

EQUIVALENT CONTINUUM MODEL FOR DEPLOYABLE FLAT LATTICE STRUCTURES

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ABSTRACT: Deployable structures can be stored in a compact, folded configuration and are easily deployed into load-bearing, open forms. Hence, they are suitable for applications where speed and ease of erection and reusability are desired. The structures investigated here are prefabricated space frames made of so called scissor-like elements, sets of two straight bars connected to each other by a pivot. These structures are stress-free and self-standing in both their folded and deployed configurations, thus overcoming major disadvantages of previous designs. This study deals with deployable structures that are flat and subjected to normal loads in their deployed configuration. Although the behavior for that loading case is linear, the availability of an equivalent continuum model for stiffness prediction is desirable because it can significantly reduce the computational effort during preliminary design. The derivation of such a model is not straightforward because of the unorthodox geometry and the rotations allowed by the hinged and pivotal connections. This problem is addressed by first applying the direct stiffness method within a symbolic manipulation framework to transform the lattice structure to an equivalent single-layer grid, and then using existing expressions to obtain the desired equivalent plate. The model exhibits good accuracy and convergence characteristics for uniform loads.

INTRODUCTION

Deployable structures are prefabricated assemblages of structural members that can be transformed from a closed, compact configuration to a predetermined, expanded form in which they are stable and can carry loads (Merchan 1987). This behavior is achieved by using kinematic connections that permit relative rotations among individual components that undergo small elastic deformations. This paper deals with strut-type deployable structures that use one-dimensional bars as basic components.

The basic module of these structures is the so-called scissor-like element (SLE) shown in Fig. 1. It consists of two bars connected to each other at an intermediate point through a pivotal connection. At their four end nodes the bars are hinged to end nodes of other SLEs to form larger units. More specifically, an improved version of scissor-type deployable structures is investigated. These structures are stress-free in their folded configurations, develop stresses and exhibit a geometrically nonlinear behavior during deployment, and experience a snap-through that "locks" them in their deployed configuration, at which point they are self-standing and stress-free, except for dead-weight effects. This is made possible by enforcing suitable geometric constraints between the member lengths. An important design requirement is that the response during deployment, even though geo-

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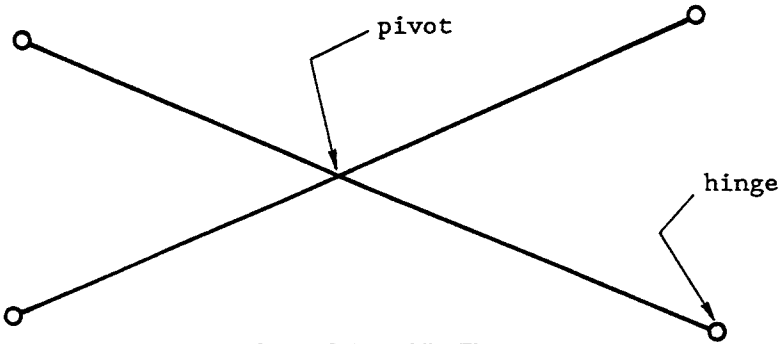


FIG. 1. Scissor-Like-Element

metrically nonlinear, remains elastic from a material point of view, so that the bars have no residual stresses in the deployed configuration. By having this response, these structures overcome major disadvantages of previous designs, such as need for external stabilization and reduced load-bearing capacity in the deployed configuration. An example of such a flat structure is shown in Fig. 2.

These structures were introduced in the department of architecture at Massachusetts Institute of Technology (MIT) by Zalewski (Krishnapillai and Zalewski 1981), who recognized their great potential and identified the need for an in-depth investigation of their geometric and structural characteristics. The writers have been carrying out such an investigation during the last few years (Rosenfeld and Logcher 1988; Logcher et al. 1989; Gantes 1988; Gantes et al. 1989; Gantes et al. 1990, 1993a, 1993b; Gantes et al. 1990, 1991a, 1991b, 1992; Rosenfeld et al. 1993), and have managed to come up with a systematic design methodology that is presented in detail in Gantes (1991).

This paper describes one of the tools derived as part of that methodology, namely an “equivalent” continuum model that helps predict the stiffness characteristics of deployable flat slabs when they are subjected to normal loads in their deployed configuration. It is true that the exact finite element analysis in the deployed configuration is not particularly expensive, since it is linear. But the storage space requirements are very large, and start becoming a serious problem as the number of units increases. This is also due to the complicated pivotal and hinged connections that require more than one nodal point to be described accurately. These drawbacks provided the motivation for the derivation of an equivalent continuum model for the prediction of deflections. This continuum approximation is particularly useful for performing parametric studies in the preliminary design stage.

In the next section, previous work on the derivation of equivalent continuum models for space trusses and frames is presented. The differences between those structures and deployable lattice structures are identified, and explain the need for a different, two-step approach. The first step, substitution of the structure by a single-layer grid of uniform beams, is described next. The second step, using existing expressions from the literature to substitute the grid by an orthotropic plate of uniform thickness, is then described. Finally, the model is tested for a specific case, and its limitations are outlined.

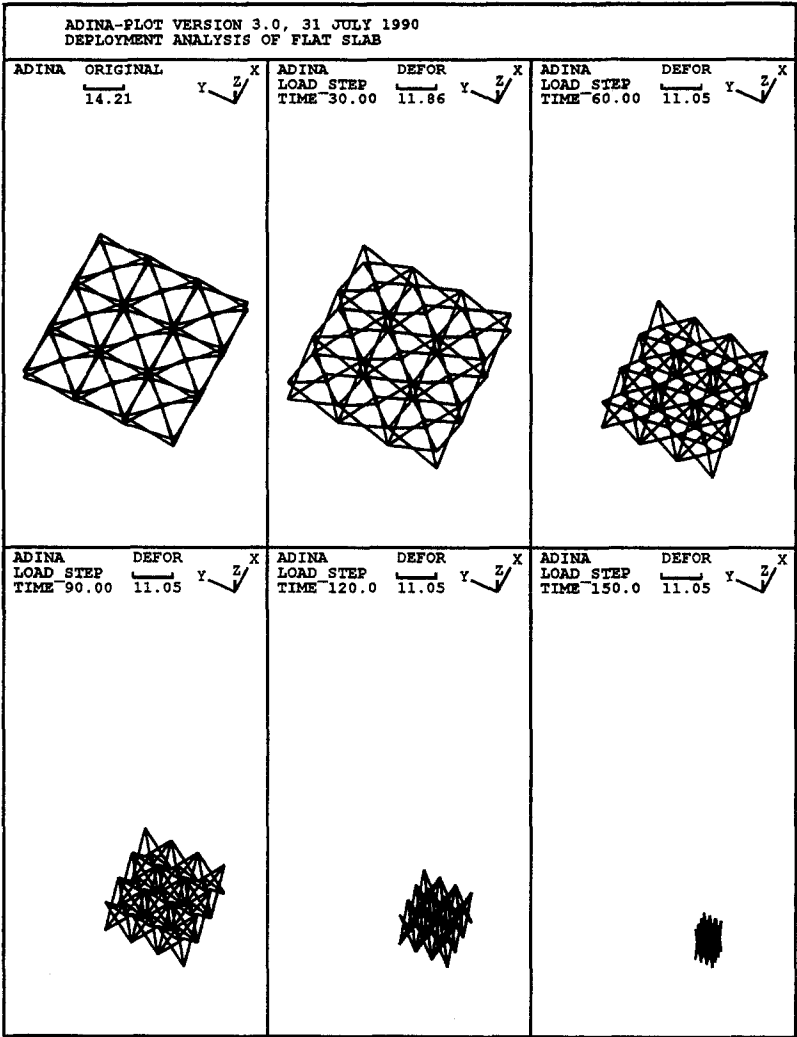


FIG. 2. Successive Deployment Stages of Flat Deployable Structure

MOTIVATION FOR TWO-STEP APPROACH

A considerable amount of work has been produced by many researchers in reducing the computational effort required for the analysis of large, repetitive structures with many degrees of freedom. One of the most active investigators in the field is A. K. Noor, who has numerous relevant publications (Noor et al. 1978; Noor and Weisstein 1981; Noor and Andersen 1981; Noor 1982; Noor 1983; Noor and Russell 1986). Noor (1983) classified the possible approaches for the analysis of large, repetitive lattice structures into four main categories. The first category is the direct method, wherein the structure is analyzed as a system of discrete finite elements. This method has the obvious drawback of high computational cost for structures with

many degrees of freedom. The second category replaces the actual structure with an equivalent continuum model. This method has the limitations that local deformation effects are typically not accounted for, and difficulties are associated with any other type of connections, except hinges. The third category is the discrete field method, which uses the regularity of the lattice to write the equilibrium and compatibility equations at a typical joint and then perform Taylor series expansion to replace these equations with differential equations for the whole structure. This method works well for simple geometries, but encounters problems for complex lattice configurations. Finally, the fourth category is the periodic structure approach, which combines finite elements and transfer matrix methods to solve structures with rotationally symmetric geometries and beam-like lattices.

Our efforts focused on deriving an "equivalent" continuum model, and, more specifically, to obtain a slab of uniform thickness made of a uniform material that behaves macroscopically in the same way as the deployable structures investigated here. Although the equivalent continuum approach does not always have a rational theoretical basis, it can provide satisfactory results for several types of problems and was chosen for its simplicity and cost effectiveness. It has been used in the past to predict stiffness (Makowski 1981; Noor et al. 1978; Tamma and Saw 1987) and stability (Noor and Weisstein 1981) characteristics of repetitive structures.

The approach used by Noor is based on energy equivalence. The equivalent continuum structure is defined as having the same amount of strain energy stored in it as that of the original lattice structure when both are deformed identically (Noor and Russell 1986). In pin-jointed trusses, an ordinary continuum is used for which the displacement field completely characterizes the motion of the structure. For structures with rigid connections, however, a so-called "micropolar continuum" has to be used, where the motion is characterized by both a displacement field and an independent rotation field (Noor 1983). Beam structures with rigidly jointed flexural members have been handled by McCallen and Romstad (1988) without micropolar elasticity by taking into account an additional strain energy term not included in the Timoshenko beam theory.

Another approach for deriving 'equivalent' continuum models is that of effective rigidities, introduced by K. Heki, another leading investigator in the field (Heki 1968, 1972, 1984, 1985, 1986). The effective rigidity of a lattice plate is defined as the rigidity of an equivalent plate that deforms in the same manner as the lattice plate. To derive the effective rigidities, the deformations of the two plates are equated to each other for some appropriate loading conditions (Heki 1972). This method has been used also by other researchers. Nayfeh and Hefzy applied Heki's approach to derive effective rigidities for several types of space trusses and space frames (Nayfeh and Hefzy 1978, 1980, 1982).

Aswani et al. (1982) applied the methodology of Nayfeh and Hefzy to derive an equivalent continuum for a lattice beam consisting of tetrahedral units with pinned connections. Their study demonstrated the satisfactory agreement of approximate and exact results for deflections and natural frequencies, and stressed the influence of transverse shear deformation on the performance of the method. Tamma and Saw (1987) compared the energy-equivalence method as proposed by Noor and the stiffness-equivalence approach as applied by Nayfeh and Hefzy to predict deflected shapes of lattice beams, and showed that the results were in very good agreement. A stiffness-equivalence approach was adopted for our study.

An additional difficulty associated with the derivation of equivalent continua for deployable structures is related to the pivotal connections. It is a different connection type than the pinned and rigid connections that had been encountered in previous research efforts, and complicates the solution by allowing for free rotations between the two bars of a scissor-like element. The procedure used to overcome this difficulty was to first substitute each SLE by an equivalent uniform beam, and thus obtain an equivalent to the

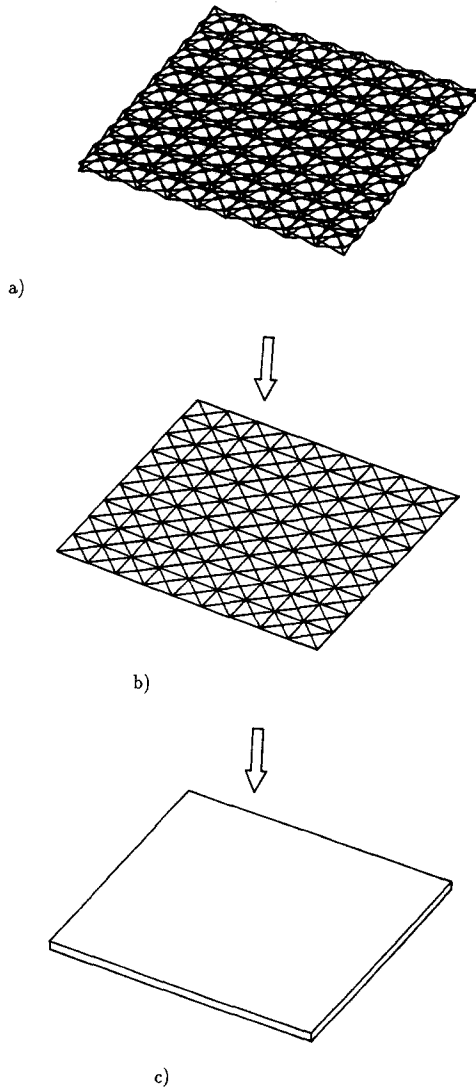


FIG. 3. Two-Step Derivation of Equivalent Slab: (a) Deployable Lattice Structure in Its Flat Deployed Configuration; (b) Equivalent Single Layer Grid; (c) Equivalent Uniform Slab

initial structure single-layer grid. This grid can be further substituted by a uniform slab. This procedure is illustrated in Fig. 3. The individual steps of the derivation will be further explained in the following sections.

SUBSTITUTION OF SLES BY UNIFORM BEAMS

The first step in the derivation of an equivalent continuum uniform plate for a deployable flat lattice structure consists of substituting all scissor-like elements of the structure with equivalent uniform beams. A stiffness equivalence is desired, hence, the equivalent beam is defined as one that deflects identically as the SLE for a given loading. This discussion focuses on flat deployable lattice structures consisting of units with square-plan view similar to the structure shown in Fig. 2. The details of such a square unit are outlined in Fig. 4. These units have two types of SLEs—symmetric (outer) SLEs that form the sides of the square and nonsymmetric (inner) SLEs that form the half-diagonals. A development of these two different SLEs on a common plane is also illustrated in Fig. 4.

Separate derivations are required for these two types of elements. Stretching and in- and out-of-plane bending are the modes of deformation addressed for each element. The terms in- and out-of-plane bending refer to deformations of the bars in and out of the local plane (xy) of the SLE, respectively (Figs. 5–9). Fig. 5 illustrates the concept of a uniform beam with equivalent in-plane-bending characteristics as a symmetric SLE. The SLE is supported in a way that simulates the boundary conditions imposed on it when it deforms as part of the overall structure. The SLE is loaded with a vertical uniformly distributed load q_{oy} , where the index o refers to the outer SLE and the index y to the local in-plane direction of the SLE. This load causes a vertical displacement δ_{oy} of the pivotal connection node. The equivalent beam is clamped at both ends, since the upper and lower end nodes of the SLE do not have any relative horizontal displacements, thus not allowing for a rotation of the end cross section as a whole. The same load q_{oy} is applied on the uniform beam, and it must cause an equal vertical displacement δ_{oy} at the center of the beam.

It is not easy, however, to obtain an analytical expression for the vertical displacement δ_{oy} and thus for the 'equivalent' moment of inertia I_{eqoy} . The independent rotations of the two bars at the pivotal connection cannot be treated with conventional structural analysis techniques, and the assumption that the connection behaves as rigid due to small overall displacements of the structure leads to erroneous results and overestimates the stiffness of the element. The method used to overcome this problem was to apply the direct stiffness method (Przemieniecki 1968) within a symbolic manipulation program, in this case MACSYMA (VAX 1985).

The idea of using symbolic manipulation for computational mechanics has been proposed a few times in the past. It has mainly been applied to derive analytical expressions of stiffness matrices for finite element analysis, and thus avoid numerical integration (Kikuchi 1989; Noor and Andersen 1981). In our case, a somewhat different use is made. The local stiffness matrices of the four beam elements of the SLE are defined within MACSYMA. Next, the transformation matrices of these elements with respect to a global coordinate system are defined, and are used to transform local to global stiffness matrices. Then, the direct stiffness method is used to assemble the global stiffness matrix for the whole structure. This process takes into account the boundary conditions by releasing the appropriate degrees of freedom. The peculiarities of the pivotal connections are also

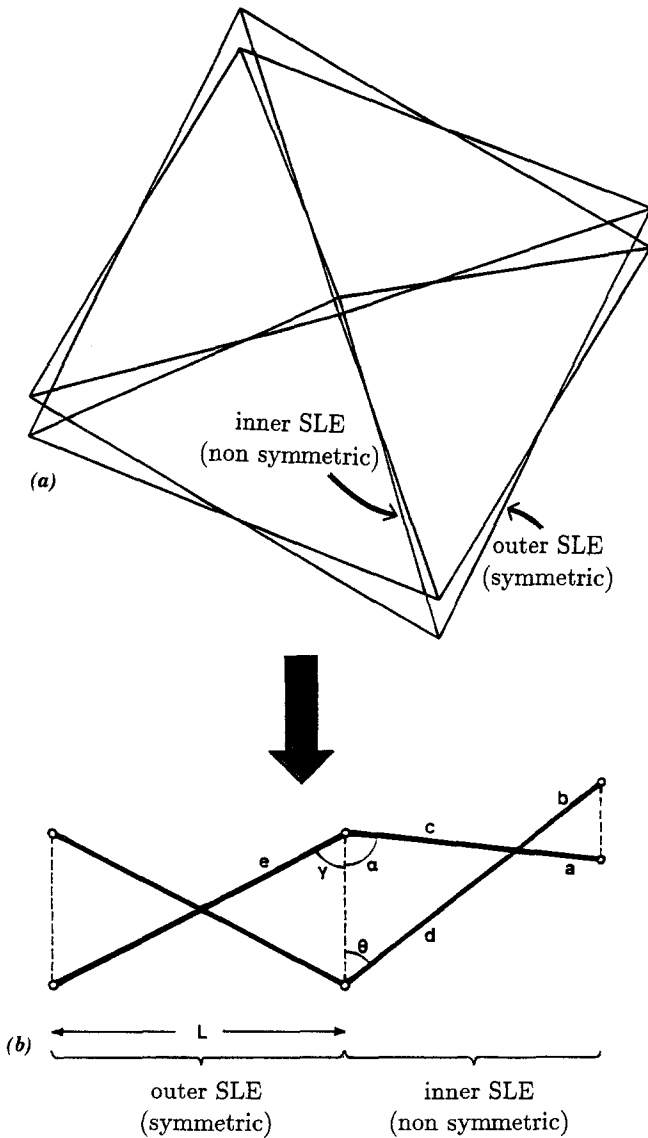


FIG. 4. Two Types of SLEs for Square Units of Flat Structure and Their Geometry in Deployed Configuration

accounted for by defining one node for each bar, and using the master node/slave node technique (Bathe 1982) to constrain the common degrees of freedom. The resulting stiffness matrix is inverted to obtain the flexibility matrix. The load is lumped on the nodes and the loading vector is defined, and multiplied to the flexibility matrix to yield the displacement vector. The following expression for the vertical displacement δ_o , at the center of the beam is obtained:

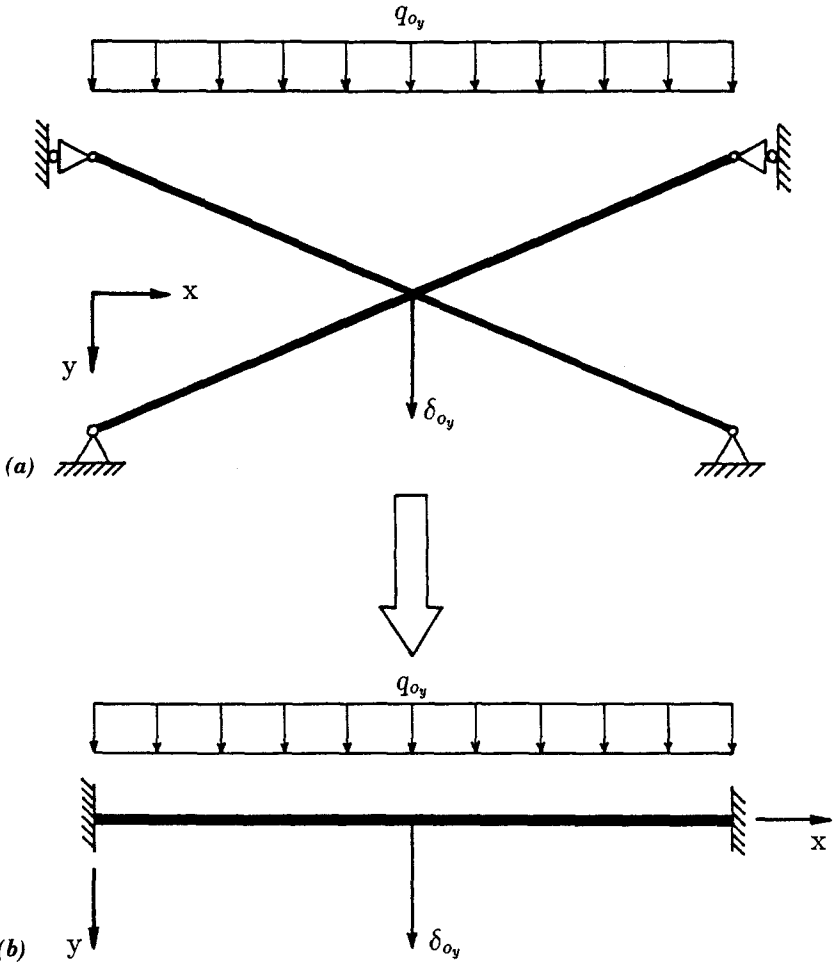


FIG. 5. In-Plane-Bending Equivalent Beams for Symmetric SLEs

$$\delta_{o_y} = \frac{q_{o_y} e^2 \sin \gamma}{E_o} (z_1 + z_2) \dots \dots \dots (1)$$

where

$$z_1 = \frac{9I_{o_y} \sin^2 \gamma}{12A_o I_{o_y} \sin^2 \gamma \cos^2 \gamma + 4A_o^2 e^2 \cos^4 \gamma} \dots \dots \dots (2)$$

and

$$z_2 = \frac{e^2}{3I_{o_y} \sin^2 \gamma + A_o e^2 \cos^2 \gamma} \dots \dots \dots (3)$$

A_o , I_{o_y} , and E_o = cross-sectional area, moment of inertia for in-plane bending, and Young's modulus of the members of the outer SLE, respectively.

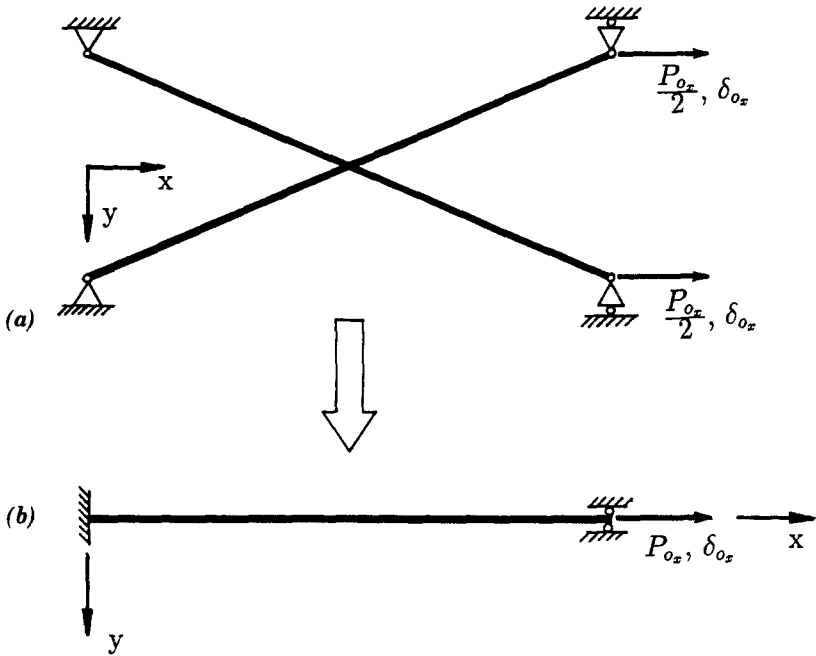


FIG. 6. Stretching-Equivalent Beams for Symmetric SLEs

The length e and the angle γ are shown in Fig. 4, and define the geometry of the outer SLE in the deployed configuration. For the equivalent beam holds

$$\delta_{o_y} = \frac{q_{o_y} L^4}{384 E_{eq_o} I_{eq_{o_y}}} \dots \dots \dots (4)$$

where E_{eq_o} and $I_{eq_{o_y}}$ = Young's modulus and the moment of inertia of the equivalent beam, respectively, and

$$L = 2e \sin \gamma \dots \dots \dots (5)$$

Combining (1) and (5), and assuming that E_{eq_o} is equal to E_o , we obtain

$$I_{eq_{o_y}} = \frac{e^2 \sin^3 \gamma}{24(z_1 + z_2)} \dots \dots \dots (6)$$

A similar procedure is followed to derive the equivalent cross-sectional area for outer SLEs. The loading and the boundary conditions are illustrated in Fig. 6. The results obtained for the deflections of the SLE and the equivalent beam, respectively, are

$$\delta_{o_x} = \frac{P_{o_x} e}{E_o A_o \sin^2 \gamma} \dots \dots \dots (7)$$

and

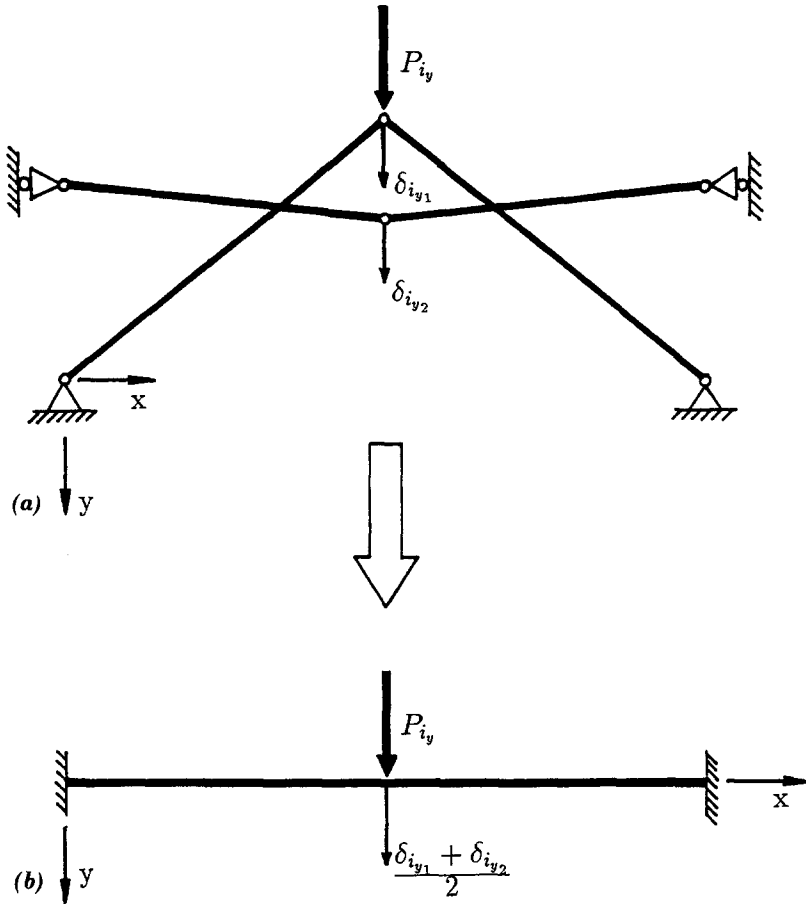


FIG. 7. In-Plane-Bending Equivalent Beams for Nonsymmetric SLEs

$$\delta_{o_x} = \frac{P_{o_x} L}{E_{eq_o} A_{eq_o}} \dots \dots \dots (8)$$

so that

$$A_{eq_o} = 2A_o \sin^3 \gamma \dots \dots \dots (9)$$

The height of an equivalent rectangular cross-section is given by

$$h_{eq_o} = \sqrt{\frac{12I_{eq_{o_y}}}{A_{eq_o}}} \dots \dots \dots (10)$$

The same process is applied to determine the effective bending and stretching rigidities of diagonal scissor-like elements. The loading and boundary conditions are shown in Figs. 7 and 8, respectively. The result obtained for the moment of inertia of the 'equivalent' beams is

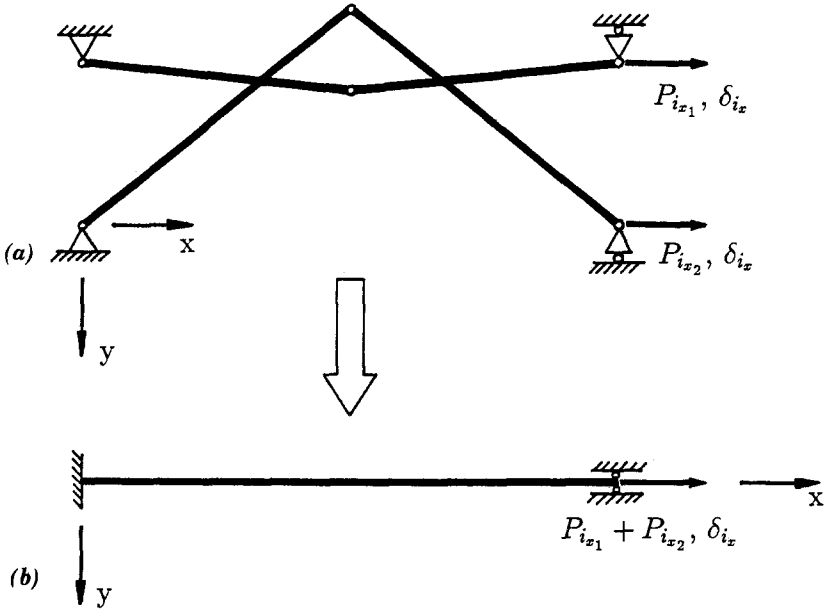


FIG. 8. Stretching-Equivalent Beams for Nonsymmetric SLEs

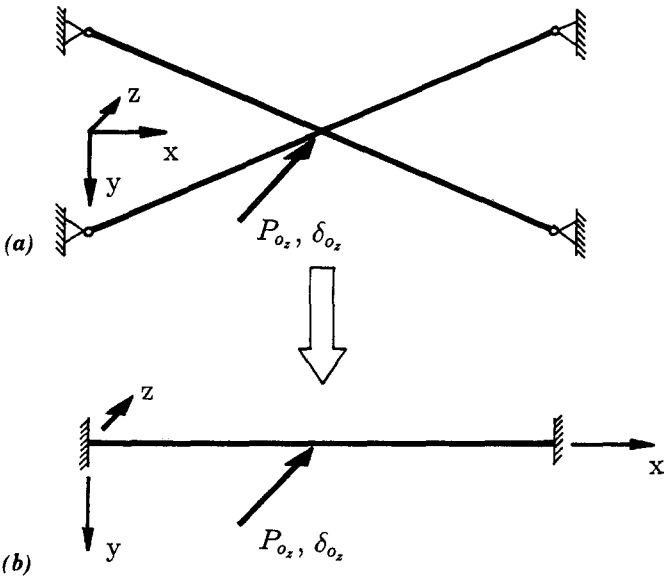


FIG. 9. Out-of-Plane-Bending Equivalent Beams for Symmetric SLEs

$$I_{eq_{iy}} = A_i \frac{(a + c)^3}{6(b + 2d)} \sin^3\alpha \cos^2\theta \dots\dots\dots (11)$$

where the lengths a , b , c , and d , and the angles α and θ are shown in Fig. 4, and define the geometry of the inner SLE in the deployed configuration.

The expression for the area A_{eq_i} of the equivalent beam is very complicated and is not given here. A very long Fortran command with that expression was generated directly by MACSYMA, and used in a subroutine that calculates the properties of the equivalent continuum model. The height of an equivalent rectangular cross section is given by

$$h_{eq_i} = \sqrt{\frac{12I_{eq_{iy}}}{A_{eq_i}}} \dots\dots\dots (12)$$

Concerning the equivalent bending rigidities of the scissor-like elements for out-of-plane loading, the pivotal connections are of no great importance, since small, if any, relative rotations between the two bars will occur due to the nature of the imposed loads and deformations. An outer SLE subjected to a concentrated out-of-plane load P_{o_z} at the middle, as shown in Fig. 9, will deflect by

$$\delta_{o_z} = \left(\frac{1}{2}\right) \frac{P_{o_z}(2e)^3}{48E_o I_{o_z}} \dots\dots\dots (13)$$

since each one of the two bars of the SLE acts as a simply supported beam. The deflection in the middle of an equivalent beam is given by

$$\delta_{o_z} = \frac{P_{o_z}L^3}{192E_{eq_o}I_{eq_{o_z}}} \dots\dots\dots (14)$$

Although the SLE as a whole will rather behave as a simply supported beam, the equivalent beam is calculated as clamped at both ends, since the structure is substituted by a grid of rigidly connected beams. Combining (13) and (14), we get

$$I_{eq_{o_z}} = \frac{1}{2} I_{o_z} \dots\dots\dots (15)$$

The out-of-plane-bending rigidity of the set of two nonsymmetric SLEs as they are shown in Figs. 7 and 8 can be neglected, since the system will behave as a mechanism for out-of-plane loading. In reality, this is not exactly the case, since another set of two inner SLEs will provide some support in the middle. The out-of-plane bending rigidity, however, will be very small due to the asymmetry of the element that will cause significant torsional deformations for any loads that are not in-plane. Therefore, it is reasonable to assume that

$$I_{eq_{iz}} = 0 \dots\dots\dots (16)$$

This completes the first step of the derivation of the equivalent continuum model, namely the substitution of all SLEs by uniform beams.

SUBSTITUTION OF EQUIVALENT GRID BY EQUIVALENT SLAB

The initial flat deployable structure [Fig. 3(a)] has now been substituted with an equivalent grid of uniform beams running in 0°, 90°, 45°, and -45° directions that are rigidly connected to each other [Fig. 3(b)]. Several expressions exist in the literature that can be used for the substitution of this grid by an equivalent uniform plate [Fig. 3(c)]. Since the derivation of the equivalent uniform beams for the SLEs was based on stiffness considerations, a stiffness-based approach was chosen for the second part of the derivation as well, namely the approach of Nayfeh and Hefzy (1982).

In Nayfeh and Hefzy (1982), expressions are derived for the effective continuum properties of several types of planar grids, among them the 0°, 90°, 45°, -45° grid encountered here. The derivation in Nayfeh and Hefzy (1982) is a three-step process. First, all basic planar lattices are identified. Then, effective continuum properties are derived for each of them in local coordinates. The direct method is used for that, namely, the nodal displacements of the basic lattice are equated to those at the corners of the continuum plate element under the same loading. Constitutive relations of the form $\sigma_{ij} = C_{ijkl}\epsilon_{kl}$ ($i, j, k, l = 1, 2, 3$) are derived. The constitutive matrices are transformed to global coordinates according to the formula $C_{ijkl} = C'_{pqrs}\beta_{pi}\beta_{qj}\beta_{rk}\beta_{sl}$, where $\beta_{ij} = \partial x'_i/\partial x_j$ is the direction cosine of the angle between the local axis x'_i and the global axis x_j . Finally, the contributions of all basic lattices are added up and averaged over the equivalent plate element. The result for the grid in question is an orthotropic model with the following constitutive matrix:

$$C = \begin{bmatrix} C_{1111} & C_{1122} & 0 & 0 & 0 & 0 \\ C_{1122} & C_{1111} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{1313} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{1313} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{1212} \end{bmatrix} \dots\dots\dots (17)$$

which has four independent constants C_{1111} , C_{1122} , C_{1313} , and C_{1212} . Directions 1 and 2 are in-plane, as defined by the orientation of symmetric SLEs, and direction 3 is perpendicular to the plane of the slab. The stress and strain vectors are defined by

$$\sigma = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} \dots\dots\dots (18a)$$

$$\epsilon = \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ \epsilon_{23} \\ \epsilon_{13} \\ \epsilon_{12} \end{bmatrix} \dots\dots\dots (18b)$$

and are related by

$$\sigma = C\epsilon \dots\dots\dots (19)$$

The four constitutive constants are given by

$$C_{1111} = \frac{E_o A_{eq_o}}{L h_{eq}} + \frac{E_i A_{eq_i}}{2\sqrt{2}L h_{eq}} + \frac{3E_i I_{eq_{iz}}}{\sqrt{2}L^3 h_{eq}} \dots\dots\dots (20)$$

$$C_{1122} = \frac{E_i A_{eq_i}}{2\sqrt{2}L h_{eq}} - \frac{3E_i I_{eq_{iz}}}{\sqrt{2}L^3 h_{eq}} \dots\dots\dots (21)$$

$$C_{1313} = \frac{3E_o I_{eq_{oy}}}{L^3 h_{eq}} + \frac{3E_i I_{eq_{iy}}}{2\sqrt{2}L^3 h_{eq}} \dots\dots\dots (22)$$

$$C_{1212} = \frac{E_i A_{eq_i}}{2\sqrt{2}L h_{eq}} + \frac{6E_o I_{eq_{oz}}}{L^3 h_{eq}} \dots\dots\dots (23)$$

where h_{eq} = thickness of the equivalent slab defined as:

$$h_{eq} = \left(\frac{1}{2}\right) (h_{eq_o} + h_{eq_i}) \dots\dots\dots (24)$$

This completes the second step of the derivation of the equivalent continuum slab, namely, the substitution of the grid by a uniform slab.

MODEL VERIFICATION AND LIMITATIONS

At this point, the derivation of an equivalent uniform plate for flat deployable structures has been completed. The properties of the continuum plate are expressed through the constitutive matrix C. To make use of the model and assess its accuracy, known analytical expressions for the calculation of deflections of uniform plates under several types of loads can be applied. The orthotropic nature of the equivalent continuum has to be taken into account during that process. Furthermore, the rotational inertia of the bars, and, particularly, shear deformation effects should be taken into account, as illustrated by the study of Aswani et al. (1982).

The expressions for deflections of anisotropic plates given by Ambartsumyan (1970) have been used here to predict the maximum deflections in the middle of rectangular deployable lattice structures when they are simply supported in their deployed configuration and subjected to normally applied loads. The deflection w of an arbitrary point (x, y) for a rectangular plate with dimensions a_1 by b_1 that is simply supported along its four edges and subjected to arbitrary normally applied loads is given by a trigonometric series

$$w = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} f_{mn} \sin \frac{m\pi x}{a_1} \sin \frac{n\pi y}{b_1} \dots\dots\dots (25)$$

where

$$f_{mn} = \frac{\Delta_{1mn}}{\Delta_{0mn}} \alpha_{mn} \dots\dots\dots (26)$$

where Δ_{1mn} and Δ_{0mn} are calculated as functions of the geometry and the

constitutive coefficients. The values of α_{mn} can be obtained by expansion of the external load function $F(x, y)$ into a double Fourier series

$$F(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \alpha_{mn} \sin \frac{m\pi x}{a_1} \sin \frac{n\pi y}{b_1} \dots\dots\dots (27)$$

For the case of a uniformly distributed load q

$$\alpha_{mn} = \begin{cases} \frac{16q}{\pi^2} \frac{1}{mn}, & m, n = 1, 3, 5, \dots \\ 0, & \text{otherwise} \end{cases} \dots\dots\dots (28)$$

For the case of a concentrated load P applied at the point (ξ, η)

$$\alpha_{mn} = \frac{4P}{a_1 b_1} \sin \frac{m\pi\xi}{a_1} \sin \frac{n\pi\eta}{b_1}, \quad m, n = 1, 2, 3, \dots \dots\dots (29)$$

First, the response to a uniformly distributed load was investigated. Fig. 10 shows the convergence characteristics of the equivalent continuum model as a function of the number of units of the structure in each direction. The error measure plotted on the vertical axis is defined as the difference of exact deflection, as obtained from finite element analysis, from predicted deflection divided by the exact deflection.

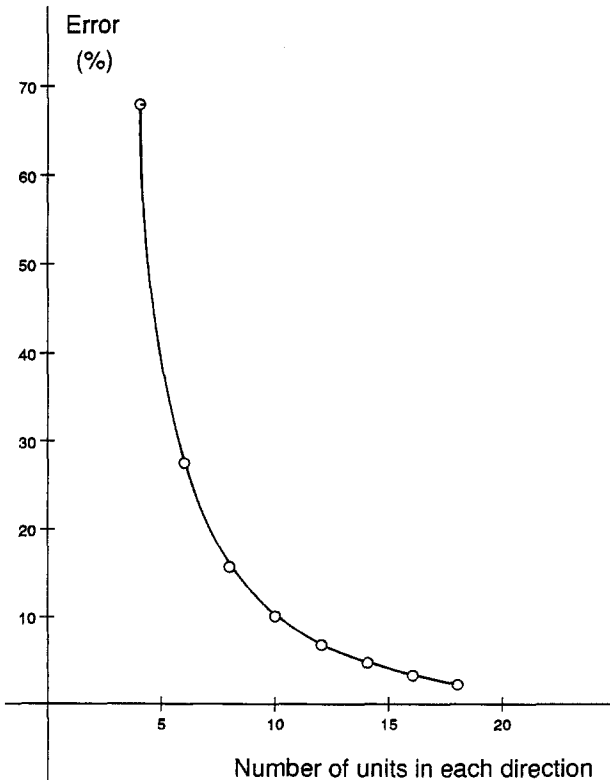


FIG. 10. Convergence Characteristics of Equivalent Continuum Model

The difference between exact and predicted results is due to partial violation of the assumptions involved in the derivation. During both steps of the derivation, substitution of SLEs by uniform beams and substitution of the grid of uniform beams by an equivalent orthotropic plate, certain assumptions were made concerning the boundary conditions and the deformation modes to which the members are subjected, and for which stiffness equivalence is guaranteed. In reality, the structure does not behave in exact accordance with these assumptions; therefore, an error is observed. Our assumptions are better satisfied for lattice structures with larger number of units, which explains the good agreement with the exact finite element results as the number of units increases.

These results refer to a square structure with specific unit configuration in terms of geometry and member properties. However, several other geometric configurations as well as other cross-sectional properties were tested with similar results. In general, it can be said that the model provides satisfactory results, except for the following cases when larger errors can be encountered:

- For bars of the outer SLEs that do not satisfy the relation $e^2/A_o \approx 1,000$ by an order of magnitude or more. Note that this relation holds for normal beams.
- For “thin” bars with a width to height ratio of two or more.
- For structures where outer SLEs have much stiffer bars than inner SLEs.

The derivation assumptions are violated much more when the structure is subjected to other types of load that result in quite different modes of deformation than those assumed. For a concentrated load applied in the middle of the structure, no consistent results could be obtained for the deflections, although the error was always less than 30% for more than eight units in each direction. This behavior is consistent with what has been reported in the literature for the performance of other ‘equivalent’ models for repetitive structures under concentrated loads (Makowski 1981). It is therefore recommended to use the ‘equivalent’ continuum model only in the preliminary design stage for the prediction of deflections due to uniformly distributed load.

In that case, the proposed model can provide considerable savings in time and required computer resources. It is meaningful to use the approximate model only for structures with a large number of units for three reasons. First, the higher number of degrees of freedom results in large computer space requirements that make finite element analysis very expensive. Second, for long spans stiffness governs the design, hence the deflections predicted by the approximate model constitute the deciding design constraint. Third, the accuracy of the deflections given by the continuum model is then acceptable for design calculations.

SUMMARY AND CONCLUSIONS

The derivation of a stiffness-equivalent continuum model for the prediction of deflections of flat deployable lattice structures that are subjected to normal loads in their deployed configuration has been described. The difficulties encountered were due to the unorthodox geometry and the rotations between members allowed by the hinged and pivotal connections. This led

to a two-step approach. First, the structure is substituted by a single-layer grid of uniform beams. Then, this grid is replaced by an orthotropic plate of constant thickness. The model demonstrated satisfactory accuracy and convergence characteristics for deployable flat lattice structures with square-plan view in the deployed configuration that are simply supported along their edges and are loaded uniformly.

Further research is required, particularly pertaining to structures of other shapes and different support conditions. Prediction of stresses by the model would, of course, be desirable, but the preceding two-step approach does not appear to be promising in that direction. From a practical design point of view, however, this is not a big obstacle, since these structures are almost always governed by stiffness (Gantes 1991).

Finally, the growing potential of symbolic manipulation for engineering problems has been demonstrated. Difficulties related to the maximum order of matrices that can be inverted will be alleviated in the future due to continuous hardware and software improvements, but can also be overcome by substructuring. Hence, further contributions can be achieved by applying this powerful tool.

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APPENDIX II. NOTATION

The following symbols are used in this paper:

- a, b, c, d, e = member lengths (illustrated in Fig. 4);
- A_o = cross-sectional area of members of symmetric (outer) SLEs;
- A_i = cross-sectional area of members of nonsymmetric (inner) SLEs;
- A_{eq_o} = cross-sectional area of equivalent beam for symmetric (outer) SLEs;
- A_{eq_i} = cross-sectional area of equivalent beam for nonsymmetric (inner) SLEs;
- a_1, b_1 = dimensions of rectangular flat slab;
- C = equivalent constitutive matrix;
- C_{ijkl} = equivalent constitutive coefficients;
- E_o = Young's modulus of material of symmetric (outer) SLEs;
- E_i = Young's modulus of material of nonsymmetric (inner) SLEs;
- E_{eq_o} = Young's modulus of material of equivalent beam for symmetric (outer) SLEs;
- E_{eq_i} = Young's modulus of material of equivalent beam for nonsymmetric (inner) SLEs;
- $F(x, y)$ = function of external loads normally applied upon flat slab;
- f_{mn} = coefficients by Ambartsumyan for calculation of slab deflections;
- h_{eq} = thickness of equivalent plate;
- h_{eq_o} = height of equivalent rectangular cross section of beam that replaces symmetric (outer) SLEs;
- h_{eq_i} = height of equivalent rectangular cross section of beam that replaces nonsymmetric (inner) SLEs;
- $I_{eq_{o_y}}$ = moment of inertia of equivalent beam for symmetric (outer) SLEs in plane of SLE;
- $I_{eq_{o_z}}$ = moment of inertia of equivalent beam for symmetric (outer) SLEs perpendicular to plane of SLE;
- $I_{eq_{i_y}}$ = moment of inertia of equivalent beam for nonsymmetric (inner) SLEs in plane of SLE;
- $I_{eq_{i_z}}$ = moment of inertia of equivalent beam for nonsymmetric (inner) SLEs perpendicular to plane of SLE;
- I_{i_y} = moment of inertia of members of nonsymmetric (inner) SLEs in plane of SLE;
- I_{i_z} = moment of inertia of members of nonsymmetric (inner) SLEs perpendicular to plane of SLE;

- I_{o_y} = moment of inertia of members of symmetric (outer) SLEs in plane of SLE;
 I_{o_z} = moment of inertia of members of symmetric (outer) SLEs perpendicular to plane of SLE;
 L = length of side of polygon in deployed configuration;
 P = concentrated load applied upon flat slab;
 P_{i_x} = concentrated load applied upon inner SLE in local x -direction;
 P_{i_y} = concentrated load applied upon inner SLE in local y -direction;
 P_{o_x} = concentrated load applied upon outer SLE in local x -direction;
 P_{o_z} = concentrated load applied upon outer SLE in local z -direction;
 q = uniformly distributed load applied upon flat slab;
 q_{o_y} = uniformly distributed load applied upon outer SLE in local y -direction;
 w = transverse displacement of slab;
 z_1 = auxiliary variable defined in text;
 z_2 = auxiliary variable defined in text;
 α, γ, θ = angles between members in deployed configuration (illustrated in Fig. 4);
 α_{mn} = Fourier expansion coefficients of load function $F(x, y)$;
 $\Delta_{0_{mn}}, \Delta_{1_{mn}}$ = coefficients by Ambartsumyan for calculation of slab deflections;
 δ_{i_x} = nodal displacement for inner SLE in local x -direction;
 δ_{i_y} = nodal displacement for inner SLE in local y -direction;
 δ_{o_x} = nodal displacement for outer SLE in local x -direction;
 δ_{o_y} = nodal displacement for outer SLE in local y -direction;
 δ_{o_z} = nodal displacement for outer SLE in local z -direction;
 ϵ = strain vector;
 ϵ_{kl} = elements of strain vector;
 σ = stress vector; and
 σ_{ij} = elements of stress vector.