mu \equiv 0$ and $\mathscr{R}(X, g)$ is dense in $C(X)$.

II. For the case in which the interior of X is not empty, $\mathscr{R}(X, g)$ is not dense in $C(X)$, if g is twice continuously differentiable. This follows since $\mathscr{R}(X, g)$ is contained in the solutions of the equation

$$(-\bar{\partial}g)\bar{\partial}^2 F + \bar{\partial}^2 g \bar{\partial} F = 0$$

in the interior of X.

A natural question is whether the module $R(X, g)$ is "local". That is, suppose that $f \in C(X)$ and for every z in X there is a neighborhood $U(z)$ of z with $f \in R(X \cap \overline{U(z)}, g)$. In this case we say that f is "locally" in $R(X, g)$. If f is "locally" in $R(X, g)$, is f in $R(X, g)$? A related question was considered in [**8**]. Again handling the set Z on which $\bar{\partial}g$ vanishes in X seems to be the primary obstacle. While we cannot prove a localization theorem in the generality of section one, we have the following result.

THEOREM 2. *Let $p(\bar{z})$ be a polynomial in $\bar{z}$. Then the module $R(X, p)$ is "local".*

The proof will follow several lemmas. The first is a partial converse of Lemma 1.

LEMMA 4. *Let g be C^1 in a neighborhood of X and let $Z_1 = \{z \in \mathbb{C} : \bar{\partial}g(z) = 0\}$. Assume that Z_1 does not contain any connected components of $\mathbb{C} - X$. If $\tilde{\mu} \equiv 0$ off X, then $\mu \perp R(X, g)$.*

(Note: the hypothesis holds if Z_1 has no interior.)

Proof. Assume that $g \in C_c^1(\mathbb{C})$ and that $\tilde{\mu} \equiv 0$ off X. As before

$$0 = \tilde{\mu}(\lambda) = [g d\mu]\hat{}(\lambda) - g(\lambda)\hat{\mu}(\lambda)$$

for $\lambda \in \mathbb{C} - X$. Differentiating with respect to $\bar{\partial}$ and using the analyticity of the Cauchy transform off X we get $0 = 0 - \bar{\partial}g(\lambda)\hat{\mu}(\lambda)$ for $\lambda \notin X$. Again by analyticity and the hypothesis on Z_1, we see that $\hat{\mu}$, and thus $[g d\mu]\hat{}$, vanish off X. Now by a theorem in ([**2**], p. 46) we have $\mu \perp R(X)$ and $\mu \perp gR(X)$.

LEMMA 5. *Assume that g is C^2 in a neighborhood of X. Let $h \in C_c^2(\mathbb{C})$ with $\dfrac{\bar{\partial}h}{\bar{\partial}g} \in C^1$ in a neighborhood of X. Let μ be a finite Borel measure on X. There exists a finite Borel measure γ with*

(1) *support $\gamma \subset$ support of h and*

(2) $\tilde{\gamma} = h\tilde{\mu}$.

(Here $\tilde{\mu}$ and $\tilde{\gamma}$ are computed with respect to g.)

Proof. By a theorem in ([2], p. 50)

$$\tilde{\mu} = [gd\mu]^{\wedge} - g\hat{\mu} = \left[\bar{\partial} g\hat{\mu}\frac{dm}{\pi}\right]^{\wedge}.$$

This fact will be used repeatedly. Define

$$d\gamma = -\bar{\partial} h\hat{\mu}\frac{dm}{\pi} + hd\mu + \bar{\partial}\left[\frac{\bar{\partial} h}{\bar{\partial} g}\right]\left[\bar{\partial} g\hat{\mu}\frac{dm}{\pi}\right]^{\wedge}\frac{dm}{\pi} - \bar{\partial} h\hat{\mu}\frac{dm}{\pi}.$$

Clearly (1) holds. Computing we derive the following sequence of equalities;

$$\hat{\gamma} = h\hat{\mu} - \frac{\bar{\partial} h}{\bar{\partial} g}\left[\bar{\partial} g\hat{\mu}\frac{dm}{\pi}\right]^{\wedge},$$

$$\bar{\partial} g\hat{\gamma}\, dm = \bar{\partial} gh\hat{\mu}\, dm - \bar{\partial} h\left[\bar{\partial} g\hat{\mu}\frac{dm}{\pi}\right]^{\wedge} dm,$$

$$[\bar{\partial} g\hat{\gamma}\, dm]^{\wedge} = h[\bar{\partial} g\hat{\mu}\, dm]^{\wedge},$$

and so $\tilde{\gamma} = h\tilde{\mu}$.

Proof of Theorem 2. Take g to be the polynomial $p(\bar{z})$. We must show that if $\mu \perp R(X, p)$, then μ annihilates all functions "locally" in $R(X, p)$. Suppose that f is "locally" in $R(X, p)$. For each z in X let $U(z)$ denote an open disc with center z for which $f \in R(X \cap \overline{U(z)}, p)$. If $p(\bar{z})$ is constant then $R(X, p) = R(X)$, which is well known to be local (see [2], p. 51). Otherwise let Z denote the finite subset of X on which $\bar{\partial} p(\bar{z})$ is zero. For values of z in $X - Z$ assume that $U(z)$ does not meet Z. By compactness we need only add a finite number of values of $U(z)$ with z in $X - Z$ to the collection of values of $U(z)$ with z in Z to obtain a finite open covering of X. Relabel this covering by $U_1, \ldots, U_n$. For $i = 1, \ldots, n$ let h_i denote a C^{∞} function, which is identically one in a neighborhood of the center of U_i, and which has compact support $\bar{U}_i$. Define

$$j_m = h_m \Big/ \sum_{i=1}^{n} h_i.$$

By our choice of j_m Lemma 5 applies and we have a measure γ_m on X with

(3) $\hat{\gamma}_m = j_m\tilde{\mu}.$

Hence

$$\sum_{m=1}^{n} \tilde{\gamma}_m = \left(\sum_{m=1}^{n} j_m\right)\tilde{\mu} = \tilde{\mu},$$

so

$$\sum_{m=1}^{n} \gamma_m = \mu.$$

By Lemma 1 and (3) $\tilde{\gamma}_m$ vanishes off $X \cap \bar{U}_m$. Lemma 4 asserts that $\gamma_m \perp R(X \cap \bar{U}_m, p)$. Therefore

$$\int_X f\,d\mu = \sum_{m=1}^{n} \int_X f\,d\gamma_m = 0\,.$$

III. Let $g_1, \ldots, g_n$ be continuous functions on X. It would be interesting to investigate the rational module $\mathscr{R}(X)+\mathscr{R}(X)g_1+\ldots+\mathscr{R}(X)g_n$. The special case when $g_1 = \bar{z}, \ldots, g_n = \bar{z}^n$ was treated in [3] and [6]. The general situation appears difficult even if X has empty interior.

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