# INFINITE DIMENSIONAL TILTING MODULES AND COTORSION PAIRS

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*Dedicated to Claus Michael Ringel on his 60th birthday*

Classical tilting theory generalizes Morita theory of equivalence of module categories. The key property – existence of category equivalences between large full subcategories of the module categories – forces the representing tilting module to be finitely generated.

However, some aspects of the classical theory can be extended to infinitely generated modules over arbitrary rings. In this paper, we will consider such an aspect: the relation of tilting to approximations (preenvelopes and precovers) of modules. As an application, we will present recent connections between tilting theory of infinitely generated modules and the finitistic dimension conjectures.

General existence theorems provide a big supply of approximations in the category Mod-$R$ of all modules over an arbitrary ring $R$. However, the corresponding approximations may not be available in the subcategory of all finitely generated modules. So the usual sharp distinction between finitely and infinitely generated modules becomes unnatural, and even misleading.

A convenient tool for the study of module approximations is the notion of a cotorsion pair. Tilting cotorsion pairs are defined as the cotorsion pairs induced by tilting modules. We will present their characterization among all cotorsion pairs, and then apply it to a classification of tilting classes in particular cases (e.g., over Prüfer domains). The point of the classification is that in the particular cases, the tilting classes are of finite type. This means that we can replace the single infinitely generated tilting module by a set of finitely presented modules; the tilting class is then axiomatizable in the language of the first order theory of modules.

Most of this paper is a survey of recent developments. We give complete definitions and statements of the results, but most proofs are omitted or replaced by outlines of the main ideas. For full details, we refer to the original papers listed in the references, or to the forthcoming monograph [51]. However, Theorems 3.4, 3.7, 4.14, and 4.15 are new, hence presented with full proofs.

In §1, we introduce cotorsion pairs and their relations to approximation theory of infinitely generated modules over arbitrary rings. In §2 and §3, we study infinitely generated tilting and cotilting modules, and characterize the induced tilting and cotilting cotorsion pairs. §4 deals with tilting classes of finite type and cotilting classes of cofinite type, and with their classification over particular rings. Finally, §5 relates tilting approximations to the first and second finitistic dimension conjectures.

We start by fixing our notation. For an (associative, unital) ring $R$, Mod-$R$ denotes the category of all (right $R$-) modules. mod-$R$ denotes the subcategory of Mod-$R$ formed by all modules possessing a projective resolution consisting of finitely generated modules. (If $R$ is a right coherent ring then mod-$R$ is just the category of all finitely presented modules).

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Let $\mathcal{C}$ be a class of modules. For a cardinal $\kappa$, we denote by $\mathcal{C}^{<\kappa}$, and $\mathcal{C}^{\leq\kappa}$, the subclass of $\mathcal{C}$ consisting of the modules possessing a projective resolution containing only $<\kappa$-generated, and $\leq\kappa$-generated, modules, respectively. For example, mod-$R$ = (Mod-$R$)$^{<\omega}$. Further, $\varinjlim \mathcal{C}$ denotes the class of all modules that are direct limits of modules from $\mathcal{C}$. (In general, $\varinjlim \mathcal{C}$ is not closed under direct limits, but it is in case $\mathcal{C} \subseteq$ mod-$R$. In that case, $\mathcal{C} = \varinjlim \mathcal{C} \cap$ mod-$R$ provided $\mathcal{C}$ is closed under finite direct sums and direct summands.)

Let $n < \omega$. We denote by $\mathcal{P}_n$ ($\mathcal{I}_n$, $\mathcal{F}_n$) the class of all modules of projective (injective, flat) dimension $\leq n$. Further, $\mathcal{P}$ ($\mathcal{I}$, $\mathcal{F}$) denotes the class of all modules of finite projective (injective, flat) dimension. The injective hull of a module $M$ is denoted by $E(M)$.

We denote by $\mathbb{Z}$ the ring of all integers, and by $\mathbb{Q}$ the field of all rational numbers. For a commutative domain $R$, $Q$ denotes the quotient field of $R$.

For a left $R$-module $N$, we denote by $N^* = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Q}/\mathbb{Z})$ the *character module* of $N$. Note that $N^*$ is a (right $R$-) module.

Let $M$ be a module. Then $M$ is a *dual module* provided that $M = N^*$ for a left $R$-module $N$. $M$ is *pure-injective* provided that $M$ is a direct summand in a dual module. $M$ is (Enochs) *cotorsion* provided that $\mathrm{Ext}^1_R(M, F) = 0$ for each $F \in \mathcal{F}_0$. Notice that any dual module is pure-injective, and any pure-injective module is cotorsion (The converses do not hold in general; however, flat cotorsion modules over left coherent rings are pure-injective, [81]). The class of all pure-injective, and cotorsion, modules is denoted by $\mathcal{PI}$, and $\mathcal{EC}$, respectively.

A module $M$ is *divisible* if $\mathrm{Ext}^1_R(R/rR, M) = 0$ for each $r \in R$, and *torsion-free* if $\mathrm{Tor}^R_1(M, R/Rr) = 0$ for each $r \in R$ (If $R$ is a commutative domain, then these notion coincide with the classical ones). The class of all divisible and torsion-free modules is denoted by $\mathcal{DI}$ and $\mathcal{TF}$, respectively.

## 1. Cotorsion pairs and approximations of modules

Cotorsion pairs are analogs of (non-hereditary) torsion pairs, with Hom replaced by Ext. They were introduced by Salce (under the name "cotorsion theories") in [69]. The analogy with the well-known torsion pairs makes it possible to derive easily some basic notions and facts about cotorsion pairs. However, the main point concerning cotorsion pairs is their close relation to special approximations of modules: cotorsion pairs provide a homological tie between the dual notions of a special preenvelope and a special precover. This tie (discovered in [69], cf. 1.8.3) is a sort of remedy for the non-existence of a duality in Mod-$R$.

Before introducing cotorsion pairs, we define various Ext-orthogonal classes.

Let $\mathcal{C} \subseteq$ Mod-$R$. Define $\mathcal{C}^\perp = \bigcap_{n<\omega} \mathcal{C}^{\perp_n}$ where $\mathcal{C}^{\perp_n} = \{M \in \text{Mod-}R \mid \mathrm{Ext}^n_R(C, M) = 0 \text{ for all } C \in \mathcal{C}\}$ for each $n < \omega$. Dually, let ${}^\perp\mathcal{C} = \bigcap_{n<\omega} {}^{\perp_n}\mathcal{C}$ where ${}^{\perp_n}\mathcal{C} = \{M \in \text{Mod-}R \mid \mathrm{Ext}^n_R(M, C) = 0 \text{ for all } C \in \mathcal{C}\}$ for each $n < \omega$.

**1.1. Cotorsion pairs.** Let $R$ be a ring. A *cotorsion pair* is a pair $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ of classes of modules such that $\mathcal{A} = {}^{\perp_1}\mathcal{B}$ and $\mathcal{B} = \mathcal{A}^{\perp_1}$. The class $\mathcal{A} \cap \mathcal{B}$ is called the *kernel* of $\mathfrak{C}$. The cotorsion pair $\mathfrak{C}$ is *hereditary* provided that $\mathrm{Ext}^i_R(A, B) = 0$ for all $A \in \mathcal{A}$, $B \in \mathcal{B}$ and $i \geq 2$.

Notice that $\mathfrak{C}$ is hereditary iff $\mathcal{A} = {}^\perp\mathcal{B}$ and $\mathcal{B} = \mathcal{A}^\perp$. The property of $\mathfrak{C}$ being hereditary can easily be expressed in terms of the properties of $\mathcal{A}$ and $\mathcal{B}$: $\mathfrak{C}$ is hereditary iff $\mathcal{A}$ is closed under kernels of epimorphisms iff $\mathcal{B}$ is closed under cokernels of monomorphisms.

Each module $M$ in the kernel of a cotorsion pair $\mathfrak{C}$ is a *splitter*, that is, $M$ satisfies $\mathrm{Ext}^1_R(M, M) = 0$. We will see that the kernel of $\mathfrak{C}$ in the tilting and cotilting cases plays an important role: it determines completely the classes $\mathcal{A}$ and $\mathcal{B}$. (This contrasts with what happens for torsion pairs: since $\mathrm{id}_M \in \mathrm{Hom}_R(M, M)$ for each module $M$, the "kernel" of any torsion pair is trivial.)

**1.2.** By changing the category, we could take a complementary point of view, working modulo the kernel rather than stressing its role. By a result of Beligiannis and Reiten [24], each complete hereditary cotorsion pair $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ in Mod-$R$ determines a torsion pair, $(\underline{\mathcal{A}}, \underline{\mathcal{B}})$, in the stable module category $\underline{\text{Mod-}R}$ (of Mod-$R$ modulo the kernel of $\mathfrak{C}$), cf. 1.8.3. Consequently, special $\mathcal{A}$-precovers and special $\mathcal{B}$-preenvelopes are functorial modulo maps factoring through the kernel, cf. [61].

The class of all cotorsion pairs is partially ordered by inclusion in the first component: $(\mathcal{A}, \mathcal{B}) \leq (\mathcal{A}', \mathcal{B}')$ iff $\mathcal{A} \subseteq \mathcal{A}'$. The $\leq$-least cotorsion pair is $(\mathcal{P}_0, \text{Mod-}R)$, the $\leq$-greatest $(\text{Mod-}R, \mathcal{I}_0)$; these are the *trivial* cotorsion pairs.

The cotorsion pairs over a ring $R$ form a complete lattice, $\mathfrak{L}_R$: given a sequence of cotorsion pairs $\mathcal{S} = ((\mathcal{A}_i, \mathcal{B}_i) \mid i \in I)$, the infimum of $\mathcal{S}$ in $\mathfrak{L}_R$ is $(\bigcap_{i\in I} \mathcal{A}_i, (\bigcap_{i\in I} \mathcal{A}_i)^{\perp_1})$, the supremum being $({}^{\perp_1}(\bigcap_{i\in I} \mathcal{B}_i), \bigcap_{i\in I} \mathcal{B}_i)$.

For any class of modules $\mathcal{C}$, there are two cotorsion pairs associated with $\mathcal{C}$: $({}^{\perp_1}\mathcal{C}, ({}^{\perp_1}\mathcal{C})^{\perp_1})$, called the cotorsion pair *generated* by $\mathcal{C}$, and $({}^{\perp_1}(\mathcal{C}^{\perp_1}), \mathcal{C}^{\perp_1})$, the cotorsion pair *cogenerated* by $\mathcal{C}$. If $\mathcal{C}$ has a representative set of elements $\mathcal{S}$, then the first cotorsion pair is generated by the single module $\prod_{S\in\mathcal{S}} S$, while the second is cogenerated by the single module $\bigoplus_{S\in\mathcal{S}} S$.

The existence of cotorsion pairs generated and cogenerated by any class of modules indicates that $\mathfrak{L}_R$ is a large class in general.

For example, the condition of all cotorsion pairs being trivial is extremely restrictive: by [74] and [39], for a right hereditary ring $R$, this condition holds iff $R = S$ or $R = T$ or $R$ is the ring direct sum $S \boxplus T$, where $S$ is semisimple artinian and $T$ is Morita equivalent to a $2 \times 2$-matrix ring over a skew-field. As another example, consider the case of $R = \mathbb{Z}$: by [49], any partially ordered set embeds in $\mathfrak{L}_{\mathbb{Z}}$; in particular, $\mathfrak{L}_{\mathbb{Z}}$ is a proper class.

**1.3.** Replacing Ext by Tor in 1.1, we can define a *Tor-torsion pair* as the pair $(\mathcal{A}, \mathcal{B})$ where $\mathcal{A} = \{A \in \text{Mod-}R \mid \mathrm{Tor}_1^R(A, B) = 0 \text{ for all } B \in \mathcal{B}\}$ and $\mathcal{B} = \{B \in R\text{-Mod} \mid \mathrm{Tor}_1^R(A, B) = 0 \text{ for all } A \in \mathcal{A}\}$. Similarly to the case of cotorsion pairs, we can define Tor-torsion pairs generated (cogenerated) by a class of left (right) $R$-modules. Tor-torsion pairs over a ring $R$ form a complete lattice; by 1.4.3 below, the cardinality of this lattice is $\leq 2^{2^\kappa}$ where $\kappa = \mathrm{card}(R) + \aleph_0$.

The well-known Ext-Tor relations yield an embedding of the lattice of Tor-torsion pairs into $\mathfrak{L}_R$ as follows: a Tor-torsion pair $(\mathcal{A}, \mathcal{B})$ is mapped to the cotorsion pair $(\mathcal{A}, \mathcal{A}^{\perp_1})$. The latter cotorsion pair is easily seen to be generated by the class $\{B^* \mid B \in \mathcal{B}\}$. In this way, Tor-torsion pairs are identified with particular cotorsion pairs generated by classes of pure-injective modules.

The following lemma says that most of the classes of modules defined above occur as first or second components of cotorsion pairs cogenerated by sets:

**Lemma 1.4.** *Let $R$ be a ring and $n < \omega$. Let $\kappa = card(R) + \aleph_0$.*

1. *$\mathfrak{C} = (\mathcal{P}_n, (\mathcal{P}_n)^\perp)$ is a hereditary cotorsion pair cogenerated by $\mathcal{P}_n^{\leq\kappa}$. If $R$ is right noetherian then $\mathfrak{C}$ is cogenerated by $\mathcal{P}_n^{\leq\omega}$.*
2. *Let $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a cotorsion pair generated by a class of pure-injective modules. Then $\mathfrak{C}$ is cogenerated by $\mathcal{A}^{\leq\kappa}$.*
3. *Let $(\mathcal{A}, \mathcal{B})$ be a Tor-torsion pair. Then $(\mathcal{A}, \mathcal{A}^{\perp_1})$ is a cotorsion pair cogenerated by $\mathcal{A}^{\leq\kappa}$, and generated by $\{B^* \mid B \in \mathcal{B}\}$. In particular, $(\mathcal{F}_n, (\mathcal{F}_n)^\perp)$ is a hereditary cotorsion pair cogenerated by $\mathcal{F}_n^{\leq\kappa}$.*

(4) $({}^{\perp}\mathcal{I}_n, \mathcal{I}_n)$ *is a hereditary cotorsion pair cogenerated by* $({}^{\perp}\mathcal{I}_n)^{\leq \lambda}$ *where* $\lambda$ *is the least infinite cardinal such that each right ideal of* $R$ *is* $\lambda$*-generated.*

(5) *Let* $R$ *be a right noetherian ring. Then the cotorsion pair cogenerated by* $\mathcal{I}_n$ *is cogenerated by a set.*

(6) $({}^{\perp_1}\mathcal{DI}, \mathcal{DI})$ *and* $(\mathcal{TF}, \mathcal{TF}^{\perp_1})$ *are cogenerated by sets of cardinality* $\leq \kappa$.

Proof. 1.[1] For $n = 0$, we apply the classical result of Kaplansky saying that each projective module is a direct sum of the countably generated ones. For $n \geq 1$, it suffices to prove that for all $M \in \mathcal{P}_n$ and $0 \neq x \in M$ there is a submodule $N \subseteq M$ such that $N, M/N \in \mathcal{P}_n$, $\operatorname{card}(N) \leq \kappa$, and $x \in N$. This is proved by a back and forth argument in a fixed projective resolution of $M$, see [1]. The noetherian case is similar, cf. [72].

2. This is proved in [40].

3. The first statement follows by part 2. and by 1.3. The second is a particular case of the first one.

4. By Baer test lemma for injectivity, we have $M \in \mathcal{I}_n$ iff $\operatorname{Ext}^1_R(N, M) = 0$ where $N$ runs over a representative set of all $n$-th syzygies of cyclic modules.

5. Since $R$ is right noetherian, there is a cardinal $\mu$ such that any injective module is a direct sum of $\leq \mu$-generated modules, and the proof proceeds in a dual way to 1., see [1].

6. The first cotorsion pair is cogenerated by the set $\{R/rR \mid r \in R\}$. The assertion concerning the second pair is a particular case of 3. □

The key property of cotorsion pairs is their relation to approximations of modules. The connection is via the notion of a special approximation, [81]:

**1.5. Special approximations.** Let $R$ be a ring, $M$ a module and $\mathcal{C}$ a class of modules. An $R$-homomorphism $f : M \to C$ is a *special* $\mathcal{C}$*-preenvelope* of $M$ provided that $f$ induces a short exact sequence $0 \to M \xrightarrow{f} C \to D \to 0$ with $C \in \mathcal{C}$ and $D \in {}^{\perp_1}\mathcal{C}$. $\mathcal{C}$ is a *special preenveloping* class if each module $M \in$ Mod-$R$ has a special $\mathcal{C}$-preenvelope.

Dually, an $R$-homomorphism $g : C \to M$ is a *special* $\mathcal{C}$*-precover* of $M$ provided that $g$ induces a short exact sequence $0 \to B \to C \xrightarrow{g} M \to 0$ with $C \in \mathcal{C}$ and $B \in \mathcal{C}^{\perp_1}$. $\mathcal{C}$ is a *special precovering* class if each module $M \in$ Mod-$R$ has a special $\mathcal{C}$-precover.

The terminology of 1.5 comes from the fact that special preenvelopes and precovers are special instances of the following more general notions, [42], [81]:

**1.6.** Let $R$ be a ring, $M$ a module, and $\mathcal{C}$ a class of modules. An $R$-homomorphism $f : M \to C$ with $C \in \mathcal{C}$ is a $\mathcal{C}$*-preenvelope* of $M$ provided that for each $C' \in \mathcal{C}$ and each $R$-homomorphism $f' : M \to C'$ there is an $R$-homomorphism $g : C \to C'$ such that $f' = gf$.

The $\mathcal{C}$-preenvelope $f$ is a $\mathcal{C}$*-envelope* of $M$ if $f$ has the following minimality property: if $g$ is an endomorphism of $C$ such that $gf = f$ then $g$ is an automorphism.

$\mathcal{C}$ is a *preenveloping (enveloping)* class provided that each module $M \in$ Mod-$R$ has a $\mathcal{C}$-preenvelope (envelope).

The notions of a $\mathcal{C}$*-precover*, $\mathcal{C}$*-cover*, *precovering* class, and *covering* class are defined dually.

A preenvelope (precover) may be viewed as a kind of weak (co-) reflection [44]; however, we do not require the assignment $M \mapsto C$ $(C \mapsto M)$ to be functorial or unique, cf. 1.2.

However, if a module $M$ has a $\mathcal{C}$-envelope (cover) then the $\mathcal{C}$-envelope (cover) is easily seen to be uniquely determined up to isomorphism; morever the $\mathcal{C}$-envelope (cover) of $M$ is isomorphic to a direct summand in any $\mathcal{C}$-preenvelope ($\mathcal{C}$-precover) of $M$, [81].

Classical examples of enveloping classes include $\mathcal{I}_0$ and $\mathcal{PI}$, see [35] and [80], and of covering classes, $\mathcal{P}_0$ in case $R$ is a right perfect ring, and $\mathcal{TF}$ in case $R$ is a domain, see [13] and [41]. We will have many more examples later in this section.

**1.7.** The definitions above can be extended to the setting of an abitrary category $\mathcal{K}$ (in place of Mod-$R$) and its subcategory $\mathcal{C} \subseteq \mathcal{K}$. In the particular case when $\mathcal{K} =$ mod-$R$, we will say $\mathcal{C}$ is *covariantly finite* (*contravariantly finite*) provided that $\mathcal{C}$ is preenveloping (precovering) in mod-$R$, cf. [12].

The following lemma connects cotorsion pairs to approximations of modules:

**Lemma 1.8.** *Let $R$ be a ring, $M$ a module, and $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ a cotorsion pair.*

(1) [79] *Assume $M$ has a $\mathcal{B}$-envelope $f$. Then $f$ is a special $\mathcal{B}$-preenvelope. So if $\mathcal{B}$ is enveloping then $\mathcal{B}$ is special preenveloping.*
(2) [79] *Assume $M$ has a $\mathcal{A}$-cover $f$. Then $f$ is a special $\mathcal{A}$-precover. So if $\mathcal{A}$ is covering then $\mathcal{A}$ is special precovering.*
(3) [69] *$\mathcal{A}$ is special precovering iff $\mathcal{B}$ is special preenveloping. In this case $\mathfrak{C}$ is called a* complete *cotorsion pair.*

Proof. 1. Since $\mathcal{I}_0 \subseteq \mathcal{B}$, $f$ is monic, so there is a short exact sequence $0 \to M \xrightarrow{f} B \xrightarrow{g} C \to 0$. Take a short exact sequence $0 \to B' \to D \xrightarrow{h} C \to 0$ with $B' \in \mathcal{B}$. Considering the pull-back of $g$ and $h$, and using the minimality of the map $f$, we obtain a splitting map for $h$, thus proving that $C \in \mathcal{A}$.

2. This is dual to 1.

3. Let $M$ be a module. Consider a short exact sequence $0 \to M \to I \xrightarrow{f} J \to 0$ where $I \in \mathcal{I}_0$. Let $g : A \to J$ be a special $\mathcal{A}$-precover of $J$. Then the pull-back of $g$ and $f$ yields a special $\mathcal{B}$-preenvelope of $M$. The proof of the converse implication is dual. □

The following example shows that in 1.8.3, we cannot claim that $\mathcal{A}$ is a covering class iff $\mathcal{B}$ is an enveloping one (however, by 1.10 below, the equivalence holds in case $\mathcal{A}$ is closed under direct limits):

**Example 1.9.** [21], [22], [23] Let $R$ be a commutative domain and $\mathfrak{C}$ be the cotorsion pair cogenerated by the quotient field $Q$. Matlis proved that $\mathfrak{C}$ is hereditary iff proj.dim$(Q) \leq 1$ (that is, $R$ is a *Matlis domain*).

The class $\mathcal{B} = \{Q\}^{\perp_1}$ is the class of all *Matlis cotorsion* modules. Since $\mathcal{B} =$ (Mod-$Q)^{\perp_1}$, $\mathcal{B}$ is an enveloping class, [81]. For example, the $\mathcal{B}$-envelope of a torsion-free reduced module $M$ coincides with the $R$-completion of $M$, cf. [46].

On the other hand, $\mathcal{A}$ (called the class of all *strongly flat* modules) is a covering class iff all proper factor-rings of $R$ are perfect. For example, if $R$ is a Prüfer domain then $\mathcal{A}$ is a covering class iff $R$ is a Dedekind domain.

Cotorsion pairs $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ such that $\mathcal{A}$ is a covering class and $\mathcal{B}$ is an enveloping class are called *perfect*. By 1.8, each perfect cotorsion pair is complete. There is a sufficient condition for perfectness of complete cotorsion pairs due to Enochs. For a proof, we refer to [42] and [81]:

**Theorem 1.10.** *Let $R$ be a ring, $M$ a module, and $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ a cotorsion pair. Assume that $\mathcal{A}$ is closed under direct limits.*

(1) *If $M$ has a $\mathcal{B}$-preenvelope then $M$ has a $\mathcal{B}$-envelope.*
(2) *If $M$ has an $\mathcal{A}$-precover then $M$ has an $\mathcal{A}$-cover.*

*In particular, $\mathfrak{C}$ is perfect iff $\mathfrak{C}$ is complete iff $\mathcal{A}$ is covering iff $\mathcal{B}$ is enveloping.*

**1.11.** Let $R$ be a ring, and $\mathcal{C}$ a subclass of mod-$R$ closed under extensions and direct summands such that $R \in \mathcal{C}$. Let $\mathcal{D} = \varinjlim \mathcal{C}$. Then $\mathcal{C} = \mathcal{D} \cap$ mod-$R$. Moreover, by [8], the Tor-torsion pair cogenerated by $\mathcal{C}$ has the form $(\mathcal{D}, \mathcal{E})$ for some $\mathcal{E} \subseteq R$-Mod.

By 1.4.3, there are two associated cotorsion pairs: $(\mathcal{A}, \mathcal{B})$ – the cotorsion pair cogenerated by $\mathcal{C}$, and $(\mathcal{D}, \mathcal{G})$ – the cotorsion pair generated by the class of all dual modules in $\mathcal{B}$. Clearly, $(\mathcal{A}, \mathcal{B}) \leq (\mathcal{D}, \mathcal{G})$.

Assume that $R$ is an artin algebra. By [61], if $\mathcal{C}$ is resolving and contravariantly finite, then $(\mathcal{D}, \mathcal{G})$ is also generated by $\mathcal{H} = \mathcal{B} \cap$ mod-$R$. So $(\mathcal{C}, \mathcal{H})$ is a complete hereditary cotorsion pair *in mod-$R$*. Moreover, $\mathcal{G} = \varinjlim \mathcal{H}$.

Let $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a complete cotorsion pair. It is an open problem whether $\mathcal{A}$ is a covering class iff $\mathcal{A}$ is closed under direct limits (The 'if' part is true by 1.10). 1.9 shows that $\mathcal{B}$ may be enveloping even if $\mathcal{A}$ is not closed under direct limits.

**1.12. Invariants of modules.** Assume $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ is a perfect cotorsion pair. Then often the modules in the kernel, $\mathcal{K}$, of $\mathfrak{C}$ can be classified up to isomorphism by cardinal invariants. There are two ways to extend this classification:

a) Any module $A \in \mathcal{A}$ determines – by an iteration of $\mathcal{B}$-envelopes (of $A$, of the cokernel of the $\mathcal{B}$-envelope of $A$, etc.) – a long exact sequence all of whose members (except for $A$) belong to $\mathcal{K}$. This sequence is called the *minimal $\mathcal{K}$-coresolution* of $A$. The sequence of the cardinal invariants of the modules from $\mathcal{K}$ occuring in the coresolution provides for an invariant of $A$. In this way, the structure theory of the modules in $\mathcal{K}$ is extended to a structure theory of the modules in $\mathcal{A}$.

b) Dually, any module $B \in \mathcal{B}$ determines – by an iteration of $\mathcal{A}$-covers – a long exact sequence all of whose members (except for $B$) belong to $\mathcal{K}$, the *minimal $\mathcal{K}$-resolution* of $B$. This yields a sequence of cardinal invariants for any module $B \in \mathcal{B}$.

For specific examples to a) and b), we consider the case when $R$ is a commutative noetherian ring:

If $\mathfrak{C} = (\text{Mod-}R, \mathcal{I}_0)$, then $\mathcal{K} = \mathcal{I}_0$, and by the classical theory of Matlis, each $M \in \mathcal{K}$ is determined up to isomorphism by the multiplicities of indecomposable injectives $E(R/p)$ ($p$ a prime ideal of $R$) occuring in an indecomposable decomposition of $M$. The cardinal invariants of arbitary modules (in $\mathcal{A} = \text{Mod-}R$) constructed in a) are called the *Bass numbers*. A formula for their computation goes back to Bass: the multiplicity of $E(R/p)$ in the $i$-th term of the minimal injective coresolution of a module $N$ is $\mu_i(p, N) = \dim_{k(p)} \operatorname{Ext}^i_{R_p}(k(p), N_p)$ where $k(p) = R_p/\operatorname{Rad}(R_p)$, and $R_p$ and $N_p$ is the localization of $R$ and $N$ at $p$, respectively, cf. [63].

If $\mathfrak{C} = (\mathcal{F}_0, \mathcal{EC})$, then $\mathcal{K}$ consists of the flat pure-injective modules: these are described by the ranks of the completions, $T_p$, of free modules over localizations $R_p$ ($p$ a prime ideal of $R$) occuring in their decomposition, [42]. The construction b) yields a sequence of invariants for any cotorsion module $N$. These invariants are called the *dual Bass numbers*. A formula for their computation is due to Xu [81]: the rank of $T_p$ in the $i$-th term of the minimal flat resolution of $N$ is $\pi_i(p, N) = \dim_{k(p)} \operatorname{Tor}^{R_p}_i(k(p), \operatorname{Hom}_R(R_p, N))$.

In view of 1.4, the following result says that most cotorsion pairs are complete, hence provide for approximations of modules.

For a module $M$ and a class of modules $\mathcal{C}$, a *$\mathcal{C}$-filtration* of $M$ is an increasing sequence of submodules of $M$, $(M_\alpha \mid \alpha < \sigma)$, such that $M = \bigcup_{\alpha<\sigma} M_\alpha$, $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$ for all limit ordinals $\alpha < \sigma$, and $M_{\alpha+1}/M_\alpha$ is isomorphic to an element of $\mathcal{C}$ for each $\alpha < \sigma$. A module possessing a $\mathcal{C}$-filtration is called *$\mathcal{C}$-filtered.*

**Theorem 1.13.** *Let $R$ be a ring and $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ a cotorsion pair cogenerated by a set of modules $\mathcal{S}$. Then $\mathfrak{C}$ is complete, and $\mathcal{A}$ is the class of all direct summands of all $\mathcal{S} \cup \{R\}$-filtered modules.*

Proof. [39] The core of the proof is a construction (by induction, using a push-out argument inspired by [48] in the non-limit steps), for each pair of modules, $(M, N)$, of a short exact sequence $0 \to M \to B \to A \to 0$ such that $A$ is $\{N\}$-filtered and $B \in \{N\}^{\perp_1}$. By assumption, $\mathfrak{C}$ is cogenerated by a single module, say $N$. For any module $M$, the short exact sequence above yields a special $\mathcal{B}$-preenvelope of $M$, proving that $\mathfrak{C}$ is complete.

For a module $X \in \mathcal{A}$, consider a short exact sequence $0 \to M \to F \to X \to 0$ with $F$ free. Let $0 \to M \to B \to A \to 0$ be as above. The push-out of the maps $M \to F$ and $M \to B$ yields a split exact sequence $0 \to B \to G \to X \to 0$, and $G$ is an extension of $F$ by $A$, hence $G$ is $\{N, R\}$-filtered. The converse is proved by induction on the length of the filtration. □

1.13 was applied by Enochs to prove the flat cover conjecture: each module has a flat cover and a cotorsion envelope, [25]. This was generalized in [40] as follows:

**Theorem 1.14.** *Let $R$ be a ring and $\mathfrak{C}$ be a cotorsion pair generated by a class of pure-injective modules. Then $\mathfrak{C}$ is perfect.*

Proof. By 1.4, $\mathfrak{C}$ is cogenerated by a set of modules. By 1.13, $\mathfrak{C}$ is a complete cotorsion pair. By a classical result of Auslander, the functor $\mathrm{Ext}^1_R(-, M)$ takes direct limits to the inverse ones for each pure-injective module $M$. In particular, ${}^{\perp_1}\{M\}$ is closed under direct limits. So 1.10 applies, proving that $\mathfrak{C}$ is perfect. □

The flat cover conjecture is the particular case of 1.14 for $\mathfrak{C}$ generated by $\mathcal{PI}$.

For Dedekind domains, we can extend 1.14 further:

**Theorem 1.15.** *Let $R$ be a Dedekind domain and $\mathfrak{C}$ be a cotorsion pair generated by a class of cotorsion modules. Then $\mathfrak{C}$ is perfect.*

Proof. Let $C$ be a cotorsion module and $f : F \to C$ be its flat cover. Then $F$ is flat and cotorsion, hence pure-injective. By [40], ${}^{\perp}C = {}^{\perp}F$. So $\mathfrak{C}$ is generated by a class of pure-injective modules, and 1.14 applies. □

However, the possibility of extending 1.14 to larger classes of modules depends on the extension of set theory (ZFC) that we work in. Here, one uses the well-developed theory studying dependence of vanishing of Ext on set-theoretic assumptions. This theory originated in Shelah's solution of the Whitehead problem, but has many more applications [36].

In the positive direction, Gödel's axiom of constructibility (V = L) is useful, or rather its combinatorial consequence called Jensen's diamond principle $\diamondsuit$:

$\diamondsuit$ Let $\kappa$ be a regular uncountable cardinal, $E$ be a stationary subset in $\kappa$, and $X$ be a set of cardinality $\kappa$ such that $X = \bigcup_{\alpha<\kappa} X_\alpha$ where $(X_\alpha \mid \alpha < \kappa)$ is an increasing chain of subsets of $X$ with $\mathrm{card}(X_\alpha) < \kappa$ for all $\alpha < \kappa$, and $X_\beta = \bigcup_{\gamma<\beta} X_\gamma$ for each limit ordinal $\beta < \kappa$.

Then there exist sets $Y_\alpha$ ($\alpha \in E$) such that $Y_\alpha \subseteq X_\alpha$ for all $\alpha \in E$, and moreover, for each $Z \subseteq X$, the set $\{\alpha \in E \mid Z \cap X_\alpha = Y_\alpha\}$ is stationary in $\kappa$.

(A subset $E \subseteq \kappa$ is *stationary* in $\kappa$ if $E$ has a non-empty intersection with each closed and unbounded subset of $\kappa$.)

The following result is proved in [40] by induction, applying $\diamondsuit$ in regular cardinals, and Shelah's Singular Compactness Theorem in the singular ones:

**Theorem 1.16.** *Assume $\diamondsuit$. Let $R$ be a right hereditary ring and $\mathfrak{C}$ a cotorsion pair generated by a set of modules. Then $\mathfrak{C}$ is complete.*

In the negative direction, Shelah's uniformization principle $\mathrm{UP}^+$ is useful. Like Gödel's axiom of constructibility, $\mathrm{UP}^+$ is relatively consistent with ZFC + GCH, but $\mathrm{UP}^+$ and $\diamondsuit$ are mutually inconsistent.

$UP^+$ Let $\kappa$ be a singular cardinal of cofinality $\omega$. There is a stationary subset $E$ in $\kappa^+$ consisting of ordinals of cofinality $\omega$, and a ladder system $\mu = (\mu_\alpha \mid \alpha \in E)$ with the following uniformization property:

For each cardinal $\lambda < \kappa$ and each system of maps $h_\alpha : \mu_\alpha \to \lambda$ ($\alpha \in E$) there is a map $f : \kappa^+ \to \lambda$ such that for each $\alpha \in E$, $f$ coincides with $h_\alpha$ in all but finitely many ordinals of the ladder $\mu_\alpha$.

(A *ladder system* $\mu = (\mu_\alpha \mid \alpha \in E)$ consists of ladders, the ladder $\mu_\alpha$ being a strictly increasing countably infinite sequence of ordinals $< \alpha$ whose supremum is $\alpha$.)

UP$^+$ can be used, for any non-right perfect ring $R$, to construct particular free modules $G \subseteq F$ such that $M = F/G$ is a non-projective module satisfying $\mathrm{Ext}^1_R(M, N) = 0$ for each module $N$ with $\mathrm{card}(N) < \lambda$. The point is that in the particular setting, a homomorphism $\varphi : G \to N$ determines a system of maps $h_\alpha$ ($\alpha \in E$) as in the premise of UP$^+$. The uniformization map $f$ can then be used to define a homomorphism $\psi : F \to N$ extending $\varphi$, thus giving $\mathrm{Ext}^1_R(M, N) = 0$.

The following is proved in [38] (cf. with 1.15):

**Theorem 1.17.** *Assume UP$^+$. Let $R$ be a Dedekind domain with a countable spectrum, and $\mathfrak{C}$ a cotorsion pair generated by a set containing at least one non-cotorsion module. Then $\mathfrak{C}$ is not cogenerated by a set of modules.*

In the particular case of $R = \mathbb{Z}$, there is a stronger result by Eklof and Shelah [37], using a much stronger version of UP$^+$ which we do not state here, but just denote by SUP (SUP is also relatively consistent with ZFC + GCH, cf. [37]):

**Theorem 1.18.** *Assume SUP. Denote by $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ the cotorsion pair generated by $\mathbb{Z}$. Then $\mathbb{Q}$ does not have an $\mathcal{A}$-precover; in particular, $\mathfrak{C}$ is not complete.*

Notice that the class $\mathcal{A}$ in 1.18 is the well-known class of all *Whitehead groups.*

We finish this section by two open problems. *Let $R$ be a ring and $\mathfrak{C}$ a cotorsion pair.*

*1. Is $\mathfrak{C}$ complete provided that $\mathfrak{C}$ is generated by a class of cotorsion modules?* This is true in the Dedekind domain case by 1.15. Notice that for right perfect rings, the question asks whether all cotorsion pairs are complete.

*2. Is the completeness of $\mathfrak{C}$ independent of ZFC in case $\mathfrak{C}$ is generated by a set containing at least one non-cotorsion module?* This is true when $R = \mathbb{Z}$ and $\mathfrak{C}$ is generated by $\mathbb{Z}$, cf. 1.16 and 1.18. The term "set" is important here, since by 1.4 and 1.13, in ZFC there are many complete cotorsion pairs $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ with $\mathcal{B}$ containing non-cotorsion modules.

## 2. Tilting cotorsion pairs

In this section, we will investigate relations between tilting and approximation theory of modules. For this purpose, it is convenient to work with a rather general definition of a tilting module. Our definition allows for infinitely generated modules, and also modules of finite projective dimension $> 1$.

**2.1. Tilting modules.** Let $R$ be a ring. A module $T$ is *tilting* provided that

(1) $\mathrm{proj.dim}(T) < \infty$;
(2) $\mathrm{Ext}^i_R(T, T^{(\kappa)}) = 0$ for any cardinal $\kappa$ and any $i \geq 1$;
(3) There are $k < \omega$, $T_i \in \mathrm{Add}(T)$ ($i \leq k$), and an exact sequence

$$0 \to R \to T_0 \to \cdots \to T_k \to 0.$$

Here, $\mathrm{Add}(T)$ denotes the class of all direct summands of arbitrary direct sums of copies of the module $T$.

Let $n < \omega$. Tilting modules of projective dimension $\leq n$ are called *n-tilting*. A class of modules $\mathcal{C}$ is *n-tilting* if there is an $n$-tilting module $T$ such that $\mathcal{C} = \{T\}^{\perp}$.

A cotorsion pair $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ is *n-tilting* provided that $\mathcal{B}$ is an $n$-tilting class.

Notice that the notions above do not change when replacing the tilting module $T$ by the tilting module $T^{(\kappa)}$ ($\kappa > 1$). It is convenient to define an equivalence of tilting modules as follows: $T$ is *equivalent* to $T'$ provided that the induced tilting classes coincide: $\{T\}^{\perp} = \{T'\}^{\perp}$ (This is also equivalent to $\mathrm{Add}(T) = \mathrm{Add}(T')$.)

Clearly, 0-tilting modules coincide with the projective generators. Finite dimensional tilting modules over artin algebras have been studied in great detail - we refer to [4], [65] and [78] in this volume for much more on this classical case. We will now give several examples of infinitely generated 1-tilting modules:

**2.2. Fuchs tilting modules.** [46], [47] Let $R$ be a commutative domain, and $S$ a multiplicative subset of $R$. Let $\delta_S = F/G$ where $F$ is the free module with the basis given by all sequences $(s_0, \dots, s_n)$ where $n \geq 0$, and $s_i \in S$ for all $i \leq n$, and the empty sequence $w = ()$. The submodule $G$ is generated by the elements of the form $(s_0, \dots, s_n)s_n - (s_0, \dots, s_{n-1})$ where $0 < n$ and $s_i \in S$ for all $1 \leq i \leq n$, and of the form $(s)s - w$ where $s \in S$.

The module $\delta = \delta_{R \setminus \{0\}}$ was introduced by Fuchs. Facchini [43] proved that $\delta$ is a 1-tilting module. The general case of $\delta_S$ comes from [47]: the module $\delta_S$ is a 1-tilting module, called the *Fuchs tilting module*. The corresponding 1-tilting class is $\{\delta_S\}^{\perp} = \{M \in \text{Mod-}R \mid Ms = M \text{ for all } s \in S\}$, the class of all *S-divisible modules*. If $R$ is a Prüfer domain or a Matlis domain, then the 1-tilting cotorsion pair cogenerated by $\delta$ is $(\mathcal{P}_1, \mathcal{DI})$.

**Example 2.3.** [5] Let $R$ be a commutative 1-Gorenstein ring. Let $P_0$ and $P_1$ denote the set of all prime ideals of height 0 and 1, respectively. By a classical result of Bass, the minimal injective coresolution of $R$ has the form

$$0 \to R \to \bigoplus_{q \in P_0} E(R/q) \xrightarrow{\pi} \bigoplus_{p \in P_1} E(R/p) \to 0.$$

Consider a subset $P \subseteq P_1$. Put $R_P = \pi^{-1}(\bigoplus_{p \in P} E(R/p))$. Then $T_P = R_P \oplus \bigoplus_{p \in P} E(R/p)$ is a 1-tilting module, the corresponding 1-tilting class being $\{T_P\}^{\perp} = \{M \mid \mathrm{Ext}^1_R(E(R/p), M) = 0 \text{ for all } p \in P\}$. In particular, if $R$ is a Dedekind domain then $\{T_P\}^{\perp} = \{M \mid \mathrm{Ext}^1_R(R/p, M) = 0 \text{ for all } p \in P\} = \{M \mid pM = M \text{ for all } p \in P\}$.

In his classical work [67], Ringel discovered analogies between modules over Dedekind domains and tame hereditary algebras. The analogies extend to the setting of infinite dimensional tilting modules:

**2.4. Ringel tilting modules.** [67], [68] Let $R$ be a connected tame hereditary algebra over a field $k$. Let $G$ be the generic module. Then $S = \mathrm{End}(G)$ is a skew-field and $\dim_S G = n < \omega$. Denote by $\mathcal{T}$ the set of all tubes. If $\alpha \in \mathcal{T}$ is a homogenous tube, we denote by $R_\alpha$ the corresponding Prüfer module. If $\alpha \in \mathcal{T}$ is not homogenous, denote by $R_\alpha$ the direct sum of all Prüfer modules corresponding to the rays in $\alpha$. Then there is an exact sequence

$$0 \to R \to G^{(n)} \xrightarrow{\pi} \bigoplus_{\alpha \in \mathcal{T}} R_\alpha^{(\lambda_\alpha)} \to 0$$

where $\lambda_\alpha > 0$ for all $\alpha \in \mathcal{T}$.

Let $P \subseteq \mathcal{T}$. Put $R_P = \pi^{-1}(\bigoplus_{\alpha \in P} R_\alpha^{(\lambda_\alpha)})$. Then $T_P = R_P \oplus \bigoplus_{\alpha \in P} R_\alpha$ is a 1-tilting module, called the *Ringel tilting module*. The corresponding 1-tilting class is the class of all modules $M$ such that $\mathrm{Ext}^1_R(N, M) = 0$ for all (simple) regular modules $N \in P$.

**2.5. Lukas tilting modules.** [60], [62] Let $R$ be a connected wild hereditary algebra over a field $k$. Denote by $\tau$ the Auslander-Reiten translation, and by $\mathcal{R}$ the class of all *Ringel divisible modules*, that is, of all modules $D$ such that $\mathrm{Ext}^1_R(M, D) = 0$ for each regular module $M$.

Let $M$ be any regular module. Then for each finite dimensional module $N$, Lukas constructed an exact sequence $0 \to N \to A_M \to B_M \to 0$ where $A_M \in M^\perp$ and $B_M$ is a finite direct sum of copies of $\tau^n M$ for some $n < \omega$. Letting $\mathcal{C}_M = \{\tau^m M \mid m < \omega\}$, we can iterate this construction (for $N = R$, $N = A_M$, etc.) and get an exact sequence $0 \to R \to C_M \to D_M \to 0$ where $D_M$ has a countable $\mathcal{C}_M$-filtration. Then $T_M = C_M \oplus D_M$ is a 1-tilting module, called the *Lukas tilting module*. The corresponding 1-tilting class is $\mathcal{R}$ (so in contrast to 2.4, $T_M$ and $T_{M'}$ are equivalent for all regular modules $M$ and $M'$).

Now, we will consider a simple example of an infinitely generated $n$-tilting module. In §5, we will see that this example is related to the validity of the first finitistic dimension conjecture for Iwanaga-Gorenstein rings.

A ring $R$ is called *Iwanaga-Gorenstein* provided that $R$ is left and right noetherian and the left and right injective dimensions of the regular module are finite, [42]. In this case, $\mathrm{inj.dim}(R_R) = \mathrm{inj.dim}({}_R R) = n$ for some $n < \omega$, and $R$ is called *$n$-Gorenstein*. Notice that 0-Gorenstein rings coincide with the quasi-Frobenius rings.

**Example 2.6.** Let $R$ be an $n$-Gorenstein ring. Let

$$0 \to R \to E_0 \to \cdots \to E_n \to 0$$

be the minimal injective coresolution of $R$. Then $T = \bigoplus_{i \leq n} E_i$ is an $n$-tilting module. The only non-trivial fact needed for this is that $\mathcal{P} = \mathcal{P}_n = \mathcal{I}_n = \mathcal{I}$ ($= \mathcal{F}_n = \mathcal{F}$) for any $n$-Gorenstein ring, cf. [42, §9].

For any tilting cotorsion pair $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$, there is a close relation among the classes $\mathcal{A}$, $\mathcal{B}$, and the kernel of $\mathfrak{C}$:

**Lemma 2.7.** *Let $R$ be a ring and $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ a tilting cotorsion pair. Let $T$ be an $n$-tilting module with $\{T\}^\perp = \mathcal{B}$. Then*

1. *$\mathfrak{C}$ is hereditary and complete. Moreover, $\mathfrak{C} \leq (\mathcal{P}_n, \mathcal{P}_n^\perp)$, and the kernel of $\mathfrak{C}$ equals Add($T$).*
2. *$\mathcal{A}$ coincides with the class of all modules $M$ such that there is an exact sequence*

$$0 \to M \to T_0 \to \cdots \to T_n \to 0$$

   *where $T_i \in$ Add($T$) for all $i \leq n$.*
3. *Let $0 \to F_n \to \cdots \to F_0 \to T \to 0$ be a free resolution of $T$ and let $\mathcal{S} = \{S_i \mid i \leq n\}$ be the corresponding set of syzygies of $T$. Then $\mathcal{A}$ coincides with the class of all direct summands of all $\mathcal{S}$-filtered modules.*
4. *$\mathcal{B}$ coincides with the class of all modules $N$ such that there is a long exact sequence*

$$\cdots \to T_{i+1} \to T_i \to \cdots \to T_0 \to N \to 0$$

   *where $T_i \in$ Add($T$) for all $i < \omega$. In particular, $\mathcal{B}$ is closed under arbitrary direct sums.*

Proof. 1. The first claim follows from $\mathcal{B} = \{T\}^\perp$ by 1.13, the second is clear from $T \in \mathcal{P}_n$. The last claim is proved in [2].

2. Since $\mathcal{A}$ is closed under kernels of monomorphisms, any $M$ possessing such exact sequence is in $\mathcal{A}$. Conversely, we obtain the desired sequence by an iteration of special $\mathcal{B}$-preenvelopes (of $M$ etc.). The fact that we can stop at $n$ follows from proj.dim$(T) \leq n$.

3. This follows by the characterization of $\mathcal{A}$ given in 1.13, since $\mathfrak{C}$ is cogenerated by $\bigoplus_{i\leq n} S_i$.

4. If $N \in \mathcal{B}$ then the long exact sequence can be obtained by an iteration of special $\mathcal{A}$-precovers (of $N$ etc.). The converse uses proj.dim$(T) \leq n$ once again. □

We arrive at the characterization of tilting cotorsion pairs in terms of approximations. We start with the case of $n = 1$ treated in [6]:

**Theorem 2.8.** *Let $R$ be a ring.*

1. *A class of modules $\mathcal{C}$ is 1-tilting iff $\mathcal{C}$ is a special preenveloping torsion class.*
2. *Let $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a cotorsion pair. Then $\mathfrak{C}$ is 1-tilting iff $\mathfrak{C}$ is complete, $\mathfrak{C} \leq (\mathcal{P}_1, \mathcal{P}_1^\perp)$, and $\mathcal{B}$ is closed under arbitrary direct sums.*

Proof. 1. Since $\{T\}^\perp$ is closed under homomorphic images and extensions for any module $T$ with proj.dim$(T) \leq 1$, the only-if part is a consequence of parts 1. and 4. of 2.7 (for $n = 1$). For the if-part, we consider a special $\mathcal{B}$-preenvelope of $R$; this yields an exact sequence $0 \to R \to B \to A \to 0$ with $B \in \mathcal{B}$ and $A \in \mathcal{A}$. Then $T = A \oplus B$ is a 1-tilting module with $\{T\}^\perp = \mathcal{C}$, cf. [6].

2. The only-if part follows directly from parts 1. and 4. of 2.7. For the if-part, note that $\mathcal{B}$ is closed under homomorphic images and extensions since $\mathcal{B} = \mathcal{A}^{\perp_1}$ and $\mathcal{A} \subseteq \mathcal{P}_1$. So $\mathcal{B}$ is a torsion class, and part 1. applies. □

We stress that the special approximations induced by 1-tilting modules may not have minimal versions in general (compare this with 3.9.1 below). For example, if $R$ is a domain and $\delta$ is the Fuchs tilting module from 2.2 then the special $\{\delta\}^\perp$-preenvelopes coincide with the special divisible preenvelopes (and also with the special FP-injective preenvelopes). However, if $R$ is a Prüfer domain with proj.dim$(Q) \geq 2$, then the regular module $R$ does not have a divisible envelope (and so it does not have an FP-injective envelope), see [75].

The characterization in the general case is due to Angeleri Hügel and Coelho [2]:

**Theorem 2.9.** *Let $R$ be a ring and $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a cotorsion pair. Then $\mathfrak{C}$ is $n$-tilting iff $\mathfrak{C}$ is hereditary and complete, $\mathfrak{C} \leq (\mathcal{P}_n, \mathcal{P}_n^\perp)$, and $\mathcal{B}$ is closed under arbitrary direct sums.*

Proof. The only-if part follows by 2.7. For the if-part, consider the iteration of special $\mathcal{B}$-preenvelopes of $R$, of Coker$(f)$ (where $f$ is a special $\mathcal{B}$-preenvelope of $R$), etc. By assumption, this yields a finite $(\mathcal{A} \cap \mathcal{B})$-coresolution of $R$, $0 \to R \to T_0 \to \cdots \to T_n \to 0$. Then $T = \bigoplus_{i\leq n} T_i$ is $n$-tilting with $\{T\}^\perp = \mathcal{B}$, cf. [2]. □

**2.10.** Many authors define a partial tilting module $P$ as the module satisfying the first two conditions of 2.1 (for $P$). However, in general, these two conditions are not sufficient for existence of a *complement* of $P$ (= a module $P'$ such that $T = P \oplus P'$ is tilting and $\{P\}^\perp = \{T\}^\perp$). For a counter-example, consider $R = \mathbb{Z}$ and $P = \mathbb{Q}$; then $\{P\}^\perp$ is the class of all cotorsion groups which is not closed under arbitrary direct sums.

The extra condition (E): "$\{P\}^\perp$ is closed under arbitrary direct sums" is clearly necessary for the existence of a complement of $P$. We define a *partial $n$-tilting* module $P$ as a module of projective dimension $\leq n$ satisfying (E), and $\mathrm{Ext}^i_R(P, P) = 0$ for all $0 < i < \omega$. Then a complement of $P$ always exists in Mod-$R$ by 2.9: $\{P\}^\perp$

is an $n$-tilting class with an $n$-tilting module $T$ such that $\{T\}^\perp = \{P\}^\perp$, so $T$ is a complement of $P$, cf. [3]. Condition (E) is of course redundant in case $P \in \text{mod-}R$.

Let $P$ be a finitely presented partial 1-tilting module. If $R$ is an artin algebra then $P$ has a finitely presented complement by a classical result of Bongartz, cf. [78]. However, a finitely presented complement of $P$ may not exists even if $R$ is a hereditary noetherian domain, cf. [32].

Rickard and Schofield constructed artin algebras and finitely presented partial 2-tilting modules with no finitely presented complements, cf. [78].

## 3. Cotilting cotorsion pairs

In this section, we will consider the dual case of cotilting modules and cotilting cotorsion pairs.

Similarly as tilting modules, the cotilting ones have first appeared in the representation theory of finite dimensional $k$-algebras. There, the finite dimensional cotilting modules coincide with the $k$-duals of the finite dimensional tilting modules, so the theory is obtained by applying the $k$-duality.

1-cotilting modules over general rings are closely related to dualities (see [30] in this volume for more details). Also, in §4, we will see that restricting to tilting modules and classes of finite type, we actually have an explicit homological duality available producing the corresponding cotilting modules and classes of cofinite type.

However, there is no explicit duality available in the general case. The problem is that the dual of the key approximation construction of 1.13 does not work in ZFC: by 1.18, there is an extension of ZGC + GCH with a cotorsion pair $\mathfrak{C}$ generated by a set such that $\mathfrak{C}$ is not complete.

Fortunately, there is a remedy. First, for $n = 1$, a fundamental result of Bazzoni says that 1-cotilting modules are pure-injective (see 3.5 below), so we can apply 1.14 directly. As shown in [2], for $n > 1$, the classical work of Auslander and Buchweitz [10] makes it possible to overcome the problem.

**3.1. Cotilting modules.** Let $R$ be a ring. A module $C$ is *cotilting* provided that

(1) $\text{inj.dim}(C) < \infty$;
(2) $\text{Ext}^i_R(C^\kappa, C) = 0$ for any cardinal $\kappa$ and any $i \geq 1$;
(3) There are $k < \omega$, $C_i \in \text{Prod}(C)$ $(i \leq k)$, and an exact sequence

$$0 \to C_k \to \cdots \to C_0 \to W \to 0,$$

where $W$ is an injective cogenerator for Mod-$R$, and $\text{Prod}(C)$ denotes the class of all direct summands of arbitrary direct products of copies of the module $C$.

Let $n < \omega$. Cotilting modules of injective dimension $\leq n$ are called *$n$-cotilting.* A class of modules $\mathcal{C}$ is *$n$-cotilting* if there is an $n$-cotilting module $C$ such that $\mathcal{C} = {}^\perp\{C\}$. A cotorsion pair $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ is *$n$-cotilting* provided that $\mathcal{A}$ is an $n$-cotilting class.

The equivalence of cotilting modules is defined as follows: $C$ is *equivalent* to $C'$ provided that the induced cotilting classes coincide: ${}^\perp\{C\} = {}^\perp\{C'\}$ (that is, $\text{Prod}(C) = \text{Prod}(C')$.)

0-cotilting modules coincide with the injective cogenerators. In 4.12 below, we will see that any resolving subclass of $\mathcal{P}_n^{<\omega}$ yields an $n$-cotilting class (of left $R$-modules), so there is a big supply of $n$-cotilting modules for $n \geq 1$ in general.

We will need the following version of a characterization of cotilting modules by Bazzoni [15]. It generalizes the case of $n = 1$ from [29].

**Lemma 3.2.** *Let $R$ be a ring, $C$ a module, and $0 < n < \omega$. Then $C$ is $n$-cotilting iff ${}^\perp\{C\}$ coincides with the class, $Cog_n(C)$, of all modules $M$ possesing an exact sequence $0 \to M \to C_0 \to \cdots \to C_n$ where $\kappa$ is a cardinal and $C_i = C^\kappa$ for all $i \leq n$.*

A class $\mathcal{C}$ of modules is *definable* provided that $\mathcal{C}$ is closed under arbitrary direct products, direct limits, and pure submodules, [34]. (Definability implies axiomatizability: definable classes are axiomatized by equality to 1 of certain of the Baur-Garavaglia-Monk invariants. Definable classes of modules correspond bijectively to closed sets of indecomposable pure-injective modules, cf. [34] and [64].)

It is an open problem whether each cotilting module is pure-injective. There is a criterion of pure-injectivity of cotilting modules, [16]:

**Lemma 3.3.** *Let $R$ be a ring and $C$ a cotilting module. Then $C$ is pure-injective iff ${}^\perp\{C\}$ is closed under direct limits iff ${}^\perp\{C\}$ is closed under pure submodules iff ${}^\perp\{C\}$ is definable.*

We will now introduce (almost) rigid systems in order to characterize pure-injective cotilting modules and the corresponding cotilting classes:

Let $n < \omega$. Consider a set $\mathcal{S} = \{M_\alpha \mid \alpha < \kappa\}$ of modules such that each $M_\alpha$ ($\alpha < \kappa$) is pure-injective with $\text{inj.dim}(M_\alpha) \leq n$, and $\text{Ext}^i_R(M_\alpha, M_\beta) = 0$ for all $\alpha, \beta < \kappa$ and $1 \leq i \leq n$ (So in particular, each $M_\alpha$ is a splitter.) Then $\mathcal{S}$ is an *$n$-rigid system* if all the elements of $\mathcal{S}$ are indecomposable. $\mathcal{S}$ is *almost $n$-rigid* if $M_0$ is superdecomposable, and all $M_\alpha$ ($0 < \alpha < \kappa$) are indecomposable.

**Theorem 3.4.** *Let $R$ be a ring, $n < \omega$, and $C$ a pure-injective $n$-cotilting module. Then there is an almost $n$-rigid system $\mathcal{S}$ such that $C' = \prod_{M \in \mathcal{S}} M$ is an $n$-cotilting module equivalent to $C$.*

Proof. By a result of Fisher [64], the pure-injective module $C$ is of the form $C = M_0 \oplus E$ where $M_0$ is zero or superdecomposable, and $E$ is zero or a pure-injective hull of a direct sum of indecomposable pure-injective modules, $E = PE(\bigoplus_{0<\alpha<\kappa} M_\alpha)$. Then $E$ is a direct summand in $P = \prod_{0<\alpha<\kappa} M_\alpha$, and $P$ is a pure submodule, hence a direct summand, in $E^\kappa$. Put $C' = M_0 \oplus P$. Then ${}^\perp\{C\} = {}^\perp\{C'\}$, and also $\text{Cog}_n(C) = \text{Cog}_n(C')$, so 3.2 gives that $C'$ is an $n$-cotilting module equivalent to $C$. It follows that $\mathcal{S} = \{M_\alpha \mid \alpha < \kappa\}$ is an almost $n$-rigid system. □

The pure-injectivity assumption in 3.4 is redundant in case $n = 1$:

**Theorem 3.5.** [14] *Let $R$ be a ring and $C$ a 1-cotilting module. Then $C$ is pure-injective. In particular, ${}^\perp\{C\}$ is a definable class.*

Being definable, 1-cotilting classes are completely characterized by the indecomposable pure-injective modules they contain, cf. [64].

**3.6.** Assume there are no superdecomposable pure-injective modules. Then the system $\mathcal{S}$ in 3.4 is $n$-rigid. So it only remains to determine which of the $n$-rigid systems indeed yield $n$-cotilting modules.

This occurs when $R$ is a Dedekind domain, or a tame hereditary algebra, for example; in fact, in these cases the structure of indecomposable pure-injective modules is well-known, see [34] and [59].

In the Dedekind domain case, 1-rigid systems contain no finitely generated modules. It follows from 3.4 that up to equivalence, cotilting modules are of the form $C_P = \prod_{p \in P} J_p \oplus \bigoplus_{q \in \text{Spec}(R) \setminus P} E(R/q)$ where $0 \notin P \subseteq \text{Spec}(R)$, and $J_p$ denotes the completion of the localization of $R$ at $p$, cf. 4.14 and 4.17 below.

For the case of tame hereditary algebras, we refer to [26] and [27], or [73] in this volume.

Notice that by 3.7 below, in the right artinian case, each 1-rigid system yields a partial 1-cotilting module in the sense of 3.11.

In the noetherian case, there is more to say for $n = 1$. We can characterize 1-cotilting classes in terms of 1-rigid systems (for a different description, in terms of torsion-free classes in mod-$R$, see 3.10 below):

**Theorem 3.7.** *Let $R$ be a right noetherian ring.*
*If $\mathcal{C}$ is a 1-cotilting class then there is a 1-rigid system $\mathcal{S}$ such that $\mathcal{C} = \bigcap_{M \in \mathcal{S}} {}^{\perp}\{M\}$.*
*Conversely, if $R$ is right artinian and $\mathcal{S}$ a 1-rigid system then $\bigcap_{M \in \mathcal{S}} {}^{\perp}\{M\}$ is a 1-cotilting class.*

Proof. Let $C$ be a 1-cotilting module such that $\mathcal{C} = {}^{\perp}\{C\}$. By a result of Ziegler, $C$ is elementarily equivalent to a pure-injective envelope of a direct sum of indecomposable pure-injective modules, hence to a direct product of indecomposable pure-injective modules, $E = \prod_{\alpha < \kappa} M_\alpha$, cf. [64]. In particular, $E$ is a direct summand in an ultrapower of $C$. Since any ultrapower of $C$ is isomorphic to a direct limit of products of copies of $C$, 3.5 yields $E \in \mathcal{C}$. For right noetherian rings, Baer test lemma shows that $\mathcal{I}_1$ is definable, so $E \in \mathcal{I}_1$ because $E$ is elementarily equivalent to $C \in \mathcal{I}_1$.

Since $\{A\}^{\perp_1}$ is definable for each finitely presented module $A$, we have $C \in \{A\}^{\perp_1}$ iff $E \in \{A\}^{\perp_1}$. By a classical result of Auslander, $\mathrm{Ext}^1_R(-, I)$ takes direct limits into inverse ones for any pure-injective module $I$. Since $R$ is right noetherian, it follows that $\mathcal{C} = {}^{\perp_1}\{C\} = {}^{\perp_1}\{E\}$. In particular, $E$ is a pure-injective splitter of injective dimension $\leq 1$, so the modules $M_\alpha$ form a 1-rigid system.

Conversely, by [26], ${}^{\perp}\{M\}$ is closed under arbitrary direct products for any $M \in \mathcal{S}$. Let $P = \prod_{M \in \mathcal{S}} M$. By 1.14 and 3.9, ${}^{\perp}\{P\}$ is a 1-cotilting class. □

Now, we turn to relations between cotilting modules and approximations. Except for part 3., the dual of 2.7 holds true – a proof making use of [10] appears in [2]. (In view of 3.5, one can proceed more directly for $n = 1$, by dualizing the proof of 2.7 with help of 1.14):

**Lemma 3.8.** *Let $R$ be a ring and $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a cotilting cotorsion pair. Let $C$ be an $n$-cotilting module with ${}^{\perp}\{C\} = \mathcal{A}$. Then*

1. *$\mathfrak{C}$ is hereditary and complete. Moreover, $({}^{\perp}\mathcal{I}_n, \mathcal{I}_n) \leq \mathfrak{C}$, and the kernel of $\mathfrak{C}$ equals Prod$(C)$.*
2. *$\mathcal{A}$ coincides with the class of all modules $M$ such that there is a long exact sequence*
$$0 \to M \to C_0 \to \cdots \to C_i \to C_{i+1} \to \dots$$
*where $C_i \in$ Prod$(C)$ for all $i < \omega$. In particular, $\mathcal{A}$ is closed under arbitrary direct products.*
3. *$\mathcal{B}$ coincides with the class of all modules $N$ such that there is an exact sequence*
$$0 \to C_n \to \cdots \to C_0 \to N \to 0$$
*where $C_i \in$ Prod$(C)$ for all $i \leq n$.*

**Theorem 3.9.** *Let $R$ be a ring.*

1. *A class of modules $\mathcal{C}$ is 1-cotilting iff $\mathcal{C}$ is a covering torsion-free class.*
2. *Let $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a cotorsion pair. Then $\mathfrak{C}$ is $n$-cotilting iff $\mathfrak{C}$ is hereditary and complete, $({}^{\perp}\mathcal{I}_n, \mathcal{I}_n) \leq \mathfrak{C}$, and $\mathcal{A}$ is closed under arbitrary direct products.*

Proof. 1. The proof is dual to the one for 2.8.1, using 3.5, and then 1.14 in place of 1.13, cf. [6].

2. This is by a dual argument to the one for 2.9, see [2]. □

In particular, 1-cotilting classes coincide with those torsion-free classes $\mathcal{C}$ that are covering. If $R$ is right noetherian, then $\mathcal{C}$ is completely determined by its subclass $\mathcal{C} \cap \text{mod-}R$, and the latter is characterized as a torsion-free class in mod-$R$ containing $R$. More precisely, we have

**Theorem 3.10.** [26] *Let $R$ be a right noetherian ring. There is a bijective correspondence between 1-cotilting classes of modules, $\mathcal{C}$, and torsion-free classes, $\mathcal{E}$, in mod-$R$ containing $R$. The correspondence is given by the mutually inverse assignments $\mathcal{C} \mapsto \mathcal{C} \cap \text{mod-}R$ and $\mathcal{E} \mapsto \varinjlim \mathcal{E}$.*

Proof. If $\mathcal{C}$ is a 1-cotilting class, then clearly $\mathcal{C} \cap \text{mod-}R$ is a torsion-free class in mod-$R$ containing $R$.

Conversely, given $\mathcal{E}$ as in the claim, let $\mathcal{C} = \varinjlim \mathcal{E}$. By [33], $\mathcal{C}$ is a torsion-free class in Mod-$R$. Since $R \in \mathcal{E}$, by 1.11, there is a Tor-torsion pair of the form $(\mathcal{C}, \mathcal{D})$. By 1.4.3 and 1.14, $\mathcal{C}$ is a covering class. By 3.9.1, $\mathcal{C}$ is 1-cotilting.

Now, $\mathcal{E} = \varinjlim \mathcal{E} \cap \text{mod-}R$. Conversely, given a 1-cotilting class $\mathcal{C}$, each $M \in \mathcal{C}$ is a directed union of the system of its finitely presented submodules, $\{M_i \mid i \in I\}$ (because $R$ is right noetherian). Since $\mathcal{C}$ is 1-cotilting, $M_i \in \mathcal{C}$ for each $i \in I$. So $\mathcal{C} = \varinjlim(\mathcal{C} \cap \text{mod-}R)$, and the assignments are mutually inverse. □

We note that the corresponding result to 3.10 does not hold for 1-tilting classes. Namely, given a right noetherian ring $R$ and a 1-tilting (torsion) class $\mathcal{T}$ in Mod-$R$, the class $\mathcal{T} \cap \text{mod-}R$ is certainly a torsion class in mod-$R$. Let $\mathcal{C} = \varinjlim(\mathcal{T} \cap \text{mod-}R)$. By [33], $\mathcal{C}$ is a torsion class in Mod-$R$ contained in $\mathcal{T}$. However, $\mathcal{C}$ is not 1-tilting in general: if $R$ is an artin algebra and $\mathcal{T} = (\mathcal{P}_1^{<\omega})^{\perp}$, then $\mathcal{C}$ is closed under arbitrary direct products iff $\mathcal{P}_1^{<\omega}$ is contravariantly finite. The latter fails for the IST-algebra [57], for example

(However, if $R$ is an artin algebra and $\mathcal{C}$ a 1-tilting class of finite type, there is a way of reconstructing $\mathcal{C}$ from $\mathcal{C} \cap \text{mod-}R$, see 4.3 below.)

**3.11.** Define a *partial* 1-*cotilting* module $P$ as a splitter of injective dimension $\leq 1$ satisfying the extra condition of ${}^{\perp}\{P\}$ being closed under arbitrary direct products. Then $P$ has a complement in the sense that there is a module $P'$ such that $C = P \oplus P'$ is 1-cotilting and ${}^{\perp}\{P\} = {}^{\perp}\{C\}$. This follows from 3.9 and [74, §6]. (Note that $P$ is pure-injective by 3.5.) By [26], the extra condition is redundant when $P$ is pure-injective and $R$ is right artinian.

We finish this section by two open problems:

*Let $R$ be a ring, $n \geq 1$, and $C$ be an $n$-cotilting module. Is $C$ pure-injective?* (By 3.5, this is always true for $n = 1$. By [20], this is also true for any $n \geq 1$ in case $R$ is countable.)

*Let $R$ be a ring. Does 3.4 hold in the stronger form, with $n$-rigid systems replacing the almost $n$-rigid ones?*

## 4. Finite type, duality, and some examples

The duality between the notions of a tilting and cotilting module can be made explicit in case the modules are of finite and cofinite type, respectively. We start with the tilting case:

**4.1. Tilting modules of finite type.** Let $R$ be a ring.

(1) Let $\mathcal{C}$ be a class of modules. Then $\mathcal{C}$ is of *finite type* (*countable type*) provided there exist $n < \omega$ and a subset $\mathcal{S} \subseteq \mathcal{P}_n^{<\omega}$ ($\mathcal{S} \subseteq \mathcal{P}_n^{\leq\omega}$) such that $\mathcal{C} = \mathcal{S}^{\perp}$.

(2) Let $T$ be a tilting module. Then $T$ is of *finite type* (*countable type*, *definable*) provided the class $\{T\}^\perp$ is of finite type (countable type, definable).

**Lemma 4.2.** [5] *Let $R$ be a ring and $\mathcal{C}$ be a class of modules of finite type. Then $\mathcal{C}$ is tilting and definable.*

Proof. By assumption, there are $n < \omega$ and a set $\mathcal{S} \subseteq \mathcal{P}_n^{<\omega}$ such that $\mathcal{C} = \mathcal{S}^\perp$.

By a classical result of Brown, the covariant functor $\mathrm{Ext}_R^n(M, -)$ commutes with direct limits for each $n \geq 0$ and each $M \in$ mod-$R$. Also, it is easy to see that $\{N\}^{\perp_1}$ is closed under pure submodules for any finitely presented module $N$. It follows that $\mathcal{C}$ is definable.

Let $\mathfrak{C}$ be the cotorsion pair cogenerated by $\mathcal{S}$. By 1.13, $\mathfrak{C} = (\mathcal{A}, \mathcal{C})$ is complete and $\mathcal{A} \subseteq \mathcal{P}_n$, so 2.9 gives that $\mathfrak{C}$ is a tilting cotorsion pair. That is, $\mathcal{C}$ is a tilting class. □

4.2 says that there is a rich supply of tilting classes in general: any subset $\mathcal{S} \subseteq \mathcal{P}_n^{<\omega}$ (for some $n < \omega$) determines one. A more precise general description appears in 4.12 below; for artin algebras, there is also the following analog of 3.10:

**Theorem 4.3.** [60] *Let $R$ be an artin algebra. There is a bijective correspondence between 1-tilting classes of finite type, $\mathcal{C}$, and torsion classes, $\mathcal{T}$, in mod-$R$ containing all finitely generated injective modules. The correspondence is given by the mutually inverse assignments $\mathcal{C} \mapsto \mathcal{C} \cap$ mod-$R$, and $\mathcal{T} \mapsto \mathrm{KerHom}_R(-, \mathcal{F})$ where $(\mathcal{T}, \mathcal{F})$ is a torsion pair in mod-$R$.*

In [5], there is a general criterion for tilting modules to be of finite type:

**Lemma 4.4.** *Let $R$ be a ring and $T$ be a tilting module. Let $\mathcal{B} = \{T\}^\perp$, and $(\mathcal{A}, \mathcal{B})$ be the corresponding tilting cotorsion pair. Then $T$ is of finite type iff $T$ is definable and $T \in \varinjlim \mathcal{A}^{<\omega}$.*

The last condition of 4.4 is always satisfied for $n = 1$:

**Lemma 4.5.** *Let $R$ be a ring and $M$ be a module of projective dimension $\leq 1$. Let $(\mathcal{A}, \mathcal{B})$ be the cotorsion pair cogenerated by $M$. Then $M \in \varinjlim \mathcal{A}^{<\omega}$.*

Proof. Since $M \in \mathcal{P}_1$, there is an exact sequence $0 \to F \subseteq G \to M \to 0$ where $F$ and $G$ are free modules. Let $\{x_\alpha \mid \alpha < \kappa\}$ and $\{y_\beta \mid \beta < \lambda\}$ be a free basis of $F$ and $G$, respectively. W.l.o.g., $\kappa$ is infinite. For each finite subset $S \subseteq \kappa$ let $S'$ be the least (finite) subset of $\lambda$ such that $F_S = \bigoplus_{\alpha \in S} x_\alpha R \subseteq G_S = \bigoplus_{\beta \in S'} y_\beta R$. Then $F$ is a directed union of its summands of the form $F_S$ where $S$ runs over all finite subsets of $\kappa$. Let $M_S = G_S/F_S$. Then $M_S \in \mathcal{P}_1^{<\omega}$, and $M = P \oplus H$ where $P$ is free and $H = \varinjlim_S M_S$. By 1.11, it suffices to prove that $H \in \varinjlim \mathcal{A}^{<\omega}$.

We will show that $M_S \in \mathcal{A}^{<\omega}$ for each finite subset $S \subseteq \kappa$. Take an arbitrary $N \in \mathcal{B} = \{M\}^\perp$. Then any homomorphism from $F$ to $N$ extends to $G$. Let $\varphi$ be a homomorphism from $F_S$ to $N$. Since $F_S$ is a direct summand in $F$, $\varphi$ extends to $F$, hence to $G$, and $G_S$. It follows that $N \in \{M_S\}^\perp$, so $M_S \in \mathcal{A}^{<\omega}$, and $H \in \varinjlim \mathcal{A}^{<\omega}$. □

For 1-tilting modules, 4.4 and 4.5 yield

**Theorem 4.6.** [19] *Let $R$ be a ring and $T$ be a 1-tilting module. Then $T$ is definable iff $T$ is of finite type iff $\{T\}^\perp$ is closed under pure submodules.*

It is open whether all 1-tilting modules are of finite type. However, they are always of countable type:

**Theorem 4.7.** [19] *Let $R$ be a ring and $T$ be a 1-tilting module. Then $T$ is of countable type.*

The proof of 4.7 uses set-theoretic methods developed by Eklof, Fuchs and Shelah for the structure theory of so called Baer modules [36]. However, 4.7 holds in ZFC. In this sense, 4.7 says that the structure of 1-tilting modules, and classes, is purely an algebraic problem, depending only on the structure of countably and finitely presented modules rather than additional set-theoretic assumptions. 4.7 is instrumental in characterizing tilting classes over Prüfer and Dedekind domains, see 4.16 and 4.17 below.

The counterpart of a tilting (right $R$-) module of finite type is a cotilting left $R$-module of cofinite type:

**4.8. Cotilting modules of cofinite type.** Let $R$ be a ring. Let $\mathcal{C} \subseteq R$-Mod. Then $\mathcal{C}$ is of *cofinite type* provided that there exist $n < \omega$ and a subset $\mathcal{S} \subseteq \mathcal{P}_n^{<\omega}$ such that $\mathcal{C} = \mathcal{S}^\intercal$, where $\mathcal{S}^\intercal = \{M \in R\text{-Mod} \mid \operatorname{Tor}_i^R(S, M) = 0 \text{ for all } S \in \mathcal{S} \text{ and all } 0 < i \leq n\}$.

Let $C$ be a cotilting left $R$-module. Then $C$ is of *cofinite type* provided that the (cotilting) class ${}^\perp\{C\}$ is of cofinite type.

Applying 3.9, we can dualize 4.2:

**Lemma 4.9.** *Let $R$ be a ring and $\mathcal{C}$ be a class of left $R$-modules of cofinite type. Then $\mathcal{C}$ is cotilting and definable.*

4.5 yields a characterization of 1-cotilting classes of cofinite type:

**Lemma 4.10.** *Let $R$ be a ring and $\mathcal{C}$ be a class of left $R$-modules. Then $\mathcal{C}$ is 1-cotilting of cofinite type iff there is a module $M \in \mathcal{P}_1$ such that $\mathcal{C} = \{M\}^\intercal$.*

Proof. For the only-if part, consider $\mathcal{S} \subseteq \mathcal{P}_n^{<\omega}$ such that $\mathcal{S}^\intercal = \mathcal{C}$. Put $M = \bigoplus_{S \in A} S$ where $A$ is a representative set of elements of $\mathcal{S}$. Then $\mathcal{C} = \{M\}^\intercal = {}^\perp\{M^*\}$, so $M^* \in \mathcal{I}_1$ (because $\mathcal{C}$ is 1-cotilting). It follows that $M \in \mathcal{F}_1$. So $S \in \mathcal{F}_1 \cap \text{mod-}R = \mathcal{P}_1^{<\omega}$ for each $S \in A$, and $M \in \mathcal{P}_1$.

For the if part, let $(\mathcal{A}, \mathcal{B})$ be the cotorsion pair cogenerated by $M$. Since $\mathcal{A} \subseteq \mathcal{P}_1$, it suffices to show that $\{M\}^\intercal = (\mathcal{A}^{<\omega})^\intercal$. By 4.5, $(\mathcal{A}^{<\omega})^\intercal \subseteq \{M\}^\intercal$. Conversely, let $N \in \{M\}^\intercal$. Then $N^* \in \mathcal{B}$, so $N \in \mathcal{A}^\intercal \subseteq (\mathcal{A}^{<\omega})^\intercal$. □

Since classes of cofinite type are closed under direct limits, any cotilting module of cofinite type is pure-injective by 3.3.

The bijective correspondence between tilting classes of finite type and cotilting classes of cofinite type is mediated by resolving subclasses of mod-$R$. It is analogous to the classical characterization of cotilting classes in mod-$R$ over artin algebras due to Auslander and Reiten [11].

**Definition 4.11.** Let $R$ be a ring and $\mathcal{S} \subseteq \text{mod-}R$. Then $\mathcal{S}$ is *resolving* provided that $\mathcal{P}_0^{<\omega} \subseteq \mathcal{S}$, $\mathcal{S}$ is closed under direct summands and extensions, and $\mathcal{S}$ is closed under kernels of epimorphisms.

Notice that a subclass $\mathcal{S} \subseteq \mathcal{P}_1^{<\omega}$ is resolving iff $\mathcal{S}$ is closed under extensions and direct summands, and $R \in \mathcal{S}$.

**Theorem 4.12.** [5] *Let $R$ be a ring and $n < \omega$. There is a bijective correspondence among*

- *$n$-tilting classes of finite type,*
- *resolving subclasses of $\mathcal{P}_n^{<\omega}$,*
- *$n$-cotilting classes of cofinite type in $R$-Mod.*

Proof. Given an $n$-tilting class of finite type $\mathcal{T} \subseteq \text{Mod-}R$, we put $\mathcal{S} = {}^\perp\mathcal{T} \cap \text{mod-}R$; conversely, given a resolving subclass $\mathcal{S} \subseteq \mathcal{P}_n^{<\omega}$, we let $\mathcal{T} = \mathcal{S}^\perp$. These assignments are mutually inverse. Similarly, given an $n$-cotilting class of cofinite type $\mathcal{C} \subseteq$

$R$-Mod, we let $\mathcal{S} = {}^{\intercal}\mathcal{C} \cap \text{mod-}R$; conversely, $\mathcal{C} = \mathcal{S}^{\intercal}$. For more details, we refer to [5]. □

Moreover, if $T$ is an $n$-tilting module of finite type then $T^*$ is an $n$-cotilting left $R$-module of cofinite type; in the correspondence of 4.12, the $n$-tilting class $\{T\}^{\perp}$ corresponds to the $n$-cotilting class ${}^{\perp}\{T^*\} = \{T\}^{\intercal}$, cf. [5].

**Lemma 4.13.** [5] *Let $R$ be a left noetherian ring and $C$ be a 1-cotilting left $R$-module. Then ${}^{\perp}\{C\} = \{C^*\}^{\intercal}$.*

Proof. By 3.5, $C$ is pure-injective, so $C$ is a direct summand in $C^{**}$. In particular, $\{C^*\}^{\intercal} = {}^{\perp}\{C^{**}\} \subseteq {}^{\perp}\{C\}$. Conversely, take $M \in R$-mod. If $\operatorname{Ext}^1_R(M, C) = 0$, then the Ext-Tor relations yield $\operatorname{Tor}^R_1(C^*, M) = 0$. Since $R$ is left noetherian, if $N \in {}^{\perp}\{C\}$ then $N$ is a directed union, $N = \bigcup_{i \in I} N_i$, of submodules of $N$ such that $N_i \in {}^{\perp}\{C\} \cap R$-mod for all $i \in I$. So $N_i \in \{C^*\}^{\intercal}$. Since Tor commutes with direct limits, we have $N \in \{C^*\}^{\intercal}$. This proves that ${}^{\perp}\{C\} = \{C^*\}^{\intercal}$. □

4.10 and 4.13 yield a partial converse of 4.9:

**Theorem 4.14.** *Let $R$ be a left noetherian ring. Assume that $\mathcal{F}_1 = \mathcal{P}_1$ (this holds when $R$ is (i) right perfect or (ii) right hereditary or (iii) 1-Gorenstein, for example). Then every 1-cotilting left $R$-module is of cofinite type.*

Proof. Let $C$ be a 1-cotilting left $R$-module. Then $C^* \in \mathcal{F}_1 = \mathcal{P}_1$. By 4.13, ${}^{\perp}\{C\} = \{C^*\}^{\intercal}$. The latter class is of cofinite type by 4.10. □

4.14 applies to the left artinian case:

**4.15. 1-cotilting classes over left artinian rings.** Let $R$ be a left artinian ring. Then 1-cotilting classes of left $R$-modules are of cofinite type, hence coincide with the classes of the form $\{M \in R\text{-Mod} \mid \operatorname{Tor}^R_1(S, M) = 0 \text{ for all } S \in \mathcal{S}\}$ for some $\mathcal{S} \subseteq \mathcal{P}_1^{<\omega}$. Moreover, by 4.12, these classes correspond bijectively to the classes $\mathcal{S}'$ closed under extensions and direct summands, and satisfying $\mathcal{P}_0^{<\omega} \subseteq \mathcal{S}' \subseteq \mathcal{P}_1^{<\omega}$. By 3.10, they also correspond to torsion-free classes in mod-$R$ containing $R$.

Assume $R$ is an artin algebra. Then it is open whether each 1-tilting class is of finite type. By 4.3, 1-tilting classes of finite type correspond bijectively to torsion classes in mod-$R$ containing all finitely generated injective modules.

In general, the converse of 4.9 does not hold: there exist Prüfer domains with 1-cotilting modules that are not of cofinite type. We are going to discuss the Prüfer and Dedekind domain cases in detail:

**4.16. Tilting and cotilting classes over Prüfer domains.** [19], [70], [71], [18] Let $R$ be a Prüfer domain. Then all tilting modules have projective dimension $\leq 1$, and they are of finite type. Moreover, for each 1-tilting class, $\mathcal{T}$, there is a set, $\mathcal{E}$, of non-zero finitely generated (projective) ideals of $R$ such that $\mathcal{T}$ consists of all modules $M$ satisfying $IM = M$ for all $I \in \mathcal{E}$ (or, equivalently, $\operatorname{Ext}^1_R(R/I, M) = 0$ for all $I \in \mathcal{E}$). This is proved in [19] and [18], using 4.7.

Moreover, tilting classes correspond bijectively to finitely generated localizing systems, $\mathcal{I}$, of $R$ in the sense of [45, §5.1]. (A multiplicatively closed filter $\mathcal{I}$ of non-zero ideals of $R$ is a *finitely generated localizing system* provided that $\mathcal{I}$ contains a basis consisting of finitely generated ideals; by [45, 5.1], finitely generated localizing systems correspond bijectively to overrings of $R$.) Given such system $\mathcal{I}$, the corresponding tilting class consists of all modules $M$ satisfying $IM = M$ for all $I \in \mathcal{I}$, cf. [71]. The notion of the Fuchs tilting module from 2.2 can be extended to give a classification of all tilting modules over Prüfer domains up to equivalence – for more details, we refer to [71].

By [20], all cotilting modules have injective dimension $\leq 1$. By (the proof of) 4.12, the cotilting classes of cofinite type coincide with the classes of the form

$\{M \mid \mathrm{Tor}_1^R(M, R/I) = 0$ for all $I \in \mathcal{I}\}$ where $\mathcal{I}$ is a finitely generated localizing system. However, by [17], there exist maximal valuation domains $R$ such that the class of all Whitehead modules is 1-cotilting, but not of cofinite type.

A complete description is available for Dedekind domains:

**4.17. Tilting and cotilting modules over Dedekind domains.** [19] Let $R$ be a Dedekind domain. By 2.3, for each set of maximal ideals, $P$, there is a tilting module $T_P = R_P \oplus \bigoplus_{p \in P} E(R/p)$ with the corresponding tilting class $\{T_P\}^\perp = \{M \mid pM = M$ for all $p \in P\}$. Since localizing systems of ideals of $R$ are determined by their prime ideals, by 4.16, any tilting module $T$ is equivalent to $T_P$ for a set of maximal ideals $P$, cf. [19]. (In the particular case when $R = \mathbb{Z}$, and $R$ is a small Dedekind domain, this result was proved assuming V = L in [50] and [77], respectively).

By 4.12, cotilting classes of cofinite type are exactly the classes of the form $\mathcal{C}_P = \{M \mid \mathrm{Tor}_1^R(M, R/p) = 0$ for all $p \in P\}$ for a set, $P$, of maximal ideals of $R$. Moreover, $\mathcal{C}_P = {}^\perp\{C_P\}$ where $C_P = \prod_{p \in P} J_p \oplus \bigoplus_{q \in \mathrm{Spec}(R) \setminus P} E(R/q)$ is a cotilting module. (Here, $J_p$ denotes the completion of the localization of $R$ at $p$).

By 4.14 (or 3.6), all cotilting classes are of the form $\mathcal{C}_P$, and all cotilting modules are equivalent to the modules of the form $C_P$, for a set, $P$, of maximal ideals of $R$, cf. [40].

The analogy between modules over Dedekind domains and over tame hereditary algebras (cf. 2.3 and 2.4) extends to the tilting and cotilting setting: we refer to [26], [27], and [73], for more details.

We finish this section by an open problem:

*Let $R$ be a ring, $n \geq 1$, and $T$ be an $n$-tilting module. Is $T$ of countable type? Is it definable? Is it of finite type?* (By 4.7, the first question has a positive answer in case $n = 1$. In that case, the second and the third questions are equivalent by 4.6.)

## 5. Tilting modules and the finitistic dimension conjectures

Let $R$ be a ring and $\mathcal{C}$ be a class of modules. The *$\mathcal{C}$-dimension* of $R$ is defined as the supremum of projective dimensions of all modules in $\mathcal{C}$.

If $\mathcal{C} = $ Mod-$R$ then the $\mathcal{C}$-dimension is called the (right) *global dimension* of $R$; if $\mathcal{C} = \mathcal{P}$, it is called the *big finitistic dimension* of $R$. If $\mathcal{C}$ is the class of all finitely generated modules in $\mathcal{P}$ then the $\mathcal{C}$-dimension is called the *little finitistic dimension* of $R$. These dimensions are denoted by gl.dim($R$), Fin.dim($R$), and fin.dim($R$), respectively.

Clearly, fin.dim($R$) $\leq$ Fin.dim($R$) $\leq$ gl.dim($R$) for any ring $R$. Moreover, if $R$ has finite global dimension, then gl.dim($R$) is attained on cyclic modules, so all the three dimensions coincide.

If $R$ has infinite global dimension, then the finitistic dimensions take the role of the global dimension to provide a fine measure of complexity of the module category. For example, if $R = \mathbb{Z}_{p^n}$ for a prime integer $p$ and $n > 1$, then $R$ has infinite global dimension, but both finitistic dimensions are 0; they certainly reflect better the fact that $R$ is of finite representation type.

In [13], Bass considered the following assertions

(I) fin.dim($R$) = Fin.dim($R$)

(II) fin.dim($R$) is finite

and proposed to investigate the validity of these assertions in dependence on the structure of the ring $R$. Later, (I) and (II) became known as the *first*, and the *second*, *finitistic dimension conjecture*, respectively.

In the case when $R$ is commutative noetherian, Bass, Raynaud and Gruson proved that Fin.dim($R$) coincides with the Krull dimension of $R$, so classical examples of Nagata can be used to provide counter-examples to the assertion (II). In case $R$ is commutative local noetherian, Auslander and Buchweitz proved that fin.dim($R$) coincides with the depth of $R$, so (I) holds iff $R$ is a Cohen-Macaulay ring.

Assume that $R$ is right artinian. Then the validity of (II) is still an open problem. However, Huisgen-Zimmermann proved that (I) need not hold even for monomial finite dimensional algebras, [53]. Smalø then constructed, for any $1 < n < \omega$, examples of finite dimensional algebras such that fin.dim($R$) = 1 and Fin.dim($R$) = $n$, [72].

There are many positive results available: (II) was proved for all monomial algebras in [52], for algebras of representation dimension $\leq 3$ in [58] etc.

Moreover, (I) and (II) were proved for all algebras such that $\mathcal{P}^{<\omega}$ is contravariantly finite in [11] and [56]. In this section, we will use tilting approximations to give a simple proof of the latter result. Then we will prove (I) for all Iwanaga-Gorenstein rings.

*In the rest of this section, $R$ will be a right noetherian ring. We will denote by $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ the cotorsion pair cogenerated by the class $\mathcal{P}^{<\omega}$.* By 1.13, $\mathfrak{C}$ is complete and hereditary; moreover, $\mathcal{P}^{<\omega} = \mathcal{A} \cap \text{mod-}R$.

The basic relation between tilting approximations and the finitistic dimension conjectures comes from [7]:

**Theorem 5.1.** *Let $R$ be a right noetherian ring. Then (II) holds iff $\mathfrak{C}$ is a tilting cotorsion pair. Moreover, if $T$ is a tilting module such that $\{T\}^\perp = \mathcal{B}$, then fin.dim($R$) = proj.dim($T$).*

Proof. Assume fin.dim($R$) $= n < \omega$. Then $\mathcal{P}^{<\omega} \subseteq \mathcal{P}_n$, so $\mathcal{B}$ is of finite type, and $\mathfrak{C}$ is a tilting cotorsion pair by 4.2. Conversely, if $\mathfrak{C}$ is $n$-tilting then $\mathcal{P}^{<\omega} \subseteq \mathcal{P}_n$, so (II) holds. Since fin.dim($R$) is the least $m$ such that $\mathcal{A} \subseteq \mathcal{P}_m$, we infer that fin.dim($R$) = proj.dim($T$). □

A dual version of 5.1 for artin algebras appears in [28].

**5.2.** The tilting module $T$ in 5.1 is unique up to equivalence, and it is clearly of finite type. In principle, $T$ can be constructed as in the proof of 2.9: that is, by an iteration of special $\mathcal{B}$-preenvelopes of $R$ etc. yielding an Add($T$)-coresolution of $R$, $0 \to R \to T_0 \to \cdots \to T_n \to 0$, and giving $T = \bigoplus_{i \leq n} T_i$. However, little is known of the (definable) class $\mathcal{B}$ in general, so this construction is of limited use. (The construction works fine for gl.dim($R$) $< \infty$. Then $\mathcal{B} = \mathcal{I}_0$, so the Add($T$)-coresolution above can be taken as the minimal injective coresolution of $R$.)

In the artinian case, we can compute fin.dim($R$) using $\mathcal{A}$-approximations of all the (finitely many) simple modules. This is proved in [76], generalizing [11]:

**Theorem 5.3.** *Let $R$ be a right artinian ring and $\{S_0, \dots S_m\}$ be a representative set of all simple modules. For each $i \leq m$, take a special $\mathcal{A}$-preenvelope of $S_i$, $f_i : A_i \to S_i$. Then fin.dim($R$) = $max_{i \leq m}$proj.dim($A_i$).*

*Moreover, all the modules $A_i$ ($i \leq m$) can be taken finitely generated iff $\mathcal{P}^{<\omega}$ is contravariantly finite. In this case (II) holds true, since $\mathcal{P}^{<\omega} = \mathcal{A} \cap$ mod-$R$.*

Now, we will relate pure-injectivity properties of the tilting module $T$ from 5.1 to closure properties of the class $\mathcal{A}$.

A module $M$ is *pure-split* if all pure submodules of $M$ are direct summands; $M$ is $\sum$-*pure-split* iff all modules in Add(M) are pure-split. For example, any $\sum$-pure-injective module is $\sum$-pure-split, [55].

A module $M$ is *product complete* if $\mathrm{Prod}(M) \subseteq \mathrm{Add}(M)$. Any product complete module is $\sum$-pure-injective, [59].

The following is proved in [7] and [8]:

**Lemma 5.4.** *Let $R$ be a right noetherian ring satisfying (II). Let $T$ be the tilting module from 5.1. Then*

(1) *$T$ is $\sum$-pure-split iff $\mathcal{A}$ is closed under direct limits.*
(2) *$T$ is product complete iff $\mathcal{A}$ is closed under products iff $\mathcal{A}$ is definable.*
(3) *$\mathcal{A} = \mathcal{P}$ iff Add($T$) is closed under cokernels of monomorphisms.*

**5.5.** The condition $\mathcal{A} = \mathcal{P}$ implies (I), since any module of finite projective dimension is then a direct summand in a $\mathcal{P}^{<\omega}$-filtered module, by 1.13. In fact, when proving the first finitistic dimension conjectures in 5.6 and 5.8 below, we will always prove that $\mathcal{A} = \mathcal{P}$. However, (I) may hold even if $\mathcal{A} \subsetneq \mathcal{P}$, see [8].

**Theorem 5.6.** [7] *Let $R$ be an artin algebra such that (II) holds. Let $T$ be the tilting module from 5.1. Then $T$ can be taken finitely generated iff $\mathcal{P}^{<\omega}$ is contravariantly finite. In this case, (I) holds.*

Proof. If $\mathcal{P}^{<\omega}$ is contravariantly finite, then $\mathcal{B}^{<\omega}$ is covariantly finite (by a version of 1.8.3 in mod-$R$). As in the proof of 2.9, an iteration of the $\mathcal{B}^{<\omega}$-envelopes of $R$ etc. yields an Add($T$)-coresolution of $R$, $0 \to R \to T_0 \to \cdots \to T_n \to 0$. Then $T' = \bigoplus_{i \leq n} T_i$ is a finitely generated tilting module equivalent to $T$. The converse implication follows from [11].

If $T$ is finitely generated then $T$ is $\sum$-pure injective, and [7] gives that Add($T$) is closed under cokernels of monomorphisms. By 5.4.3, $\mathcal{A} = \mathcal{P}$, so (I) holds true. □

5.3 and 5.6 now give

**Corollary 5.7.** [11], [56] *Let $R$ be an artin algebra such that $\mathcal{P}^{<\omega}$ is contravariantly finite. Then (I) and (II) hold for $R$.*

Note that all right serial artin algebras satisfy the assumption of 5.7, see [54]. However, there are finite dimensional algebras $R$ with $\mathrm{fin.dim}(R) = \mathrm{Fin.dim}(R) = 1$ such that $\mathcal{P}^{<\omega}$ is not contravariantly finite, for example the IST-algebra [57]; for those algebras, $T$ is an infinitely generated 1-tilting module.

Finally, we turn to Iwanaga-Gorenstein rings (see 2.6). Let $n < \omega$ and $R$ be $n$-Gorenstein. Then $\mathcal{P} = \mathcal{I} = \mathcal{P}_n = \mathcal{I}_n$. In particular, there exist cotorsion pairs $\mathfrak{D} = (\mathcal{P}, \mathcal{GI})$ and $\mathfrak{E} = (\mathcal{GP}, \mathcal{I})$. The modules in $\mathcal{GI}$ are called *Gorenstein injective*, the ones in $\mathcal{GP}$ *Gorenstein projective*. The kernel of $\mathfrak{D}$ equals $\mathcal{I}_0$, the kernel of $\mathfrak{E}$ is $\mathcal{P}_0$, cf. [42]. Clearly, $\mathrm{Fin.dim}(R) = n$, so (II) holds.

By [5], also (I) holds:

**Theorem 5.8.** *Let $R$ be an Iwanaga-Gorenstein ring. Then (I) holds true. Moreover, the tilting module $T$ from 5.1 can be taken of the form $T = \bigoplus_{i \leq n} I_i$ where $0 \to R \to I_0 \to \cdots \to I_n \to 0$ is the minimal injective coresolution of $R$.*

Proof. By 1.4.1, the cotorsion pair $\mathfrak{D} = (\mathcal{P}, \mathcal{GI})$ is of countable type. By [19], for each $C \in \mathcal{P}^{\leq\omega}$ there is a $\mathcal{P}^{<\omega}$-filtered module $D$ such that $D = C \oplus P$ where $P \in \mathcal{P}_0$. So $C \in \mathcal{A}$, that is, $\mathcal{A} = \mathcal{P}$, and (I) holds. Since the minimal $\mathcal{GI}$-coresolution of $R$ is actually its minimal injective coresolution, and $\mathfrak{E} = \mathfrak{D}$, $T$ can be taken as claimed by 5.2. □

If $R$ in 5.8 is an artin algebra, then $T$ is finitely generated. So by 5.6, Iwanaga-Gorenstein artin algebras give yet another example of algebras with $\mathcal{P}^{<\omega}$ contravariantly finite, [11].

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