# Basic Tradeoffs for Energy Management in Rechargeable Sensor Networks

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***Abstract*—As many sensor network applications require deployment in remote and hard-to-reach areas, it is critical to ensure that such networks are capable of operating unattended for long durations. Consequently, the concept of using nodes with energy replenishment capabilities has been gaining popularity. However, new techniques and protocols must be developed to maximize the performance of sensor networks with energy replenishment. Here, we analyze limits of the performance of sensor nodes with limited energy, being replenished at a variable rate. We provide a simple localized energy management scheme that achieves a performance close to that with an unlimited energy source, and at the same time keeps the probability of complete battery discharge low. Based on the insights developed, we address the problem of energy management for energy-replenishing nodes with finite battery and finite data buffer capacities. To this end, we give an energy management scheme that achieves the optimal utility asymptotically while keeping both the battery discharge and data loss probabilities low.**

## I. Introduction

Advances in wireless networking combined with data acquisition have enabled us to remotely sense our environment [1], [2]. As these applications may require deployment in hard-to-reach areas, it is critical to ensure that such networks are capable of operating with full autonomy for long durations. The lack of a continuous power source in most scenarios and the limited lifetime of batteries have hindered the deployment of such networks. However, developments in renewable energy sources [3]–[9] suggest that it is feasible for sensor networks to operate unattended for extended periods. These renewable sources of energy typically provide energy replenishment at a rate that could be variable and dependent on the surroundings. Examples include, self-powered sensors that rely on harvesting strain and vibration energies from their working environment [4], as well as sensors with solar cells [5]–[7].

In this paper, we analyze the *limits* of the *performance* of networks comprised of sensor nodes with limited energy, being replenished at a variable rate. We provide a simple localized *energy management scheme* that achieves a performance, close to the optimal scheme that has access to an unlimited energy reservoir. Indeed, we show that, if the performance can be measured by a general utility function of the energy, under mild assumptions on the replenishment process, it is possible to observe a polynomial decay for the probability of complete battery discharge, and at the same time achieve a $\Theta\left(\frac{(\log M)^2}{M^2}\right)$ convergence to the optimal achievable utility[1]. Here $M$ is the total capacity of the energy source. Based on the insights developed, we address the problem of energy management in the presence of a finite data buffer. We modify our basic energy management scheme to achieve a $\Theta\left(\frac{(\log K)^2}{K^2}\right)$ convergence to the maximum utility achievable by a scheme that has access to an infinite data and energy buffers. Here $K$ is the data buffer size. In addition, this scheme achieves an exponential decay with $M$ for the battery discharge probability and a polynomial decay with $K$ for the data loss probability. To evaluate these decay rates, the main tools we use are the *large deviations theory* and *diffusion approximations*.

The added dimension of renewable energy makes the problem of energy management in sensor networks substantially different from its non-replenishment counterpart. For nodes with replenishment, conservative energy expenditure may lead to missed recharging opportunities due to battery capacity limitations. On the other hand, aggressive usage of energy may cause battery outages that leads to lack of coverage or connectivity for certain time periods. Thus, new techniques must be developed to balance these seemingly contradictory goals to maximize performance. Here, our *main goal* will be to identify the performance limits of sensor nodes with energy replenishment and provide guidelines to approach these limits.

Many fundamental wireless communication and networking problems can be stated as *utility maximization* problems, subject to energy constraints. The utility function can be the throughput (e.g., in energy efficient routing), the probability of detection of an intruder (e.g., in coverage) or the network lifetime (e.g., in sleep-wake scheduling) or the achievable rate of reliable transmission in basic wireless communication. These problems have been mainly addressed for stations with unlimited and/or non-replenishing energy sources. Here, we address the problem of maximizing a utility function of the data transmission rate in the presence of energy replenishment. The solution of the optimization problem requires stochastic optimization techniques involving high computational overheads that might be unsuitable for sensor nodes. Consequently, we will focus our attention on simple localized solutions that achieve near-optimal or asymptotically optimal performance. We use tools from large deviations theory and diffusion approximation to find closed-form expressions for the data loss and the battery discharge probabilities. These techniques allow us to analyze our schemes under mild assumptions on the battery charging and data arrival processes.

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[1]The following notations will be used to compare rates of convergence: $a_n = \mathrm{O}(b_n)$ if $a_n$ goes to zero at least as fast as $b_n$; $a_n = \mathrm{o}(b_n)$ if $a_n$ goes to zero strictly faster than $b_n$; $a_n = \Theta(b_n)$ if $a_n$ and $b_n$ go to zero at the same rate; $a_n = \Omega(b_n)$ if $a_n$ goes to zero no faster than $b_n$.

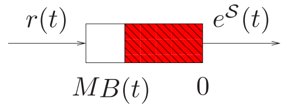



Fig. 1. Energy source with a replenishment rate $r(t)$.

There have been recent works that have studied different problems in networks with energy replenishment. Kar, et. al., [10] have proposed an activation scheme for rechargeable sensors that maximizes the network-level utility of sensing networks. The utility function in [10] depends on the number of active sensors. Gatzianas, et. al., [11] use back pressure policies to maximize the network flow of information in networks with energy replenishment. While [10] and [11] look at the total system utility, we will focus on the analyzing node-level performance leading to localized energy management schemes. Liu, et. al., [12] have derived a battery control scheme similar to the one described in this work. In addition to providing stronger convergence results than the one in [12] with sole battery control, we also consider the effect of a finite data buffer in this paper. Kansal, et. al., [13] introduce the concept of energy neutral operation, wherein the energy consumed by a node is less than or equal to the energy harvested. Vigorito, et. al., [14] extend the idea of energy neutral operation to propose an algorithm that attempts to keep the battery state close to a fixed level and at the same time stabilizes the duty cycle in order to maximize system performance. Sharma, et. al., [15] have proposed a throughput optimal energy management scheme for energy harvesting nodes. However, these works [13]–[15] do not contain an analytical evaluation of the battery discharge or the data loss probabilities for their energy management schemes.

The outline of this paper is as follows. We first state the general form of the utility maximization problem in Section II and show ways to achieve the maximum achievable utility with replenishing sources. In Section III we add a finite buffer to the problem and study energy management schemes that achieve optimal utility asymptotically while keeping the probabilities of battery discharge and data loss low. We numerically evaluate the performance of our energy management schemes in Section IV. We wrap up with conclusions in Section V.

## II. Achieving Maximum Utility With a Finite-Battery Constraint

### A. System Model and Problem Statement

Fig. 1 shows the energy source (or the battery) of a node. The total capacity of this battery is $M$ units of energy. We denote the total available energy in the battery as $B(t)$, where $t$ is the discrete time index. The battery replenishes at a rate $r(t)$. The replenishment process $\{r(t), t \geq 1\}$ is assumed to be an ergodic stochastic process with a long term mean $\lim_{\tau\to\infty} \frac{1}{\tau}\sum_{t=1}^{\tau} r(t) = \mu$. A power allocation policy $\mathcal{S}$ draws energy from this battery at a rate $e^{\mathcal{S}}(t)$ to achieve certain tasks. The success of the node in achieving these tasks is measured in terms of a concave and non-decreasing utility function[2] $U(e^{\mathcal{S}}(t))$ of the consumed energy $e^{\mathcal{S}}(t)$. We define the time average utility,

$$\bar{U}^{\mathcal{S}}(\tau) = \frac{1}{\tau}\sum_{t=1}^{\tau} U(e^{\mathcal{S}}(t)). \tag{1}$$

We consider the optimization problem in which a node tries to maximize its long-term average utility, $\bar{U}^{\mathcal{S}} = \lim_{\tau\to\infty} \bar{U}^{\mathcal{S}}(\tau)$, subject to battery constraints:

$$\begin{aligned} \max_{\{e^{\mathcal{S}}(t),\ t\geq 1\}} \quad & \bar{U}^{\mathcal{S}} \qquad\qquad (2)\\ \text{subject to} \quad & B(t) = \max\big\{0, \min\{M, B(t-1) \\ & \qquad + r(t) - e^{\mathcal{S}}(t-1)\}\big\} \\ \text{and} \quad & e^{\mathcal{S}}(t) \leq B(t). \end{aligned}$$

One approach to solving this optimization problem is by using Markov decision process (MDP) techniques. Since solving MDPs is computationally intensive, these methods may not be suitable for computationally-limited sensor nodes. Consequently, we seek schemes that are easy to implement and yet achieve close to optimal performance. The next lemma gives an upper bound for the asymptotic time-average utility achieved over all ergodic energy management policies.

**Lemma 1.** *Let $\bar{U}^{*\mathcal{S}}$ be the solution to Problem (2). Then, $\bar{U}^{*\mathcal{S}} \leq U(\mu)$.*

The proof of this lemma, given in Appendix A, uses Jensen's inequality and conservation of energy arguments. Lemma 1 tells us that for any ergodic energy management scheme $\mathcal{S}$, $\bar{U}^{\mathcal{S}} \leq U(\mu)$. With an unlimited energy reservoir (i.e., $M = \infty$) and average energy replenishment rate $\mu$, if one uses $e^{\mathcal{S}}(t) = \mu$ for all $t \geq 1$, this upper bound can be achieved. However, if $M < \infty$, achieving $\bar{U}^{\mathcal{S}} = U(\mu)$ using this simple scheme is not possible. Indeed, due to finite energy storage and variability in $r(t)$, $B(t)$ will occasionally get discharged completely. At such instances, $e^{\mathcal{S}}(t)$ has to be set to $0$, which will reduce the time-average utility.

The question we answer next is, "how close can the average utility $\bar{U}^{\mathcal{S}}$ get to the upper bound $U(\mu)$ asymptotically, as $M \to \infty$, while keeping the long-term battery discharge rate low?"

### B. An Asymptotically Optimal Power Allocation Scheme

In this section we show that there is a trade-off between achieving maximum utility and keeping the discharge rate low. First, we make some weak assumptions on the replenishment process $r(t)$, which we will be using throughout this paper. In particular, we assume that the asymptotic semi-invariant log moment generating function,

$$\bar{\Lambda}_r(s) = \lim_{\tau\to\infty} \frac{1}{\tau}\log \mathbb{E}\left[\exp\left(s\sum_{t=1}^{\tau} r(t)\right)\right], \tag{3}$$

of $r(t)$ exists for $s \in (-\infty, s_{\max})$, for some $s_{\max} > 0$. We also assume that the asymptotic variance $\bar{\sigma}_r^2 \triangleq \lim_{\tau\to\infty} \frac{1}{\tau}\mathrm{var}\left(\sum_{t=1}^{\tau} r(t)\right)$ of $r(t)$ exists. Note that, in practice, the recharging process is not necessarily stationary. While this assumption does allow the possibility that the statistics of

[2]Note that, in many practical scenarios, it is reasonable to assume that the utility function is non-decreasing and concave, since there is diminishing returns for increasing power.

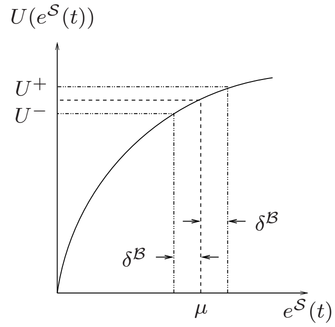



Fig. 2. With scheme $\mathcal{B}$, utility alternates between $U^+$ and $U^-$

$r(t)$ has variations (e.g., due to clouds and the solar power at different times of the day), it rules out the possibility of long-range dependencies in $r(t)$.

From the discussion in previous section, we can infer that by choosing a battery drift that goes to zero with increasing battery size, one might achieve a long-term average utility that is close to $U(\mu)$ as $M$ increases. However, smaller drift away from the empty battery state implies a more frequent occurrence of the complete battery discharge event. In the following theorem, we quantify this tradeoff between the achievable utility and the battery discharge rate, asymptotically in the large battery regime. In this regime, the battery size $M$ is large enough for the variations in $r(t)$ to average out nicely over the time scale that $B(t)$ changes significantly. Consequently, we now define the long-term battery discharge rate as the probability of discharge, i.e., $p_{\text{discharge}}(M) \triangleq \lim_{\tau\to\infty} \frac{1}{\tau}\sum_{t=1}^{\infty} \mathcal{I}_0^B(t)$, where the indicator variable $\mathcal{I}_0^B(t) = 1$ if $B(t) = 0$ and is identical to 0 otherwise.

Next, we show that one can achieve a battery discharge probability that exhibits a polynomial decay with the battery size, and at the same time achieves a utility that approaches the maximum achievable utility as $(\log M)^2/M^2$.

**Theorem 1.** *Consider any continous, concave and non-decreasing utility function $U(e(t))$ such that $\left|\frac{\partial^2 U(e)}{\partial e^2}\right| < \infty$ for all $e > 0$. Given any $\beta \geq 2$, there exists a power allocation scheme $\mathcal{B}$ such that the associated battery discharge probability $p^{\mathcal{B}}_{\text{discharge}}(M) = \Theta(M^{-\beta})$ and $U(\mu) - \bar{U}^{\mathcal{B}} = \Theta\left(\left(\frac{\log M}{M}\right)^2\right)$.*

We give a brief sketch of the proof, details of which can be found in Appendix B. Our proof is constructive as we show a strategy that achieves the asymptotic convergence rates given in Theorem 1. Our scheme is motivated by the buffer control strategy introduced in [16] to achieve the near-optimal distortion for variable rate lossy compression. Consider the allocation scheme $\mathcal{B}$ in which

$$e^{\mathcal{B}}(t) = \begin{cases} \mu - \delta^{\mathcal{B}}, & B(t) < M/2 \\ \mu + \delta^{\mathcal{B}}, & B(t) \geq M/2 \end{cases}, \tag{4}$$

for some $\delta^{\mathcal{B}} > 0$. As shown in Fig. 2, the instantaneous utility associated with Scheme $\mathcal{B}$ alternates between $U^-$ and $U^+$, depending on the battery state. By choosing $\delta_1^{\mathcal{B}} = \beta\bar{\sigma}_r^2 \frac{\log M}{M}$ for some $\beta \geq 2$, we show that long-term maximum utility $U(\mu)$ can be achieved asymptotically while achieving decay, as a polynomial of arbitrarily high order, for the battery discharge probability. We note that the while the order of the polynomial decay $\beta$ can be made arbitrarily large, it comes at the expense of slower convergence (by some constant factor) to the maximum utility function.

Here, we illustrated that with a simple scheme, it is possible to achieve desirable scaling laws for the performance of a given task, under the assumption that the asymptotic moment generating function of the replenishment process exists. To illustrate the theorem we consider a specific example.

**Example 1 Capacity of an AWGN Channel**

We study the basic limits of point to point communication with finite but replenishing energy sources. For simplicity, we consider the additive white Gaussian noise (AWGN) channel. At time $t$, the transmitter transmits a complex valued block (vector of symbols) $\mathbf{X}(t)$ and the receiver receives $\mathbf{Y}(t)$. We have,

$$\mathbf{Y}(t) = h\mathbf{X}(t) + \mathbf{W}(t), \tag{5}$$

where the channel gain $h$ is a complex constant and $\mathbf{W}(t)$ is additive white (complex) Gaussian noise with two sided power spectral density $N_0/2$. We define the channel SNR as $\gamma = \mathbb{E}\left[|h|^2\right]/N_0$. The maximum amount of data that could be reliably communicated [17] over this channel with an amount of energy $e(t)$ at time $t$ is,

$$C(e(t)) = \log_2(1 + e(t)\gamma) \text{ bits/channel use}, \tag{6}$$

assuming the block size is long enough so that sufficient averaging of additive noise is possible. Thus, the rate at which reliable communication can be achieved is a concave non-decreasing function of the transmit power and it can be viewed as our utility function. Consequently, using a constant power $\mu$, the maximum utility of $\bar{C} = C(\mu)$ can be achieved, which is the famous AWGN channel capacity result.

Now, we generalize the AWGN capacity result to the case with finite energy sources. Suppose that we want to transmit the maximum amount of data over the AWGN channel, using a battery of energy capacity $M$ and a replenishment rate $r(t)$. We assume that each time slot is long enough for sufficiently long code blocks to be formed. We substitute $U(\cdot)$ with $C(\cdot)$ in Eq. (1) to get the relevant optimization problem. With an unlimited energy source ($M = \infty$) of limited average power $\mu$, the maximum achievable long term average rate, i.e., the channel capacity is $C(\mu) = \log_2(1 + \mu\gamma)$ bits/channel use. By using the energy management scheme $\mathcal{B}$ given in Eq. (4), an average rate $\bar{C}^{\mathcal{B}}$ can be achieved such that $C(\mu) - \bar{C}^{\mathcal{B}} = \Theta\left(\frac{(\log M)^2}{M^2}\right)$ while the battery discharge probability follows $p^{\mathcal{B}}_{\text{discharge}}(M) = \Theta(M^{-\beta})$ for some $\beta \geq 2$.

### C. Basic Limits of Power Allocation Schemes

To understand the strength of Theorem 1, we note that it is not trivial to achieve decaying discharge probability and maximum utility with increasing battery size. In fact, an ergodic[3] energy management scheme cannot achieve exponential decay in discharge probability and convergence (even asymptotically)

[3]An ergodic energy management scheme $e^{\mathcal{S}}(t)$ is the one that satisfies $\lim_{\tau\to\infty} \frac{1}{\tau}\sum_{t=1}^{\tau} e^{\mathcal{S}}(t) = \mathbb{E}\left[e^{\mathcal{S}}(t)\right]$

to the maximum average utility function simultaneously. We formalize this statement in the following theorem.

**Theorem 2.** *Consider any continous, concave and non-decreasing utility function $U(\cdot)$. If an ergodic energy management scheme $\mathcal{S}$ has a discharge probability $p^{\mathcal{S}}_{\text{discharge}}(M) = \Theta(\exp(-\alpha_c M))$ for some constant $\alpha_c > 0$, then the time average utility, $\bar{U}^{\mathcal{S}}$, for Scheme $\mathcal{S}$ satisfies $U(\mu) - \bar{U}^{\mathcal{S}} = \Omega(1)$.*

The proof of this theorem is provided in Appendix C and it is similar to that of Theorem 1. We apply *large deviations* technique to the net drift of the battery process to find the decay rate of $p^{\mathcal{S}}_{\text{discharge}}(M)$ with $M$. Jensen's inequality is then used to lower bound the difference between $U(\mu)$ and $\bar{U}^{\mathcal{S}}$.

So far, we have shown how to maximize a concave non-decreasing utility function subject to battery constraints. At every point in time, one should choose a power level as close to the replenishment rate as the battery constraints allow and this way one can asymptotically achieve a performance very close to that with unlimited energy sources. The main limitation of this approach is that it may not be feasible for some applications in practice. For instance in many sensor network applications, data is stored in finite buffers for transmission. Since scheme $\mathcal{B}$ does not adapt to the buffer state, this may lead to data losses. To overcome these limitations, in the next section, we investigate energy management schemes with finite buffer and battery constraints.

## III. Achieving Maximum Utility With Finite Buffer and Battery Constraints

### A. System Model and Problem Statement

In this section, we extend the problem introduced in Section II to the case when data packets arrive at a node and are kept in a finite buffer before transmission. Hence, the task is to transmit packets arriving at the data buffer without dropping them due to exceeding the buffer capacity. We define $Q(t)$ as the data queue state at time $t$, and the data buffer size is $K < \infty$. The data arrival process $a(t)$, represents the amount of data (in bits) arriving at the data buffer in the time slot $t$. The process $\{a(t), t \geq 1\}$ is an ergodic process independent of the energy replenishment process $\{r(t), t \geq 1\}$ and $\mathbb{E}[a(\tau)] = \lambda$. We assume that the process $a(t)$ has a finite asymptotic variance $\bar{\sigma}_a^2 = \lim_{\tau\to\infty} \frac{1}{\tau}\text{var}\left(\sum_{t=1}^{\tau} a(t)\right)$. The energy replenishment model is the same as used previously. We use $C(\cdot)$ as given in Eq.(6) as the rate-power function for the wireless channel and assume that data is served at that rate as a function of the consumed energy $e(t)$ at time $t$. We also assume that $\lambda < C(\mu)$, which is a necessary condition for system stability [15]. Without this condition, there exists no joint energy and data buffer control policy that can simultaneously keep the long-term battery discharge and data loss rates arbitrarily low.

The objective of an efficient energy management scheme in this case is to maximize the average utility function of the data transmitted subject to battery and data buffer constraints:

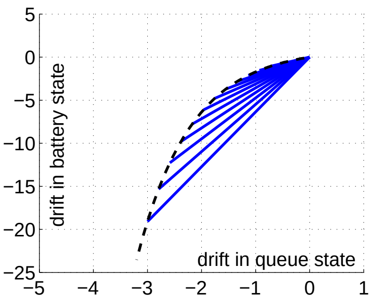



Fig. 3. Possible drift directions for $(Q(t), B(t))$ for an AWGN channel of channel SNR 0 dB. Here, at time $t$, $r(t) = 0,\ a(t) = 0$.

$$
\begin{aligned}
\max_{e(t),\ t\geq 1} \quad & \lim_{\tau\to\infty} \frac{1}{\tau} \sum_{t=1}^{\tau} U_D(C(e(t))) \qquad (7)\\
\text{subject to} \quad & B(t) = \max\{0, \min\{M, B(t-1) \\
& \qquad + r(t) - e(t-1)\}\},\\
& Q(t) = \max\{0, \min\{K, Q(t-1) \\
& \qquad + a(t) - C(e(t-1))\}\},\\
& e(t) \leq B(t) \quad \text{and} \quad C(e(t)) \leq Q(t).
\end{aligned}
$$

Here $U_D(C(e))$ is a non-decreasing concave utility gained by transmitting $C(e)$ bits. Since $\lambda < C(\mu)$, we know that $U_D(\lambda)$ is an upper bound on the achievable long-term utility with any energy management scheme. This statement can be proved using Jensen's inequality, following identical steps as the proof of Lemma 1 and we skip it to avoid repetition.

### B. An Asymptotically Optimal Energy Management Scheme

Solution of Problem (7) jointly controls the data queue state and the battery state to avoid energy outage and data overflow while maximizing the utility. The main complexity in such an approach stems from the fact that the drifts of $Q(t)$ and $B(t)$ are dependent. In Fig. 3 we illustrate the connection between the service rate and the energy consumed at a time slot $t$ for an AWGN channel with SNR 0 dB and $a(t) = 0,\ r(t) = 0$. For instance, to provide 3 units of service, the node needs to consume $\sim 18$ units of energy.

With this dependence, a critical factor one needs to take into consideration is the relative "size" of the data buffer with respect to the battery. In the sequel, we assume a *large battery regime*, where the time scale at which changes occur in $B(t)$ is much larger than the time scale at which $Q(t)$ varies. Namely, within the duration that some change occurs in $B(t)$, $Q(t)$ may fluctuate significantly. Technically, for an AWGN channel with an SNR $\gamma$, this assumption is equivalent to $M \gg \frac{1}{\gamma}(2^{\lambda}-1)K$, i.e., the total amount of energy in the battery is much larger than that required to serve a full data buffer worth of packets.

Intuitively, in large battery regime, an energy control algorithm should give "priority" to adjusting the queue state to achieve a high performance. Consequently, it should choose $e(t)$ such that the drift of $Q(t)$ is always toward a desired queue state even though this may cause battery drift to be negative. Since battery size is large, such temporary negative drifts are expected to affect the battery discharge rate only

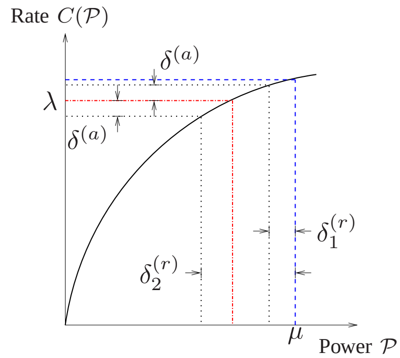



Fig. 4. Relation between $\delta^{(a)}$, $\delta_1^{(r)}$ and $\delta_2^{(r)}$.

minimally. With these observations, we state the following theorem, which indeed verifies our intuitions. This theorem shows an asymptotic tradeoff between the achieved utility and the long-term rates of discharge and data loss as $K \to \infty$. In this regime, the data buffer size $K$ is large enough for the variations in $a(t)$ to average out nicely over the time scale that $Q(t)$ changes significantly. Consequently, we now define the long-term data loss rate as the data loss probability, i.e., $p_{\text{loss}}(K) \triangleq \lim_{\tau\to\infty} \frac{1}{\tau}\sum_{t=1}^{\infty} \mathcal{I}_K^Q(t)$, where the indicator variable $\mathcal{I}_K^Q(t) = 1$ if $Q(t) = K$ and is identical to 0 otherwise.

**Theorem 3.** *Consider any non-decreasing concave utility function $U_D(\cdot)$ such that $\left|\frac{\partial^2 U_D(C(e))}{\partial e^2}\right| < \infty$ for all $e > 0$. For any $\lambda < C(\mu)$, given any $\beta_Q \geq 2$, there exists an energy management scheme $\mathcal{Q}$ that achieves a data loss probability $p_{loss}^{\mathcal{Q}}(K) = O(K^{-\beta_Q})$, battery discharge probability $p_{discharge}^{\mathcal{Q}}(M) = O(\exp(-\alpha_Q M))$ for some $\alpha_Q > 0$ and $U_D(\lambda) - \bar{U}^{\mathcal{Q}} = \Theta\left(\frac{(\log K)^2}{K^2}\right)$ under the large battery regime.*

Theorem 3 states that it is possible to have an exponential decay (with $M$) for the battery discharge probability and a polynomial decay (with $K$) for the data loss probability and at the same time achieve a time average utility that approaches the upper bound on the achievable long-term utility, $U_D(\lambda)$, as $(\log K)^2/K^2$. Note that $U_D(\lambda)$ can only be achieved with an infinite battery and data buffer sizes. We provide an outline for the proof, a full version of which can be found in Appendix D. The proof for this theorem is also constructive as we first present Scheme $\mathcal{Q}$, and then derive the performance metrics for this scheme.

Consider the energy management scheme $\mathcal{Q}$, where

$$e^{\mathcal{Q}}(t) = \begin{cases} \mu - \delta_1^{(r)}, & Q(t) \geq K/2 \\ \mu - \delta_2^{(r)}, & Q(t) < K/2 \end{cases}, \tag{8}$$

and the drifts $\delta_1^{(r)}$ and $\delta_2^{(r)}$ are chosen to satisfy the relationship

$$C(\mu - \delta_1^{(r)}) - \lambda = \lambda - C(\mu - \delta_2^{(r)}) = \beta_Q \bar{\sigma}_a^2 \frac{\log K}{K}. \tag{9}$$

From Fig. 4, we note that this choice of energy drifts correspond to a queue drift of $|\delta^{(a)}| = \beta_Q \bar{\sigma}_a^2 \frac{\log K}{K}$, toward the state $K/2$, regardless of the queue state $Q(t)$. The queue and battery drifts with scheme $\mathcal{Q}$ are illustrated in Fig. 5. We observe that even though Scheme $\mathcal{Q}$ regulates the data queue to a desired state (i.e., $K/2$), the battery is always regulated towards full state (i.e., $M$). State equation for $Q(t)$ is given by,

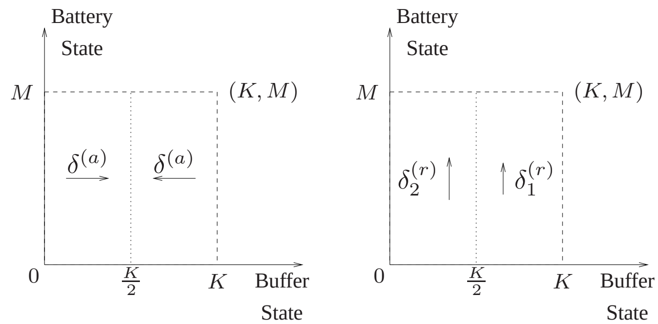



Fig. 5. Data queue and battery drifts for scheme $\mathcal{Q}$.

$$Q(t+1) = \begin{cases} \min\{K, Q(t) + a(t) - \lambda - \delta^{(a)}\}, & Q(t) \geq K/2 \\ \max\{0, Q(t) + a(t) - \lambda + \delta^{(a)}\}, & Q(t) < K/2 \end{cases}, \tag{10}$$

and the state equation for $B(t)$ is given by,

$$B(t+1) = \begin{cases} \{\min\{M, B(t) + r(t) - \mu + \delta_1^{(r)}\}\}^+, & Q(t) \geq K/2 \\ \{\min\{M, B(t) + r(t) - \mu + \delta_2^{(r)}\}\}^+, & Q(t) < K/2 \end{cases}, \tag{11}$$

where $\{a\}^+ = \max\{0, a\}$.

While $p_{\text{loss}}^{\mathcal{Q}}(K)$ can be found using a method similar to the one used in Theorem 1 (this time for the data queue) to calculate $p_{\text{discharge}}^{\mathcal{B}}(M)$, calculation of $p_{\text{discharge}}^{\mathcal{Q}}(M)$ requires a careful analysis. More specifically, the battery drift in a particular time slot depends on the data queue state, which makes the application of large deviation techniques for calculating $p_{\text{discharge}}^{\mathcal{Q}}(M)$ difficult. To show the desired order results, we transform the problem in two steps as follows:

**(T1)** We construct a sequence of arrival rates such that $\lambda \uparrow C(\mu)$. In this limiting regime, from Eq. (9), $\delta_1^{(r)}, \delta_2^{(r)} \downarrow 0$ and hence $\delta^{(a)} \downarrow 0$, for which we also use a sequence of $K$ values that increase to $\infty$. As a result, both the battery and the data queue will operate in the *heavy traffic* limit. Consequently, we apply the *diffusion approximation* to the joint queue and battery state process. The underlying continuous process is a 2-dimensional *reflected Brownian motion* (RBM), where we label $\{(\mathbf{Q}(t), \mathbf{B}(t)),\ t \geq 0\}$ as the diffusion approximation for the joint queue length and battery process. The reflections occur at the empty and full queue states, i.e., $\mathbf{Q}(t) = 0$ and $K$ and at the empty and full battery states, i.e., $\mathbf{B}(t) = 0$ and $M$. Note that, the probabilities for hitting the boundaries calculated for the associated RBM, $\mathbb{P}(\mathbf{Q}(t) = K)$ and $\mathbb{P}(\mathbf{B}(t) = 0)$ are identical to $\mathbb{P}(Q(t) = K)$ and $\mathbb{P}(B(t) = 0)$ respectively in the heavy traffic limit (Chapter 5 [18]). Furthermore, since the heavy traffic limit poses a worst case for the probabilities under consideration, the order results of the form O$(\cdot)$ shown in the heavy traffic limit hold for all $\lambda < C(\mu)$.

As we show in Appendix D, with Scheme $\mathcal{Q}$, $\mathbf{Q}(t)$ is has a symmetric distribution around state $\mathbf{Q}(t) = K/2$. Further, whenever it reaches $\mathbf{Q}(t) = K/2$, it can go above or below $\mathbf{Q}(t)$ with equal probability. Consequently, we define $\mathbf{Q}_u(t) \triangleq$

$|\mathbf{Q}(t) - K/2| + K/2$, i.e., the RBM associated with the queue process in the upper half of the buffer. Process $\mathbf{Q}_u(t)$ has reflections at $\mathbf{Q}_u(t) = K/2$ and $K$, and one can see that the probability of loss $\mathbb{P}(\mathbf{Q}(t) = K) = \frac{1}{2}\mathbb{P}(\mathbf{Q}_u(t) = K)$.

**(T2)** Closed-form analysis of the stationary distribution for the state of the RBMs is not possible except in some special cases given in [19]. Since our model does not fall in that category, the steady state reflection probabilities cannot be derived in closed form. To address this issue, we remove the upper and lower boundaries for the data buffer and the battery respectively, and allow $\{\mathbf{Q}(t),\ t \geq 0\}$ (and hence $\{\mathbf{Q}_u(t),\ t \geq 0\}$) to take on values in $[K/2, \infty)$ and $\{\mathbf{B}(t),\ t \geq 0\}$ to take on values in $(-\infty, M]$. Then, under scheme $\mathcal{Q}$ as given in (8), we define $p^{\mathcal{Q}}_{\text{overflow}}(K) \triangleq \frac{1}{2}\lim_{\tau\to\infty} P(\mathbf{Q}_u(\tau) > K)$ and $p^{\mathcal{Q}}_{\text{underflow}}(M) \triangleq P(\mathbf{B}(\tau) < 0)$. Using Theorems 1 and 2 in [20], one can see that $p^{\mathcal{Q}}_{\text{loss}}(K) = \mathrm{O}(K^{-\beta_Q})$ for some $\beta_Q > 0$ if and only if there exists some $\beta'_Q > 0$ such that $p^{\mathcal{Q}}_{\text{overflow}}(M) = \mathrm{O}(K^{-\beta'_Q})$. Similarly, Theorem 1 and 2 in [20] imply that $p^{\mathcal{Q}}_{\text{discharge}}(M) = \mathrm{O}(\exp(-\alpha_Q M))$ if and only if $p^{\mathcal{Q}}_{\text{underflow}}(M) = \mathrm{O}(\exp(-\alpha_Q M))$. Thus, it suffices to show the desired scaling laws for the aforementioned unbounded queue state and battery state processes.

After transforming the problem as described above to one involving threshold crossing with Brownian motions, we derive the individual probabilities using the following techniques in Appendix D. First, we define a reward of one unit every time $\mathbf{Q}_u(t)$ goes above $K$. Using renewal-reward theory, we find

$$p^{\mathcal{Q}}_{\text{overflow}}(K) = \frac{{\delta^{(a)}}^2}{\bar{\sigma}_a^2}\exp\left(-\frac{\delta^{(a)}K}{\bar{\sigma}_a^2}\right). \tag{12}$$

By substituting $\delta^{(a)} = \beta'_Q\bar{\sigma}_a^2\frac{\log K}{K}$, we have the desired scaling law for the queue overflow probability. Next, we define two power allocation policies that lead to a higher and lower battery discharge probabilities, respectively, for the same energy replenishment process. The steady-state probability distribution of the battery state for these policies is an exponentially-distributed random variable. Applying the well-known squeeze theorem in the limit $M \to \infty$, we obtain the desired scaling law for the battery discharge probability. Finally, proof for the convergence of the time average utility follows the same line of argument to that for Theorem 1.

### *C. Exploring Tradeoffs Between Battery Discharge and Buffer Overflow Probabilities*

So far, we focused on achieving performance that was close to the optimal while keeping the probabilities of discharge and data loss low. In this section we will look at quantifying tradeoff between the probabilities of battery discharge and data loss.

**Theorem 4.** *For a sensor node with energy replenishment and a wireless channel with a rate-power function $C(\cdot)$, there exists an energy management scheme $\mathcal{E}$ that achieves $\lim_{M\to\infty}\frac{1}{M}\log p^{\mathcal{E}}_{\mathit{discharge}}(M) = -\frac{2\delta^{(r)}}{\bar{\sigma}_r^2}$ while $\lim_{K\to\infty}\frac{1}{K}\log p^{\mathcal{E}}_{\mathit{loss}}(K) = -\frac{2(C(\mu-\delta^{(r)})-\lambda)}{\bar{\sigma}_a^2}$ for any $0 < \delta^{(r)} < \mu - C^{-1}(\lambda)$.*

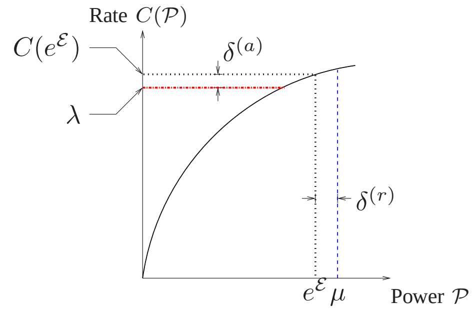



Fig. 6. Relation between $\delta^{(a)}$ and $\delta^{(r)}$.

To prove this theorem (given in Appendix E), we consider a power allocation scheme $\mathcal{E}$ that is given by,

$$e^{\mathcal{E}}(t) = \mu - \delta^{(r)} \tag{13}$$

for all $t$. The mean drifts for the battery state and the data queue state are given by $\delta^{(r)}$ and $C(\mu-\delta^{(r)})-\lambda$, respectively. Applying the diffusion limits on these processes, we get the required probability results.

Fig. 6 illustrates the relationship between the parameters of the system. Here, we note that $\delta^{(a)} = C(\mu - \delta^{(r)}) - \lambda$. Any increase in $\delta^{(r)}$ would lead to a corresponding decrease in $\delta^{(a)}$. Since $\delta^{(r)}$ is proportional to the discharge probability decay exponent and $\delta^{(a)}$ is proportional to the data loss probability decay exponent, we will get the given tradeoff.

Theorem 4 shows that, while it is possible to achieve exponential decay rates for battery discharge and data loss probabilities, there is a tradeoff in the decay exponents. More specifically, by varying $\delta^{(r)}$, it is possible to increase (or decrease) the decay rate exponent for the data loss probability. However this will directly result a proportional decrease (or increase) in the decay rate exponent for the battery discharge probability.

## IV. Performance Evaluation

Our theorems illustrate tradeoffs for energy management schemes in the buffer and battery size asymptotic regimes and showed optimality of some simple energy management schemes. In this section, we conduct simulations to evaluate the performance of those schemes in the presence of a finite battery and a finite data buffer. We construct the energy replenishment process $r(t)$ using the real solar radiation measurements collected at the Solar Radiation Research Laboratory [21]. The data set used is the global horizontal radiation or the total solar radiation using a Precision Spectral Pyranometer. We use data from January 1999 to July 2010 collected at 1 minute intervals. We assume that the energy replenishment process is proportional to the total solar radiation.

In our simulations, we choose the battery size in the range of $10^3 - 10^6$ mAh, as in [12]. Fig. 7(a) shows a sample of the replenishment process $r(t)$ over a 48 hour period. Also, we used the utility function $U(e) = \log(1 + \gamma e)$.

To compare our scheme, we use the Throughput-Optimal (TO) policy given in Eq. (4) of [15]. The TO policy is given by,

$$e^{TO}(t) = \min\{B(t), \mu - \epsilon\}, \tag{14}$$

where $\epsilon$ is a constant such that $C(\mu - \epsilon) > \lambda$.

### A. Battery Constraints with Infinitely Backlogged Buffer

In Fig. 7, we revisit the power allocation scheme $\mathcal{B}$ discussed in Example 1 for an infinitely backlogged data buffer. The communication channel is AWGN with SNR $\gamma = 1$, and we choose the polynomial decay exponent $\beta = 2$. From Theorems 2 and 1 we know that the TO policy should achieve an exponential decay for discharge probability compared to the quadratic decay for scheme $\mathcal{B}$. On the other hand, the TO policy can not achieve the maximum utility while scheme $\mathcal{B}$ should achieve maximum utility as $(\log M)^2/M^2$. Fig. 7(b) plots the battery discharge probability as a function of the battery size $M$. As expected, the TO policy performs better than scheme $\mathcal{B}$. However, the advantage of using policy $\mathcal{B}$ is evident in Fig. 7(c), which compares the time average utilities achieved by each scheme. It can be seen that, for the choice of parameters and data used in this simulation, scheme $\mathcal{B}$ achieves the maximum utility $U(\mu)$ for a battery size of $10^7$ mAh, whereas the TO scheme does not achieve the maximum utility even asymptotically.

### B. Buffer and Battery Constraints

Fig. 8 compares the performance of power allocation schemes when both battery and buffer constraints are present. We simulate the data arrival process by generating a Markov-modulated Poisson process with mean $\lambda = 7.44$ bits in every time slot. We use a two-state Markov chain to generate a bursty data arrival process. One state of the Markov chain generates a Poisson random variable with mean 25 bits and the other state generates a Poisson random variable with mean 1. The mean of the energy replenishment process $\mu = 9.58$ mAh per time slot and we choose $\beta_Q = 2$. In Fig. 8(a), we fix the buffer size to $10^5$ bits and plot the battery discharge probability as a function of the battery size $M$. The discharge probabilities for both schemes should decay exponentially for both schemes. However, the decay exponent for scheme $\mathcal{Q}$ should be larger than the decay exponent for the TO scheme. In fact, the decay exponent for the scheme $\mathcal{Q}$ should be proportional to $\mu - C^{-1}(\lambda)$, whereas the decay exponent for the TO scheme is proportional to $\epsilon < \mu - C^{-1}(\lambda))$ (given in Eq. (14). As expected, the discharge probability for scheme $\mathcal{Q}$ decays faster than that of the TO scheme. In Fig. 8(b), we plot the data loss probability as a function of the buffer size while keeping the battery size fixed at $10^5$ mAh. We observe that the loss probability for the TO scheme decays faster than that for scheme $\mathcal{Q}$. This trend is expected as the TO scheme should have an exponential decay compared to a quadratic decay for scheme $\mathcal{Q}$. Fig. 8(c) compares the convergence of the time average utilities to the maximum utility function for the two schemes. We observe that scheme $\mathcal{Q}$ converges to the maximum utility for moderately large buffer sizes ($\sim 10^4$ bits). On the other hand, TO scheme does not achieve the optimal utility even asymptotically.

Fig. 9 compares the performance of power allocation schemes with increasing traffic intensity. We define traffic intensity as $\rho \triangleq \frac{\lambda}{C(\mu)} = \frac{\lambda}{\log_2(1+\gamma\mu)}$. We fix the buffer length at $10^5$ bits and battery capacity is set at $10^7$ mAh. In Fig. 9(a) we observe that the discharge probability increases with traffic intensity. For values of $\rho < 0.97$, scheme $\mathcal{Q}$ performs almost an order of magnitude better than the TO scheme in terms of the discharge probability. For traffic intensities close to unity the scheme $\mathcal{Q}$ degenerates to the TO scheme and their performances converge. Fig. 9(b) shows that the data loss probabilities for both schemes also increases with increasing traffic intensity. Similar to the discharge probability, for values of $\rho < 0.97$, the loss probability for scheme $\mathcal{Q}$ is almost an order of magnitude lower than that for the TO scheme. This can be explained by the higher discharge probability for the TO scheme leading to severe performance degradation. Finally, we observe in Fig. 9(c) that the time average utility decreases with increasing $\rho$. This is a direct consequence of increasing battery discharge and data loss probabilities leading to sub-optimal performance in both schemes. As $\rho \to 1$, the performances of the two energy management schemes degrade highly.

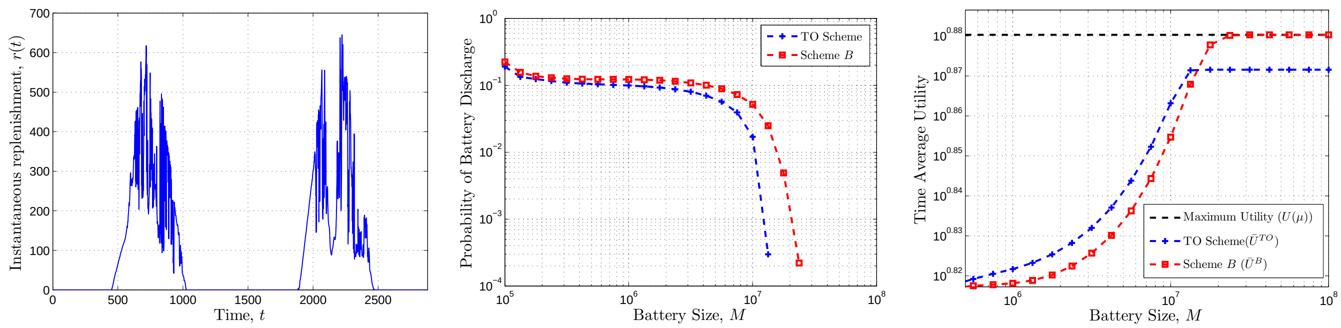
### C. Trade-offs Between Buffer Overflow and Battery Discharge Probabilities

In Fig. 10, we numerically evaluate the trade-off between battery discharge and data loss given in Theorem 4. We use the data arrival and energy replenishment process used previously. Fig 10(a) illustrates that in order to increase the decay exponent for the battery discharge probability, the energy management scheme has to decrease the exponent for the data loss probability. We choose three operating points on this curve and evaluate the battery discharge and data loss scaling for these points. As we go from operating point 1 to 3, the data loss probability decay exponent increases and the battery discharge probability decay exponent decreases. In Fig. 10(b), we observe that the quickest decay for the loss probability is for operating point 1. In Fig. 10(c), as expected, we see the opposite effect wherein the discharge probability decays fastest for the operating point 3.

## V. Conclusions

In this paper, we studied the basic limits and associated tradeoffs for energy management schemes in energy replenishing sensor networks. We showed that it is possible to observe a polynomial decay for the discharge probability with increased battery size, and at the same time achieve $\Theta((\log M)^2/M^2)$ convergence to the maximum achievable utility using a simple energy management scheme. We showed the strength of this result by showing that it is not possible to simultaneously observe an exponential decay for the discharge probability and achieve maximum utility. With the insights drawn, we addressed the problem of energy management with buffer and battery constraints. We showed that, in addition to achieving $\Theta((\log K)^2/K^2)$ convergence to the optimum utility, it is possible to achieve a polynomial decay for the data loss probability and exponential decay for the the battery discharge probability using a simple energy management scheme.

To analyze the buffer and battery processes we made use of large deviations theory and diffusion approximations. The main advantage of using these tools in our work is that it

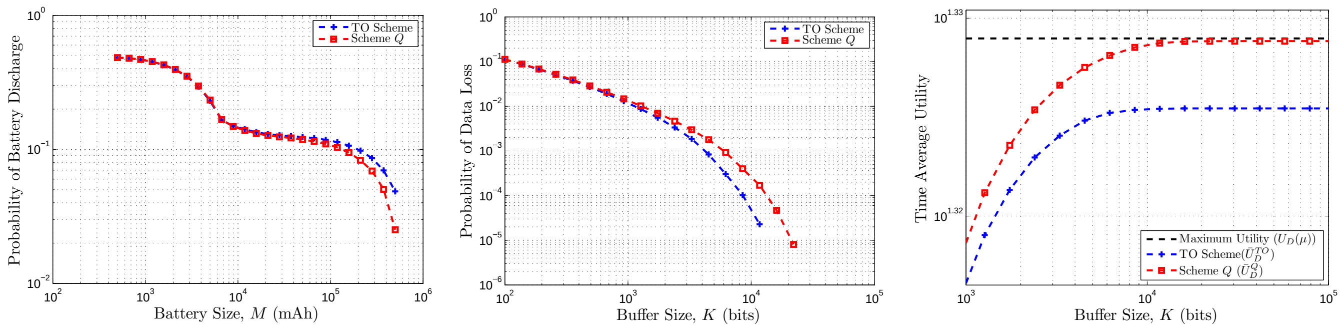



(a) Sample of the replenishment process $r(t)$ over a 48 hour period.

(b) Battery discharge probability scaling with battery size $M$.

(c) Time average utility scaling with battery size $M$.

Fig. 7. Performance evaluation for the AWGN channel example.

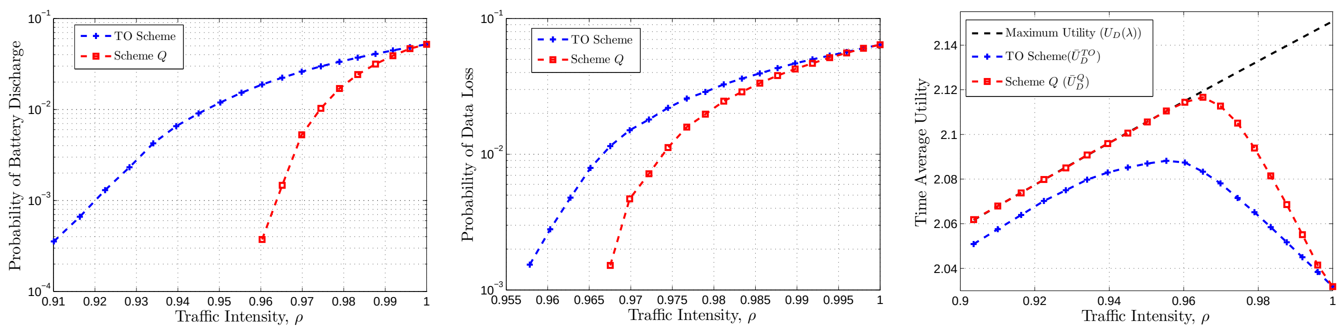



(a) Battery discharge probability scaling with battery size $M$.

(b) Data loss probability scaling with buffer size $K$.

(c) Time average utility scaling with buffer size $K$.

Fig. 8. Performance evaluation for energy management schemes under buffer and battery constraints.

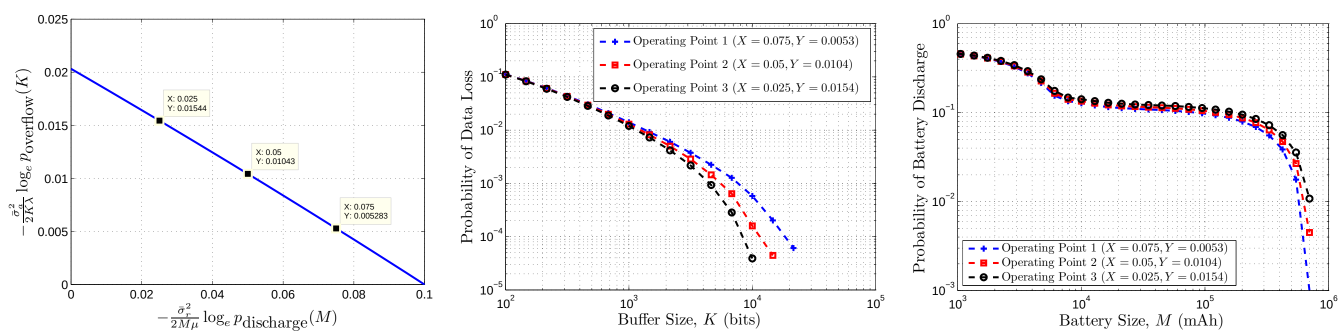



(a) Battery discharge probability scaling with traffic intensity $\rho$.

(b) Data loss probability scaling with traffic intensity $\rho$.

(c) Time average utility scaling with traffic intensity $\rho$.

Fig. 9. Performance evaluation of energy management schemes under buffer and battery constraints with increasing traffic intensities.




(a) Trade-off between battery discharge and data loss probability.

(b) Data loss probability scaling for different operating points.

(c) Battery discharge probability scaling for different operating points.

Fig. 10. Performance evaluation of energy management schemes under buffer and battery constraints with increasing traffic intensities.

allows analytical tractability while keeping the system model fairly general in nature. Finally, we numerically illustrated the performance of the our simple energy management schemes along with that of another existing scheme, and demonstrated that our scheme can perform up to an order of magnitude better in terms of outage probabilities while achieving the maximum utility asymptotically.

One possible future direction of research is the design of optimal or near-optimal practical energy management solutions in the presence of channel fading and multi-user interference and to build distributed algorithms to realize these schemes.

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## APPENDIX A
## PROOF OF LEMMA 1

To prove this lemma, we first use the finite form of Jensen's inequality to establish,

$$\frac{1}{\tau}\sum_{t=1}^{\tau} U(e^{\mathcal{S}}(t)) \le U\left(\frac{1}{\tau}\sum_{t=1}^{\tau} e^{\mathcal{S}}(t)\right).$$

Since this inequality holds for any finite $\tau$, passing the limit $\tau \to \infty$, the inequality is preserved,

$$\begin{aligned}\lim_{\tau\to\infty}\frac{1}{\tau}\sum_{t=1}^{\infty} U(e^{\mathcal{S}}(t)) &\le \lim_{\tau\to\infty} U\left(\frac{1}{\tau}\sum_{t=1}^{\infty} e^{\mathcal{S}}(t)\right)\\ &= U\left(\lim_{\tau\to\infty}\frac{1}{\tau}\sum_{t=1}^{\infty} e^{\mathcal{S}}(t)\right),\end{aligned} \tag{15}$$

where (15) follows since $U(\cdot)$ is a continuous function [22]. From conservation of energy, we have

$$\lim_{\tau\to\infty}\frac{1}{\tau}\sum_{t=1}^{\tau} e^{\mathcal{S}}(t) \le \lim_{\tau\to\infty}\frac{1}{\tau}\sum_{t=1}^{\tau} r(t) = \mu, \tag{16}$$

since $M < \infty$. Combining Eqs. (15) and (16), we have the required result,

$$\lim_{\tau\to\infty}\frac{1}{\tau}\sum_{t=1}^{\infty} U(e^{\mathcal{S}}(t)) = \bar{U}^{\mathcal{S}} \le U(\mu). \tag{17}$$

## APPENDIX B
## PROOF OF THEOREM 1

In this appendix, we prove that the power allocation scheme $\mathcal{B}$ achieves the scaling properties given in Theorem 1. First, consider a general form of Scheme $\mathcal{B}$:

$$e^{\mathcal{B}}(t) = \begin{cases}\mu - \delta^-, & B(t) \le M/2\\ \mu + \delta^+, & B(t) > M/2\end{cases}, \tag{18}$$

for some pair $\delta^-, \delta^+$, that will be chosen later. We will show that the desired solution involves $\delta^- = \delta^+ = \delta^{\mathcal{B}}$.

Depending on whether the battery state is less than (or more than) half full, the expected drift of the battery state becomes positive (or negative). Given $B(t) \le M/2$, the asymptotic semi-invariant log-moment generating function of the battery state drift, $d^-(t) \triangleq r(t) - (\mu - \delta^-)$, is

$$\begin{aligned}\bar{\Lambda}_{d^-}(s) &= \lim_{\tau\to\infty}\frac{1}{\tau}\log\mathbb{E}\left[\exp\left(s\sum_{t=1}^{\tau} d^-(t)\right)\right]\\ &= \bar{\Lambda}_r(s) - s(\mu - \delta^-).\end{aligned} \tag{19}$$

Where $\bar{\Lambda}_r(s)$ is given by Eq. (3). Let $s^*_{d-}$ be the negative root[4] of $\bar{\Lambda}_{d^-}(s)$, i.e., $\bar{\Lambda}_{d^-}(s^*_{d-}) = \bar{\Lambda}_r(s^*_{d-}) - s^*_{d^-}(\mu - \delta^-) = 0$. Also as $\delta^- \to 0$, $s^*_{d-} \to 0$.

Before we prove Theorem 1, we state and prove the following lemmas. Lemma 2 gives the rate of decay of the probability of battery discharge with respect to the battery size $M$ for the Scheme $\mathcal{B}$. Lemma 3 expresses the rate decay exponent $s^*_{d-}$ for scheme $\mathcal{B}$ in terms of the asymptotic variance of energy replenishment process $r(t)$.

**Lemma 2.** *The probability of battery discharge under Scheme $\mathcal{B}$ with battery size $M$ follows $p^{\mathcal{B}}_{\text{discharge}}(M) = \Theta\left(\exp\left(\frac{s^*_{d-}M}{2}\right)\right)$, where $s^*_{d-}$ is the negative root of $\bar{\mu}_{d^-}(s)$.*

*Proof:* Fix a constant $A > 0$ and decompose the time line into intervals, such that each interval is of length $\lceil \frac{M}{2A} \rceil$ and the $i$th interval ends at time slot $t_i = i\lceil \frac{M}{2A} \rceil$. Assume that the system has been active since $t = -\infty$. We define $E_i$ as the event that the battery is empty at the end of time slot 0 and the last time the battery was half full (i.e., $M/2$) is some instant during the interval $-i = \left[-(i+1)\lceil \frac{M}{2A} \rceil + 1, -i\lceil \frac{M}{2A} \rceil\right]$. The event of an empty battery at time slot 0 can be decomposed as a union of events $E_i$,

$$p^{\mathcal{B}}_{\text{discharge}}(M) = \sum_{i=0}^{\infty} \mathbb{P}\left(E_i\right) \tag{20}$$

A necessary condition for event $E_i$ to occur is,

$$\sum_{t=-(i+1)\lceil \frac{M}{2A} \rceil + 1}^{0} \left(e^{\mathcal{B}}(t) - r(t)\right) > \frac{M}{2} \tag{21}$$

Using Chernoff's bound, for any $\theta_i \geq 0$,

$$\begin{aligned}
&\mathbb{P}\left(\sum_{t=-(i+1)\lceil \frac{M}{2A} \rceil + 1}^{0} (e^{\mathcal{B}}(t) - r(t)) > \frac{M}{2}\right) \\
&\leq \mathbb{E}\left[\exp\left(\theta_i \sum_{t=-(i+1)\lceil \frac{M}{2A} \rceil + 1}^{0} (e^{\mathcal{B}}(t) - r(t))\right)\right] \exp\left(-\theta_i \frac{M}{2}\right) \\
&= \mathbb{E}\left[\exp\left(-\theta_i \sum_{t=-(i+1)\lceil \frac{M}{2A} \rceil + 1}^{0} r(t)\right)\right] \\
&\quad \times \exp\left(\theta_i (i+1)\left\lceil \frac{M}{2A} \right\rceil (\mu - \delta^-)\right) \exp\left(-\theta_i \frac{M}{2}\right) \\
&= \exp\Bigg( -\frac{M}{2}\Bigg[\theta_i\left(1 - \frac{i+1}{A}(\mu - \delta^-)\right) - \frac{i+1}{A}\bar{\Lambda}_r(-\theta_i) \\
&\qquad + \epsilon_i(M, \theta_i)\Bigg]\Bigg),
\end{aligned} \tag{22}$$

where $\epsilon_i(M, \theta_i) \to 0$ as $M \to \infty$.

[4]Note that $\bar{\Lambda}_{d^-}(0) = 0$ and $\left.\frac{\partial \bar{\Lambda}_{d^-}(s)}{\partial s}\right|_{s=0} = \lim_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} \mathbb{E}\left[d^-(t)\right] = \delta^- > 0$. Consequently $s^*_{d-} < 0$ will exist.

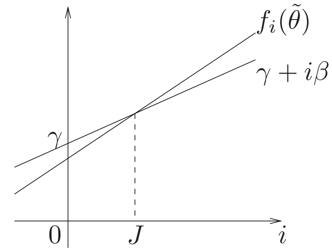


Fig. 11. A geometric proof for the existence of $J$ and $\delta > 0$ such that for every $i > J$, $f_i(\tilde{\theta}) > \gamma + i\delta$.

In order to find the tightest bound for each $i$, we choose $\theta^*_i \geq 0$ to maximize,

$$f_i(\theta) \triangleq \theta\left(1 - \frac{i+1}{A}(\mu - \delta^-)\right) - \frac{i+1}{A}\bar{\Lambda}_r(-\theta), \tag{23}$$

over all $\theta > 0$ and let $\gamma = \inf_{i\geq 0}\sup_{\theta\geq 0} f_i(\theta) = \inf_{i\geq 0} f_i(\theta^*_i)$. We can rewrite $f_i(\theta)$ as,

$$f_i(\theta) = \theta - \frac{\mu - \delta^-}{A}\theta - \frac{\bar{\Lambda}_r(-\theta)}{A} - i\left(\frac{(\mu - \delta^-)\theta + \bar{\Lambda}_r(-\theta)}{A}\right)$$

Since $\lim_{\tau\to\infty} \frac{1}{\tau}\sum_{t=1}^{\tau} \mathbb{E}\left[r(t)\right] = \mu > \mu - \delta^-$, the function $(\mu - \delta^-)\theta + \bar{\Lambda}_r(-\theta)$ has a negative slope at $\theta = 0$. Hence, we can choose some $\tilde{\theta} > 0$, such that $(\mu - \delta^-)\tilde{\theta} + \bar{\Lambda}_r(-\tilde{\theta}) < 0$. This implies that there exists a $J$ and a $\beta > 0$ such that for every $i > J$,

$$f_i(\tilde{\theta}) > \gamma + i\beta \tag{24}$$

as illustrated in Fig. 11. Returning to Eq. (20),

$$\begin{aligned}
&p^{\mathcal{B}}_{\text{discharge}}(M) = \sum_{i=0}^{\infty} \mathbb{P}\left(E_i\right) \\
&\leq \sum_{i=0}^{\infty} \mathbb{P}\left(\sum_{k=-(i+1)\lceil \frac{M}{2A} \rceil + 1}^{0} (e^{\mathcal{B}}(k) - r(k)) > \frac{M}{2}\right) \\
&\leq \sum_{i=0}^{J} \exp\left(-\frac{M}{2}\left[f_i(\theta^*_i) + \epsilon_i(M, \theta^*_i)\right]\right) \\
&\qquad + \sum_{i=J+1}^{\infty} \exp\left(-\frac{M}{2}\left[f_i(\tilde{\theta}) + \epsilon_i(M, \tilde{\theta})\right]\right) \\
&\leq \sum_{i=0}^{J} \exp\left(-\frac{M}{2}\left[\gamma + \min_{0\leq i\leq J} \epsilon(M, \theta^*_i)\right]\right) \\
&\qquad + \sum_{i=J+1}^{\infty} \exp\left(-\frac{M}{2}\left[\gamma + i\beta + \inf_{i>J} \epsilon_i(M, \tilde{\theta})\right]\right) \\
&= \exp\left(-\frac{M}{2}\gamma\right)\Bigg[(J+1)\exp\left(\min_{0\leq i\leq J} \epsilon(M, \theta^*_i)\right) \\
&\qquad + \frac{\exp\left(-\frac{M}{2}\left((J+1)\beta + \inf_{i>J} \epsilon_i(M, \tilde{\theta})\right)\right)}{1 - \exp\left(-\beta\frac{M}{2}\right)}\Bigg].
\end{aligned} \tag{25}$$

As $M \to \infty$,

$$\limsup_{M\to\infty} \frac{2}{M} \log p^{\mathcal{B}}_{\text{discharge}}(M) \leq -\gamma$$

Since this inequality holds for any $A>0$, we let $A\to\infty$ as follows:

$$\begin{aligned}&\limsup_{M\to\infty}\frac{2}{M}\log p^{\mathcal{B}}_{\text{discharge}}(M)\\&\le -\inf_{i\ge 0}\sup_{\theta\ge 0}\left[\theta\left(1-\frac{i}{A}(\mu-\delta)\right)-\frac{i}{A}\bar{\Lambda}_r(-\theta)\right]\\&=-\inf_{T\ge 0}\sup_{\theta\ge 0}\left[\theta\left(1-T(\mu-\delta)\right)-T\bar{\Lambda}_r(-\theta)\right]\\&=-\inf_{T\ge 0}T\sup_{\theta\ge 0}\left[-\theta\left(\mu-\delta-\frac{1}{T}\right)-\bar{\Lambda}_r(-\theta)\right]\end{aligned}\tag{26}$$

Next, we find the lower bound. For some $T\ge 0$, a sufficient condition for the battery to be empty at some time slot in the interval $[-\lceil TM/2\rceil,0]$ is that,

$$\sum_{t=-\lceil\frac{TM}{2}\rceil+1}^{0}(e^{\mathcal{B}}(t)-r(t))>M.\tag{27}$$

We can lower bound $p^{\mathcal{B}}_{\text{discharge}}(M)$ using the union bound,

$$\begin{aligned}&\mathbb{P}\left(B(t)=0\text{ within some }t\in\left[-\left\lceil\frac{TM}{2}\right\rceil,0\right]\right)\\&=\mathbb{P}\left(\bigcup_{t=-\lceil TM/2\rceil}^{0}B(t)=0\right)\le\sum_{t=-\lceil TM/2\rceil}^{0}\mathbb{P}\left(B(t)=0\right)\\&=\left\lceil\frac{TM}{2}\right\rceil p^{\mathcal{B}}_{\text{discharge}}(M).\end{aligned}\tag{28}$$

We also have

$$\begin{aligned}&\mathbb{P}\left(\sum_{t=-\lceil\frac{TM}{2}\rceil+1}^{0}(e^{\mathcal{B}}(t)-r(t))>\frac{M}{2}\right)\\&=\mathbb{P}\left(\sum_{t=-\lceil\frac{TM}{2}\rceil+1}^{0}(\mu-\delta^{-}-r(t))>\frac{M}{2}\right).\end{aligned}\tag{29}$$

We define, $Z_{M,T}\triangleq\frac{2}{TM}\sum_{k=-\lceil\frac{TM}{2}\rceil+1}^{0}(\mu-\delta^{-}-r(k))$. Consequently,

$$\mathbb{P}\left(\sum_{t=-\lceil\frac{TM}{2}\rceil+1}^{0}(\mu-\delta^{-}-r(t))>\frac{M}{2}\right)=\mathbb{P}\left(Z_{M,T}>\frac{1}{T}\right).$$

Now, $\lim_{M\to\infty}\mathbb{E}\left[Z_{M,T}\right]=-\delta^{-}<0<\frac{1}{T}$ for all $T>0$. Applying the Gärtner-Ellis Theorem, we get,

$$\begin{aligned}&\lim_{M\to\infty}\frac{2}{M}\log\mathbb{P}\left(Z_{M,T}>\frac{1}{T}\right)\\&=-\sup_{s\ge 0}\left[\frac{1}{T}s-s\left(\mu-\delta\right)+T\bar{\Lambda}_r\left(-\frac{s}{T}\right)\right]\\&=-T\sup_{s\ge 0}\left[-\frac{s}{T}\left(\mu-\delta-\frac{1}{T}\right)-\bar{\Lambda}_r\left(-\frac{s}{T}\right)\right]\\&=-T\sup_{\theta\ge 0}\left[-\theta\left(\mu-\delta-\frac{1}{T}\right)-\bar{\Lambda}_r(-\theta)\right].\end{aligned}\tag{30}$$

Combining Eqs. (28) and (30), we have,

$$\begin{aligned}&\liminf_{M\to\infty}\frac{2}{M}\log p^{\mathcal{B}}_{\text{discharge}}(M)\\&\ge-\inf_{T\ge 0}T\sup_{\theta\ge 0}\left[-\theta\left(\mu-\delta-\frac{1}{T}\right)-\bar{\Lambda}_r(-\theta)\right].\end{aligned}\tag{31}$$

From Eqs. (26) and (31) we have,

$$\begin{aligned}&\lim_{M\to\infty}\frac{2}{M}\log p^{\mathcal{B}}_{\text{discharge}}(M)\\&=-\inf_{T\ge 0}T\sup_{\theta\ge 0}\left[-\theta\left(\mu-\delta-\frac{1}{T}\right)-\bar{\Lambda}_r(-\theta)\right]\\&=s^{*}_{d^{-}}.\end{aligned}\tag{32}$$

This gives us $p^{\mathcal{B}}_{\text{discharge}}(M)=\Theta\left(\exp\left(s^{*}_{d^{-}}\frac{M}{2}\right)\right)$. ■

**Lemma 3.** *The asymptotic variance of $r(t)$, $\bar{\sigma}_r^2\triangleq\lim_{T\to\infty}\frac{1}{T}var\left(\sum_{t=1}^{T}r(t)\right)$ satisfies*

$$\left.\frac{\partial s^{*}_{d^{-}}}{\partial\delta^{-}}\right|_{\delta^{-}=0}=-\frac{2}{\bar{\sigma}_r^2}\tag{33}$$

*Proof:* First, we define $\bar{\Lambda}^{(n)}_{d^{-}}(0)=\left.\frac{\partial^n\bar{\Lambda}_{d^{-}}(s)}{\partial s^n}\right|_{s=0}$. The Taylor series expansion of $\bar{\Lambda}_{d^{-}}(s^{*}_{d^{-}})$ about $s=0$ gives,

$$\begin{aligned}0=\bar{\Lambda}_{d^{-}}(s^{*})&=\sum_{n=0}^{\infty}\bar{\Lambda}^{(n)}_{d^{-}}(0)\frac{(s^{*}_{d^{-}})^n}{n!}\\&=\underbrace{\bar{\Lambda}_{d^{-}}(0)}_{=0}+\bar{\Lambda}^{(1)}_{d^{-}}(0)s^{*}_{d^{-}}+\bar{\Lambda}^{(2)}_{d^{-}}(0)\frac{(s^{*}_{d^{-}})^2}{2!}+\cdots\\&=\sum_{n=1}^{\infty}\bar{\Lambda}^{(n)}_{r}(0)\frac{(s^{*}_{d^{-}})^n}{n!}-(\mu-\delta^{-})s^{*}_{d^{-}}\\&=\mu s^{*}_{d^{-}}+\sum_{n=2}^{\infty}\bar{\Lambda}^{(n)}_{r}(0)\frac{(s^{*}_{d^{-}})^n}{n!}-(\mu-\delta)s^{*}_{d^{-}}.\end{aligned}$$

Rearranging the terms, we have

$$\sum_{n=2}^{\infty}\bar{\Lambda}^{(n)}_{r}(0)\frac{(s^{*}_{d^{-}})^{n-1}}{n!}=-\delta^{-}.\tag{34}$$

Differentiating with respect to $\delta^{-}$, we have,

$$\frac{\partial s^{*}_{d^{-}}}{\partial\delta^{-}}\sum_{n=2}^{\infty}\bar{\Lambda}^{(n)}_{r}(0)\frac{(n-1)(s^{*}_{d^{-}})^{n-2}}{n!}=-1.$$

As $\delta^{-}\to 0$, $s^{*}_{d^{-}}\to 0$ the above expression reduces to,

$$\left.\frac{\partial s^{*}_{d^{-}}}{\partial\delta^{-}}\right|_{\delta^{-}=0}\bar{\Lambda}^{(2)}_{r}(0)\frac{1}{2}=-1.\tag{35}$$

Since $\bar{\Lambda}^{(2)}_{r}(0)=\bar{\sigma}_r^2$, Eq. (35) becomes,

$$\left.\frac{\partial s^{*}_{d^{-}}}{\partial\delta^{-}}\right|_{\delta^{-}=0}=-\frac{2}{\bar{\sigma}_r^2}.$$

■

Lemma 3 implies

$$\frac{\partial s^{*}_{d^{-}}}{\partial\delta^{-}}=-\frac{2}{\bar{\sigma}_r^2}+\mathrm{o}(\delta^{-})$$

and hence,

$$s^{*}_{d^{-}}=-\frac{2}{\bar{\sigma}_r^2}\delta^{-}+\mathrm{o}(\delta^{-}),\tag{36}$$

where $\mathrm{o}(\delta^{-})/\delta^{-}\to 0$ as $\delta^{-}\to 0$.

Substituting this in Eq. (32), we have,

$$p^{\mathcal{B}}_{\text{discharge}}(M)=\mathrm{O}\left(\exp\left[\left(-\frac{2}{\bar{\sigma}_r^2}\delta^{-}+\mathrm{o}\left(\delta^{-}\right)\right)\frac{M}{2}\right]\right).$$

By choosing $\delta^- = \alpha\frac{\log M}{M}$ and $\alpha = \beta\bar{\sigma}_r^2$ we have $p^{\mathcal{B}}_{\text{discharge}}(M) = \text{O}(M^{-\beta})$.

Next we show that with $\delta^+ = \alpha\frac{\log M}{M}$ the scheme achieves an average utility $\bar{U}^{\mathcal{B}}$ such that $U(\mu) - \bar{U}^{\mathcal{B}} = \Theta\left(\frac{(\log M)^2}{M^2}\right)$. The instantaneous utility $U(e^{\mathcal{S}}(t))$ is zero with an $\text{O}(M^{-\beta})$ probability. For the remaining time, the utility alternates between $U^+$ and $U^-$ as illustrated in Fig. 2. The Taylor series expansion of the utility functions about $\mu$ will be,

$$U^+ = U(\mu) + U^{(1)}(\mu)\delta^+ + U^{(2)}(\mu)(\delta^+)^2 + \text{o}((\delta^+)^2),$$

and,

$$U^- = U(\mu) - U^{(1)}(\mu)\delta^- + U^{(2)}(\mu)(\delta^-)^2 + \text{o}((\delta^-)^2).$$

We define $\rho^+$ as the fraction of time that $B(t) > M/2$ and $\rho^- = 1 - \rho^+$ as the fraction of time that $B(t) \leq M/2$. The average utility $\bar{U}^{\mathcal{B}}$ can be written as,

$$\begin{aligned}\bar{U}^{\mathcal{B}} &= (\rho^+U^+ + \rho^-U^-)(1 - p^{\mathcal{B}}_{\text{discharge}}(M)) \\ &= U(\mu) + U^{(1)}(\mu)(\rho^+\delta^+ - \rho^-\delta^-) + \Theta\left(\frac{(\log M)^2}{M^2}\right),\end{aligned} \tag{37}$$

where Eq. (37) follows from the fact that $\delta^-, \delta^+ = \alpha\frac{\log M}{M}$ and $p^{\mathcal{B}}_{\text{discharge}}(M) = \text{O}(M^{-\beta})$ where $\beta \geq 2$.

From conservation of energy, the replenishment energy is consumed completely except for the amount lost due to battery overflows. Thus,

$$\begin{aligned}\rho^+(\mu + \delta^+) + \rho^-(1 - p^{\mathcal{B}}_{\text{discharge}}(M))(\mu - \delta^-) \\ = \mu(1 - p^{\mathcal{B}}_{\text{overflow}}(M)),\end{aligned} \tag{38}$$

where $p^{\mathcal{B}}_{\text{overflow}}(M)$ is the probability of the battery being full under the power allocation scheme $\mathcal{B}$. By a trivial extension of Lemmas 2 and 3, it can be shown that $p^{\mathcal{B}}_{\text{overflow}}(M) = \Theta\left(M^{-\beta}\right)$. We can simplify Eq. (38) as,

$$\begin{aligned}\rho^+\delta^+ - \rho^-\delta^- &= (\rho^-(\mu - \delta^-) - \mu)\Theta\left(M^{-\beta}\right) \\ &= \Theta\left(M^{-\beta}\right).\end{aligned} \tag{39}$$

By substituting Eq. (39) in the first-order term of Eq. (37), we observe that the scheme $\mathcal{B}$ achieves $U(\mu) - \bar{U}^{\mathcal{B}} = \Theta\left(\frac{(\log M)^2}{M^2}\right)$. Choosing $\delta^{\mathcal{B}} = \delta^+ = \alpha\frac{\log M}{M}$ in Eq. (4) completes the proof of Theorem 1.

## Appendix C
## Proof of Theorem 2

Consider any ergodic energy management scheme $\mathcal{S}$ that uses $e^{\mathcal{S}}(t)$ units of energy in the time slot $t$. Note that scheme $\mathcal{S}$ can be deterministic or randomized. The asymptotic semi-invariant log moment generating function of the net battery drift $d^{\mathcal{S}} \triangleq e^{\mathcal{S}}(t) - r(t)$ is given by $\bar{\Lambda}_{d^{\mathcal{S}}}(s)$. First, we state a lemma that gives the discharge probability scaling for scheme $\mathcal{S}$. Since this lemma is a minor modification of Lemma 2, we omit the proof in this paper. We direct the reader to [23] for the detailed proof of this lemma.

**Lemma 4.** *The probability of battery discharge under Scheme $\mathcal{S}$ with battery size $M$ follows $p^{\mathcal{S}}_{\text{discharge}}(M) = \Theta(\exp(-s^*_{d^{\mathcal{S}}}M))$, where $s^*_{d^{\mathcal{S}}}$ is the positive root of $\bar{\mu}_{d^{\mathcal{S}}}(s)$.*

Note that $s^*_{d^{\mathcal{S}}} > 0$ exists[5] if and only if $\mathbb{E}\left[d^{\mathcal{S}}(t)\right] < 0$. Therefore, for $p^{\mathcal{S}}_{\text{discharge}}(M)$ to decay exponentially with $M$, we require,

$$\mathbb{E}\left[e^{\mathcal{S}}(t)\right] < \mathbb{E}\left[r(t)\right] = \mu. \tag{40}$$

On the other hand, if $\mathbb{E}\left[d^{\mathcal{S}}(t)\right] \geq 0$, there exists no rate $s > 0$ at which the battery discharge probability decays exponentially with $M$, i.e., $p^{\mathcal{S}}_{\text{discharge}}(M) = \Omega(\exp(-sM))$ for all $s > 0$. By substituting $\alpha_c = s^*_{d^{\mathcal{S}}}$ in Lemma 4, we get the required scaling law $p^{\mathcal{S}}_{\text{discharge}}(M) = \Theta(\exp(-\alpha_c M))$.

The difference between the utilities is given by,

$$\begin{aligned}U(\mu) - \bar{U}^{\mathcal{S}} &= U(\mu) - \lim_{\tau\to\infty}\frac{1}{\tau}\sum_{t=1}^{\tau}U(e^{\mathcal{S}}(t)) \\ &\overset{(a)}{\geq} U(\mu) - U\left(\lim_{\tau\to\infty}\frac{1}{\tau}\sum_{t=1}^{\tau}e^{\mathcal{S}}(t)\right) \\ &\overset{(b)}{=} U(\mu) - U\left(\mathbb{E}\left[e^{\mathcal{S}}(t)\right]\right) \overset{(c)}{=} \Omega(1).\end{aligned} \tag{41}$$

Where $(a)$ follows from Eq. (15), $(b)$ follows from the ergodicity of $e^{\mathcal{S}}(t)$ and $(c)$ follows from the fact that $\mu > \mathbb{E}\left[e^{\mathcal{S}}(t)\right]$ and $U(\cdot)$ is an increasing function. This completes the proof for Theorem 2.

## Appendix D
## Proof of Theorem 3

As discussed in Section III-B, our proof is constructive. We use the energy management scheme $\mathcal{Q}$ given in Eq. (8). Also, in Step (T1) we show in the heavy traffic limit that, the data loss and the battery discharge probabilities can be found using the appropriate reflection probabilities of the associated 2-dimensional RBM. Further, we showed in Step (T2) that the scaling laws for the data loss and the battery discharge probabilities are preserved when they are approximated by, respectively, the associated overflow and the underflow probabilities of the Brownian motion without reflections. We find the individual probabilities in the following lemmas.

**Lemma 5.** *For energy management scheme $\mathcal{Q}$, given any $\beta'_Q \geq 2$, $p^{\mathcal{Q}}_{\text{overflow}}(K) = O(K^{-\beta'_Q})$.*

*Proof:* If we assume the starting state of the Brownian motion to be $\mathbf{Q}(0) = K/2$. Note that due to the strong Markovian property of a Brownian motion [24], the instants $\{T_n,\ n = 1, 2, \ldots\}$ at which the system returns to state $K/2$ (i.e., $\mathbf{Q}(T_i) = K/2$) is probabilistically equal to the starting state. Hence we can study these renewal epochs[6] to obtain steady state properties for the data queue process. Now, consider the random variables $T_u \triangleq \arg\min_{t>0}\{\mathbf{Q}(t) = K/2, \mathbf{Q}(0) = K/2, \mathbf{Q}(0^+) > K/2\}$ and $T_l \triangleq \arg\min_{t>0}\{\mathbf{Q}(t) = K/2, \mathbf{Q}(0) = K/2, \mathbf{Q}(0^+) <$

[5] Since $\bar{\Lambda}_{d^{\mathcal{S}}}(0)$ and $\left.\frac{\partial\bar{\Lambda}_{d^{\mathcal{S}}}(s)}{\partial s}\right|_{s=0} = \lim_{\tau\to\infty}\mathbb{E}\left[d^{\mathcal{S}}(\tau)\right] < 0$, $s^*_{d^{\mathcal{S}}} > 0$ will exist.

[6] If we assume the starting state to be $\mathbf{Q}(0) \neq 0$, we can simply consider the process to be a delayed renewal process. The steady state properties in the resulting analysis will not change.

$K/2\}$. The hitting time distributions can be calculated as [24],

$$\begin{aligned}\mathbb{P}&\left(T_u > t+\tau \,\middle|\, \mathbf{Q}(\tau) = \frac{K}{2}+\epsilon\right)\\ &= \Phi\left(\frac{\epsilon - \delta^{(a)}\tau}{\bar{\sigma}_a\sqrt{\tau}}\right) - \exp\left(\frac{2\delta^{(a)}\epsilon}{\bar{\sigma}_a^2}\right)\Phi\left(\frac{-\epsilon-\delta^{(a)}\tau}{\bar{\sigma}_a\sqrt{\tau}}\right)\\ &= \mathbb{P}\left(T_l > t+\tau \,\middle|\, \mathbf{Q}(\tau) = \frac{K}{2}-\epsilon\right), \end{aligned} \quad (42)$$

where $\Phi(y) \triangleq \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y}\exp\left(-\frac{x^2}{2}\right)dx$. Since Eq. (42) holds for all $\tau > 0$ and $\epsilon > 0$, the random variables $T_u$ and $T_l$ will have the same distribution. Furthermore, once the process $\mathbf{Q}(t) = K/2$, it can go above or below $K/2$ with equal probability. Consequently, we need to study the renewals associated with $\mathbf{Q}_u(t) \triangleq |\mathbf{Q}(t) - K/2| + K/2$ and can find $\mathbb{P}(\mathbf{Q}(t) > K) = \frac{1}{2}\mathbb{P}(\mathbf{Q}_u(t) > K)$, which is identical to $p^{\mathcal{Q}}_{\text{overflow}}(K)$.

If we define a unit reward (i.e., $R(t) = 1$) for every time $t$ that the process $\mathbf{Q}_u(t) > K$ then,

$$\lim_{t\to\infty} P(\mathbf{Q}_u(t) > K) = \lim_{t\to\infty}\mathbb{E}\left[R(t)\right]. \quad (43)$$

From renewal-reward theory [25] we can write,

$$\lim_{t\to\infty}\mathbb{E}\left[R(t)\right] = \frac{\mathbb{E}\left[R_n\right]}{\mathbb{E}\left[X\right]}, \quad (44)$$

where $\mathbb{E}[R_n]$ is the expected award accumulated in one renewal period, and $\mathbb{E}[X]$ is the expected length of the renewal period. To get the correct expression for $\lim_{t\to\infty}\mathbb{E}[R(t)]$, we need to write the expressions for $\mathbb{E}[R_n]$ and $\mathbb{E}[X]$ carefully. We define $\mathbb{E}[X(\epsilon)]$ as the expected time for process $\mathbf{Q}(t)$ to return to $K/2$ given that it starts at $K/2+\epsilon$. The expression for $\mathbb{E}[X(\epsilon)]$ is given by [26],

$$\mathbb{E}\left[X(\epsilon)\right] = \frac{\epsilon}{\delta^{(a)}}. \quad (45)$$

Similarily, we define $\mathbb{E}[R_n(\epsilon)]$ as the probability of reaching $K$ before $K/2$ starting at $K/2+\epsilon$. Passing the limit $\epsilon \downarrow 0$ will give the expected reward accumulated in one renewal period. Applying the expression for this probability from [26],

$$\begin{aligned}\mathbb{E}\left[R_n(\epsilon)\right] &= \frac{\exp\left(\frac{2\delta^{(a)}}{\bar{\sigma}_a^2}\epsilon\right)-1}{\exp\left(\frac{2\delta^{(a)}}{\bar{\sigma}_a^2}\frac{K}{2}\right)-1}\\ &= \frac{\frac{2\delta^{(a)}}{\bar{\sigma}_a^2}\epsilon + o(\epsilon)}{\exp\left(\frac{2\delta^{(a)}}{\bar{\sigma}_a^2}\frac{K}{2}\right)-1}. \end{aligned} \quad (46)$$

Dividing Eqs. (46) by (45) and passing the limit $\epsilon \downarrow 0$, we have,

$$\lim_{t\to\infty}\mathbb{E}\left[R(t)\right] = \lim_{\epsilon\downarrow 0}\frac{\left(\frac{2\delta^{(a)}}{\bar{\sigma}_a^2} + \frac{o(\epsilon)}{\epsilon}\right)\delta^{(a)}}{\exp\left(\frac{2\delta^{(a)}}{\bar{\sigma}_a^2}\frac{K}{2}\right)-1}. \quad (47)$$

Evaluating the limit $\epsilon \downarrow 0$, and noting that the stochastic process will be in the upper half of the buffer with probability 1/2, for large $K$ we have,

$$p^{\mathcal{Q}}_{\text{overflow}}(K) = \frac{{\delta^{(a)}}^2}{\bar{\sigma}_a^2}\exp\left(-\frac{\delta^{(a)}K}{\bar{\sigma}_a^2}\right).$$

By choosing $\delta^{(a)} = \beta'_Q\bar{\sigma}_a^2\frac{\log K}{K}$, we have,

$$\begin{aligned}p^{\mathcal{Q}}_{\text{overflow}}(K) &= {\beta'_Q}^2\bar{\sigma}_a^2\left(\frac{\log K}{K}\right)^2\exp\left(-\beta'_Q\log K\right)\\ &= \mathrm{O}\left(K^{-\beta'_Q}\right). \end{aligned} \quad (48)$$

■

**Lemma 6.** *For the energy management scheme $\mathcal{Q}$, $\lim_{M\to\infty}\frac{1}{M}\log(p^{\mathcal{Q}}_{\text{underflow}}(M)) = -\frac{2(\mu - C^{-1}(\lambda))}{\bar{\sigma}_r^2}$, where $C^{-1}(\cdot)$ is the inverse of the rate-power function $C(\cdot)$.*

*Proof:* To calculate $p^{\mathcal{Q}}_{\text{underflow}}(M) = \lim_{t\to\infty}P(\mathbf{B}(t) < 0)$, first we define power allocation policies $\mathcal{Q}_1$,

$$e^{\mathcal{Q}_1}(t) = \mu - \delta_1^{(r)}, \quad \forall\, t, \quad (49)$$

and $\mathcal{Q}_2$,

$$e^{\mathcal{Q}_2}(t) = \mu - \delta_2^{(r)}, \quad \forall\, t. \quad (50)$$

Let the battery underflow probabilities associated with schemes $\mathcal{Q}_1$ and $\mathcal{Q}_2$ be $p^{\mathcal{Q}_1}_{\text{underflow}}(M)$ and $p^{\mathcal{Q}_2}_{\text{underflow}}(M)$, respectively. For the same energy replenishment process $\{r(t), t \geq 0\}$, the net battery drift is defined as $d^{\mathcal{S}}(t) \triangleq r(t) - e^{\mathcal{S}}(t)$, for each scheme $\mathcal{S} \in \{\mathcal{Q},\, \mathcal{Q}_1,\, \mathcal{Q}_2\}$. The net drifts will have the following relation,

$$d^{\mathcal{Q}_2}(t) \geq d^{\mathcal{Q}}(t) \geq d^{\mathcal{Q}_1}(t) \quad \forall\, t. \quad (51)$$

It follows that,

$$p^{\mathcal{Q}_2}_{\text{underflow}}(M) \leq p^{\mathcal{Q}}_{\text{underflow}}(M) \leq p^{\mathcal{Q}_1}_{\text{underflow}}(M). \quad (52)$$

The Brownian approximation for the battery process under policies $\mathcal{Q}_1$ and $\mathcal{Q}_2$ is an exponential random variable, and the undeflow probabilities are given by [18], [27],

$$p^{\mathcal{Q}_1}_{\text{underflow}}(M) = \exp\left(-\frac{2\delta_1^{(r)}}{\bar{\sigma}_r^2}M\right), \quad (53)$$

and,

$$p^{\mathcal{Q}_2}_{\text{underflow}}(M) = \exp\left(-\frac{2\delta_2^{(r)}}{\bar{\sigma}_r^2}M\right), \quad (54)$$

Substituting Eqs. (53) and (54) in Eq. (52) we have,

$$\exp\left(-\frac{2\delta_2^{(r)}}{\bar{\sigma}_r^2}M\right) \leq p^{\mathcal{Q}}_{\text{underflow}}(M) \leq \exp\left(-\frac{2\delta_1^{(r)}}{\bar{\sigma}_r^2}M\right). \quad (55)$$

For a rate-power function $C(\cdot)$, we have $\delta_1^{(r)} = \mu - C^{-1}(\lambda) + \Theta(\delta^{(a)})$ and $\delta_2^{(r)} = \mu - C^{-1}(\lambda) + \Theta(\delta^{(a)})$. With $\delta^{(a)} = \beta_Q\bar{\sigma}_a^2\frac{\log K}{K} \to 0$ as $K \to 0$, we can apply the squeeze theorem in Eq. (55) to get the required result.

■

To illustrate Lemma 6, we consider the AWGN channel capacity given in Eq. (6). We have, $\delta_1^{(r)} = \mu - \frac{\exp\left(\left(\lambda+\delta^{(a)}\right)\log 2\right)-1}{\gamma}$ and $\delta_2^{(r)} = \mu - \frac{\exp\left(\left(\lambda-\delta^{(a)}\right)\log 2\right)-1}{\gamma}$. We can use the power series expansion of the exponential function to get,

$$\delta_1^{(r)} = \mu - \frac{1}{\gamma}\left((\lambda + \delta^{(a)})\log 2 + \frac{(\lambda + \delta^{(a)})^2(\log 2)^2}{2} + \cdots\right)$$
$$= \mu - \frac{1}{\gamma}\exp(\lambda \log 2) + \Theta(\delta^{(a)}), \quad (56)$$

and,

$$\delta_2^{(r)} = \mu - \frac{1}{\gamma}\left((\lambda - \delta^{(a)})\log 2 + \frac{(\lambda - \delta^{(a)})^2(\log 2)^2}{2} + \cdots\right)$$
$$= \mu - \frac{1}{\gamma}\exp(\lambda \log 2) + \Theta(\delta^{(a)}). \quad (57)$$

Substituting these expressions in Eq. (55), we get $p_{\text{discharge}}^{\mathcal{Q}}(M) = \Theta(\exp(-\alpha_Q M))$.

Finally, we show that Scheme $\mathcal{Q}$ achieves an average utility $\bar{U}_D^{\mathcal{Q}}$ such that $U_D(\lambda) - \bar{U}_D^{\mathcal{Q}} = \Theta\left(\frac{(\log K)^2}{K^2}\right)$. The instantaneous utility will be zero when the queue is empty or when the battery is discharged. Due to symmetry of the Brownian approximation, the empty buffer probability will be the same as the data loss (i.e., full buffer) probability. Since $p_{\text{discharge}}^{\mathcal{Q}}(M) = \mathrm{O}(\exp(-M))$, under the large battery regime we can ignore the discharge term. The average utility $\bar{U}_D^{\mathcal{Q}}$ can be written as,

$$\bar{U}_D^{\mathcal{Q}} = \frac{1}{2}\left(U_D(\lambda + \delta^{(a)}) + U_D(\lambda - \delta^{(a)})\right)(1 - p_{\text{loss}}^{\mathcal{Q}}(K))$$
$$= U_D(\lambda) + U_D^{(2)}(\lambda)(\delta^{(a)})^2 + \mathrm{o}\left((\delta^{(a)})^2\right) \quad (58)$$
$$= U_D(\lambda) + \Theta\left(\frac{(\log K)^2}{K^2}\right), \quad (59)$$

where Eq. (58) follows from the fact that $p_{\text{loss}}^{\mathcal{Q}}(K) = \mathrm{O}(K^{-\beta_Q})$ for some $\beta_Q \geq 2$ and Eq. (59) comes from choosing $\delta^{(a)} = \beta_Q \bar{\sigma}_a^2 \frac{\log K}{K}$. This completes the proof of Theorem 3.

## Appendix E
## Proof for Theorem 4

To prove this theorem, first consider a power allocation scheme $\mathcal{E}$ that is given by,

$$e^{\mathcal{E}}(t) = \mu - \delta^{(r)}. \quad (60)$$

We define the battery drift as $d_{\text{battery}}^{\mathcal{E}}(t) \triangleq r(t) - e^{\mathcal{E}}(t)$. The mean drift for the battery process is equal to $\delta^{(r)}$. Applying the diffusion limit, we can write the discharge probability for this energy management scheme as [18], [27],

$$p_{\text{underflow}}^{\mathcal{E}}(M) = \lim_{t\to\infty} \mathbb{P}(\mathbf{B}(t) < 0) = \exp\left(-\frac{2\delta^{(r)}}{\bar{\sigma}_r^2}M\right). \quad (61)$$

Similarly, by applying the diffusion limit to the data buffer process we can find the buffer overflow probability as,

$$p_{\text{overflow}}^{\mathcal{E}}(K) = \lim_{t\to\infty} \frac{1}{2}\mathbb{P}(\mathbf{Q}_u(t) > K) = \exp\left(-\frac{2\delta^{(a)}}{\bar{\sigma}_a^2}K\right), \quad (62)$$

where $\delta^{(a)} = C(\mu - \delta^{(r)}) - \lambda$. Substituting this value of $\delta^{(a)}$ in Eq. (62), we get the desired result.

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