An Analytical Expression of the Probability of Error for Relaying with Decode-and-forward

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***Abstract*—For a three-node relay network that employs binary linear codes and the decode-and-forward strategy, we derive analytic upper and lower bounds for the probability of error at the destination. The bounds are based on union-bound techniques and the input-output weight enumerator of the encoder. As an application, we use this bound to optimise the decoder at the destination. Our approach is verified by simulation results.**

I. Introduction

In a three-node relay network, a source communicates to a destination over a wireless channel, assisted by a cooperating relay (see Fig. 1). Such 'relay channels' were introduced by van der Meulen [1], and their capacities were studied in detail by Cover and El Gamal [2]. Among the various cooperation schemes proposed and investigated [3, 4], decode-and-forward is one of the most practical. In this scheme, the relay decodes the data transmitted by the source, and forwards the estimate to the destination.

The error rate of decode-and-forward in uncoded systems has been analized in [5, 6]. For this analysis and also for the decoding algorithms at the destination, the source-to-relay-to-destination channel may be modelled as an equivalent one-hop link with an equivalent signal-to-noise ratio (SNR) [7, 8]. The value of this equivalent SNR has been investigated in [7, 9]. The decoding thresholds of coded systems with iterative decoding at the destination has been analysed with EXIT charts [8, 10].

Published results for coded systems have some major shortcomings. First, decoding errors at the relay introduce memory in the virtual source-to-relay-to-destination channel (in contrast to the often applied assumption); there are no analytical methods available to analyse how this affects the probability of error at the destination. Second, error floors with respect to the source-to-destination SNR have been observed [11], but no theoretical explanations are available. And third, current methods to determine the equivalent SNR for coded systems do not consider that the source-to-relay-to-destination channel has memory.

In the present paper we develop novel analytical bounds for the probability of error, taking into account decoding errors at the relay. The upper bound is based on a union bound approach

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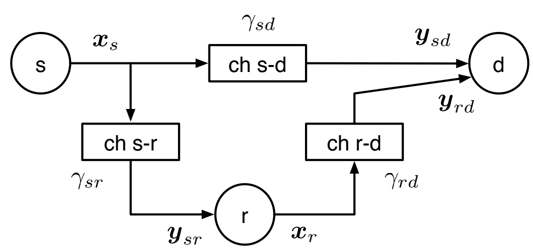

Fig. 1. Relay network consisting of source s, relay r and destination d, and the three channels (ch) in between.

and the lower bound on the dominant term, both under the mild and reasonable assumption that the SNR of the source-to-relay is not too low. As an example application of these bounds, we optimise the equivalent SNR such that the error rate is minimised.

The outline of the paper is as follows. In Section III we define the system model, the relaying strategy, and the decoding strategy at the destination. Section IV deals with the bound on the probability of error. Section V describes the optimization of the relay-to-destination SNR assumed by the destination. Numerical results are presented in Section VI. Conclusions and an outlook to future work are provided in Section VII

II. Notation

Throughout the paper, we write vectors in boldface letters, and the i-th element of a vector $\boldsymbol{a}$ as a_i. The Hamming weight of a vector $\boldsymbol{a}$ is denoted by $w(\boldsymbol{a})$, and the Hamming distance between two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is denoted by $d(\boldsymbol{a},\boldsymbol{b})$; for convenience we may simply speak of weight and distance. The support of a vector $\boldsymbol{a}$ is denoted by $\mathcal{S}(\boldsymbol{a}) = \{i : a_i \neq 0\}$, and its complement by $\mathcal{S}^c(\boldsymbol{a}) = \{i : a_i = 0\}$. Notice that $|\mathcal{S}(\boldsymbol{a})| = w(\boldsymbol{a})$.

The BPSK modulated symbol of a bit $x_i \in \{0, 1\}$ is written as $\tilde{x}_i \in \{-1, +1\}$, and we use the BPSK mapping $0 \mapsto +1$ and $1 \mapsto -1$. The signal-to-noise ratio of an AWGN channel is denoted by $\gamma = E_s/N_0$, where E_s is the received signal energy and N_0 is the single-sided noise power density. We use the complementary error function $\mathrm{erfc}(z) = 2/\sqrt{\pi} \cdot \int_z^\infty e^{-s^2} ds$.

III. System Model

We consider the wireless relay channel depicted in Fig. 1: source s communicates with destination d with the help of relay r, which uses the decode-and-forward strategy. The source-to-destination channel, the source-to-relay channel and the relay-to-destination channel are modelled as binary input AWGN channels with signal-to-noise ratio (SNR) γ_{sd}, γ_{sr} and γ_{rd}, respectively, and BPSK modulation.

The source employs a binary linear code $\mathcal{C} \subset \{0,1\}^N$ of length N and rate R; and an encoder $\mathcal{E}$ mapping user data $\mathbf{u} \in \{0,1\}^K$ to codewords $\mathbf{x}_s \in \mathcal{C}$. The user data is assumed to be uniformly distributed, and thus also the codewords.

A codeword $\mathbf{x}_s \in \mathcal{C}$ is transmitted over a wireless channel. Due to the broadcast nature of the wireless channel both the destination and the relay receive a noisy observation of $\mathbf{x}_s$, denoted by $\mathbf{y}_{sd}$ and $\mathbf{y}_{sr}$, respectively. The relay decodes $\mathbf{y}_{sr}$ and generates the estimate $\mathbf{x}_r \in \mathcal{C}$ of the transmitted codeword $\mathbf{x}_s$. It then cooperates with the source by forwarding $\mathbf{x}_r$ to the destination. We assume that the source and the relay transmit through orthogonal channels. Note that the relay may not be able to decode $\mathbf{y}_{sr}$ correctly, and therefore $\mathbf{x}_r$ and $\mathbf{x}_s$ may differ. Based on the two noisy observations $\mathbf{y}_{sd}$ and $\mathbf{y}_{rd}$, the destination estimates the codeword $\mathbf{x}_s$ that was transmitted by the source; this estimate is denoted by $\hat{\boldsymbol{x}}_s$.

A true maximum-likelihood (ML) estimation would take into account that the relay may make erroneous decisions and the corresponding statistics. Indeed, the relay introduces errors with memory and thus this should be considered by the destination. As this is by far too complex for practical implementations, we use the following decoder (cf. [8, 9, 11]).

The source-to-destination SNR γ_{sd} is assumed to be known by the destination (Assumption 1). The observation $\mathbf{y}_{rd}$ is assumed to be the output of a (virtual) memoryless AWGN channel with input $\mathbf{x}_s$ and SNR γ'_{rd} (Assumption 2). Assumption 1 and 2 are commonly used, though not explicitly stated [7–9]. Based on this model, the destination computes the L-values

$$L_{s,i} = 4\gamma_{sd} y_{sd,i}, \qquad L_{r,i} = 4\gamma'_{rd} y_{rd,i}, \tag{1}$$

$i = 1, 2, \ldots, N$, where y_i denotes the i-th element of vector $\mathbf{y}$. For the analysis in Section IV, we will need the conditional distributions of these L-values. Given $\tilde{x}_{s,i} = \pm 1$, where $\tilde{x}_{s,i} \in \{\pm 1\}$ denotes the BPSK modulated symbol of the bit $x_{s,i} \in \{0,1\}$ (see above), $L_{s,i}$ has mean $\pm 4\gamma_{sd}$ and variance $8\gamma_{sd}$. Similarly, given $x_{r,i} = \pm 1$, $L_{r,i}$ has mean $\pm 4\gamma'_{rd}$ and variance $8\gamma'^2_{rd}/\gamma_{rd}$. Using above L-values, the decoding rule is

$$\hat{\mathbf{x}}_s = \operatorname*{argmax}_{\mathbf{x} \in \mathcal{C}} \sum_{i=1}^{N} \tilde{x}_i (L_{s,i} + L_{r,i}), \tag{2}$$

Note that this decoding rule is optimal if Assumption 1 and Assumption 2 hold. While Assumption 1 is reasonable, Assumption 2 cannot be true as decode-and-forward introduces errors with memory. If the destination has some knowledge of the source-to-relay channel, it can exploit this information by properly weighing the relayed information by γ'_{rd}. In general, the optimum value of γ'_{rd} for above decoder is a function of all three SNRs γ_{sr}, γ_{rd}, and γ_{sd}.

In the remainder of this paper, we analyse the probability of error at the decoder and the effect of the value of γ'_{rd}.

IV. A bound on the error probability

The error events at the relay and at the destination are defined by

$$e_r := \{\mathbf{x}_r \neq \mathbf{x}_s\}, \qquad e := \{\hat{\mathbf{x}}_s \neq \mathbf{x}_s\}, \tag{3}$$

respectively. The complement of e_r is denoted by $\bar{e}_r$. We also define the bit error event at the destination by

$$e^b := \{\hat{x}_{s,i} \neq x_{s,i} \quad \text{for any } i\}. \tag{4}$$

For the analysis we assume without loss of generality that the all-zero codeword was transmitted, i.e. $\mathbf{x}_s = \mathbf{0}$. The error event e can then equivalently be written as

$$e \equiv \left\{ \sum_{i \in \mathcal{S}(\mathbf{x})} (L_{s,i} + L_{r,i}) < 0 \quad \text{for any } \mathbf{x} \in \mathcal{C} \text{ , } \mathbf{x} \neq \mathbf{0} \right\}, \tag{5}$$

where $\mathcal{S}(\mathbf{x})$ denotes the support of $\mathbf{x}$ (see above).

Let $A_{w,d}$ be the input-output weight enumerator (IOWE) of encoder $\mathcal{E}$, giving the number of codewords of weight d generated by input weight w; let further $A_d = \sum_{i=1}^{K} A_{w,d}$ be the weight enumerator (WE) of encoder $\mathcal{E}$, giving the number of codewords of weight d. Also, denote by $d_{\min}$ the minimum distance of code $\mathcal{C}$.

A. A bound on the frame error probability

The probability of error at the destination can be written as

$$p(e) = p(e|e_r)p(e_r) + p(e|\bar{e_r})p(\bar{e_r}), \tag{6}$$

where we distinguish between the two cases where the relay makes an error and where it does not. Using the union bound, the probability of error at the relay can be upper bounded by

$$p(e_r) \leq \frac{1}{2} \sum_{d=d_{\min}}^{N} A_d \operatorname{erfc}\left(\sqrt{d\gamma_{sr}}\right). \tag{7}$$

The probability of no error at the relay is upper-bounded by $p(\bar{e_r}) \leq 1$. We will now analyse the two conditional probabilities of error in (6).

Consider first the case that no error occurs at the relay, i.e. the term $p(e|\bar{e_r})$. The relay network of Fig. 1 is then equivalent to a system where $\mathbf{x}_s$ is transmitted over two independent parallel channels with SNR γ_{sd} and γ_{rd}. Let $\mathbf{b}$ be a nonzero codeword of $\mathcal{C}$ with support $\mathcal{S}(\mathbf{b})$ (as defined above). Splitting up the error event in (5) and taking the union bound, we obtain the upper bound

$$\begin{aligned} p(e|\bar{e_r}) &= p(e|\mathbf{x}_s = \mathbf{0}, \mathbf{x}_r = \mathbf{0}) \leq \\ &\leq \sum_{\substack{\mathbf{b} \in \mathcal{C} \\ \mathbf{b} \neq \mathbf{0}}} p\left(\sum_{i \in \mathcal{S}(\mathbf{b})} (L_{s,i} + L_{r,i}) < 0 \middle| \mathbf{x}_s = \mathbf{0}, \mathbf{x}_r = \mathbf{0} \right) \\ &= \frac{1}{2} \sum_{d=d_{\min}}^{N} A_d \operatorname{erfc}\left(\sqrt{\frac{d(\gamma_{sd} + \gamma'_{rd})^2}{\gamma_{sd} + \frac{\gamma'^2_{rd}}{\gamma_{rd}}}} \right). \end{aligned} \tag{8}$$

The last line is obtained by the following considerations: Given $\mathbf{x}_s = \mathbf{0}$ and $\mathbf{x}_r = \mathbf{0}$, $L_{s,i}$ and $L_{r,i}$ have positive mean values. Therefore, $\sum_{i\in\mathcal{S}(\mathbf{b})}(L_{s,i}+L_{r,i})$ is Gaussian with mean $4w(\mathbf{b})(\gamma_{sd}+\gamma'_{rd})$ and variance $8w(\mathbf{b})(\gamma_{sd}+\gamma'^2_{rd}/\gamma_{rd})$ (see Section III); notice that $w(\mathbf{b}) = |\mathcal{S}(\mathbf{b})|$.

Consider now the case that an error occurs at the relay, i.e. the term $p(e|e_r)$. This probability can be written as

$$p(e|e_r) = p(e|e_r, \mathbf{x}_s = \mathbf{0}) = \\ = \sum_{\substack{\mathbf{a}\in\mathcal{C}\\ \mathbf{a}\neq\mathbf{0}}} p(e|\mathbf{x}_s = 0, \mathbf{x}_r = \mathbf{a})p(\mathbf{x}_r = \mathbf{a}|e_r, \mathbf{x}_s = \mathbf{0}) \quad (9)$$

(Notice that $p(\mathbf{x}_r = \mathbf{0}|e_r, \mathbf{x}_s = \mathbf{0}) = 0$ and thus $\mathbf{a} = \mathbf{0}$ may as well be included in the summation.)

The computation of (9) is cumbersome. To simplify the computation, we make the following assumption: whenever the relay makes an error, it decodes to a codeword that has minimum distance to the transmitted codeword (Assumption 3). Notice that this assumption is justified if γ_{sr} is high, which is required for the decode-and-forward scheme to be useful.

Define $\mathcal{C}_{\min} = \{\mathbf{x}\in\mathcal{C} : w(\mathbf{x}) = d_{\min}\}$ the set of minimum weight codewords. Then, assuming $\mathbf{x}_s = \mathbf{0}$ as before, we have $\mathbf{x}_r = \mathbf{0}$ (if the relay decodes correctly) or $\mathbf{x}_r \in \mathcal{C}_{\min}$ (if the relay makes an error). Correspondingly, (9) can be re-written as

$$\begin{aligned} &p(e|e_r, \mathbf{x}_s = \mathbf{0}) = \\ &= \sum_{\mathbf{a}\in\mathcal{C}_{\min}} p(e|\mathbf{x}_s = \mathbf{0}, \mathbf{x}_r = \mathbf{a})p(\mathbf{x}_r = \mathbf{a}|e_r, \mathbf{x}_s = \mathbf{0}) \\ &= p(e|\mathbf{x}_s = \mathbf{0}, \mathbf{x}_r = \mathbf{a}), \end{aligned} \quad (10)$$

for any $\mathbf{a}\in\mathcal{C}_{\min}$. Here we used that $p(e|\mathbf{x}_s = \mathbf{0}, \mathbf{x}_r = \mathbf{a})$ takes the same value for all $\mathbf{a}\in\mathcal{C}_{\min}$, as the code is linear; and that $\sum_{\mathbf{a}\in\mathcal{C}_{\min}} p(\mathbf{x}_r = \mathbf{a}|e_r, \mathbf{x}_s = \mathbf{0}) = 1$ by Assumption 3. Similar to the case of no error at the relay, we split up the error event in (5) and take the union bound:

$$p(e|\mathbf{x}_s = \mathbf{0}, \mathbf{x}_r = \mathbf{a}) \leq \\ \leq \sum_{\substack{\mathbf{b}\in\mathcal{C}\\ \mathbf{b}\neq\mathbf{0}}} \underbrace{p\left(\sum_{i\in\mathcal{S}(\mathbf{b})}(L_{s,i}+L_{r,i}) < 0 \middle| \mathbf{x}_s = \mathbf{0}, \mathbf{x}_r = \mathbf{a}\right)}_{p_2(\hat{\mathbf{x}}_s = \mathbf{b}|\mathbf{x}_s = \mathbf{0}, \mathbf{x}_r = \mathbf{a})}, \quad (11)$$

for any $\mathbf{a}\in\mathcal{C}_{\min}$.

Each term in the sum denotes a pairwise error probability $p_2(\cdot)$, namely the probability that the destination decides for $\mathbf{b}$ instead of $\mathbf{0}$ if $\mathbf{x}_s = \mathbf{0}$ and $\mathbf{x}_r = \mathbf{a}$ were transmitted. We will now write these probabilities using the complementary error function by determining the mean and the variance of the sum of the L-values, similar to the (8).

Since $\mathbf{x}_s = \mathbf{0}$, all values $L_{s,i}$ have positive mean. Since $\mathbf{x}_r = \mathbf{a}$, the values $L_{r,i}$ have positive mean for $i\in\mathcal{S}(\mathbf{b})\cap\mathcal{S}^c(\mathbf{a})$, and they have negative mean for $i\in\mathcal{S}(\mathbf{b})\cap\mathcal{S}(\mathbf{a})$; $\mathcal{S}^c(\mathbf{a})$ denotes the complement set of $\mathcal{S}(\mathbf{a})$. Using the mean values of these L-values (see Section III) and the number of terms in each set, the mean value of the overall sum can be written as

$$\begin{aligned} &|\mathcal{S}(\mathbf{b})|\cdot 4\gamma_{sd} + |\mathcal{S}(\mathbf{b})\cap\mathcal{S}^c(\mathbf{a})|\cdot 4\gamma'_{rd} - |\mathcal{S}(\mathbf{b})\cap\mathcal{S}(\mathbf{a})|\cdot 4\gamma'_{rd} \\ &= |\mathcal{S}(\mathbf{b})|\cdot 4(\gamma_{sd}+\gamma'_{rd}) - 2\cdot|\mathcal{S}(\mathbf{b})\cap\mathcal{S}(\mathbf{a})|\cdot 4\gamma'_{rd} \\ &= 4w(\mathbf{b})(\gamma_{sd}+\gamma'_{rd}) - 8w(\mathbf{a}*\mathbf{b})\gamma'_{rd}, \end{aligned} \quad (12)$$

where $\mathbf{a}*\mathbf{b}$ denotes the element-wise product of the two vectors; notice that $|\mathcal{S}(\mathbf{b})\cap\mathcal{S}(\mathbf{a})| = w(\mathbf{a}*\mathbf{b})$. The variance of the overall sum is $8w(\mathbf{b})(\gamma_{sd}+\gamma'^2_{rd}/\gamma_{rd})$, as in the case where the relay decodes error free. Using this mean and variance, the pairwise error probability can be written as

$$p_2(\hat{\mathbf{x}}_s = \mathbf{b}|\mathbf{x}_s = \mathbf{0}, \mathbf{x}_r = \mathbf{a}) = \\ = \frac{1}{2}\operatorname{erfc}\left(\frac{w(\mathbf{b})(\gamma_{sd}+\gamma'_{rd}) - 2w(\mathbf{a}*\mathbf{b})\gamma'_{rd}}{\sqrt{w(\mathbf{b})\left(\gamma_{sd}+\frac{\gamma'^2_{rd}}{\gamma_{rd}}\right)}}\right). \quad (13)$$

For $\mathbf{a} = \mathbf{0}$, we obtain the terms of the sum in (8).

For $w(\mathbf{a}*\mathbf{b}) = 0$, the argument in (13) is maximum, and thus the error probability is minimum. On the other hand, with increasing $w(\mathbf{a}*\mathbf{b})$ (increasing 'overlap' of $\mathbf{a}$ and $\mathbf{b}$), the argument decreases and thus the error probability increases. The overlap between $\mathbf{a}$ and $\mathbf{b}$ is maximum when $\mathbf{b} = \mathbf{a}$. In this case we have that $w(\mathbf{a}*\mathbf{b}) = d_{\min}$. If $\mathbf{b}\neq\mathbf{a}$, however, the maximum value of $w(\mathbf{a}*\mathbf{b})$ is given by $\min\left(d_{\min}, \left\lfloor\frac{w(\mathbf{b})}{2}\right\rfloor\right)$; we call this value $w_m(\mathbf{b})$. Correspondingly, the pairwise error probability can be upper bounded by

$$p_2(\hat{\mathbf{x}}_s = \mathbf{b}|\mathbf{x}_s = \mathbf{0}, \mathbf{x}_r = \mathbf{a}) \leq \\ \leq \frac{1}{2}\operatorname{erfc}\left(\frac{w(\mathbf{b})(\gamma_{sd}+\gamma'_{rd}) - 2w'_m(\mathbf{b})\gamma'_{rd}}{\sqrt{w(\mathbf{b})\left(\gamma_{sd}+\frac{\gamma'^2_{rd}}{\gamma_{rd}}\right)}}\right), \quad (14)$$

where $w'_m(\mathbf{b}) = d_{\min}$ if $\mathbf{b} = \mathbf{a}$, and $w'_m(\mathbf{b}) = w_m(\mathbf{b})$ otherwise.

Finally, using (14), (11) and (10) in (9), $p(e|e_r)$ can be upper bounded by

$$\begin{aligned} p(e|e_r) &\leq \sum_{\substack{\mathbf{b}\in\mathcal{C}\\ \mathbf{b}\neq\mathbf{0}}} p_2(\hat{\mathbf{x}}_s = \mathbf{b}|\mathbf{x}_s = \mathbf{0}, \mathbf{x}_r = \mathbf{a}) \leq \\ &\leq \sum_{\substack{\mathbf{b}\in\mathcal{C}\\ \mathbf{b}\neq\mathbf{0}}} \frac{1}{2}\operatorname{erfc}\left(\frac{w(\mathbf{b})(\gamma_{sd}+\gamma'_{rd}) - 2w'_m(\mathbf{b})\gamma'_{rd}}{\sqrt{w(\mathbf{b})\left(\gamma_{sd}+\frac{\gamma'^2_{rd}}{\gamma_{rd}}\right)}}\right) = \\ &= \frac{1}{2}\operatorname{erfc}\left(\frac{d_{\min}(\gamma_{sd}-\gamma'_{rd})}{\sqrt{d_{\min}\left(\gamma_{sd}+\frac{{\gamma'_{rd}}^2}{\gamma_{rd}}\right)}}\right) + \\ &+ \frac{1}{2}\sum_{d=d_{\min}}^{N} A'_d \operatorname{erfc}\left(\frac{d(\gamma_{sd}+\gamma'_{rd}) - 2w_m(\mathbf{b})\gamma'_{rd}}{\sqrt{d\left(\gamma_{sd}+\frac{\gamma'^2_{rd}}{\gamma_{rd}}\right)}}\right), \end{aligned} \quad (15)$$

where $A'_d = A_d - 1$ for $d = d_{\min}$, and $A'_d = A_d$ otherwise; notice that the first term corresponds to the case where $\mathbf{b} = \mathbf{a}$.

This completes the computation of an upper bound on the probability of error in (6). The corresponding lower bound is obtained by considering only the terms with minimum distance (with the corresponding multiplicities) in (7), (8) and (15).

B. A bound on the bit error probability

The bit error probability $p(e^b)$ can be bounded in a similar way to the frame error probability. We first write it as

$$p(e^b) = p(e^b|e_r)p(e_r) + p(e^b|\bar{e_r})p(\bar{e_r}). \tag{16}$$

We then use the upper bound on $p(e_r)$ as given in (7), $p(\bar{e_r}) \leq 1$, and

$$p(e^b|\bar{e_r}) \leq \frac{1}{2} \sum_{d=d_{\min}}^{N} A_d^{(b)} \operatorname{erfc}\left(\sqrt{\frac{d(\gamma_{sd} + \gamma'_{rd})^2}{\gamma_{sd} + \frac{{\gamma'_{rd}}^2}{\gamma_{rd}}}}\right), \tag{17}$$

with the bit multiplicity $A_d^{(b)} = \sum_{w=1}^{K} \frac{w}{K} A_{w,d}$.

Assuming again that the relay decodes on a codeword at minimum distance (Assumption 3), we have

$$\begin{aligned} p(e^b|e_r) &= p(e^b|e_r, \mathbf{x}_s = \mathbf{0}) = \\ &= \sum_{\mathbf{a} \in \mathcal{C}_{\min}} p(e^b|\mathbf{x}_s = \mathbf{0}, \mathbf{x}_r = \mathbf{a}) p(\mathbf{x}_r = \mathbf{a}|e_r, \mathbf{x}_s = \mathbf{0}) \\ &= \frac{1}{A_d} \sum_{\mathbf{a} \in \mathcal{C}_{\min}} p(e^b|\mathbf{x}_s = \mathbf{0}, \mathbf{x}_r = \mathbf{a}). \end{aligned} \tag{18}$$

Note that in contrast to the frame error probability, the probability $p(e^b|\mathbf{x}_s = \mathbf{0}, \mathbf{x}_r = \mathbf{a})$ depends on $\mathbf{a}$ and not only on its weight $w(\mathbf{a})$, as several input weights may lead to a codeword of minimum weight. However, we can consider the worst case, which is given by the maximum weight of information words that generate a codeword of minimum weight. We denote the corresponding bit multiplicity by $A_{d_{\min}}^{(b),\max}$. Then, using the same approach as for the frame error probability, we obtain the upper bound

$$\begin{aligned} p(e^b|e_r) &= \frac{1}{A_d} \sum_{\mathbf{a} \in \mathcal{C}_{\min}} p(e^b|\mathbf{x}_s = \mathbf{0}, \mathbf{x}_r = \mathbf{a}) \leq \\ &\leq \frac{1}{2} A_{d_{\min}}^{(b),\max} \operatorname{erfc}\left(\frac{d_{\min}(\gamma_{sd} - \gamma'_{rd})}{\sqrt{d_{\min}\left(\gamma_{sd} + \frac{{\gamma'_{rd}}^2}{\gamma_{rd}}\right)}}\right) + \\ &+ \frac{1}{2} \sum_{d=d_{\min}}^{N} A_d'^{(b)} \operatorname{erfc}\left(\frac{d(\gamma_{sd} + \gamma'_{rd}) - 2w_m(\mathbf{b})\gamma'_{rd}}{\sqrt{d\left(\gamma_{sd} + \frac{\gamma'^2_{rd}}{\gamma_{rd}}\right)}}\right), \end{aligned} \tag{19}$$

where $A_d'^{(b)} = A_d^{(b)} - A_{d_{\min}}^{(b),\max}$ for $d = d_{\min}$, and $A_d'^{(b)} = A_d^{(b)}$ otherwise. Note that by using $A_{d_{\min}}^{(b),\max}$, this upper bound does not depend on $\mathbf{a}$.

V. Optimization of γ'_{rd}

Since a true ML decoder is far too complex, a common approach in the literature is to model the source-to-relay-to-destination channel as a virtual memoryless channel with SNR γ_{eq}, where γ_{eq} depends on γ_{sr} and γ_{rd} [7, 9]. At the destination γ'_{rd} is then set to γ_{eq}. In general, however, γ'_{rd} is also a function of γ_{sd}. Moreover, due to decoding at the relay, the noise on this virtual channel is not linear, not Gaussian, and not memoryless. Therefore the decoder is suboptimal, and the performance depends on the value chosen for γ'_{rd}.

In this paper, we propose to optimize γ'_{rd} as a function of the triplet $(\gamma_{sd}, \gamma_{sr}, \gamma_{rd})$. Notice that in the decoding rule (2), the value of γ'_{rd} is used to trade-off the reliability of the source-to-destination channel, depending on γ_{sd}, to the reliability of the virtual source-to-relay-to-destination channel, depending on γ_{sr} and γ_{rd}.

We use the lower bound on the error probability, derived above, to perform this optimization. In particular, for each triplet $(\gamma_{sd}, \gamma_{sr}, \gamma_{rd})$, we determine numerically the value of γ'_{rd} which minimizes the lower bound on the error probability; we denote this optimum value by γ_{opt}.

VI. Numerical results

In this Section, we evaluate the tightness of the bound on the error probability derived in Section IV by comparing the bounds to simulation results. For the examples in this Section, we have chosen a rate-$1/2$ 4-states convolutional encoder with generator polynomials $(1, 5/7)$ in octal form, and we use the information word length $K = 64$.

We consider two different scenarios. In Scenario 1, the SNRs γ_{sr} and γ_{rd} are fixed, and γ_{sd} varies. In Scenario 2, γ_{sr} and γ_{rd} vary with γ_{sd} as:

$$\gamma_{sr} = \gamma_{sd} g_{sr}, \qquad \gamma_{rd} = \gamma_{sd} g_{rd}, \tag{20}$$

where the gains $g_{sr} = (d_{sd}/d_{sr})^{\alpha}$ and $g_{rd} = (d_{sd}/d_{rd})^{\alpha}$ are due to shorter distances [12]. The values d_{sd}, d_{sr} and d_{rd} denote the distances between source and destination, source and relay, and relay and destination, respectively; usually $2 \leq \alpha \leq 6$; here we assume $\alpha = 2$.

In Fig. 2 we plot the bounds on the frame error rate (dashed curves with empty markers) together with the simulation results (solid curves with solid markers) for Scenario 1 as a function of γ_{sd}^b (in dB), where $\gamma_{sd}^b = \gamma_{sd}/R$ denotes the SNR per information bit. We assume the SNRs $\gamma_{sr}^b = \gamma_{rd}^b = 5$ dB. When two dashed curves with the same markers are plotted, the lower curves correspond to the lower bound, while the upper curves correspond to the upper bound. Several values for γ'_{rd} are considered. In all cases the bounds match very well with the simulation results. For high γ_{sd}^b the simulation results are on the upper bounds. Due to the use of the union bound, the upper bound diverges for low SNRs. On the other hand the lower bound gives also a good approximation of the performance. Note that for $\gamma_{sr}^b = 5$ dB, the probability of error at the relay is $p(e_r) \approx 3.5 \times 10^{-3}$, and thus not negligible. Very good matching is also observed for higher error probabilities at the relay.

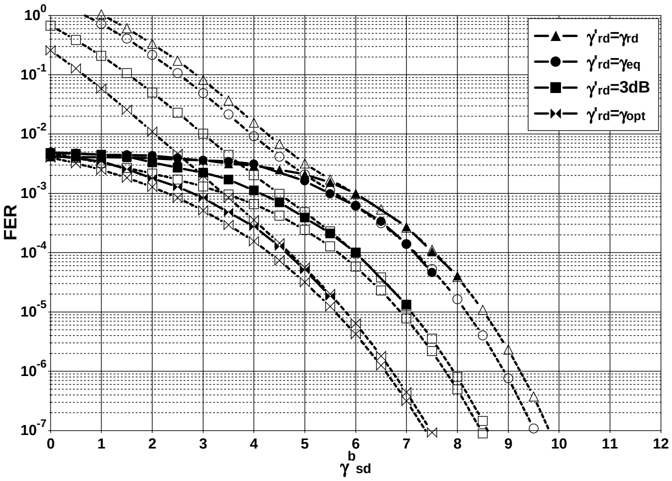

Fig. 2. FER bounds (dashed curves with empty markers) and simulation results (solid curves with solid markers) for the relay network in Fig. 1. Scenario 1 with $\gamma_{sr}^{b} = 5$ dB and $\gamma_{rd}^{b} = 5$ dB.

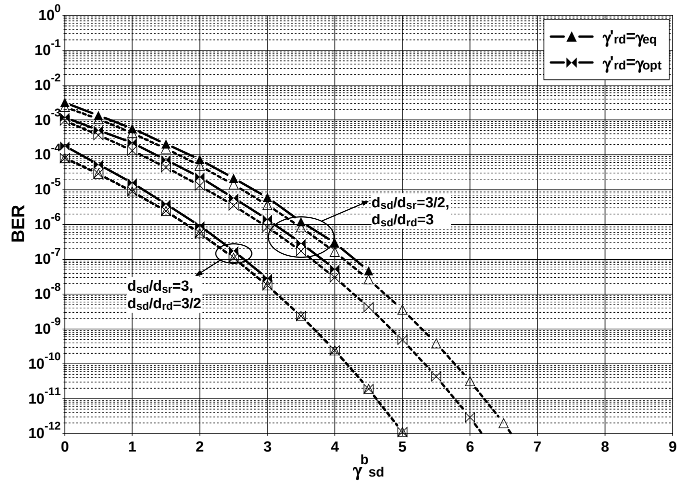

Fig. 3. BER bounds (dashed curves with empty markers) and simulation results (solid curves with solid markers) for the relay network in Fig. 1. Scenario 2, case 1 (relay closer to source) and case 2 (relay closer to destination).

The worst performance is obtained if we set $\gamma'_{rd} = \gamma_{rd}$, as no information about the source-to-relay channel is exploited. Slightly better results are achieved if we model the source-to-relay-to-destination channel as a virtual memoryless AWGN channel with SNR γ_{eq}. However, performance can significantly be improved if other values for γ'_{rd} are used. For instance, if we set $\gamma'_{rd} = 3$ dB a gain of 1 dB is achieved at FER=10^{-5}. The optimization of γ'_{rd} using the procedure described in Section V yields the best results. A gain of 2.4 dB is achieved at FER=10^{-5} with respect to the curve with $\gamma'_{rd} = \gamma_{\text{eq}}$. We remark that the optimal value γ_{opt} of γ'_{rd} decreases with increasing values of γ_{sd}.

In Fig. 3 we plot the lower bounds on the bit error rate (dashed curves with empty markers) together with the simulation results (solid curves with solid markers) for Scenario 2 as a function of γ_{sd}^{b}. Two cases are considered: in case 1, the relay is closer to the source than to the destination, and therefore we use $d_{sd}/d_{sr} = 3$ and $d_{sd}/d_{rd} = 3/2$; in case 2, the relay is closer to the destination than to the source, and therefore we use $d_{sd}/d_{sr} = 3/2$ and $d_{sd}/d_{rd} = 3$.

For both cases, the curve for $\gamma'_{rd} = \gamma_{\text{eq}}$ and the one for $\gamma'_{rd} = \gamma_{rd}$ are indistinguishable, therefore only one is plotted. Indeed, it turns out that γ_{eq} is very close to γ_{rd}. For both cases the bounds are tight. Note that the lower bound is also tight when the probability of error at the relay $p(e_r)$ is high. For instance, for $\gamma_{sd}^{b} = 0$ dB and case 2, we have $p(e_r) \approx 5 \times 10^{-2}$, and the lower bound predicts still well the actual performance. For case 1, γ_{opt} is almost identical to γ_{eq} and γ_{rd}, thereby no difference is observed in the plotted curves. On the other hand for case 2 the optimization of γ'_{rd} according to the procedure described in the previous Section yields a gain of around 0.5 dB. For other values of d_{sd}/d_{sr} and d_{sd}/d_{rd}, we observed a similar behaviour.

VII. Conclusions and Future Work

In the present paper we have developed a new analytical method to upper and lower bound the error rate of relaying with decode-and-forward. The bounds are based on union-bound approaches and the weight enumerators of the code. As an application we have considered the optimisation of the equivalent SNR of the source-to-relay-to-destination link at the destination.

In future work we will extend our bounding approach to schemes with re-encoding at the relay, which leads to distributed turbo-like codes, and to scenarios with multiple users and network coding at the relay.

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