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A critical review of established methods of structural topology optimization

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Abstract The aim of this article is to evaluate and compare established numerical methods of structural topology optimization that have reached the stage of application in industrial software. It is hoped that our text will spark off a fruitful and constructive debate on this important topic.

Keywords Topology optimization · SIMP method · Stress constraints · Compliance constraints · Checkerboard control · Perforated plates · Trusses · Michell structures · SERA method · ESO method

1 Introduction

Topology optimization is a relatively new but extremely rapidly expanding research field, which has interesting theoretical implications in mathematics, mechanics, multi-physics and computer science, but also important practical applications by the manufacturing (in particular, car and aerospace) industries, and is likely to have a significant role in micro- and nanotechnologies. At the recent ISSMO World Congress (WCSMO 7, 2007, Seoul), more than 160 papers discussed topology optimization (out of some 400 submitted papers).

The first paper on topology optimization was published over a century ago by the versatile Australian inventor Michell (1904), who derived optimality criteria for the least-weight layout of trusses. Some 70 years later, the author and his research group extended Michell's theory to grillages (beam systems) in a number of papers (starting with Rozvany 1972a and b). Based on these applications, Prager and Rozvany (e. g. 1977, presented in Gainesville in 1976) formulated the first general theory of topology optimization, termed "optimal layout theory" (for a review, see Rozvany 1993 or Rozvany et al. 1995). They applied this primarily to exact analytical optimization of grid-type structures, but it has also important implications for numerical methods and continuum-type structures. A number of papers deal with extensions of this theory, the most recent ones discussing exact solutions of popular benchmark problems (e.g., Lewinski and Rozvany 2007, 2008a and b).

Starting with the landmark paper of Bendsoe and Kikuchi (1988), numerical methods for topology optimization have been investigated extensively since the late 1980s. As explained in a review article (Rozvany 2001a), in numerical finite element (FE)-based optimization, we may deal with various types of topologies involving possibly several materials that may be isotropic, anisotropic, and/or porous. For simplicity, here we restrict our attention to so-called ISE topologies with **I**sotropic **S**olid or **E**mpty ground elements of fixed boundaries. In other words, any "ground element" of an ISE topology is either filled completely by a given isotropic material or contains no material. Each ground element may consist of one or several finite elements.

In the following, two methods of numerical topology optimization, namely SIMP and ESO (SERA) will be discussed in detail (see Sections 3 and 4), although the latter has been used only in isolated cases by the industry.

Topological derivative-based and level-set methods (e.g., Sokolowski and Zochowski 1999; Sethian and Wiegman 2000; Allaire et al. 2002, 2004; Wang et al. 2003, 2004,

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ranging to Norato et al. 2007) show tremendous promise, but have not reached the stage of regular industrial applications as yet. Non-gradient methods have also been proposed for topology optimization. The disadvantage of genetic algorithms for topology optimization (e.g., Hajela and Lee 1995) is that they become prohibitively expensive for large systems (with 10^4 to 10^6 ground elements).

2 The SIMP method

The presently most popular numerical FE-based topology optimization method is the SIMP method, which was developed in the late eighties. It is sometimes called "material interpolation", "artificial material", "power law", or "density" method, but "SIMP" is now used fairly universally. The term "SIMP" stands for **S**olid **I**sotropic **M**icrostructure (or **M**aterial) with **P**enalization for intermediate densities. The basic idea of this approach was proposed by Bendsoe (1989), while the term "SIMP" was coined later by the author and first introduced in a paper by Rozvany et al. (1992).

2.1 Basic concept of SIMP

In practical problems of ISE type topology optimization, we usually have a very large number of ground elements. Because of the size of this discrete value (0–1) problem, direct search methods would be prohibitively expensive, and it is therefore advisable to use a continuous variable formulation. Considering, for example, topology optimization of a perforated plate in plane stress, we may denote the thickness of the plate by ρ, which could also represent density or specific cost in other topology problems. If we optimized a variable thickness plate for compliance or displacement constraints, for example, the normalized stiffness (s)–plate thickness (ρ) relation becomes

$$\rho = s \tag{1}$$

which is shown in continuous line in Fig. 1. The plate thickness is normalized (scaled) by dividing over the actual plate thicknesses ($\overline{\rho}$) by the prescribed maximum plate thickness (ρ_0), giving $\rho = \overline{\rho}/\rho_0$. Similarly, the normalized stiffness equals the actual stiffness divided over by the stiffness corresponding to the prescribed maximum thickness. Then, empty (white) and solid (black) elements, respectively, will have ρ=0 and ρ=1.

However, optimal solutions based on relation (1) for compliance or displacement problems would consist of mostly "grey" elements with $0<\rho<1$. Such a result would be very far from a (0–1) solution required in topology optimization, and therefore, it would not help much with the considered problem class. Nearly "black-and-white" topologies can be obtained, however, if we penalize grey elements having "intermediate" densities of $0<\rho<1$. For this, various functions can be used, but one of the simplest is the so-called power law proposed by Bendsoe (1989), with

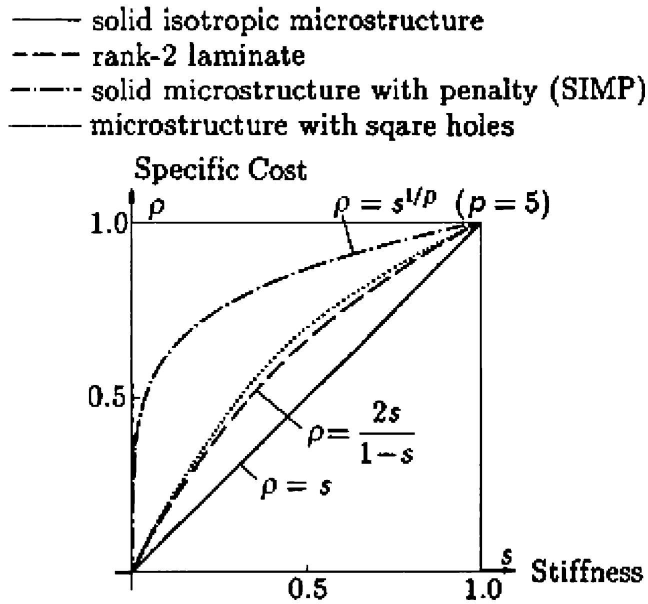

Fig. 1 Normalized stiffness (s)–specific cost or density (ρ) relations in topology optimization methods (after Rozvany et al. 1994)

$$\rho = s^{1/p} (p > 1), \tag{2}$$

see the dash-dot line in Fig. 1.

Alternative methods for suppressing intermediate densities were proposed by Stolpe and Svanberg (2001), Fuchs et al. (2005), and Sigmund (2007).

2.2 Rationale for or physical modeling of the SIMP method

To justify the relation in (2), we may employ physical or computational arguments. Physical justifications in the past were based on either (a) taking fictitious manufacturing costs for intermediate thicknesses into consideration (e.g., Rozvany and Zhou 1991, presented in 1990; Zhou and Rozvany 1991) or (b) non-homogeneous plate elements of a suitable microstructure (Bendsoe and Sigmund 1999).

In case (a) above, ρ in Fig. 1 denotes the specific cost (cost per unit area) of the plate, including manufacturing costs, details of which are reviewed in a paper by Rozvany (2001a). The optimization problem consists of minimizing the total cost for a given compliance or vice versa. A conceptual difficulty may arise from the fact that we are switching from the usual volume or weight minimization to cost minimization. On the other hand, in case (b), we can justify (2) only within certain bounds on the p values.

Computational justification does not need a physical model, penalization being a standard technique of discrete value optimization.

2.3 SIMP vs other gradient-type methods

As explained in a review article (Rozvany 2001a), mainly the author's research group explored the details of the SIMP method during the nineties because others concentrated on homogenization methods, such as optimal microstructures with penalization (OMP; e.g., Allaire and Kohn 1993b; Allaire 2002) or near-optimal microstructure (NOM; e.g., Bendsoe and Kikuchi 1988; Bendsoe et al. 1993). The latter used optimized square or rectangular holes and homogenization. It took some time for the author's research school to convince the optimization community that both these methods are somewhat uneconomical, without significant advantages. Moreover, the NOM method may lead to some grey elements due to insufficient penalization (see the dotted line for square holes in Fig. 1). Optimal microstructures would give too weak penalization (dash-line in Fig. 1) without adding extra penalty in a second stage of the computation. Both OMP and NOM methods use several variables per ground element, while SIMP uses only one. In addition, SIMP requires no homogenization. However, at the early stages of the history of topology optimization, OMP and NOM methods represented a significant progress in this field, and their exploration was therefore justified. A great advantage of the OMP method is that its side-product is a solution with optimal microstructures (e.g., Jog et al. 1994), which gives a useful insight into optimal topologies.

2.4 Brief history of the SIMP method

The idea of FE-based topology optimization is due to Rossow and Taylor (1973) who used the unpenalized relation (1) (continuous line in Fig. 1) and therefore obtained solutions with intermediate thicknesses ("grey" elements). In a pioneering publication, Bendsoe and Kikuchi (1988) suggested homogenization based on square or rectangular holes, which also produced some degree of penalization (dotted line in Fig. 1). A year later Bendsoe, in another milestone paper, considered a SIMP-like method based on the power law relation in (2), but expressed as yet preference for homogenization with square holes. This he justified by referring to mesh dependence and fictitiousness of material properties represented by (2) above, which was quite reasonable from his viewpoint. Mlejnek (e.g., 1992) proposed later an independently conceived SIMP-like procedure. Some outstanding homogenization papers discussed applications to multiple loading (Diaz and Bendsoe 1992), eigenvalue problems (Diaz and Kikuchi 1992), and 3D continua under single and multiple loads (Diaz and Lipton 1997, 2000).

As mentioned, the author and Zhou based the relation (2) on combined cost of material and manufacturing, assuming that only one plate thickness is available originally, and other thicknesses must be obtained by an expensive machining process. Zhou developed the full computational details of the SIMP method, with suitable redesign formulae for thicknesses (densities) and Lagrange multipliers (for a detailed review, see Gaspar et al. 2002). At a Karlsruhe conference in 1990, Zhou and the author presented already advanced solutions (Rozvany and Zhou 1991, 1993; Zhou and Rozvany 1991), one of which is reproduced (once more) in Fig. 2a. The corresponding analytical solution is shown in Fig. 2b (Lewinski et al. 1994). A lower resolution solution for this problem was obtained via homogenization by Olhoff et al. (1991).

Zhou and the author initially applied the DCOC optimality criteria method (Zhou and Rozvany 1992/1993) to solve the optimization problem under SIMP formulation. While DCOC offers unique capability in the handling of stress constraints, general optimization codes such as CONLIN and MMA (Fleury 1989; Svanberg 1987) based on dual method of convex programming offer robust black-box solutions to a general multiply constrained optimization problem. In fact, due to its public availability, Svanberg's MMA code has almost become a standard tool within the topology optimization community.

SIMP was gradually accepted by other topology groups as well, particularly after Sigmund, who started working in the author's research group in Essen on topology optimization, moved back to Denmark and contributed much to explaining SIMP to the optimization community. Sigmund's (2001a) educational article with a 99-line SIMP-code and his web-based topology optimization program (Tcherniak and Sigmund 2001) played an important role in SIMP's general acceptance. Sigmund's versatile applications of SIMP, amongst many others, included compliant mechanisms (e.g., Sigmund 1997), geometrically nonlinear structures

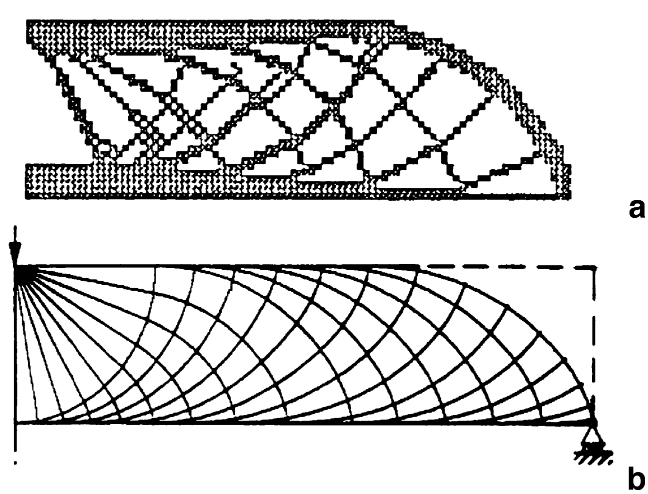

Fig. 2 **a** A very early SIMP solution for the "MBB beam", presented in 1990 (Rozvany and Zhou 1991). **b** The corresponding exact analytical truss solution (Lewinski et al. 1994)

(Buhl et al. 2000), multi-physics actuators (Sigmund 2001b), photonic crystal structures (Jensen and Sigmund 2004, 2005), and phononic band gap materials (Sigmund and Jensen 2003). As mentioned, Bendsoe and Sigmund (1999) also generated microstructures corresponding to the power law in (2) above.

Olhoff has also been extremely active in applying SIMP to reliability-based problems (Kharmanda and Olhoff 2004), vibrating continua (Olhoff and Du 2006; Du and Olhoff 2007a and b), and design-dependent loads (Hammer and Olhoff 1999, 2000; Du and Olhoff 2004a and b). Maute extended topology optimization to shells and elastoplastic structures (Maute and Ramm 1997, 1998; Maute et al. 1998).

The complete acceptance of SIMP can be seen also from the fact that the second edition of Bendsoe's trendsetting book (Bendsoe and Sigmund 2003) uses almost exclusively SIMP for its main part (on what we call here ISE topologies), while its earlier edition (Bendsoe 1995) mentioned the "penalized variable thickness approach" only in one sentence under "Biographical notes" (p. 230).

Topology optimization has become increasingly popular in a wide range of industries including automotive, aerospace, heavy industry, etc. partly because of development and promotion through commercial FEA software (e.g., OptiStruct, Genesis, MSC/Nastran, Ansys, Tosca, etc.). To the knowledge of the author, all software have implemented the SIMP method, except Tosca that used an ESO type method. Recent publications (e.g., Pedersen and Allinger 2006) indicate that Tosca has also started to adopt the SIMP approach combined with MMA (Svanberg 1987).

2.5 Is SIMP really a heuristic method?

Many researchers outside the topology field seem to believe that SIMP and ESO are equally heuristic methods, and therefore, it is only a question of luck if we get the correct solution by either of them. This author strongly disagrees with this view for the following reasons.

Taking the formulation of SIMP on the basis of fictitious manufacturing costs for intermediate densities, for example, we have a physical problem, even if the assumed cost structure may not possibly correspond to true manufacturing costs. For this real problem, we have derived rigorously redesign formulae based on sensitivities. In optimality criteria (OC) methods, formulae for element densities and Lagrange multipliers are available for combinations of various (stress, displacement, natural frequency, stability, etc.) constraints. The method does not become more heuristic either if we use some rigorously derived nonlinear programming method (e.g., MMA, see Svanberg 1987, or CONLIN, see Fleury 1989) for the considered physical problem.

With the above penalization, even originally convex problems (e.g., minimization of compliance for given volume) become non-convex, and therefore, it is theoretically possible to finish up in a local optimum. However, from the very beginning, the author's team used an unpenalized problem ($p=1$) in the first computational cycle, increasing the penalization factor (p) progressively in small steps in subsequent cycles. This procedure was first described in a book chapter (Rozvany et al. 1994) where R. V. Kohn was thanked for the idea. Later, this technique was called by others "continuation method" (e.g., Petersson and Sigmund 1998). This way, for originally convex problems, we get a global optimum for $p=1$, and later, grey regions change locally into black-and-white regions of the same average density, and thereby, we do not move away too far from the global optimum. At least, this is found by comparisons with exact analytical solutions for the same problem.

In practice, all real-world engineering problems are non-convex, and hence, an optimization process typically leads to a local optimum. This feature does not render the entire field of structural optimization 'heuristic'.

The so-called "continuation method" itself may be classed heuristic, but it only determines the initial design for each subsequent iteration for an increased p value. The actual optimization procedure is based on rigorously derived sensitivities.

From the very start, a serious problem with SIMP was the erroneous appearance of corner contacts between solid ground elements in the solution (checkerboards, diagonal element chains, isolated hinges). This problem was first noticed in connection with homogenization methods (Bendsoe et al. 1993, see also a review in Bendsoe 1995). A highly efficient but partially heuristic solution to this problem, the filtering method, was suggested by Sigmund (1994, 2001a), see also Diaz and Sigmund (1995); for comprehensive reviews of filtering methods, see Sigmund and Petersson (1998) and Sigmund (2007). We must understand that checkerboarding is merely a discretization error of the FE method, which has nothing to do with SIMP. This error results in an overestimation of the stiffness, if for example, each ground element consist of a single four-node FE and solid (material-filled) elements have only a corner contact. If the computer time is not a primary consideration (e.g. in some basic research projects), then checkerboarding can be controlled without heuristics, if e.g. we use higher order elements (Sigmund and Petersson 1998) or more FEs per ground element (Zhou and Rozvany 1991). Rigorously derived sensitivities are used also, if we employ a checkerboard prohibiting constraint (Poulsen 2002), perimeter constraint (Haber et al. 1996), slope constraints (Petersson and Sigmund 1998), adaptive side constraints (Zhou et al. 2001), or penalization for corner contacts (COSIMP, Rozvany et al. 2004; Pomezanszki et al. 2005).

2.6 The concept of "extended optimality"

We are discussing extended optimality here briefly because an equivalent objective function was proposed for ESO by Ling and Steven (2002) under the term "performance index". This was also used by Edwards et al. (2007) and in an attempt to justify ESO by Tanskanen (2002), see Sections 3 and 12.

The above concept was explained in a paper by Rozvany et al. (2002a), see also an erratum (Rozvany et al. 2005b). Considering, for example, a perforated plate in plane stress, traditional topology optimization may consider minimization of the compliance for a given limit on material volume (or on volume fraction) or minimization of the volume for a given limiting compliance value. In both problems, the uniform plate thickness is restricted to a given value.

Considering "extended optimality" (Rozvany et al. 2002a), neither the uniform plate thickness nor the volume fraction is fixed, and we minimize the total volume (weight) for given constraints on the response variables (the simplest being compliance). In other words, in extended topology optimization, we optimize the plate thickness and topology simultaneously. As explained in the above paper, this is a much wider problem than traditional topology optimization, and it may give entirely different results. For perforated plates in plane stress, an equivalent problem is minimization of the product $(C.V)$ of the compliance C and the material volume V (see Rozvany et al. 2002a, 2005b) for variable uniform plate thickness and variable volume fraction.

The implications of extended optimality and a case study are presented in a separate brief note (Rozvany 2008), which shows that an extended optimum can be very much different from, and much more economical than, the conventional one for perforated plates. The above brief note is based on previous basic findings by Kohn and Strang 1986; Lurie and Cherkaev 1986; and Rozvany et al. 1985, 1987).

2.7 Numerical verification of solutions obtained by the SIMP method

The author has repeatedly expressed concern about the lack of "quality control" in topology optimization (Rozvany et al. 2005a, 2006a and b). His method of verifying numerical solutions in topology optimization is based on the fact that the optimal topology of perforated plates in plane stress under a compliance constraint tends to that for plane trusses, if the volume fraction of the plate approaches zero (Rozvany et al. 1985, 1987; Bendsoe and Haber 1993; Allaire and Kohn 1993a; Rozvany 2008). Most of the authors in numerical topology optimization simply compare their solutions visually with the exact optimal truss topology and are satisfied with a vague resemblance. This is a very subjective method for verifying topology optimization methods and solutions.

To enable a reliable check on numerical solutions in topology optimization, Lewinski and the author are deriving analytically a series of exact truss topologies for popular benchmark problems (Rozvany 1998; Lewinski and Rozvany 2007, 2008a and b). The actual verification of a solution consists of the following procedure (Rozvany et al. 2005a, b, 2006a and b).

(a) For a given set of response (or behavioural) constraints, derive numerically the optimal topology for various volume fractions and various numbers of elements.
(b) Calculate the structural volume (weight) for each solution.
(c) Extrapolate the volume (weight) value for zero volume fraction and infinite number of elements.
(d) Compare this extrapolated value with that calculated analytically for the exact benchmarks.

For compliance constraints, the above method reduces to the one represented graphically in Fig. 3. In the latter, we indicate that the "efficiency" of a topology should tend to unity as we decrease the volume fraction and increase the number of elements. In Fig. 3, C = compliance of a numerical solution, C_M = compliance of Michell truss, $E = C_M V_M / CV$ = efficiency of a numerical solution, F = volume fraction, N = element number, V = structural volume of a numerical solution, and V_M = structural volume of Michell truss.

As can be seen from the next section, certain numerical difficulties had to be overcome (Rozvany et al. 2006a), but the above procedure gives a quantitative confirmation of SIMP solutions.

2.8 Quality control in topology optimization: difficulties and how to overcome them

The procedure outlined in Section 2.7 (Fig. 3) may encounter computational difficulties. In this section, we explain how to handle these pitfalls.

(a) *Mesh-independent methods*

Mesh independence for methods of checkerboard control is a mixed blessing because it prevents getting higher resolution solutions with mesh refinement. This means that in Fig. 3, we get stuck on the lowest curve (marked "N lower") and cannot get onto the higher curves. It follows that we cannot extrapolate to get an estimate for an infinitely dense system. The solution is simple: not to use mesh independent methods in the proposed procedure. For getting increasing resolution and checkerboard control, we may use, e.g., higher order elements or several FEs per ground element. In quality control, computer time is of secondary importance because we check on the efficiency of a particular method relatively infrequently.

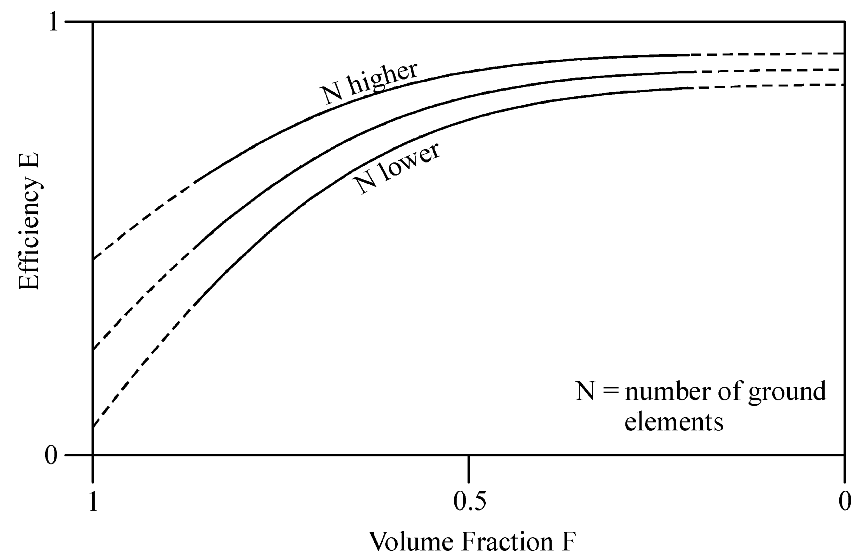

Fig. 3 Quantitative verification of a numerical topology solution by using exact truss topologies as benchmarks

This is in contrast to industrial applications where computer time is of primary consideration. Manufacturability is of course important in practice and may justify low resolution solutions, but high-resolution solutions will be increasingly important with high-tech manufacturing methods, mass production, and micro/nanotechnology. Of course, the author agrees with Sigmund (2007) that one could get a series of solutions of different resolutions with mesh-independent filters, decreasing the filter radius progressively.

(b) *Erroneously higher stiffness for coarser meshes*

It was found in earlier studies (e.g., Rozvany et al. 2005a, b) that contrary to the theoretical trends in Fig. 3, the "efficiency" decreased with element refinement in numerical experiments using SIMP. As it was realized later, this was because the number of FEs and the number of ground elements were increased simultaneously, keeping the number of FEs per ground element constant (e.g., 4). The coarser net of larger size elements increased the discretization error, erroneously increasing the stiffness and decreasing the compliance (even without corner contacts in the solution). This phenomenon is well known from the finite element literature.

The above problem was overcome in improved experiments (e.g., Rozvany et al. 2006a and b) in which the total number of FEs was kept constant in all calculations, but the number of ground elements was progressively increased (i.e., the number of FEs per ground element decreased). After this essential correction, the expected trend was observed.

(c) *Sudden drop of the efficiency curve at low volume fractions*

It was also found that the efficiency curves in Fig. 3. suddenly drop (take on very low values) at low volume fractions. This is because for a given number of ground elements, a very low volume fraction allows too few "black" elements for an efficient load transmission. In Michell's "bicycle wheel problem", for example, a low volume fraction would allow only too few "spokes" in the solution, resulting in a less efficient layout causing higher compliance values (see Rozvany et al. 2003). At even lower volume fraction, the allowed number of elements is insufficient for transmitting the prescribed loads at all, and the compliance value goes to infinity (only theoretically, in practice we use a very low finite thickness for "white" elements).

The solution to the above difficulty is to ignore systematically designs with such "forced" incorrect topologies when we extrapolate for near-zero volume fraction and near-infinite FE number (see Rozvany et al. 2006b).

(d) *Poisson's ratio*

In Michell structures, the members are in uniaxial stress, and the volume of member intersections is negligible. For this reason, we have found that we get the best convergence to the corresponding Michell truss if we use zero Poisson's ratio for the perforated plate in the proposed quality test.

(e) *The choice of the test problem*

It is shown in a brief note (Rozvany 2008) that in some cases, the Michell truss (with a near-zero volume fraction) gives a much *higher* value for the objective function $C.V$ than perforated plate solutions for other volume fractions. In such benchmark problems, the efficiency curves in Fig. 3 approach the line with $E=1$ from above. However, it is preferable to select benchmark problems for which a Michell truss corresponds to the extended optimum.

2.9 The role of tuning parameters in topology optimization

In many topology optimization papers, particularly in those on heuristic methods, the results depend largely on the value of certain arbitrarily chosen constants termed "tuning parameters". The problem is that authors must often experiment with lots of tuning parameter values before they get a reasonably looking solution, and they do not state this in their research papers. Sometimes, they do not even list the tuning parameter values used, and therefore, their results are not replicable. For any respectable quantitative verification of the results, all the above information should be fully reported in the literature.

Undoubtedly, the optimal topologies obtained by SIMP also depend on the value of certain parameters, but this can be fully justified for the reasons explained below. As an example, we usually set the following values in OC-based SIMP programs:

(a) initial value of p;
(b) steps in the p value;
(c) simultaneous convergence criteria for stepping up the p value, such as
   1. prescribed highest number of ground elements changing from active to passive or vice versa,
   2. prescribed highest change in total volume or weight,
   3. prescribed highest change in the thickness or density of any one ground element.

In the above SIMP application, broadly speaking, we increase the accuracy of the results if the initial p value is closer to unity (it should be preferably unity), and the other values listed above are made smaller. On the other hand, greater accuracy is costing more computer time. For this reason, the selection of the tuning parameters actually decides how much extra computer time we want to invest for greater accuracy of the results. They are, therefore, not as much "tuning parameters", but rather "accuracy parameters".

In many heuristic methods, we get quite absurd results if the tuning parameters are not set optimally, and therefore, the method may lead to entirely non-optimal solutions.

In OC-based SIMP programs, we usually also employ step-size parameters, multiplying the calculated element thickness variation by a constant (e.g., 0.6) and limiting the maximum thickness variation of elements. This again represents a trade-off between computer time and stability of convergence. We are getting more stable convergence at the price of extra computer time if we decrease the above "step size parameters".

3 The ESO (or SERA) method

So-called "hard-kill" methods introduce finite changes in a design on the basis of certain heuristic criteria, which may not be based on sensitivities. One of these is inappropriately called "ESO" (**E**volutionary **S**tructural **O**ptimization) because "evolutionary" usually refers to Darwinian processes (as in genetic algorithms) and "optimization" implies computation of a truly optimal solution, which has been shown not to be the case with ESO. An appropriate term for this method would be "SERA" (**S**equential **E**lement **R**ejections and **A**dmissions), suggested by the author (e.g., Rozvany and Querin 2002a, b). In "ESO", a certain parameter value (we call it here "criterion function", e.g., Mises stress or energy density) is calculated for each element, and in each iteration some element(s) with the lowest criterion function value are eliminated (changed from material-filled "black" elements to empty "white" elements). In the so-called "BESO" (bi-directional ESO, e.g., Yang et al. 1998) method, new elements are added in locations next to those elements with a high criterion function value.

In more recent versions of ESO (e.g., Ling and Steven 2002; Edwards et al. 2006, 2007), a two-stage procedure is used for improving the results of ESO. After deriving a large number of solutions by the usual ESO method, the value of a "performance index" or "objective function" is calculated for each solution, and then the "global optimum" with the highest (or lowest) value of the performance index (or objective function) is found by enumeration (i.e., numerical comparison).

Although ideas similar to ESO were presented earlier by Schnack (e.g. Schnack et al. 1988) and by Mattheck (e.g. Mattheck and Burkhardt 1990), the literature on ESO is most extensive, with well over hundred publications (starting with Xie and Steven 1992, including a book, Xie and Steven 1997), a publicity that probably exceeds the merits of this method. In fact, the Xie and Steven 1992 paper is one of the most cited papers on structural optimization in Google Scholar (no doubt, most of the non-critical citations by the ESO group).

3.1 Usual criticisms of ESO

Most critics of ESO mention the following shortcomings.

(a) ESO is fully heuristic, that is, there exists no rigorous proof that element eliminations or admissions on the above basis do give an optimal solution. It will be explained in Section 12 that Tanskanen's (2002) paper does not justify ESO's computational procedure either.

(b) There may not be any rational relation between the above "criterion function" and the "performance index" or "objective function".
(c) It is not particularly efficient if we have to select the best solution by comparison out of a very large number of intuitively generated solutions (enumeration method).
(d) Although ESO usually requires a much greater number of iterations than gradient-type methods, it may yield an entirely non-optimal solution even with respect to ESO's objective function (see Zhou and Rozvany 2001).
(e) In the above two-stage procedure, we have no control over the final volume fraction, we just have to accept the one given by ESO. If we do pre-assign a volume fraction, then we may finish up with a highly non-optimal value of the objective function.
(f) The ESO procedure cannot be easily extended to other constraints, or to multi-load, multi-constraint problems. This was also remarked by Tanskanen (2002, p 5495).

3.2 Example showing complete breakdown of ESO

In a very brief note, Zhou and Rozvany (2001) verified criticism (d) above on a simple example shown in Fig. 4a. This indicates a ground structure with volume fraction of 100%, having a compliance value of C=388.5. They showed that ESO removes element "a" from the above ground structure, giving the design in Fig. 4b with volume fraction of 99%, but a huge compliance value of C=4,371. To demonstrate how non-optimal this design is, they calculated that if element "c" in Fig. 4a were to be removed, then the compliance would increase insignificantly to 396. ESO gives 11 times as much increase by removing element "a".

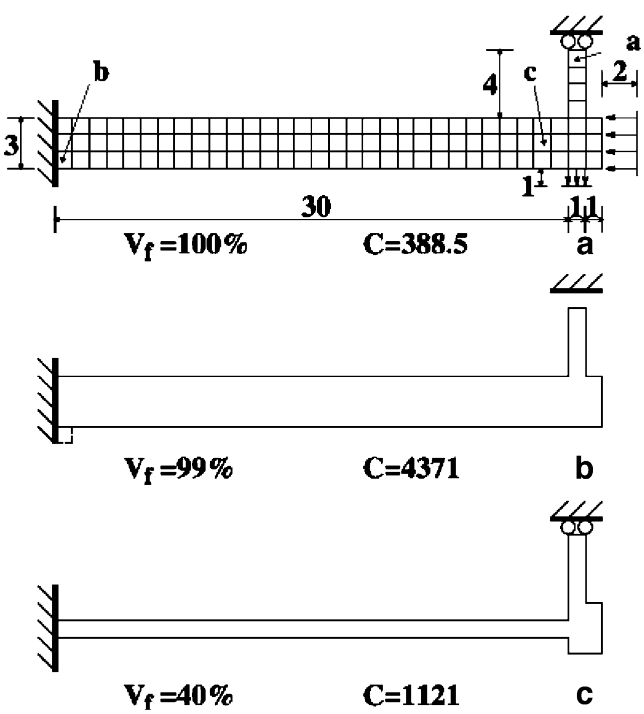

Fig. 4 The example in a brief note by Zhou and Rozvany (2001). **a** Problem statement, **b** highly non-optimal solution for a volume fraction of 99% by the ESO method, **c** intuitively "good" design for a volume fraction of 40%

They also compared the ESO design in Fig. 4b with an intuitive design in Fig. 4c, which has a low volume fraction of 40%, but a compliance value of only C=1,121. Even if we accept an "objective function" of volume fraction times compliance ($V.C$), we have $V.C$=432,729 for the ESO design and only $V.C$=44,840 for the intuitive design. The latter is a mere upper bound on the true optimum, which may have an even lower objective function value than that of the intuitive design. We can conclude that ESO gives about ten times higher "objective function" value than the true optimum.

Moreover, if we employ BESO for adding elements to the structure in Fig. 4b, we find that the new element would be the one shown in broken line at the bottom left corner because the element "b" (Fig. 4a) has the highest stress or energy density value after the removal of element "a". This means that BESO fails to correct the disastrous mistake in the first iteration of ESO.

Zhou and Rozvany (2001) proposed the solution in Fig. 4c as an intuitively good design, but in retrospect, this design seems to be very near optimal for a volume fraction of 40%. This is because in the solution in Fig. 4c, the four elements in contact with the external load cannot be removed, and moving any other element would either (a) break the horizontal or the vertical member or (b) introduce load transmission by corner contact (checkerboard) only, both of which produce a very high compliance value if a more accurate FE analysis is used. For the more accurate analysis of topologies with corner contacts, see Rozvany et al. (2002b, 2003), and Pomezanski et al. (2005).

In a very interesting paper, Stolpe and Bendsoe (2007) generated global optima for the considered problem by both a nonlinear branch and cut method and by simple enumeration. The compliance value for their global optimum for a volume fraction of 40% is a little lower than that for the "intuitive" solution in Fig. 4c, but the latter assumed that no checkerboarding or removal of loaded elements is permitted.

4 Attempts to defend ESO from the above criticisms

Unquestionably, any discussion on the above issue is very useful for the topology community, and therefore, any attempted rebuttal of the criticisms under Sections 3.1 and 3.2 are of considerable interest. Such discussions may lead to great improvements of ESO (SERA) by joint research efforts from both sides.

Counter-arguments about the above criticisms, and also suggestions for improving ESO (SERA), have appeared in the literature. Out of these, we will examine the three best ones by the ESO group (Edwards et. al. 2006, 2007; Huang and Xie 2007; Tanskanen 2002) and those by the author and Querin (e.g., Rozvany and Querin 2002a and b, 2004).

5 First version of paper by Edwards et al. (2006)

This text was originally presented at (and published in the proceedings of) a very large US (AIAA/ASME/ASCE/AHS/ASC) meeting in 2006. Its modified version was published in the journal Structural and Multidisciplinary Optimization. First, we discuss here the conference version in detail because it has probably been exposed to more readers/participants than the journal version ever will be.

The considered paper is rather useful because it may open up a constructive debate about ESO. As will be seen, in its conference form, it rather confirms the shortcomings listed under Section 3.1, but its authors may express a different opinion. Here, we list some implications for both ESO and SIMP based on the evidence arising from this earlier version.

5.1 Implications of the considered paper for ESO

(A) The objective function is the normalized standard deviation of the element Mises stresses from the fully stressed design. This may not make much sense because fully stressed designs are often not optimal. This was pointed out by Save and Prager (1985) who called fully stressed design criteria "questionable optimality criteria", referring to some results by Chern and Prager (1972). Similar conclusions were reached by Haftka and Gürdal (1992), Section 9.1.1 and Rozvany (2001b).

(B) The above objective function cannot be used for other types of constraints, whilst SIMP formulations are available for any combination of a wide range of constraints. Mises stress would be an entirely inappropriate criterion, e.g., for a displacement constraint.

(C) As mentioned above, it is not very efficient to generate heuristically a very large number of solutions and then picking the one with the lowest objective function value (enumeration).

(D) The ESO examples require a very high iteration number (e.g. 1400), much higher than the SIMP examples in the same paper.

(E) The plots of the objective function values jump all over the place (Fig. 5), with sharp local minima (points d, e, f). This is nothing like the monotonic convergence of SIMP (Fig. 6).

6 Implications of the considered paper for SIMP

1. Iteration history is usually monotonic (Fig. 6). Note that in this diagram, the broken line (penalized compliance)

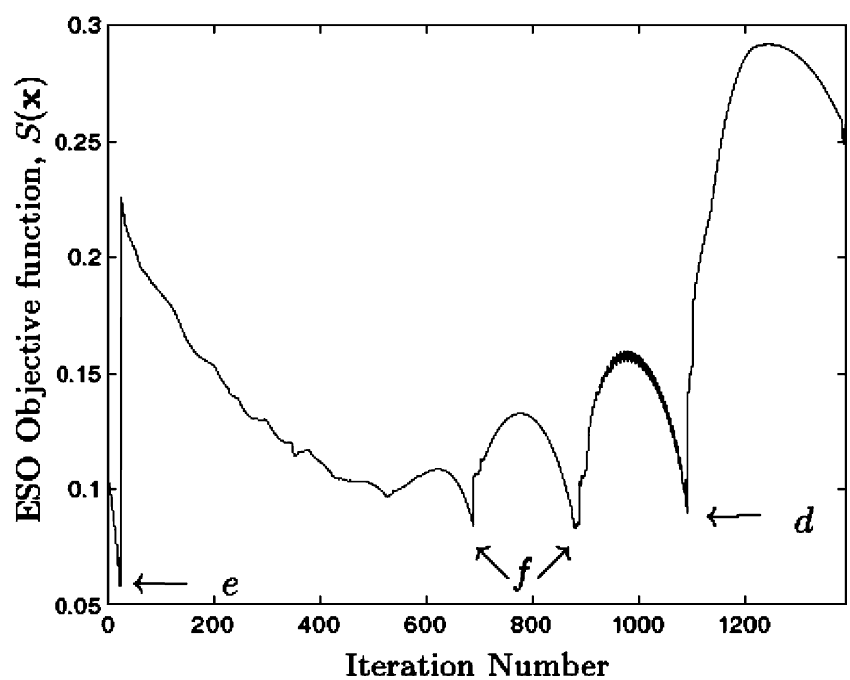

Fig. 5 Variation of the objective function for the ESO method (after Edwards et al. 2006)

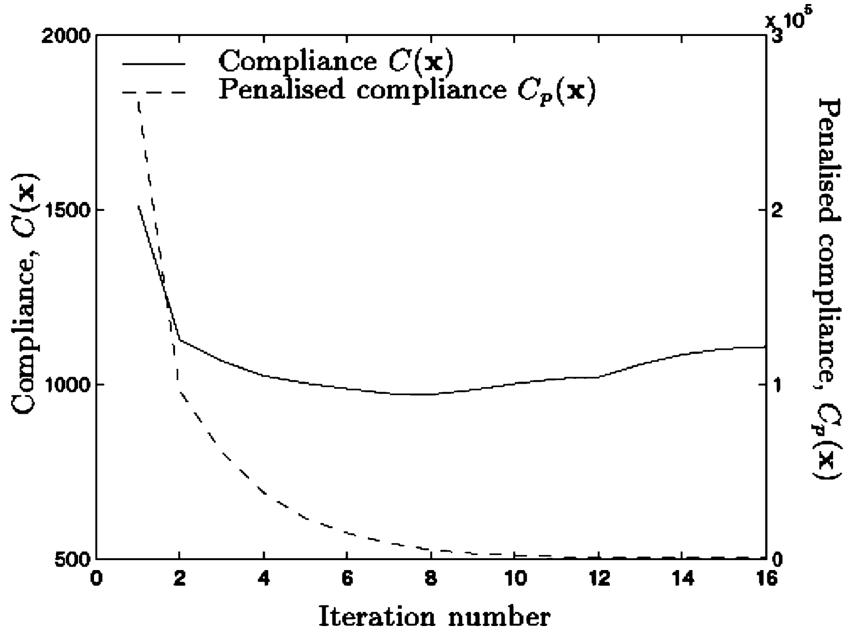

Fig. 6 Variation of the objective function (*broken line*) for the SIMP algorithm (after Edwards et al. 2006)

is the objective function. The unpenalized compliance increases because penalization forces black-and-white solutions, which are less efficient than grey solutions.

2. Optimal solution is at the end of the iterative procedure.
3. Low iteration number.
4. It does not follow from the paper that SIMP generates non-optimal solutions (see also par. 6 below).
5. The paper uses a heuristic method for checkerboard control, which may cause anomalies in SIMP solutions.
6. For the penalty factor in SIMP, the authors used a constant value of 4, which is more likely to give a non-global (local) minimum than the "continuation method" (i.e., increasing its value of p gradually from unity).

7 Note by Huang and Xie (2007)

This is a most reasonable communication. In fact, the author hopes to do some collaborative work with the writers of this note in the future. Three possible methods are proposed for eliminating the problem outlined by Zhou and Rozvany (2001).

1. Preventing changes in support conditions by "freezing" elements next to supports. The writers of the note correctly point out that this can cause computational complications. This author could easily produce an example with many possible supports, some of which are obviously uneconomical to use. The very aim of topology optimization for multi-support problems is to select by computer the supports that are economical to use, and this is not possible if we freeze all elements at supports.
2. Replacing removed elements with low density elements, which may be recovered in the next cycle. This method was actually proposed by Rozvany and Querin (e.g. 2001, 2002a and b, 2004) under "virtual material" approach (see also Section 11.1). Oscillation can be avoided by indeed freezing those elements that have been reinstated by the virtual material approach.
3. Mesh refinement. This is discussed in Section 9.

8 Second version of paper by Edwards et al. (2007)—remaining deficiencies

Possibly following suggestions of the reviewers, the authors have revised their paper significantly. For example, the somewhat sweeping title of the conference paper "Investigation on (sic) the validity of topology optimization methods" has been toned down to "An evaluative study on ESO and SIMP for optimizing a cantilever tie-beam", which is more factual. The previously used unrealistic objective function (normalized standard deviation of the Mises stress field) has been replaced with the product of the compliance and volume. Instead of using a constant penalty factor of p=4 (which is likely to give the wrong local optimum), the "continuation method" of gradually increasing the penalty factor was employed. However, a number of fundamental weaknesses remained in the paper.

8.1 Mixing up the results of two distinctly different problems

In addition to considering the problem in Fig. 4a of this paper, the above authors considered also a different problem in which the previous ground structure is embedded into a larger rectangular ground structure with 224 elements (instead of 100, see Fig. 4 of Edwards et al. 2007). This is not a different initial design, but an entirely different problem, and as expected, the optimal ESO topology for 101 elements even contains elements outside the ground structure of the original problem (see top right four elements in Fig. 5 of Edwards et al. 2007). It is well known that both in SIMP and in ESO, the “initial design” consists of the entire ground structure with only “black” elements or (in SIMP) uniformly grey elements. It follows that in this case, not the initial designs are different, but we are considering two entirely different problems (a frequent error in evaluative studies), the optimal design for the second one even containing elements outside the design domain of the first problem. However, the ESO *CV* history of the second problem also crashes at the 99th iteration (Fig. 6, Edwards et al. 2007), so the introduction of a different problem only delays the breakdown of ESO.

8.2 ESO producing only insignificant improvements of their objective function ($C.V$)

For the ground structure in Fig. 4a, we have $C.V$=38,850 (38,986 according to Edwards et al.). The ESO runs in Table 1 in Edwards et al. (2007) give final results of $C.V$= 39,997, 38,671, 41,699, 41,469. Even if we accept the value of 38.986 for the ground structure, this means an improvement of only 0.8% for one solution and a worsening of the objective function in the other three ESO solutions. In fact, the average final objective function for the four solution is 40,456, almost 4% higher than that of the initial design. Is this optimization?

8.3 ESO’s highly inefficient and somewhat irrational optimization procedure

Gradient methods try to improve the solution in each iteration, and the required (possibly local) optimum is obtained with the last iteration when conditions for optimality (the Kuhn–Tucker conditions) are satisfied. ESO does not use such a condition for evaluation. It first generates by a fully heuristic procedure a very large number of solutions (e.g., 1,600 solutions in Fig. 13, Edwards et al. 2007). One would hope that the optimal solution is found after all this effort. But no, this set of solutions must be searched through to find the best objective function value from it. The authors call this solution “global minimum”, but it is certainly not the global minimum in the accepted sense of this term.

It is surprising that Tanskanen (2002, p 5487) claims that ESO requires relatively little FEA time. The example in Fig. 13 required 1,600 analyses for a small system; SIMP can converge with a fraction of this iteration number.

8.4 ESO may produce highly non-optimal solutions and there is no proof that *any* of its solutions are optimal

The Zhou and Rozvany (2001) note showed that ESO may fail completely. It will be explained in Section 9 that mesh refinement is no remedy against ESO’s breakdown. Although both Tanskanen (2002, p 5487) and Edwards et al. (2007) try to verify ESO by claiming that comparisons with Michell structures are “quite promising”, one must point out that no quantitative comparisons have been carried out for ESO (they have been for SIMP, see Section 2.7). A vague subjective resemblance is no proof, Michell’s problem being a convex one, for which even the crudest methods produce some similarity to the corresponding Michell layout.

8.5 Comparison of objective function values for ESO and SIMP are meaningless because they use different objective functions

As the objective function for SIMP is either compliance (C) or volume (V), and for ESO it is $C.V$, they aim at different objectives and produce different results. A comparison of the numerical results for these is similar to a fictitious car rally in which one car goes for speed and the other one tries to minimize fuel consumption. It is shown in a separate note (Rozvany 2008) that compliance minimization and that of $C.V$ produce quite different optimal solutions.

8.6 Chaotic convergence histories for ESO

It can be seen from Fig. 7a that the objective function history for ESO jumps around quite irregularly, lacking any sign of convergence. The monotonic volume curve (broken line) should not mislead the reader; the ESO program enforces volume reduction in each iteration by taking out some element(s).

In the objective function history of SIMP (Fig. 7b), the penalized volume value (top curve) naturally steps up, as it should, with each increment of the penalty factor p, but within each iteration (with a constant p value), the convergence is perfectly monotonic. The volume curve is increasing most of the time because black-and-white solutions are less efficient than grey solutions. However, we do require in topology optimization black-and-white solutions. These curves were obtained for a fairly complex structure (Michell cantilever).

8.7 Misapplication of SIMP

In the considered paper (Edwards et al. 2007), four SIMP solutions were computed with the original mesh size, two for the original problem and two for a modified problem with rectangular ground structure. Each used two different initial

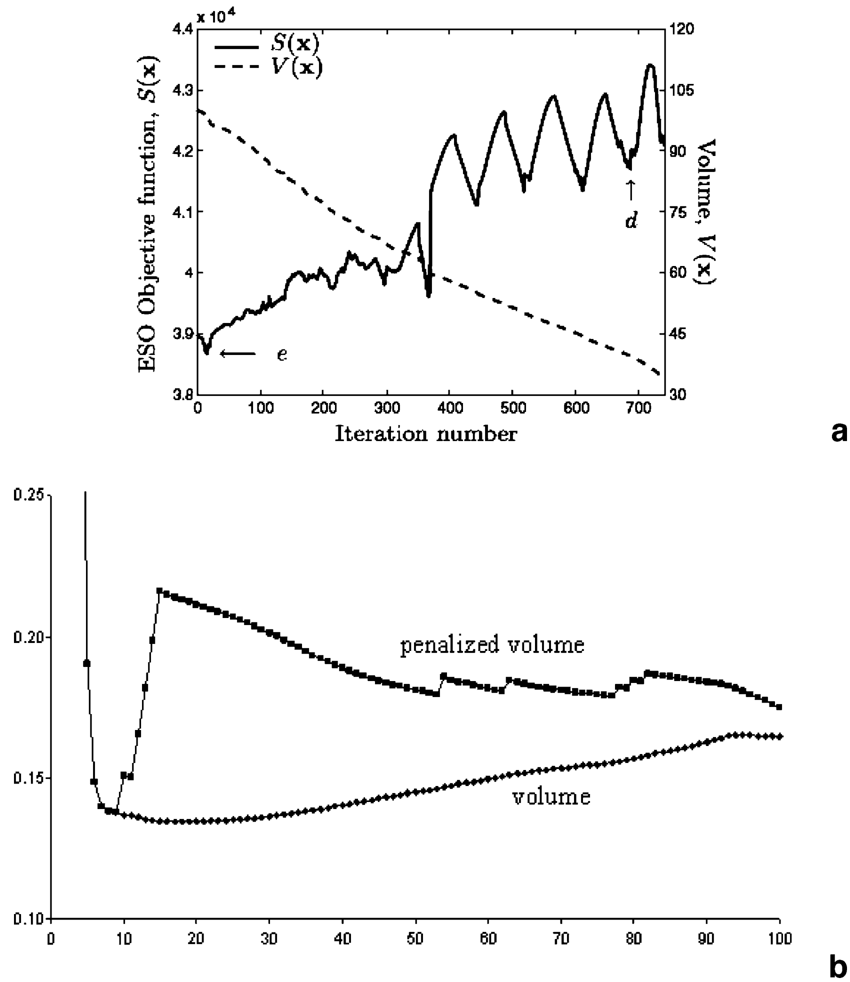

Fig. 7 Iteration histories for **a** ESO (after Edwards et al. 2007) and **b** SIMP (after Querin). In the latter, *top curve*: penalized volume, *bottom curve*: volume

p values ($p=1$ and $p=1.5$). First, it is not surprising that $p=1.5$ gives different results, we have known about "*p*-dependence" of SIMP solutions for more than 15 years. On the basis of many years experience, one should start with $p=1$, use small increments of *p*, and one should also employ more refined convergence criteria than the one used in the above paper (see Section 2.9 above).

There are two things wrong with the above SIMP solutions. First, there are too many *grey elements* in the results. As was shown by Rietz (2001, consistently misspelled as "Reitz" by the authors), grey elements can be eliminated (in practice, nearly eliminated). The same was shown by Stolpe and Svanberg (2001) for their alternative interpolation scheme. Second, there is only a *corner contact* between two black elements in three solutions, which makes these solution a false optimum.

According to the paper under consideration (Section 2.3), the authors used a Bartlett filter, which is a heuristic method with an unknown objective function that may modify the original Michell-like layouts, but does not suppress single hinges (corner-contacts) in the solution. Combining such a heuristic method with different starting values of the penalty factor (*p*) may produce unpredictable changes in the solution.

It should be noted, however, that filtering is a heuristic but highly economical method in practical applications. In basic research, however, computer time is usually not important, and therefore, it is possible to use less heuristic techniques. The wide scatter of solutions by filtering methods was shown in an excellent review article by Sigmund (2007).

As mentioned, four-node elements grossly underestimate the compliance value for corner contacts. The compliance value even reaches infinity if we calculate its value analytically (Rozvany et al. 2002b, 2003). This error can be avoided or reduced if we use higher order elements, or several FEs per ground element, or the constrained slope method (Petersson and Sigmund 1998), or penalization against corner contact (COSIMP method, Rozvany et al. 2004; Pomezanski et al. 2005), or Poulsen's (2002) constraint. The above methods are based on rigorous sensitivities and, therefore, do not interfere significantly with the Michell-type optimal topologies. However, they usually suppress even isolated hinges. A more accurate FE analysis would have given highly non-optimal compliance values for solutions with hinges (even with some light-grey elements around them) in Figs. 16, 18, and 19 of Edwards et al. (2007).

It may make sense, however, that the beam tries to lean downwards to the left because this reduces the force in the vertical tie. This tendency can be observed both in solutions with mesh refinement in this paper and in a very carefully derived solution for the Zhou–Rozvany problem (Fig. 4a) by Norato et al. (2007).

8.8 Comparing results with an "intuitive" solution

In both versions of their paper (Edwards et al. 2006, 2007), the authors point out that some of their solutions have a lower $C.V$ or C value than that of the "intuitive" solution in Fig. 4c in this article. As a question of basic logic, any optimization method should indeed do in general better than a solution picked intuitively. Such a finding, therefore, proves nothing in relation to ESO or SIMP. In fact, Stolpe and Bendsoe (2007) obtained global optima with a lower compliance value for the Zhou-Rozvany (2001) problem than the intuitive solution in Fig. 4c.

8.9 Gross error in strain energy calculation

Zhou and Rozvany (2001) showed that both finite element analysis and analytical results from a two-bar truss model give almost the same value for the compliance of their initial design (difference of 0.18%). Edwards et al. (2007) tried to calculate the optimal cross-section values for the Zhou–Rozvany problem using the same two-bar truss model. Leaving the cross-section of the horizontal bar unchanged (A_x=3), they obtain from the uniform energy density principle a cross-sectional area of A_y=1/4 for the vertical bar. Even according to Edwards et al. (2007), the force in the horizontal bar is 6, and in the vertical bar, it is 1. Their "optimal" cross-sections would give, therefore, a stress value of 2 in the horizontal bar and a stress of 4 in the vertical bar.

It has been known for almost 40 years (e.g., Hegemier and Prager 1969) that in trusses uniform energy density and uniform stress are equivalent for a single load condition. In fact, Edwards et al. (2007) have forgotten in their Eq. (16) that the number of (constant size) elements (n_y) in the vertical bar does depend on the cross-section of that member and this oversight caused the above gross error. The correct value for the cross sectional area of the vertical bar would be A_y=1/2, which is 100% higher than the erroneous value by Edwards et al. (2007).

8.10 Fallaciousness of the argumentation using uniform energy density

Irrespective of the incorrectly calculated cross-section, an optimization method should not produce a 1,000% error in the objective function just because it cannot attain an analytical optimal solution with a given FE net. The user of ESO would generally have no idea what the analytical optimal solution is, and would believe blindly in the computer output.

8.11 Mesh refinement

It is claimed by almost all authors from the ESO group that failure of ESO may be prevented by using a finer FE mesh. This question is examined in the next section.

9 Is it really possible to avoid breakdown of ESO by mesh refinement?

The answer is probably yes–no ("jein" in slangy German). Mesh refinement has been proposed in both texts above. In the Edwards et al. (2006) text, 3,600 elements were used in the Zhou–Rozvany (2001) example (instead of 100), in the Huang and Xie (2007) note 10,000. One can only mention that for any of the above mesh refinements, ESO would crash if we used higher values for the horizontal load in the above examples. For example, for the 10,000 element case, we may increase the horizontal load intensity from 2 to 12. Then, even when only one (out of ten) element remains in place at the top support, in this element, the stress will be about 10, while in the horizontal beam, it will be roughly 12. Then, of course, ESO will take out the element next to the top supports, and the structure will become again highly non-optimal.

Summarizing, for any mesh density, we can specify a load for which ESO breaks down. Defenders of ESO may reply that for any load, they can propose a mesh density for which ESO does not break down. This may be true, but then the user of an ESO program can never know if, for the considered supports and loads and a given mesh, he does not get a breakdown of ESO with an extremely uneconomical solution.

10 Deeper reasons for the breakdown of and a possible rigorous basis for ESO (SERA)

It was explained by Zhou and Rozvany (2001) that the compliance sensitivity of the element "a" in Fig. 4a changes from unity to almost 160 million, as the element thickness varies from unity to 10^{-6}. This accounts for the disastrously wrong choice of ESO in eliminating an element.

As indicated by the author (e.g., Rozvany and Querin 2002b), ESO would give a correct "iteration-wise optimal element change", if for all rejected elements of that iteration the relevant sensitivities did not change significantly as their thickness varies from unity to zero. This could be checked in each iteration by comparing the sensitivity value with the actual change caused by a unit change in the density of the rejected elements. If the difference is large, the corresponding elements could be stopped from being eliminated. This would help ESO (SERA) much more in the considered example than the suggestions in the above three texts.

It was correctly pointed out by Zhou at the ISSMO congress in Seoul that the proposed check may become uneconomical for very large systems used in practice. For example, if hundred elements are rejected (out of, say, hundred thousand) in each iteration, then it is expensive to assess which elements caused the disparity between sensitivities for infinitesimal and finite changes in densities.

11 Proposed improvements of ESO (SERA) by Rozvany and Querin

Apart from the suggestion in the previous section, Rozvany and Querin (2001, 2002a and b, 2004) have proposed the following improvements.

11.1 Virtual material

Where a solid element has been taken out, an element of low density (e.g., 10^{-6}) is inserted (this is usually already done in ESO). Then, BESO should be based on the stress or energy density in such an element of "virtual material" and not on those in the adjacent solid element. This would recover the "wrongly removed" element "a" in the considered example (see Fig. 4 above).

11.2 Use of correct Lagrange multiplier values and sensitivities

For several load conditions or several design constraints, one does not know which constraints are active, and to which extent, unless the Lagrange multipliers are evaluated. This would also be necessary for ESO. A simple example for two load conditions was shown at the Dalian meeting (Rozvany and Querin 2001).

ESO may avoid the use of sensitivities for compliance design under a single load condition, because for that case compliance values and first-order compliance sensitivities are proportional to each other (if all elements have the same density). For any other problem, ESO should use at least first-order sensitivities, which would also increase significantly the computational effort. Of course, the use of second-order sensitivities would prevent the failure experienced in the Zhou–Rozvany (2001) example, but its use in practice would become prohibitively expensive.

12 Comments on Tanskanen's "proof" of ESO's computational procedure

Tanskanen's (2002) paper has some interesting ideas. However, it will be shown subsequently that it does not justify the use of ESO in topology optimization.

As mentioned before, Tanskanen (2002) claims that (1) ESO "requires relatively small amount of FE time" and (2) it has been verified by comparisons with Michell trusses. These assertions do not stand up to a more objective scrutiny. The claim under (2) is based on rough visual comparisons, while SIMP has been verified by quantitative procedures (see Section 2.7). Michell's problem being convex and self-adjoint, even the crudest methods produce vaguely Michell-like configurations for the same supports and loading. Tanskanen's claim that ESO is "simple to program" is true only because ESO deals mostly with the two simplest problems, which would be student exercises for other methods.

Tanskanen (2002) also states (p 5494): "...the objective function cannot reach the minimum until...all the structural members are thoroughly fully stressed." "...if there occurs bending moment in the structural members, the fully stressed state cannot be reached." What does Tanskanen mean by "thoroughly" fully stressed. A structure is either fully stressed or not. Moreover, all authors solving problems for perforated plates in plane stress know that for higher volume fractions, very wide members also develop in the solution (e.g., along the edges of a "Michell cantilever"). These cannot behave as pinjointed bars; they are subject to considerable bending as well. Moreover, it was found by Pedersen (2000) that for the shape design of plates in plane stress "...minimum compliance shape design will have uniform energy density...." and "the stiffest design will also be the strongest design..." (i.e., optimal for stress constraints). Pedersen's conclusions were referring to structures that were not at all truss-like.

Tanskanen declares "it was assumed that ESO minimizes the $C.V$ product". In actual fact, ESO uses Mises stress or element compliance as criterion for element elimination in most papers. Out of the large number of solutions generated by ESO, the "global optimum" (?) could be located by using any *arbitrarily* selected objective function or "performance index". The choice of the latter has nothing to do with the ESO algorithm used in its element elimination procedure.

Two main shortcomings of Tanskanen's (2002) paper will be discussed in the next two subsections.

12.1 First weakness: formulation in terms of trusses and absurdly complicated optimization procedure

Tanskanen states that "ESO should be applied to structural problems having pinjointed connections. For other types of problems ESO should be studied further."

Apart from the fact that structural *problems* do not have pinjointed connections (but *structures* may have), ESO is applied mainly to perforated plates, which can behave like pinjointed systems (i.e., trusses) *only if* the volume fraction is very low. For higher volume fractions, they behave quite differently. A "proof" based on Michell trusses is therefore not valid for ESO's usual problem classes. Apart from this, Tanskanen's treatment of Michell trusses is absurdly complicated for the following reasons.

It is well known (e.g., Sved 1954; Hemp 1973) that discretized Michell trusses (with finite number of joints in the ground structure) can be derived by standard linear programming methods. When the author did his postdoctoral work in Oxford around 1970 with William Hemp, discretized Michell trusses were derived by linear programming for ground structures with several hundred potential members (see, e.g., McConell 1974). This project started almost four decades ago when computer compatibilities were incomparably more limited than now.

One can see that Michell1's problem is a linear programming one, by virtue of Sved's (1954) theorem that at least one optimal discretized Michell truss must be statically determinate (i.e., does not require conditions of compatibility). As the cross sectional area (A)–member force (F) relation is then piecewise linear (c is the reciprocal value of the permissible stress)

$$A = c|F|, \tag{3}$$

and the joint equilibrium equations (zero sum of horizontal and vertical force components) are linear in the member forces (F), both Sved (1954) and Hemp (1973) concluded correctly that discretized Michell trusses constitute a linear programming problem. The above argument is for stress-constrained Michell structures, but the same topology is valid for a compliance constraint (e.g., Hegemier and Prager 1969).

Considering the above facts, it is rather remarkable that Tanskanen (2002) suggests a very complicated and cumbersome method for Michell trusses (which he does not even verify with a single numerical example). After a long and superfluous description of the meaning of "compliance" and of a general form of optimization problems with equality, inequality, and side constraints, he shows that Michell structures minimize the compliance–volume product ($C.V$). All this had been well known before. Then, he uses a logarithmic problem transformation

$$\min \ln(C.V) = \min(\ln C + \ln V), \tag{4}$$

which he proposes to solve by linearization and sequential linear programming (SLP). Finally, he describes the well-known Simplex algorithm for solving sequentially the SLP problem. No numerical example is given to show that the proposed method actually works.

The use of such a complicated procedure is absurd for a problem, which has been known for over 50 years as a simple linear programming one!

12.2 Second weakness: use of sensitivities for finite design modifications

It was explained in Section 10 that ESO breaks down in the Zhou–Rozvany (2001) example because the compliance sensitivity varies from unity to many millions as the element thickness changes from unity to near zero. As Tanskanen (2002) uses compliance sensitivities, his otherwise doubtful proof could only be valid if sensitivities did not vary significantly during such finite design changes. Possible remedies for this problem were given in Section 10, but they may be prohibitively expensive for larger real-world systems.

13 Other methods with finite design changes in the recent literature

13.1 An efficient and thoroughly verified method using SILP (Svanberg and Werme 2006a)

For optimizing the topology of perforated plates in plane stress, Svanberg and Werme (2006a) used sequential integer linear programming. This method is clearly and rigorously derived and verified by examples (unlike the paper by Tanskanen 2002). The results for their benchmark problem (L-shaped domain) are consistent with other papers by these authors (Svanberg and Werme 2005, 2006b, 2007; Werme 2006) and have also been confirmed by comparison with exact analytical solutions by Lewinski and Rozvany (2008a).

13.2 Method of element removal and reintroduction by Bruns and Tortorelli (2003)

Bruns and Tortorelli (2003) discussed ways of overcoming computational difficulties in topology optimization by element removal and reintroduction, with particular reference to large deformations.

They use a Gaussian (bell-shaped) measure for density filtering. Their method is basically similar to SIMP, with a significant improvement by means of removing low-density elements, which can be reintroduced later. The proposed method achieves significant savings in computer time.

It should be emphasized that the method of Bruns and Tortorelli (2003) differs significantly from ESO because it is basically a continuous variable, gradient-type method, but it reduces the problem size by iteratively decreasing the design domain. The design changes are based on rigorously derived sensitivities, unlike in ESO.

14 Conceptual and terminological ambiguities and misconceptions in some ESO publications

As we were discussing Tanskanen's (2002) paper, we would like to point out some other fallacies in it.

> "In the literature, topology optimization is most often applied to truss ground structures..." This was possibly the situation in the seventies. These days much more attention is paid to two- and three-dimensional continua.

> "Compared with MP algorithms, the OC methods are efficient in large optimization problems, but lack generality in various kinds of minimization problems." This statement is about thirty years out of date. The reconciliation of OC and MP was concluded by Claude Fleury by the end of 1970s (see, e.g., Fleury 1982). The so-called OC school never correctly, mathematically, solved the issue which constraints are active. Dual MP methods, e.g., CONLIN (Fleury 1989) or MMA (Svanberg 1987) are nothing but mathematically correct versions of OC methods. Moreover, what is the difference by Tanskanen between "optimization problems" and "minimization problems"?

> "It cannot be said that the objective function in...(2.11) (i.e., min $(C.V)$ for Michell trusses) is convex"

As mentioned, Michell's problem is known to be convex (e.g., Hemp 1973) if suitably formulated.

On p 5494 Tanskanen (2002) claims that once an optimal topology is obtained for a compliance constraint, this design could be made to satisfy other constraints by scaling the cross-sectional areas. It was shown, e.g., by Sankaranarayanan et al. (1992) and Rozvany (2001b) that this procedure can lead to entirely non-optimal designs. It is appropriate to remark here that Tanskanen (2002) quotes somewhat inaccurately this author (Rozvany 1993) that "structural optimization is divided into three classes: *sizing*, *geometrical*, and *topology optimization*." This author, in fact, emphasized that these must be "simultaneous operations", and the order of the three was, logically, topology, geometry, and sizing.

Everywhere in the literature, "compliance" is defined as the virtual work (i.e., scalar product) of external forces and displacements. This can also be used in dual methods for calculating the volume of a structure. For some unknown reason, Tanskanen (2002) deviates from this established practice and makes compliance equal to the total strain energy, which is half the magnitude of the usual compliance value.

When discussing "optimality criteria methods", Tanskanen (2002) writes mostly about the fully stressed design method, which is called by Save and Prager (1985) a "questionable" optimality criterion. This is because Chern and Prager (1972) showed it much earlier that it may lead to non-optimal solutions. In relation to rigorously derived optimality criteria, Tanskanen (2002) only mentions that "OC algorithms are also discussed by Save and Prager". One should remark that the cited book consists of a brief review of optimality criteria methods and was written mostly after William Prager's death. It is on analytical methods, does not mention even once finite elements or matrix notation. In the chapters on topology optimization (termed "design" of trusses and grillages), mostly the joint work of Prager and this author is summarized at an elementary level, with references to 22 publications of the author (some with Prager, whose only research partner he was during the last decade of Prager's life). For a more advanced study of optimality criteria methods, the reader is referred to the author's books (Rozvany 1976, 1989) or the review articles (Rozvany and Zhou 1994; Rozvany et al. 1995).

Referring now to the more recent paper by Edwards et al. (2007), the author would like to comment on the following points.

Checkerboard patterns are called several times a "numerical instability". This and other explanations of checkerboarding were widespread in the nineties. We know it now (see also Section 2.5) that simple (e.g., four-node) FEs grossly overestimate the stiffness of solid ground elements having only a corner contact. SIMP cannot "know" that the information coming from the FE program is incorrect, and therefore, it uses this wrong data, producing checkerboard patterns, diagonal element chains, and/or single hinges. These turn out to be highly non-optimal if we use a more accurate FE analysis (e.g., several or higher order elements for each ground element). However, the iterative procedure resulting in checkerboards etc. is quite stable, so we should not speak about "numerical instabilities" (but "discretization errors").

"The SIMP formulation is based on...minimizing the total compliance". In actual fact, SIMP can optimize the structural volume for a variety of constraints and multiple loads. The above assertion is therefore incorrect. It is ESO which tries to optimize mostly for compliance (or Mises stress).

"This investigation has identified some of the difficulties associated with the two methods and the reasons behind the said difficulties". Edwards et al. (2006, 2007) have not pointed out anything new about SIMP, we have almost 20 years of experience with it. It has been known for long that if the objective function surface is very flat (as in the considered example), then starting with a non-unit p value may change the computed topology significantly. However, SIMP has never produced a more than 1,000% error, as ESO did in the Zhou–Rozvany (2001) example. Moreover, it is no wonder that SIMP gives different solutions if the problems are different (see Section 8.1).

Edwards et al. (2008) finish their "Conclusions" with the sentence "When the volume constraint was specified as $V(x) \leq 99$ and a fine computational mesh was used, ESO and SIMP produced similar results; however, ESO produced the overall optimum solution". This very final statement could be interpreted by the reader that ESO is actually a better method than SIMP. We would like to point out that no designer of sound mind would optimize a real world structure for a volume fraction of 99%. Moreover, the comparison is unrealistic because the SIMP algorithm was not used at its best by Edwards et al. (2007), see Section 8.9.

14.1 Comments on Tanskanen's (2006) second ESO paper

In his more recent paper, Tanskanen (2006) reiterates that ESO is "very simple to program" and "requires relatively small amount of FE time". These claims were already discussed in Sections 12 and 14. He also admits that "different constraints cannot be added into the problem" and "ESO is not capable of handling general stress and displacement constraints" He therefore states that "these constraints do not need (sic) to be included in topology optimization" and can be dealt with in a second stage of ("sizing") optimization. It has been well established that such a two-stage procedure may lead to a highly non-optimal solution (e.g., Sankaranarayanan et al. 1992; Rozvany 2001b).

Tanskanen (2006) also observes that in the case of additional constraints, ESO may lead to a design which is not fully stressed. For these problems, he proposes a "modified" ESO (termed "MESO"), which makes the solution more fully stressed. The problem is that for such problems, the real optimal solution is mostly not fully stressed, and therefore, the suggested method takes the solution away from the true optimum.

15 Comments on priorities in topology optimization

Sigmund (2007) quite correctly points out that the priorities for desirable characteristics of numerical topology optimization methods differ considerably in industrial applications and in academic research. This author fully agrees. In the latter, we try to get as close as possible to the theoretical optimal solution by means of rigorously derived algorithms and stable convergence, with little use of tuning parameters. In industrial applications, the dominating preferences are (1) low CPU time (2) generality of applicability, (3) reliability, (4) simplicity of implementation, and (5) simplicity of the topologies obtained. As mentioned before, priority (5) may change with time due to improved manufacturing capabilities. Priority (4) above is due to the fact that very few industrial users would have the time for studying exact optimal solutions or to check for possible errors in the derivation of numerical optimization techniques. For this reason, they are very often satisfied with any major cost saving, although they do not have the slightest clue if the solutions produced by a method are really close to the real optimum. This is why they sometimes fall for misleading publicity based on "simple" and "commonsense" arguments whose understanding does not require much knowledge of higher mathematics or mechanics. In reality, only exact benchmark solutions "furnish information which is useful in testing the validity, accuracy, and convergence of numerical methods and in assessing the efficiency of practical designs", a 30-year-old truth (Prager and Rozvany 1977). Critical reviews like the current one may help to some extent because they make industrial users aware of the possibility of false claims by the promoters of questionable methods.

This does not mean at all that the author is against heuristic methods in general. They play an important role in structural optimization, particularly in approximations. However, heuristic methods should be verified quantitatively by correctly planned numerical experiments.

16 Conclusions

The following conclusions can be drawn from the above investigation

16.1 SIMP

SIMP is a reasonably rigorously derived gradient method for topology optimization, which usually gives a solution near the correct global optimum if the problem is originally convex (e.g. in compliance problems), and the penalty factor p is increased gradually from unity. However, SIMP is used in practice for highly complex non-convex problems, and therefore, a global optimum cannot be guaranteed in general, but this is so for all gradient-based methods. Some theoretical convergence properties of SIMP have also been explored (Rietz 2001; Martinez 2005; Stolpe and Svanberg 2001). SIMP requires relatively few iterations and is suitable for a combination of a wide range of design constraints, multiple load conditions, multi-physics problems, and extremely large (often 3D) systems. SIMP has been verified quantitatively by showing numerical convergence to Michell topologies (see Section 2.7). It is used extensively in industrial software.

16.2 ESO (SERA)

The author has always had an open mind about ESO and has even been working on possible improvements of this method (under the term "SERA", e.g., Rozvany and Querin (2001, 2002a and b, 2004). However, in spite of his "neutral" position in the SIMP/ESO controversy, he cannot overlook compelling factual evidence of shortcomings in the *present versions* of ESO. Particularly due to unresolved problems arising from the Zhou–Rozvany (2001) counter-example and significant fundamental flaws in some (but not all) recent papers defending ESO (see above), the author must report with regret the following findings of facts.

ESO is presently fully heuristic, computationally rather inefficient, methodologically lacking rationality, occasionally unreliable, with highly chaotic convergence curves. Unlike the quantitative verification of SIMP (see Section 2.7), ESO has only been "verified" by vague visual comparisons with Michell topologies. In its present form (as discussed by Edwards et al. 2006, 2007 and Tanskanen 2002, 2006), ESO is only trying to solve problems with a single constraint (such as compliance or stress) and single load condition, which would be nowadays undergraduate exercises for other methods. ESO is now therefore hardly ever used in industrial applications, although greatly over-publicized in the literature. Recent papers defending ESO ignore the fact that ESO breaks down if the sensitivity with respect to element density changes rapidly over finite density variations (e.g., from unity to zero, see Zhou and Rozvany 2001). The author believes that the same problem, and some other serious deficiencies (see Section 12), render Taskanen's (2002) "proof" of ESO's algorithm invalid in its present form. Moreover, the use of the objective function ($C.V$) makes some sense in 2D problems (e.g., simultaneous optimization of topology and plate thickness for perforated plates under compliance constraint; see Rozvany 2008), but it does not represent any real-world problem in 3D topology optimization.

It follows that there is scope for very much research to improve and justify ESO so that one day, it may possibly constitute a useful alternative to gradient-type topology optimization methods. This author hopes that the authors of the papers reviewed here find his comments constructive enough to be of benefit to them, or if they disagree, they will submit discussions on this forum article.

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