Least squares periodic signal modeling using orbits of nonlinear ODE's and fully automated spectral analysis ⋆

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Abstract

Periodic signals can be modeled by means of second-order nonlinear ordinary differential equations (ODE's). The right hand side function of the ODE is parameterized in terms of known basis functions. The least squares algorithm developed for estimating the coefficients of these basis functions gives biased estimates, especially at low signal to noise ratios. This is due to noise contributions to the periodic signal and its derivatives evaluated using finite difference approximations. In this paper a fully automated spectral analysis (ASA) technique is used to eliminate these noise contributions. A simulation study shows that using the ASA technique significantly improves the performance of the least squares estimator.

Key words: Identification; Least squares; Nonlinear systems; Periodic motion; Spectral analysis.

1 Introduction

Modeling of periodic signals is a fundamental problem in many applications. Examples include vibration analysis, overtone analysis in power networks and measurement of linearity in electronic power amplifiers, see (Abd-Elrady 2002, Abd-Elrady 2004, Stoica & Moses 1997). Many systems that generate periodic signals can be described by second-order nonlinear ODE's with polynomial right hand sides. Examples include tunnel diodes, pendulums, negative-resistance oscillators and predator-prey systems, see (Khalil 2002, Perko 1991).

In this paper the periodic signal is modeled as a function of the states of a general second-order ODE, and by introducing a polynomial parameterization of the right hand side of this ODE. Therefore, the approach of this paper is expected to obtain highly accurate models by estimating only a few parameters. Estimators based on Kalman filter and extended Kalman filter (EKF) were developed in (Wigren, Abd-Elrady & Söderström 2003*a*). A least squares (LS) estimation algorithm was derived in (Wigren, Abd-Elrady & Söderström 2003*b*). Also, an algorithm based on the Markov estimate was introduced in (Abd-Elrady, Söderström & Wigren 2004).

As mentioned in (Abd-Elrady et al. 2004, Wigren et al. 2003*b*), the LS estimator is expected to give a biased estimates especially at low signal to noise ratios (SNRs). This is due to two reasons. First, the assumption on the noise to be white is not valid. In (Abd-Elrady et al. 2004) this problem was solved by estimating the noise covariance matrix and implementing a Markov estimate based algorithm. Second, derivatives of the modeled signal evaluated using finite difference approximations are highly contaminated with noise. Hence, the regressors and regressed variable of the linear regression equation are also contaminated with noise, and the problem becomes an error-in-variables (EIV) problem. A weighted total least squares (TLS) algorithm was studied in (Abd-Elrady et al. 2004) for this EIV problem but it did not lead to improvement compared to the LS estimate at moderate SNRs.

In this paper the automated spectral analysis (ASA) technique, see (Pintelon & Schoukens 2001, Schoukens, Rolain, Simon & Pintelon 2003), is used to eliminate the noise contribution in the modeled signal and evaluating the signal derivatives in the frequency domain. This is expected to avoid noise amplification in the differentia-

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tion phase extending the operating region of the LS estimation algorithm toward low SNRs.

The paper is organized as follows. Section 2 gives the details on the model. Section 3 introduces the LS estimation algorithm. The ASA technique is presented in Section 4. Section 5 gives a comparative simulation study between the LS algorithm using the ASA technique (LS-ASA) and the LS algorithm using finite difference approximation. Conclusions appear in Section 6.

2 The model

2.1 Measurements

The discrete time measured signal $u(kh)$ is given by

$$u(kh) = y(kh) + e(kh), \tag{1}$$

where $y(t)$ is the continuous time *periodic* signal to be modeled, $y(kh)$ its sampled value, $e(kh)$ is the discrete time measurement noise and h the sampling interval.

2.2 Model Structures

The idea here is to model the generation of the signal $y(t)$ by means of an unknown parameter vector $\boldsymbol{\theta}$ an a nonlinear ODE, i.e.

$$\dot{\boldsymbol{x}} = f(\boldsymbol{x}, \boldsymbol{\theta}), \tag{2}$$
$$y = h(\boldsymbol{x}). \tag{3}$$

As shown in (Wigren et al. 2003*b*), and proved rigorously in (Wigren & Söderström 2003), it can often be assumed that the second order ODE

$$\ddot{y}(t) = f\big(y(t), \dot{y}(t), \boldsymbol{\theta}\big) \tag{4}$$

generates the periodic signal that is measured. Thus choosing the state variables as follows

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} y(t) \\ \dot{y}(t) \end{pmatrix}, \tag{5}$$

the model given in (2)-(3) becomes

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} x_2(t) \\ f\big(x_1(t), x_2(t), \boldsymbol{\theta}\big) \end{pmatrix}, \tag{6}$$

$$y(t) = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}. \tag{7}$$

This model depends only on the parameters of the second right hand side function of (6), a fact that should be advantageous from a computational and performance point of view, see (Wigren et al. 2003*b*) for more details.

2.3 Parameterization

The right hand side of the second state equation of (6) is expanded in terms of known basis functions, modeling the right hand side as a truncated superposition of these functions. In case of a polynomial model, a suitable parameterization is

$$f\big(x_1(t), x_2(t), \boldsymbol{\theta}\big) = \sum_{l=0}^{L} \sum_{m=0}^{M} \theta_{l,m} x_1^l(t) x_2^m(t), \tag{8}$$

$$\boldsymbol{\theta} = \begin{pmatrix} \theta_{0,0} \cdots \theta_{0,M} \cdots \theta_{L,0} \cdots \theta_{L,M} \end{pmatrix}^T. \tag{9}$$

3 The least squares algorithm

In order to formulate the model in a linear regression form, note that (6), (8) and (9) result in the model

$$\dot{x}_1(t) = x_2(t), \tag{10}$$

$$\dot{x}_2(t) = \boldsymbol{\phi}^T\big(x_1(t), x_2(t)\big)\ \boldsymbol{\theta}, \tag{11}$$

where

$$\begin{aligned} &\boldsymbol{\phi}^T\big(x_1(t), x_2(t)\big) = \\ &\begin{pmatrix} 1 & \cdots & x_2^M(t) & \cdots & x_1^L(t) & \cdots & x_1^L(t) x_2^M(t) \end{pmatrix}. \end{aligned} \tag{12}$$

To estimate the parameter vector $\boldsymbol{\theta}$ from (11), some approximations are needed. Since $x_1(t)$, $x_2(t)$ and $\dot{x}_2(t)$ are not known, their estimates should be used. In this case, the second state equation (11) results in (at $t = kh$)

$$\widehat{\dot{x}}_2(kh) = \boldsymbol{\phi}^T\big(\widehat{x}_1(kh), \widehat{x}_2(kh)\big)\ \boldsymbol{\theta} + \varepsilon(kh). \tag{13}$$

The expression (13) follows by performing a Taylor series expansion of the regression vector $\boldsymbol{\phi}^T\big(x_1(kh), x_2(kh)\big)$ around $\big(\widehat{x}_1(kh) \quad \widehat{x}_2(kh)\big)^T$. In (13) the combined regression error, $\varepsilon(kh)$, has been introduced. It can not be expected to be white since the Taylor series expansion in (13) produces sum of noise samples that are measured at different sampling times. This is because the Taylor series terms will contain differences between $\widehat{x}_1(kh)$, $\widehat{x}_2(kh)$ and $x_1(kh)$, $x_2(kh)$, respectively, *i.e.* $e(kh)$, $e(kh-h)$ and/or $e(kh+h)$ depending on which derivative approximation is used. This means that the LS estimator will, in the end, be biased as mentioned in Section 2. However, for good estimates of $x_1(kh)$, $x_2(kh)$ and $\dot{x}_2(kh)$, and/or relatively high SNRs the accuracy is expected to be good.

Assuming that data are available at times $kh - Nh, \cdots, kh$ and defining the vectors and matrices

$$\boldsymbol{Z}_N = \begin{pmatrix} \widehat{\dot{x}}_2(kh) & \cdots & \widehat{\dot{x}}_2(kh - Nh) \end{pmatrix}^T, \tag{14}$$

$$\mathbf{\Phi}_N = \begin{pmatrix} \boldsymbol{\phi}^T\left(\widehat{x}_1(kh), \widehat{x}_2(kh)\right) \\ \vdots \\ \boldsymbol{\phi}^T\left(\widehat{x}_1(kh-Nh), \widehat{x}_2(kh-Nh)\right) \end{pmatrix}, \quad (15)$$

$$\boldsymbol{\varepsilon}_N = \Big(\varepsilon(kh) \; \cdots \; \varepsilon(kh-Nh) \Big)^T, \quad (16)$$

the following regression equation results

$$\boldsymbol{Z}_N = \mathbf{\Phi}_N \, \boldsymbol{\theta} + \boldsymbol{\varepsilon}_N. \quad (17)$$

Hence the LS estimate $\widehat{\boldsymbol{\theta}}_N^{LS}$ is given by, see (Söderström & Stoica 1989)

$$\widehat{\boldsymbol{\theta}}_N^{LS} = (\mathbf{\Phi}_N^T \mathbf{\Phi}_N)^{-1} \, \mathbf{\Phi}_N^T \, \boldsymbol{Z}_N. \quad (18)$$

The LS estimate (18) has been studied in (Wigren et al. 2003*b*) using $\widehat{x}_1(kh) = u(kh)$ in addition to $\widehat{x}_2(kh)$ and $\widehat{\dot{x}}_2(kh)$ evaluated using finite difference approximation. It is shown in (Wigren et al. 2003*b*) that the LS algorithm gives considerably accurate models at high SNRs and further research is needed to extend the operating region toward low SNRs. This is because of noise amplification during the differentiation process. In this paper the ASA technique (Schoukens et al. 2003) is used to find more accurate estimates for the noise free periodic signal $x_1(t)$ and its derivatives $x_2(t)$ and $\dot{x}_2(t)$. An accurate estimate $\widehat{x}_1(t)$ is found by eliminating noise corruption on the modeled signal and more accurate estimates $\widehat{x}_2(t)$ and $\widehat{\dot{x}}_2(t)$ are found by differentiating $\widehat{x}_1(t)$ in the frequency domain. The ASA technique is discussed in the next section.

4 The fully automated spectral analysis

In order to estimate the spectrum of the periodic signal using the ASA technique the data record should contain at least 2 full periods + 48 samples (The 48 samples are needed to account for the shift in the FIR filter used in the algorithm). The spectrum of the signal will be calculated up to $0.4 f_s$ ($f_s = 1/h$) with a relative error smaller than 10^{-5} of the peak value of the spectrum. The algorithm is explained in all details in (Schoukens et al. 2003). Here only a brief introduction is given.

Consider the periodic signal $u(t)$ given in (1) with basic frequency $f_0 = 1/T$:

$$u(t) = \sum_{r=-F}^{F} U_r \, e^{j2\pi r f_0 t} \quad (19)$$

sampled at the time instances $t = kh$, with $f_0 F \leq 0.4 f_s$, $U_r = \bar{U}_{-r}$ is the Fourier coefficient of the r^{th} component (where $\bar{\cdot}$ denotes the complex conjugate). Note that T/h is not required to be a rational number. N equidistant measurements of this signal $u(kh)$ are made over more than 2 periods once the transients disappeared (the steady state solution is reached). Under these conditions the ASA technique allows to obtain estimates $\widehat{U}_r$ of the Fourier coefficients U_r, together with an estimate of the variance

$$\sigma_U^2(r) = \mathbf{E}\left[\, (\widehat{U}_r - \mathbf{E}[\widehat{U}_r])\overline{(\widehat{U}_r - \mathbf{E}[\widehat{U}_r])} \, \right], \quad (20)$$

where $\mathbf{E}$ is the expectation operator. The procedure consists of two parts. In a first step, an initial estimate $\widehat{T}_0$ of the period length T is made using correlation methods. In a second step this initial estimate is improved by minimizing a cost function $V(T)$ (cf. (23)), and eventually the corresponding Fourier coefficients are calculated using an FFT.

4.1 Initial estimate $\widehat{T}_0$

The initial estimate of the period length is based on the autocorrelation $R_{uu}(\tau)$ of $u(t)$. The basic idea is to detect the distance between successive peaks in $R_{uu}(\tau)$. If a wide band signal with a flat amplitude spectrum is analyzed, this simple method gives a good estimate. However it fails in practice for a number of special cases. Since the method should be robust, it is refined in (Schoukens et al. 2003) to deal also with beat signals.

4.2 Improved estimate of the period length

The improved period will be obtained by minimizing a cost function $V(T)$, that is defined below step by step.

1- Assume that the period of the signal is T (to be estimated). From the initial estimate we know that the measurements cover more than $\mathcal{M} \in \mathbb{N}$ periods of the signal.

2- In the next step, we interpolate the samples with an equidistant grid with $\mathcal{L}$ samples per period, such that we get $\mathcal{ML} = 2^n$ points in the processed record (fast FFT calculations). The number of data points is also chosen high enough to avoid aliasing, $\mathcal{L} > 2.5F$. The interpolated signal

$$\widehat{z}(q, T), \; q = 0, \cdots, \mathcal{ML} - 1 \quad (21)$$

is an estimate of $u(qT/\mathcal{L})$. It is calculated starting from the measurements $u(kh)$ using classical upsampling (Crochiere & Rabiner 1983) and interpolation techniques (Rolain, Schoukens & Vandersteen 1998).

3- Calculate the DFT spectrum (using the FFT)

$$\widehat{\mathcal{Z}}(s, T) = \frac{1}{\mathcal{ML}} \sum_{q=0}^{\mathcal{ML}-1} \widehat{z}(q, T) \, e^{-j2\pi \frac{sq}{\mathcal{ML}}}. \quad (22)$$

4- Define the cost function

$$V(T) = \frac{\sum_{s=1}^{\mathcal{L}} \left(|\widehat{\mathcal{Z}}(\mathcal{M}s+1,T)|^2 + |\widehat{\mathcal{Z}}(\mathcal{M}s-1,T)|^2 \right)}{\sum_{s=1}^{\mathcal{L}} |\widehat{\mathcal{Z}}(\mathcal{M}s,T)|^2}. \quad (23)$$

This can be interpreted as the ratio of the power on the nonexcited frequency lines to that on the excited lines.

5- Define the estimate $\widehat{T}$ as

$$\widehat{T} = \arg\min_{T} V(T). \quad (24)$$

Remark 1: The idea behind the choice of the cost function in (23) is simple: if there is no leakage any more (the period fits perfectly), all the power of the signal should be at the multiples $\mathcal{M}s$. If there is leakage, a fraction of this power leaks to the neighboring lines. The cost function measures the ratio of this fraction and the algorithm minimizes it.

Remark 2: The minimization problem in (24) is nonlinear in T. A nonlinear line search is used, that is initialized from $\widehat{T}_0$. Since the cost function has many local minima, the search is split in a coarse search, scanning the cost function around the initial guess, followed by a fine search (based on parabolic interpolation) to get eventually the final estimate.

4.3 Estimation of the Fourier coefficients and their variance

Once an estimate $\widehat{T}$ is available, the full record is resampled according to this period length and split in $\mathcal{M}$ subrecords. For each subrecord the DFT spectrum is calculated. The final estimates of the Fourier coefficients and their variance are then obtained as the sample mean and sample variance of the spectra of these subrecords.

Remark 3: It can be noted that the sensitivity of the algorithm to the noise is very low and seems to be almost the same as that of a classical DFT (FFT) approach where the period length would be known a priori. So, almost no price is paid for the fact that the basic period had to be extracted from the data instead of being given a priori.

4.4 Estimation of $x_1(t)$, $x_2(t)$ and $\dot{x}_2(t)$

Once the fundamental Fourier coefficients of the periodic signal are estimated, the estimates of the noise free periodic signal, $\widehat{x}_1(t)$, and its derivatives, $\widehat{x}_2(t)$ and $\widehat{\dot{x}}_2(t)$, are evaluated as

$$\widehat{x}_1(t) = \sum_{r=-F}^{F} \widehat{U}_r \, e^{j\frac{2\pi rt}{\widehat{T}}}, \quad (25)$$

$$\widehat{x}_2(t) = \sum_{r=-F}^{F} j\frac{2\pi r}{\widehat{T}} \widehat{U}_r \, e^{j\frac{2\pi rt}{\widehat{T}}}, \quad (26)$$

$$\widehat{\dot{x}}_2(t) = \sum_{r=-F}^{F} \left(j\frac{2\pi r}{\widehat{T}} \right)^2 \widehat{U}_r \, e^{j\frac{2\pi rt}{\widehat{T}}}. \quad (27)$$

Remark 4: Note that the LS-ASA algorithm is *not* more efficient than the LS algorithm due to the internal upsampling. This is because the cutoff frequency of the filter used is at $0.4f_s$, so almost the complete frequency band passes. The reason why the LS-ASA is more efficient is twofold:

- Only one frequency line in $\mathcal{M}$ lines is used, the others are put to zeros ($\mathcal{M}$ periods measured). This leads to a reduction of the noise power by $\mathcal{M}$.
- Only the significant Fourier coefficients are used in the estimates $\widehat{x}_1(t)$, $\widehat{x}_2(t)$ and $\widehat{\dot{x}}_2(t)$. Hence, an enormous noise reduction is obtained.

5 Numerical examples

Example 1: The Van der Pol oscillator (Khalil 2002) was selected as the underlying system in this example. The oscillator is described by

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -x_1 + \varepsilon(1-x_1^2)x_2 \end{pmatrix}. \quad (28)$$

The Matlab routine `ode45` was used to solve (28) for $\varepsilon = 2$. The initial state of (28) was selected as $\left(x_1(0) \quad x_2(0)\right)^T = (0 \quad 1)^T$. All results below are based on data runs of length $N = 10^4$ with a sampling interval $h = 0.1$ s. The measured signal $u(t)$ selected as the first state with white noise added. The differentiated signals $\widehat{x}_2(t)$ and $\widehat{\dot{x}}_2(t)$ for the LS estimation algorithm were obtained using first order difference (Euler center approximation) with $\widehat{x}_1(t) = u(t)$. On the other hand, these signals were evaluated using (25)-(27) for the LS-ASA estimation algorithm.

In this example the estimated model used second degree polynomials ($L = M = 2$). Both the LS and the LS-ASA algorithms were run for different SNRs. As a measure of performance,

$$V = \frac{\|\widehat{\boldsymbol{\theta}}_N - \boldsymbol{\theta}^o\|_2}{\|\boldsymbol{\theta}^o\|_2} \quad (29)$$

was computed and plotted as a function of the SNR in Fig. 1. In (29), $\boldsymbol{\theta}^o$ denotes the true parameter vector. The phase plots for the estimated model at SNR of 20 dB in addition to the true system are given in Fig. 2. Also, the estimated periodic signal compared to the true signal are given in Fig. 3. The LS algorithm using Euler center approximation did not give stable limit cycles for

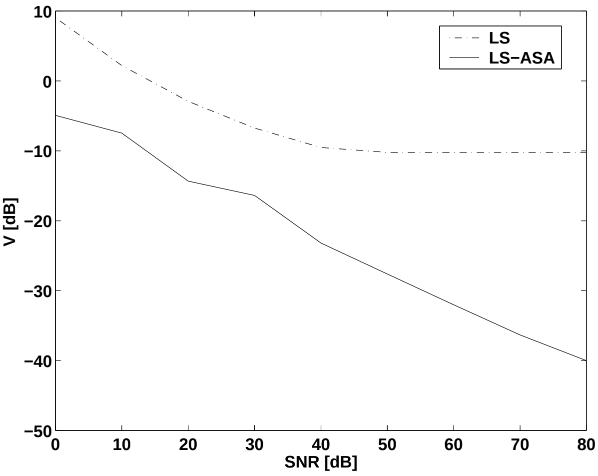

Fig. 1. Comparison between LS-ASA estimates and LS estimates for different SNRs.

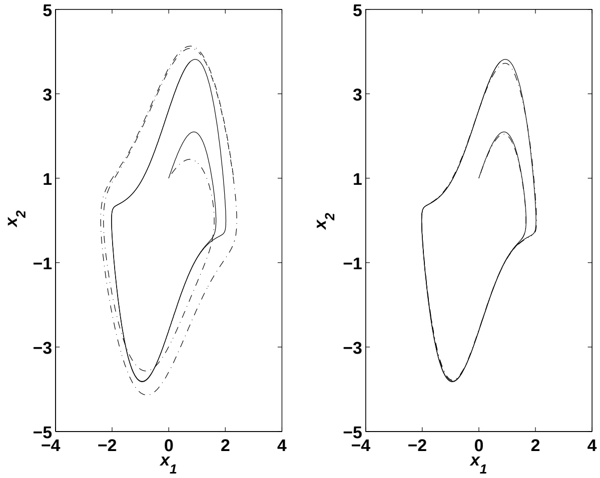

Fig. 2. True (solid) and estimated (dashed) phase plots. LS estimate (left) and LS-ASA estimate (right) [SNR=20 dB].

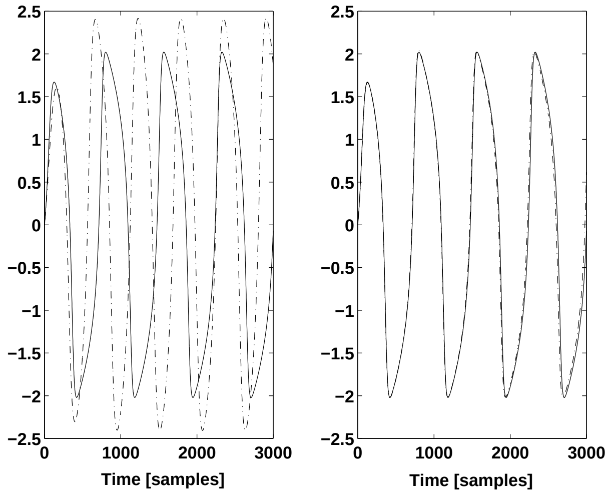

Fig. 3. True (solid) and estimated (dashed) signals. LS estimate (left) and LS-ASA estimate (right) [SNR=20 dB].

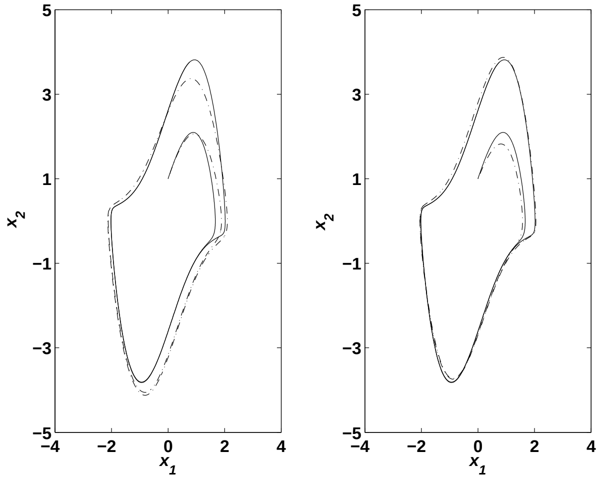

Fig. 4. True (solid) and estimated LS-ASA (dashed) phase plots. 0 dB (left) and 10 dB (right).

SNRs below 20 dB. The LS-ASA algorithm still gives good models for low SNRs as shown in Figures 4-5.

It can be concluded from Figures 1-5 that the LS-ASA estimation algorithm gives significantly better estimates than the LS algorithm using finite difference approximation.

Example 2: In this example the suggested approach of this paper is used to model a piece of a bell sound extracted from a CD in .wav format with a sampling frequency of 22.05 kHz. The LS-ASA algorithm introduced was applied to model 200 samples of the acoustic signal. The model of (6)-(9) with $L = M = 2$ was used. The following model was obtained:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -6.604 \times 10^7 x_1 + 0.306 \times 10^7 x_1^2 \end{pmatrix}. \quad (30)$$

The real data, the model output, the true and the estimated phase plots are given in Fig 6. The results of Fig. 6 show that the approach of this paper models the real acoustic signal very well.

6 Conclusions

A LS estimation algorithm based on fully automated spectral analysis (ASA) technique has been introduced for the modeling of periodic signals using a second-order nonlinear ODE model. The ASA technique removes the noise contributions on the modeled signal and its derivatives. The suggested algorithm results in significantly improved performance as compared to the LS estimation algorithm using finite difference approximation.

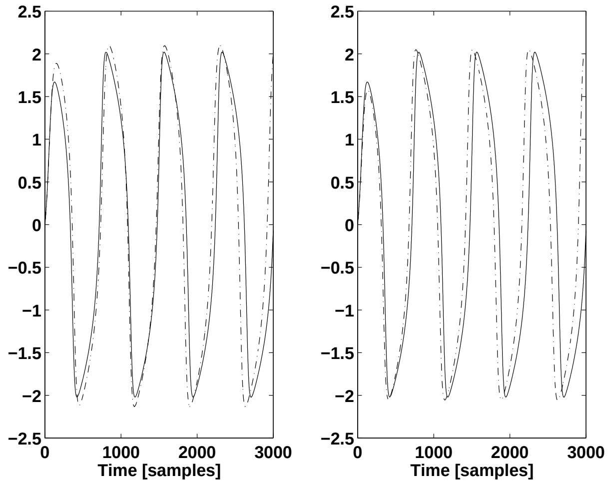

Fig. 5. True (solid) and estimated LS-ASA (dashed) signals. 0 dB (left) and 10 dB (right).

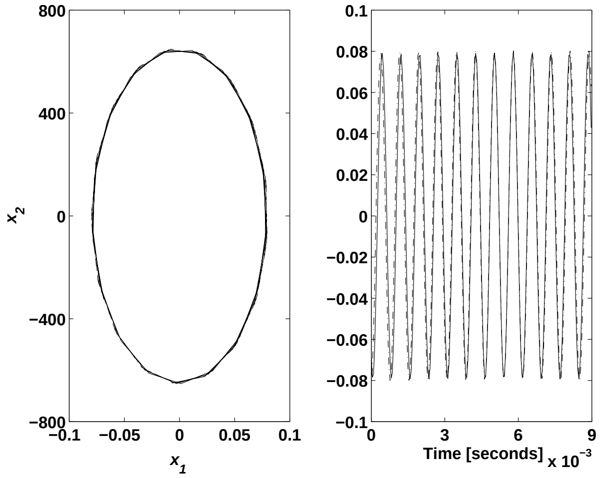

Fig. 6. Real data (solid) and estimated model (dashed). Phase plots (left) and signals (right).

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