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STRONGLY COTORSION (TORSION-FREE) MODULES AND COTORSION PAIRS

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Abstract. In this paper, strongly cotorsion (torsion-free) modules are studied and strongly cotorsion (torsion-free) dimension is introduced. It is shown that every module has a special $\mathcal{SC}_n$-preenvelope and an $\mathcal{STF}_n$-cover for any $n \in \mathbb{N}$ based on some results of cotorsion pairs from [9]. Some characterizations of strongly cotorsion (torsion-free) dimension of a module are given.

1. Introduction

Throughout this paper, all rings are associative with identity and all modules are unitary. For a ring R, we write Mod-R for the category of all right R-modules. For a module M, $M^+ = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ will denote the character module of M and $\mathfrak{I}^{-i}(M)$ ($\mathfrak{P}^i(M)$) will denote the i-th cosyzygy (syzygy) of M in $\mathfrak{I}$ ($\mathfrak{P}$), where $\mathfrak{I}$ ($\mathfrak{P}$) is an injective (a projective) resolution of M.

We first recall some known notions and facts which we need in the latter sections.

(1) A right R-module M is called (Enochs) *cotorsion* [7] if $\mathrm{Ext}^1_R(F, M) = 0$ for every flat right R-module F. M is called *strongly cotorsion* [15] if $\mathrm{Ext}^1_R(F,M) = 0$ for every right R-module F of finite flat dimension. A left R-module N is called *strongly torsion-free* [15] if $\mathrm{Tor}^R_1(F, N) = 0$ for every right R-module F of finite flat dimension.

(2) Let M be a right R-module and $\mathcal{C}$ be a class of right R-modules. A homomorphism $\phi : M \to C$ with $C \in \mathcal{C}$ is called a *$\mathcal{C}$-preenvelope* of M [6, 8] if for any homomorphism $f : M \to C'$ with $C' \in \mathcal{C}$, there is a homomorphism $g : C \to C'$ such that $g\phi = f$. Moreover, if the only such g are automorphisms of C when $C = C'$ and $f = \phi$, the $\mathcal{C}$-preenvelope ϕ is called a *$\mathcal{C}$-envelope* of M. $\mathcal{C}$ is a (pre)enveloping class provided that each module has a $\mathcal{C}$-(pre)envelope. Dually, $\mathcal{C}$-precovers, $\mathcal{C}$-covers, and covering classes of modules can be defined.

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(3) Let $\mathcal{C}^{\perp} = \text{Ker Ext}^1_R(\mathcal{C}, -) = \{N \in \text{Mod-}R \mid \text{Ext}^1_R(C, N) = 0$ for all $C \in \mathcal{C}\}$ and ${}^{\perp}\mathcal{C} = \text{Ker Ext}^1_R(-, \mathcal{C}) = \{N \in \text{Mod-}R \mid \text{Ext}^1_R(N, C) = 0$ for all $C \in \mathcal{C}\}$. A $\mathcal{C}$-preenvelope $\phi : M \to C$ is called *special* if ϕ is monic and $\text{coker}\phi \in {}^{\perp}\mathcal{C}$. Dually, the notion of a special $\mathcal{C}$-precover of a module M can be defined.

(4) Let $\mathscr{A}, \mathscr{B} \subseteq \text{Mod-}R$. The pair $(\mathscr{A}, \mathscr{B})$ is called a *cotorsion pair* (or *cotorsion theory*) [8, 9, 11] if $\mathscr{A} = {}^{\perp}\mathscr{B}$ and $\mathscr{B} = \mathscr{A}^{\perp}$. Let $\mathscr{C}$ be a class of R-modules. Following [9], $\mathfrak{C}_{\mathscr{C}} = ({}^{\perp}(\mathscr{C}^{\perp}), \mathscr{C}^{\perp})$ is called the cotorsion pair *generated* by $\mathscr{C}$, and $\mathfrak{V}_{\mathscr{C}} = ({}^{\perp}\mathscr{C}, ({}^{\perp}\mathscr{C})^{\perp})$ is called the cotorsion pair *cogenerated* by $\mathscr{C}$. A cotorsion pair $\mathfrak{C} = (\mathscr{A}, \mathscr{B})$ is called *complete* if each module has a special $\mathscr{B}$-preenvelope ($\mathscr{A}$-precover) and *hereditary* if $\text{Ext}^i_R(A, B) = 0$ for all $i \geq 1, A \in \mathscr{A}$ and $B \in \mathscr{B}$. $\mathfrak{C}$ is called *perfect* provided that $\mathscr{A}$ is a covering class and $\mathscr{B}$ is an enveloping class. $\mathfrak{C}$ is called *closed* provided that the class $\mathscr{A}$ is closed under direct limits in Mod-R. Let $\mathcal{S}$ be a set of modules, the cotorsion pair $({}^{\perp}(\mathcal{S}^{\perp}), \mathcal{S}^{\perp})$ is complete [9].

(5) Let $\mathscr{C}$ be a class of R-modules. $\mathscr{C}$ is *coresolving* provided that $\mathscr{C}$ contains all injective modules, is closed under extensions and $C \in \mathscr{C}$ whenever $0 \to A \to B \to C \to 0$ is a short exact sequence such that $A, B \in \mathscr{C}$. Dually, $\mathscr{C}$ is *resolving* provided that $\mathscr{C}$ contains all projective modules, is closed under extensions and $A \in \mathscr{C}$ whenever $0 \to A \to B \to C \to 0$ is a short exact sequence such that $B, C \in \mathscr{C}$. For a cotorsion pair $\mathfrak{C} = (\mathscr{A}, \mathscr{B})$, it is well known that $\mathfrak{C}$ is hereditary if and only if $\mathscr{A}$ is resolving or $\mathscr{B}$ is coresolving.

Let $\mathscr{C}$ be a class of R-modules. For a cardinal κ, we denote by $\mathscr{C}^{\leq \kappa}$ the subclass of $\mathscr{C}$ consisting of the modules possessing a projective resolution containing only $\leq \kappa$-generated modules. We denote by $\mathcal{F}_n$ $(\mathcal{I}_n, \mathcal{P}_n)$ the class of all right R-modules of flat (injective, projective) dimension $\leq n$, where n is a nonnegative integer. Furthermore, $\mathcal{F}$ $(\mathcal{P})$ will denote the class of all right R-modules of finite flat (projective) dimension. We use $fd(M)$ and $id(M)$ to denote the usual flat and injective dimensions of a module M respectively. $rFPD(R)$ will denote the supremum of the projective dimensions of all right R-modules of finite projective dimension.

In Section 2, we introduce the concept of strongly cotorsion dimension. We show that every module has a special $\mathcal{SC}_n$-preenvelope for any $n \in \mathbb{N}$. Some characterizations of strongly cotorsion dimension of a module are given.

In Section 3, we introduce the concept of strongly torsion-free dimension. We show that every module has an $\mathcal{STF}_n$-cover for any $n \in \mathbb{N}$. Some characterizations of strongly torsion-free dimension of a module are given.

For unexplained concepts, notions and facts, we refer the reader to [1, 3, 4, 5, 10].

2. Strongly cotorsion modules

We start with the following definition.

Definition 2.1. Given a right R-module M. Let $scd(M) = \inf\{n$: there exists an exact sequence $0 \to M \to C_0 \to C_1 \to \cdots \to C_n \to 0$ of right R-modules, where each C_i is strongly cotorsion$\}$ and call $scd(M)$ the strongly cotorsion dimension of M. If no such n exists, set $scd(M) = \infty$.

In what follows, $\mathcal{SC}_n$ denotes the class of all right R-modules of strongly cotorsion dimension $\leq n$, where n is a nonnegative integer. Clearly, $\mathcal{SC}_0$ is the class of all strongly cotorsion right R-modules.

Lemma 2.2. *Let R be a ring and M a right R-module. Then M is strongly cotorsion if and only if* $\mathrm{Ext}_R^i(F, M) = 0$ *for every $F \in \mathcal{F}$ and $i \geq 1$.*

Proof. The assertion is clear, since all syzygies of modules in $\mathcal{F}$ are again in $\mathcal{F}$. □

Proposition 2.3. *Let R be a ring. The class $\mathcal{SC}_0$ is coresolving.*

Proof. By Lemma 2.2, the cotorsion pair $({}^{\perp}\mathcal{SC}_0, \mathcal{SC}_0)$ is hereditary, so the claim follows by [9, Lemma 2.2.10]. □

Lemma 2.4 ([13, Lemma 1.5(3)])**.** *Let R be a ring and $n \in \mathbb{N}$. Let $\kappa =$ card$(R)+\aleph_0$. Then $(\mathcal{F}_n, (\mathcal{F}_n)^{\perp})$ is a cotorsion pair generated by $\mathcal{F}_n^{\leq\kappa}$.*

Theorem 2.5. *Let R be a ring. Then $({}^{\perp}\mathcal{SC}_0, \mathcal{SC}_0)$ is a hereditary complete cotorsion pair and every right R-module has a special $\mathcal{SC}_0$-preenvelope. In particular, $({}^{\perp}\mathcal{SC}_0, \mathcal{SC}_0)$ is a cotorsion pair generated by $\bigcup_{n\in\mathbb{N}} \mathcal{F}_n^{\leq\kappa}$, where $\kappa =$*card$(R)+\aleph_0$.

Proof. By definition, $\mathcal{SC}_0 = \mathcal{F}^{\perp}$, thus $({}^{\perp}\mathcal{SC}_0, \mathcal{SC}_0) = ({}^{\perp}(\mathcal{F}^{\perp}), \mathcal{F}^{\perp})$ is a cotorsion pair. Since $\mathcal{SC}_0$ is a coresolving class by Proposition 2.3, the cotorsion pair $({}^{\perp}\mathcal{SC}_0, \mathcal{SC}_0)$ is hereditary. By Lemma 2.4, $\mathcal{SC}_0 = \mathcal{F}^{\perp} = (\bigcup_{n\in\mathbb{N}} \mathcal{F}_n)^{\perp} = \bigcap_{n\in\mathbb{N}} \mathcal{F}_n^{\perp} = \bigcap_{n\in\mathbb{N}} \mathcal{F}_n^{\leq\kappa\perp} = (\bigcup_{n\in\mathbb{N}} \mathcal{F}_n^{\leq\kappa})^{\perp}$, where $\kappa =$ card$(R)+\aleph_0$. So $({}^{\perp}\mathcal{SC}_0, \mathcal{SC}_0)$ is a cotorsion pair generated by $\bigcup_{n\in\mathbb{N}} \mathcal{F}_n^{\leq\kappa}$. It is easy to see that $\bigcup_{n\in\mathbb{N}} \mathcal{F}_n^{\leq\kappa}$ has a representative set of elements, then the cotorsion pair $({}^{\perp}\mathcal{SC}_0, \mathcal{SC}_0)$ is complete, and hence every right R-module has a special $\mathcal{SC}_0$-preenvelope. □

We are now ready to compute the strongly cotorsion dimension of a non-zero module. We start with the following definition.

Definition 2.6. Let M be a non-zero right R-module, $\delta(M) = \inf\{n \in \mathbb{N} \mid \mathrm{Ext}_R^{n+1}(F, M) = 0$ for all $F \in \mathcal{F}\}$, and $\lambda(M) = \sup\{n \in \mathbb{N} \mid \mathrm{Ext}_R^n(F, M) \neq 0$ for some $F \in \mathcal{F}\}$.

Proposition 2.7. *Let R be a ring and M a non-zero right R-module. Then $scd(M) = \delta(M) = \lambda(M)$.*

Proof. The equality $\delta(M) = \lambda(M)$ is obvious. If $scd(M) = \infty$, $\delta(M) \leq scd(M)$ is obvious. Suppose $scd(M) = n$, i.e., there is an exact sequence $0 \to M \to C_0 \to \cdots \to C_n \to 0$ of right R-modules with each C_i strongly cotorsion.

For any $F \in \mathcal{F}$, since $\mathrm{Ext}_R^j(F, C_i) = 0$ for any i and $j \geq 1$, by dimension shifting, $\mathrm{Ext}_R^{n+1}(F, M) \cong \mathrm{Ext}_R^1(F, C_n) = 0$, $\delta(M) \leq n = scd(M)$ in this case. So $\delta(M) \leq scd(M)$ always holds. To prove $scd(M) \leq \delta(M)$, it is enough to prove $scd(M) \leq \delta(M)$ if $\delta(M)$ is finite. Let $\delta(M) = m$, by dimension shifting, $\mathrm{Ext}_R^1(F, \mathfrak{I}^{-m}(M)) \cong \mathrm{Ext}_R^{m+1}(F, M) = 0$ for every $F \in \mathcal{F}$, thus $\mathfrak{I}^{-m}(M)$ is strongly cotorsion. Hence M has an exact sequence $0 \to M \to E_0 \to E_1 \to \cdots \to E_{m-1} \to C_m \to 0$ such that C_m is strongly cotorsion and E_i is injective for any $i \in \{0, 1, \ldots, m-1\}$. Since $\mathcal{I}_0 \subseteq \mathcal{SC}_0$, $scd(M) \leq m$ by definition, i.e., $scd(M) \leq \delta(M)$. So $scd(M) = \delta(M)$. □

Let M be a module and $\mathcal{C}$ be a class of modules. Recall that M is $\mathcal{C}$-*filtered*, provided that there are an ordinal μ and a continuous chain of modules, $(M_\alpha \mid \alpha \leq \mu)$, consisting of submodules of M such that $M = M_\mu$, and each of the modules $M_{\alpha+1}/M_\alpha$ $(\alpha < \mu)$ is isomorphic to an element of $\mathcal{C}$. A continuous chain of modules $(M_\alpha \mid \alpha \leq \mu)$ is a sequence of modules satisfying $M_0 = 0, M_\alpha \subseteq M_{\alpha+1}$ for all $\alpha < \mu$ and $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$ for all limit ordinals $\alpha \leq \mu$ (see [9, Definition 3.1.1]).

For example, if $\mathcal{C}$ is the class of all simple R-modules, then the $\mathcal{C}$-filtered R-modules coincide with the semiartinian R-modules.

The following lemma due to Eklof gives an important sufficient condition for the vanishing of Ext.

Lemma 2.8 (Eklof Lemma)**.** *Let N be a module and M be a ${}^\perp N$-filtered module. Then $M \in {}^\perp N$.*

Lemma 2.9 ([9, Corollary 3.2.4])**.** *Let R be a ring and $\mathcal{S}$ a set of modules containing R. Then the class ${}^\perp(\mathcal{S}^\perp)$ consists of all direct summands of $\mathcal{S}$-filtered modules.*

Theorem 2.10. *Let R be a ring and $n \in \mathbb{N}$, and $\kappa = \mathrm{card}(R)+\aleph_0$. Then the following are equivalent for a non-zero right R-module M:*

1. *$scd(M) \leq n$.*
2. *$\delta(M) \leq n$.*
3. *$\lambda(M) \leq n$.*
4. *$\inf\{m \mid \mathrm{Ext}_R^{m+1}(F, M) = 0$ for all $F \in \bigcup_{i\in\mathbb{N}} \mathcal{F}_i^{\leq\kappa}\} \leq n$.*
5. *$\sup\{m \mid \mathrm{Ext}_R^{m}(F, M) \neq 0$ for some $F \in \bigcup_{i\in\mathbb{N}} \mathcal{F}_i^{\leq\kappa}\} \leq n$.*
6. *$\mathfrak{I}^{-n}(M)$ is strongly cotorsion.*
7. *There is an exact sequence $0 \to M \to C_0 \to \cdots \to C_n \to 0$ of right R-modules with each C_i strongly cotorsion.*
8. *If $0 \to M \to L_0 \to \cdots \to L_{n-1} \to W \to 0$ is an exact sequence of right R-modules with each L_i strongly cotorsion, then W is strongly cotorsion.*

Proof. (1) $\Longleftrightarrow$ (2) $\Longleftrightarrow$ (3) follows from Proposition 2.7.

(1) $\Longleftrightarrow$ (6) is easy to see from the proof of Proposition 2.7.

(1) $\Longleftrightarrow$ (7) is obvious by definition.

(6) $\Longrightarrow$ (7). Let $0 \to M \to E_0 \to \cdots \to E_{n-1} \to \mathfrak{I}^{-n}(M) \to 0$ be an exact sequence with $E_0, E_1, \dots, E_{n-1}$ injective, by hypothesis, the exact sequence $0 \to M \to E_0 \to \cdots \to E_{n-1} \to \mathfrak{I}^{-n}(M) \to 0$ satisfies the condition of (7).

(7) $\Longrightarrow$ (8). By hypothesis, $scd(M) \leq n$ and so $\mathrm{Ext}_R^{n+1}(F, M) = 0$ for every $F \in \mathcal{F}$. Let $0 \to M \to L_0 \to \cdots \to L_{n-1} \to N \to 0$ be an exact sequence of right R-modules with each L_i strongly cotorsion. For any $F \in \mathcal{F}$, since $\mathrm{Ext}_R^k(F, L_i) = 0$ for any $k \geq 1$, by dimension shifting, we have $\mathrm{Ext}_R^1(F, N) \cong \mathrm{Ext}_R^{n+1}(F, M) = 0$, i.e., N is strongly cotorsion.

(8) $\Longrightarrow$ (6) is obvious because M has an exact sequence $0 \to M \to E_0 \to \cdots \to E_{n-1} \to \mathfrak{I}^{-n}(M) \to 0$ with $E_0, E_1, \dots, E_{n-1}$ injective.

(2) $\Longrightarrow$ (4) is trivial because $\bigcup_{i \in \mathbb{N}} \mathcal{F}_i^{\leq \kappa} \subseteq \mathcal{F}$.

(4) $\Longrightarrow$ (2). For any $F \in \mathcal{F}$, there exists $m \in \mathbb{N}$ such that $F \in \mathcal{F}_m$. By Lemmas 2.4 and 2.9, F is a direct summand of an $\mathcal{F}_m^{\leq \kappa}$-filtered module. By hypothesis and Lemma 2.8, $\mathrm{Ext}_R^{n+1}(F, M) = 0$ and so $\delta(M) \leq n$.
The proof of (3) $\Longleftrightarrow$ (5) is similar to that of (2) $\Longleftrightarrow$ (4). □

Proposition 2.11. *Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be an exact sequence of right R-modules. Assume $scd(M_1) = n$. Then, for any integer $m \geq n$, $scd(M_2) \leq m \Longleftrightarrow scd(M_3) \leq m$.*

Proof. If $M_1 = 0$, we are done. Suppose $M_1 \neq 0$. Consider a following exact and commutative diagram of right R-modules:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & M_1 & \to & M_2 & \to & M_3 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & H_0 & \to & H_0 \bigoplus I_0 & \to & I_0 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & H_{m-2} & \to & H_{m-2} \bigoplus I_{m-2} & \to & I_{m-2} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & H_{m-1} & \to & H_{m-1} \bigoplus I_{m-1} & \to & I_{m-1} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & K_1 & \to & K_2 & \to & K_3 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

where H_i and I_i are injective $(0 \leq i \leq m-1)$. It follows from $scd(M_1) = n$ and Theorem 2.10 that K_1 is strongly cotorsion. Hence K_2 is strongly cotorsion if and only if K_3 is strongly cotorsion by Proposition 2.3. This completes the proof. □

The following proposition shows that the notion of strongly cotorsion dimension is similar to that of injective dimension.

Proposition 2.12. *Let R be a ring and $0 \to A \to B \to C \to 0$ a short exact sequence of right R-modules. Then*

(1) *$scd(C) \leq \max\{scd(A)-1, scd(B)\}$. If $scd(B) < scd(A)$, then $scd(C) = scd(A) - 1$; if $scd(B) > scd(A)$, then $scd(C) = scd(B)$.*
(2) *$scd(B) \leq \max\{scd(A), scd(C)\}$. If $scd(A) \leq scd(C)$, then $scd(B) = scd(C)$; if $scd(C) \leq scd(A) - 2$, then $scd(B) = scd(A)$. In particular, $scd(C) = scd(B)$ if A is strongly cotorsion.*
(3) *$scd(A) \leq \max\{scd(C)+1, scd(B)\}$. If $scd(B) < scd(C)$, then $scd(A) = scd(C) + 1$; if $scd(B) > scd(C)$, then $scd(A) = scd(B)$.*

Proof. For any $F \in \mathcal{F}$, we have the long exact sequence

$$\begin{aligned}\mathrm{Ext}^1_R(F,A) \to \mathrm{Ext}^1_R(F,B) \to \mathrm{Ext}^1_R(F,C) \to \mathrm{Ext}^2_R(F,A)\\ \to \cdots \to \mathrm{Ext}^i_R(F,C) \to \mathrm{Ext}^{i+1}_R(F,A) \to \cdots.\end{aligned}$$

If one of A, B and C is zero, we are done. If each of A, B and C is not zero, it is easy to get (1), (2), (3) by Proposition 2.7 and the above long exact sequence. □

By Proposition 2.12, we immediately have the following corollary.

Corollary 2.13. *Let R be a ring and $0 \to A \to B \to C \to 0$ a short exact sequence of right R-modules. If two of $scd(A), scd(B)$ and $scd(C)$ are finite, then so is the third.*

Obviously, $\mathcal{F} \subseteq {}^{\perp}\mathcal{SC}_0$. The following proposition gives a criterion when the equality holds.

Proposition 2.14. *Let R be a ring. Then $\mathcal{F} = {}^{\perp}\mathcal{SC}_0$ if and only if $\mathcal{F}$ is closed under direct sums.*

Proof. "$\Longrightarrow$" is trivial because ${}^{\perp}\mathcal{SC}_0$ is closed under direct sums.

"$\Longleftarrow$." Since $\mathcal{F}$ is closed under direct sums, there exists $n \in \mathbb{N}$ such that $\mathcal{F} = \mathcal{F}_n$. Thus $({}^{\perp}\mathcal{SC}_0, \mathcal{SC}_0) = ({}^{\perp}\mathcal{SC}_0, \mathcal{F}^{\perp}) = ({}^{\perp}\mathcal{SC}_0, \mathcal{F}_n^{\perp})$, i.e., $({}^{\perp}\mathcal{SC}_0, \mathcal{F}_n^{\perp})$ is a cotorsion pair. Since $(\mathcal{F}_n, \mathcal{F}_n^{\perp})$ is a cotorsion pair by Lemma 2.4, ${}^{\perp}\mathcal{SC}_0 = {}^{\perp}(\mathcal{F}_n^{\perp}) = \mathcal{F}_n = \mathcal{F}$. □

Proposition 2.15. *Let R be a ring and $\{M_i\}_{i\in I}$ a family of right R-modules. Then $scd(\prod_{i\in I} M_i) = \sup\{scd(M_i)\}_{i\in I}$.*

Proof. Without loss of generality, we may assume that each M_i is non-zero. Since $\mathrm{Ext}^j_R(F, \prod_{i\in I} M_i) \cong \prod_{i\in I} \mathrm{Ext}^j_R(F, M_i)$ for any $F \in \mathcal{F}$ and $j \geq 0$, $\sup\{scd(M_i)\}_{i\in I} \leq scd(\prod_{i\in I} M_i)$ by Proposition 2.7. If $\sup\{scd(M_i)\}_{i\in I}$ is infinite, we are done. Suppose that $\sup\{scd(M_i)\}_{i\in I}$ is finite, let $\sup\{scd(M_i)\}_{i\in I}$

$= n$, then $scd(M_i) \leq n$ for every $i \in I$, and so $\mathrm{Ext}_R^{n+1}(F, M_i) = 0$ for every $F \in \mathcal{F}$ and $i \in I$. Thus $\mathrm{Ext}_R^{n+1}(F, \prod_{i\in I} M_i) = 0$, i.e., $scd(\prod_{i\in I} M_i) \leq n$ by Theorem 2.10. Hence $scd(\prod_{i\in I} M_i) \leq \sup\{scd(M_i)\}_{i\in I}$ and so $scd(\prod_{i\in I} M_i) = \sup\{scd(M_i)\}_{i\in I}$ in this case. So the assertion holds. □

Proposition 2.16. *Let R be a ring and $\kappa = \mathrm{card}(R)+\aleph_0$. Then every right strongly cotorsion R-module is injective if and only if every right R-module is an $\bigcup_{n\in\mathbb{N}} \mathcal{F}_n^{\leq\kappa}$-filtered module.*

Proof. " $\Longrightarrow$ ". By hypothesis, we have ${}^{\perp}\mathcal{SC}_0 = {}^{\perp}\mathcal{I}_0$ =Mod-R. Since ${}^{\perp}\mathcal{SC}_0 = {}^{\perp}((\bigcup_{n\in\mathbb{N}} \mathcal{F}_n^{\leq\kappa})^{\perp})$ by Theorem 2.5, for any $H \in {}^{\perp}\mathcal{SC}_0$, H is a direct summand of an $\bigcup_{n\in\mathbb{N}} \mathcal{F}_n^{\leq\kappa}$-filtered module by Lemma 2.9. So every right R-module is a direct summand of an $\bigcup_{n\in\mathbb{N}} \mathcal{F}_n^{\leq\kappa}$-filtered module. Then every right R-module is an $\bigcup_{n\in\mathbb{N}} \mathcal{F}_n^{\leq\kappa}$-filtered module by [9, Lemma 4.2.10] (applied for κ^+).

" $\Longleftarrow$ ". By hypothesis and the proof of " $\Longrightarrow$ ", ${}^{\perp}\mathcal{SC}_0$ =Mod-R by Lemma 2.9. Since $({}^{\perp}\mathcal{SC}_0, \mathcal{SC}_0)$ is a cotorsion pair, $\mathcal{SC}_0 = ({}^{\perp}\mathcal{SC}_0)^{\perp}$ =(Mod-R)$^{\perp} = \mathcal{I}_0$, i.e., every right strongly cotorsion R-module is injective. □

Now, we define the global strongly cotorsion dimension of a ring R.

Definition 2.17. Let R be a ring. Define $rSCD(R) = \sup\{scd(M) \mid M \in$ Mod-$R\}$. $rSCD(R)$ is called the right global strongly cotorsion dimension of R.

Proposition 2.18. *Let R be a ring. Then the following are equivalent*:

(1) $rSCD(R) = 0$.
(2) *Every right R-module M is strongly cotorsion.*
(3) $rFPD(R) = 0$.

Proof. (1) $\Longleftrightarrow$ (2) follows from definition.

(2) $\Longrightarrow$ (3). By hypothesis, ${}^{\perp}\mathcal{SC}_0 = {}^{\perp}$(Mod-$R$) $= \mathcal{P}_0$. Since $\mathcal{SC}_0 = \mathcal{F}^{\perp}, \mathcal{F} \subseteq {}^{\perp}\mathcal{SC}_0 = \mathcal{P}_0$. Obviously $\mathcal{P}_0 \subseteq \mathcal{F}$, thus $\mathcal{F} = \mathcal{P}_0$. Therefore, every flat right R-module is projective, i.e., R is right perfect, and so $\mathcal{F} = \mathcal{P}$. Thus $\mathcal{P} = \mathcal{P}_0$, i.e., $rFPD(R) = 0$.

(3) $\Longrightarrow$ (2). Since $rFPD(R) = 0$, R is right perfect by [2, Theorem 6.3]. Hence $\mathcal{F} = \mathcal{P} = \mathcal{P}_0$. Thus $\mathcal{SC}_0 = \mathcal{F}^{\perp} = \mathcal{P}_0^{\perp}$ =Mod-R, i.e., every right R-module is strongly cotorsion. □

Proposition 2.19. *Let R be a ring. Then the following are equivalent*:

(1) $rSCD(R) \leq 1$.
(2) *$\mathcal{SC}_0$ is closed under factor modules.*
(3) *$\mathfrak{I}^{-1}(M)$ is strongly cotorsion for every right R-module M.*

Proof. (1) $\Longrightarrow$ (2). For any $L \in \mathcal{SC}_0$ and $K \leq L$, we have the short exact sequence $0 \to K \to L \to L/K \to 0$ of right R-modules. Since $scd(K) \leq 1$ by hypothesis and $scd(L) = 0$, $scd(L/K) = 0$ by Proposition 2.12, i.e., L/K is strongly cotorsion. So $\mathcal{SC}_0$ is closed under factor modules.

(2) $\Longrightarrow$ (3) is trivial.

(3) $\Longrightarrow$ (1) follows from Theorem 2.10. □

By Proposition 2.19, we immediately have the following corollary.

Corollary 2.20. *Let R be a ring. If $\mathcal{F} \subseteq \mathcal{P}_1$, then $rSCD(R) \leq 1$.*

Theorem 2.21. *Let R be a ring and $n \geq 1$. Then $({}^{\perp}\mathcal{SC}_n, \mathcal{SC}_n)$ is a hereditary complete cotorsion pair. In particular, it is generated by $\mathfrak{P}^n(\bigcup_{n\in\mathbb{N}} \mathcal{F}_n^{\leq\kappa})$, where $\kappa = \mathrm{card}(R)+\aleph_0$.*

Proof. By Theorem 2.10, we have $\mathcal{SC}_n = \mathrm{Ker}\ \mathrm{Ext}_R^{n+1}(\mathcal{F}, -) = (\mathfrak{P}^n(\mathcal{F}))^{\perp}$. Thus $({}^{\perp}\mathcal{SC}_n, \mathcal{SC}_n) = ({}^{\perp}((\mathfrak{P}^n(\mathcal{F}))^{\perp}), (\mathfrak{P}^n(\mathcal{F}))^{\perp})$ is a cotorsion pair. By Propositions 2.3 and 2.12, it is easy to see that $\mathcal{SC}_n$ is a coresolving class, so $({}^{\perp}\mathcal{SC}_n, \mathcal{SC}_n)$ is hereditary. Obviously $\mathcal{SC}_n \subseteq (\mathfrak{P}^n(\bigcup_{n\in\mathbb{N}} \mathcal{F}_n^{\leq\kappa}))^{\perp}$ for $\kappa = \mathrm{card}(R)+\aleph_0$. For any $C \in (\mathfrak{P}^n(\bigcup_{n\in\mathbb{N}} \mathcal{F}_n^{\leq\kappa}))^{\perp}$, we have $\mathrm{Ext}_R^1(M, C) = 0$ for every $M \in \mathfrak{P}^n(\bigcup_{n\in\mathbb{N}} \mathcal{F}_n^{\leq\kappa})$, and so $\mathrm{Ext}_R^{n+1}(W, C) = 0$ for every $W \in \bigcup_{n\in\mathbb{N}} \mathcal{F}_n^{\leq\kappa}$. Hence $\mathrm{Ext}_R^1(W, \mathfrak{I}^{-n}(C)) = 0$ by dimension shifting and so $C \in \mathcal{SC}_n$ by Theorem 2.10. Then $(\mathfrak{P}^n(\bigcup_{n\in\mathbb{N}} \mathcal{F}_n^{\leq\kappa}))^{\perp} \subseteq \mathcal{SC}_n$. Therefore, $\mathcal{SC}_n = (\mathfrak{P}^n(\bigcup_{n\in\mathbb{N}} \mathcal{F}_n^{\leq\kappa}))^{\perp}$. Since $\bigcup_{n\in\mathbb{N}} \mathcal{F}_n^{\leq\kappa}$ has a representative set of elements, $\mathfrak{P}^n(\bigcup_{n\in\mathbb{N}} \mathcal{F}_n^{\leq\kappa})$ also has a representative set of elements. Thus $({}^{\perp}\mathcal{SC}_n, \mathcal{SC}_n)$ is complete by [9, Theorem 3.2.1]. □

Proposition 2.22. *Let R and S be rings. If C is a strongly cotorsion right S-module and ${}_RM_S$ is a bimodule with M a flat S-module. Then $\mathrm{Hom}_S(M, C)$ is a strongly cotorsion right R-module.*

Proof. The proof is modeled on that of [12, Lemma 2.14]. For any right R-module F with finite flat dimension, it is easy to see that the flat dimension of $F \otimes_R M$ is also finite as a right S-module. By hypothesis, we have $\mathrm{Ext}_S^1(F \otimes_R M, C) = 0$. Furthermore, for any projective right R-module P, we have $\mathrm{Ext}_S^i(P \otimes M, C) = 0$ for any $i > 0$ by Lemma 2.2. Then there exists the exact sequence $0 \to \mathrm{Ext}_R^1(F, \mathrm{Hom}_S(M, C)) \to \mathrm{Ext}_S^1(F \otimes_R M, C)$ by the Grothendieck spectral sequence theorem [14, Theorem 5.8.3]. So

$$\mathrm{Ext}_R^1(F, \mathrm{Hom}_S(M, C)) = 0$$

and hence $\mathrm{Hom}_S(M, C)$ is a strongly cotorsion right R-module. □

By Proposition 2.22, we immediately have the following corollaries.

Corollary 2.23. *Let $f : R \to S$ be a ring homomorphism. If C is a strongly cotorsion right S-module, then C is a strongly cotorsion right R-module.*

Corollary 2.24. *Let R be a commutative ring and S a multiplicative set of R. If C is a strongly cotorsion $S^{-1}R$-module, then C is a strongly cotorsion R-module.*

Corollary 2.25. *Let R be a commutative ring. If M is a flat R-module and C is a strongly cotorsion R-module, then* $\mathrm{Hom}_R(M, C)$ *is a strongly cotorsion R-module.*

Corollary 2.26. *If R is a commutative Noetherian ring and $\hat{R}$ is the I-adic completion of R, where I is a non-trivial ideal of R. Assume that M is a strongly cotorsion $\hat{R}$-module, then M is a strongly cotorsion R-module.*

3. Strongly torsion-free modules

We start with the following definition.

Definition 3.1. Given a left R-module N. Let $stfd(M) = \inf\{n :$ there exists an exact sequence $0 \to T_n \to \cdots \to T_0 \to N \to 0$ of left R-modules, where each T_i is strongly torsion-free$\}$ and call $stfd(M)$ the strongly torsion-free dimension of N. If no such n exists, set $stfd(N) = \infty$.

In what follows, $\mathcal{STF}_n$ denotes the class of all left R-modules of strongly torsion-free dimension $\leq n$, where n is a nonnegative integer. Clearly, $\mathcal{STF}_0$ is the class of all strongly torsion-free left R-modules.

The following proposition shows that the notion of strongly cotorsion right R-modules can be seen as the dual of strongly torsion-free left R-modules in some sense.

Proposition 3.2. *Let R be a ring and N a left R-module. Then N is strongly torsion-free if and only if N^+ is strongly cotorsion.*

Proof. The result follows from the isomorphism $(\mathrm{Tor}_1^R(F, N))^+ \cong \mathrm{Ext}_R^1(F, N^+)$ for every $F \in \mathcal{F}$. □

Lemma 3.3. *Let R be a ring and N a left R-module. Then N is strongly torsion-free if and only if* $\mathrm{Tor}_i^R(F, N) = 0$ *for any $F \in \mathcal{F}$ and $i \geq 1$.*

Proof. The proof is similar to that of Lemma 2.2. □

Proposition 3.4. *Let R be a ring. The class $\mathcal{STF}_0$ is resolving.*

Proof. To prove that $\mathcal{STF}_0$ is resolving, it suffices to show that $\mathcal{STF}_0$ is closed under kernels of epimorphisms because $\mathcal{STF}_0 = \mathrm{Ker}\ \mathrm{Tor}_1^R(\mathcal{F}, -)$ by definition. Let $0 \to A \to B \to C \to 0$ be a short exact sequence of left R-modules with B and C strongly torsion-free. For any $F \in \mathcal{F}$, we have the exact sequence $0 = \mathrm{Tor}_2^R(F, C) \to \mathrm{Tor}_1^R(F, A) \to \mathrm{Tor}_1^R(F, B) = 0$ by Lemma 3.3, then $\mathrm{Tor}_1^R(F, A) = 0$ and hence $A \in \mathcal{STF}_0$. Thus $\mathcal{STF}_0$ is resolving. □

Proposition 3.5. *Let R be a ring. The class $\mathcal{STF}_0$ is closed under direct limits, pure-epimorphic images and pure submodules.*

Proof. Since the functor $\mathrm{Tor}_1^R(F, -)$ commutes with direct limits, $\mathcal{STF}_0$ is closed under direct limits. Let B be a strongly torsion-free left R-module and C a pure-epimorphic image of B, we have the pure exact sequence $0 \to A \to$

$B \to C \to 0$. For any $F \in \mathcal{F}$, we have the exact sequences $0 = \mathrm{Tor}_1^R(F, B) \to \mathrm{Tor}_1^R(F, C) \to F \otimes A \to F \otimes B \to F \otimes C \to 0$ and $0 \to F \otimes A \to F \otimes B \to F \otimes C \to 0$. Thus $\mathrm{Tor}_1^R(F, C) = 0$ and so C is strongly torsion-free, i.e., $\mathcal{STF}_0$ is closed under pure-epimorphic images. By Proposition 3.4, $\mathcal{STF}_0$ is closed under pure submodules. □

Lemma 3.6 ([9, Theorem 3.2.9]). *Let $(\mathcal{A}, \mathcal{B})$ be the cotorsion pair cogenerated by a subclass of pure injective modules. Then $(\mathcal{A}, \mathcal{B})$ is perfect and closed.*

Theorem 3.7. *Let R be a ring. Then $(\mathcal{STF}_0, (\mathcal{STF}_0)^\perp)$ is a closed hereditary perfect cotorsion pair, hence $\mathcal{STF}_0$ is a covering class.*

Proof. For any $F \in \mathcal{F}$, we have $(\mathrm{Tor}_1^R(F, N))^+ \cong \mathrm{Ext}_R^1(N, F^+)$. So N is strongly torsion-free if and only if $N \in {}^\perp(\mathcal{F}^+)$. Thus $\mathcal{STF}_0 = {}^\perp(\mathcal{F}^+)$, and so $(\mathcal{STF}_0, (\mathcal{STF}_0)^\perp) = ({}^\perp(\mathcal{F}^+), ({}^\perp(\mathcal{F}^+))^\perp)$ is a cotorsion pair cogenerated by $\mathcal{F}^+$. Since $\mathcal{STF}_0$ is resolving, $(\mathcal{STF}_0, (\mathcal{STF}_0)^\perp)$ is hereditary. Since $\mathcal{F}^+$ is a subclass of pure injective modules, $(\mathcal{STF}_0, (\mathcal{STF}_0)^\perp)$ is closed and perfect by Lemma 3.6. □

Now, we are ready to compute the strongly torsion-free dimension of a non-zero module. We start with the following definition.

Definition 3.8. Let N be a non-zero left R-module,

$$\zeta(N) = \inf\{n \in \mathbb{N} \mid \mathrm{Tor}_{n+1}^R(F, N) = 0 \text{ for all } F \in \mathcal{F}\}, \text{ and}$$

$$\nu(N) = \sup\{n \in \mathbb{N} \mid \mathrm{Tor}_n^R(F, N) \neq 0 \text{ for some } F \in \mathcal{F}\}.$$

Proposition 3.9. *Let R be a ring and N a non-zero left R-module. Then $stfd(N) = \zeta(N) = \nu(N)$.*

Proof. The proof is similar to that of Proposition 2.7. □

Theorem 3.10. *Let R be a ring and $n \in \mathbb{N}$. Then the following are equivalent for a non-zero left R-module N:*

(1) *$stfd(N) \leq n$.*
(2) *$\zeta(N) \leq n$.*
(3) *$\nu(N) \leq n$.*
(4) *$\mathfrak{P}^n(N)$ is strongly torsion-free.*
(5) *There is an exact sequence $0 \to T_n \to \cdots \to T_0 \to N \to 0$ of left R-modules with each T_i strongly torsion-free.*
(6) *If $0 \to K \to H_{n-1} \to \cdots \to H_0 \to N \to 0$ is an exact sequence of left R-modules with each H_i strongly torsion-free, then K is strongly torsion-free.*

Proof. The proof is similar to that of Theorem 2.10. □

The following proposition shows that the notion of strongly torsion-free dimension is similar to that of flat dimension.

Proposition 3.11. *Let R be a ring and $0 \to A \to B \to C \to 0$ a short exact sequence of left R-modules. Then*

(1) $stfd(C) \leq \max\{stfd(A)+1, stfd(B)\}$. *If* $stfd(B) < stfd(A)$, *then* $stfd(C) = stfd(A)+1$; *if* $stfd(B) > stfd(A)$, *then* $stfd(C) = std(B)$.

(2) $stfd(B) \leq \max\{stfd(A), stfd(C)\}$. *If* $stfd(C) \leq stfd(A)$, *then* $stfd(B) = std(A)$; *if* $stfd(A) \leq stfd(C) - 2$, *then* $stfd(B) = stfd(C)$. *In particular,* $stfd(B) = stfd(A)$ *if* C *is strongly torsion-free.*

(3) $stfd(A) \leq \max\{stfd(C)-1, stfd(B)\}$. *If* $stfd(B) < stfd(C)$, *then* $stfd(A) = stfd(C)-1$; *if* $stfd(B) > stfd(C)$, *then* $stfd(A) = stfd(B)$.

Proof. For any $F \in \mathcal{F}$, we have the long exact sequence $\cdots \to \mathrm{Tor}_{i+1}^R(F,C) \to \mathrm{Tor}_i^R(F,A) \to \cdots \to \mathrm{Tor}_2^R(F,C) \to \mathrm{Tor}_1^R(F,A) \to \mathrm{Tor}_1^R(F,B) \to \mathrm{Tor}_1^R(F,C)$. If one of A, B and C is zero, we are done. If each of A, B and C is not zero, it is easy to get $(1), (2), (3)$ by Proposition 3.9 and the above long exact sequence. $\square$

By Proposition 3.11, we immediately have the following corollary.

Corollary 3.12. *Let R be a ring and $0 \to A \to B \to C \to 0$ a short exact sequence of left R-modules. If two of $stfd(A), stfd(B)$ and $stfd(C)$ are finite, then so is the third.*

Proposition 3.13. *Let R be a ring and $\{N_i\}_{i\in I}$ a family of left R-modules. Then $stfd(\bigoplus_{i\in I} N_i) = \sup\{stfd(N_i)\}_{i\in I}$.*

Proof. Without loss of generality, we may assume that each N_i is non-zero. Since $\mathrm{Tor}_j^R(F, \bigoplus_{i\in I} N_i) \cong \bigoplus_{i\in I} \mathrm{Tor}_j^R(F, N_i)$ for any $F \in \mathcal{F}$ and $j \geq 0$, $\sup\{stfd(N_i)\}_{i\in I} \leq stfd(\bigoplus_{i\in I} M_i)$ by Proposition 3.9. If $\sup\{stfd(N_i)\}_{i\in I}$ is infinite, we are done. Suppose that $\sup\{stfd(N_i)\}_{i\in I}$ is finite. Let

$$\sup\{stfd(N_i)\}_{i\in I} = n.$$

Then $stfd(N_i) \leq n$ for every $i \in I$, and so $\mathrm{Tor}_{n+1}^R(F, N_i) = 0$ for every $F \in \mathcal{F}$ and $i \in I$. Thus $\mathrm{Tor}_{n+1}^R(F, \bigoplus_{i\in I} N_i) = 0$, $stfd(\bigoplus_{i\in I} N_i) \leq n$ by Theorem 3.10. So $stfd(\bigoplus_{i\in I} N_i) \leq \sup\{stfd(N_i)\}_{i\in I}$. Hence $stfd(\bigoplus_{i\in I} N_i) = \sup\{stfd(N_i)\}_{i\in I}$ in this case. So the assertion holds. $\square$

Lemma 3.14 ([9, Lemma 3.2.10])**.** *Let R be a ring. Then the class of all pure injective modules is cosyzygy closed.*

Theorem 3.15. *Let R be a ring and $n \geq 1$. Then $(\mathcal{STF}_n, (\mathcal{STF}_n)^{\perp})$ is a closed hereditary perfect cotorsion pair, hence, $\mathcal{STF}_n$ is a covering class.*

Proof. By Theorem 3.10, we have $\mathcal{STF}_n = \mathrm{KerTor}_{n+1}^R(\mathcal{F}, -)$. Therefore, $\mathcal{STF}_n = \mathrm{KerExt}_R^{n+1}(-, \mathcal{F}^+) = {}^{\perp}(\mathfrak{I}^{-n}(\mathcal{F}^+))$ follows from the isomorphism $(\mathrm{Tor}_{n+1}^R(\mathcal{F}, N))^+ \cong \mathrm{Ext}_R^{n+1}(N, \mathcal{F}^+)$ for every $F \in \mathcal{F}$ and $N \in R$-Mod and dimension shifting. By Lemmas 3.6 and 3.14, $(\mathcal{STF}_n, (\mathcal{STF}_n)^{\perp}) = ({}^{\perp}(\mathfrak{I}^{-n}(\mathcal{F}^+)), ({}^{\perp}(\mathfrak{I}^{-n}(\mathcal{F}^+)))^{\perp})$ is a perfect and closed cotorsion pair. By Propositions 3.4 and

3.11, it is easy to get that $\mathcal{STF}_n$ is resolving, thus $(\mathcal{STF}_n, (\mathcal{STF}_n)^{\perp})$ is hereditary. □

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