

# Morphisms represented by monomorphisms

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## Abstract

Every homomorphism of modules is projective-stably equivalent to an epimorphism but is not always to a monomorphism. We prove that a map is projective-stably equivalent to a monomorphism if and only if its kernel is torsionless, that is, a first syzygy. If it occurs although, there can be various monomorphisms that are projective-stably equivalent to a given map. But in this case there uniquely exists a "perfect" monomorphism to which a given map is projective-stably equivalent.

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## 1 Introduction

Let  $R$  be a semiperfect ring. A morphism  $f : A \rightarrow B$  and  $f' : A' \rightarrow B'$  in  $\text{mod } R$  are said to be projective-stably equivalent if they are isomorphic in  $\underline{\text{mod}} R$ ; if there exist morphisms  $\alpha : A \rightarrow A'$  and  $\beta : B \rightarrow B'$  such that  $\underline{\alpha}$  and  $\underline{\beta}$  are isomorphisms and  $\underline{\beta \circ f} = \underline{f' \circ \alpha}$  in  $\underline{\text{mod}} R$ . We say a morphism  $f$  is represented by monomorphisms ("rbm" for short) if there exists a monomorphism in  $\text{mod } R$  that is projective-stably equivalent to  $f$ .

For any homomorphism  $f : A \rightarrow B$  of  $R$ -modules,  $(f \ \rho_B) : A \oplus P_B \rightarrow B$  is surjective with a projective cover  $\rho_B : P_B \rightarrow B$ . Thus every morphism is represented by epimorphisms. The choice of epimorphism is unique; if an epimorphism  $f'$  is projective-stably equivalent to  $f$ , then  $f'$  is isomorphic to  $(f \ \rho_B)$  up to direct sum of projective modules.

On the other hand, every morphism is not always represented by monomorphisms. Even if a morphism  $f$  is rbm, the choice of monomorphism is not unique; there would be two monomorphisms that are not isomorphic up to direct sum of projective modules and both of which are projective-stably equivalent to  $f$ .

The purpose of the paper is finding a condition of a given map to be rbm. The problem was posed by Auslander and Bridger [1]. They proved that a map is rbm if and only if it is projective-stably equivalent to a "perfect" monomorphism. An exact sequence of  $R$ -modules is called perfect if its  $R$ -dual is also exact. A perfect monomorphism refers to a monomorphism

whose  $R$ -dual is an epimorphism. This is our first focal point. We studied the situation where a map is rbm, especially the structure of monomorphisms into which a given map is modified. And we obtained the obstruction for a given map to be rbm. In the case that a map is rbm, the choice of a monomorphism is not unique, but then a perfect monomorphism projectively-stably equivalent to the given map is uniquely determined up to direct sum of projective modules. (Theorem 3.9.)

Our next focus is an analogy to the homotopy category  $\mathcal{K}(\text{mod } R)$  of  $R$ -complexes. In [5] Theorem 2.6, the author showed a category equivalence between  $\underline{\text{mod}} R$  and a subcategory of  $\mathcal{K}(\text{mod } R)$ . Due to this equivalence, we describe the obstruction of being rbm with a homology of a complex associated to the given map.

Looking at Theorem 3.9, we see that when a morphism is rbm, its pseudo-kernel is always the first syzygy of its pseudo-cokernel. So it is tempting to ask if torsionlessness of the kernel is equivalent to rbm condition. This is our third point. Actually, for this we need Gorensteinness.

Theorem 4.10 : Suppose the total ring of fractions  $Q(R)$  of a ring  $R$  is Gorenstein. A morphism  $f$  is rbm if and only if  $\text{Ker } f$  is torsionless, equivalently, a first syzygy.

## 2 Preliminaries

We shall fix the notations and give some review on the correspondence between stable module category and homotopy class category of complexes. We omit the proofs for results that are in [5].

Throughout the paper,  $R$  is a commutative semiperfect ring, equivalently a finite direct sum of local rings; that is, each finite module has a projective cover ( see [6] for semiperfect rings). The category of finitely generated  $R$ -modules is denoted by  $\text{mod } R$ , and the category of finite projective  $R$ -modules is denoted by  $\text{proj } R$ . By an  $R$ -module we mean "a finitely generated  $R$ -module". For an  $R$ -module  $M$ ,  $\rho_M : P_M \rightarrow M$  denotes a projective cover of  $M$ . For an abelian category  $\mathcal{A}$ ,  $\mathcal{K}(\mathcal{A})$  stands for the category of the homotopy equivalence class of complexes in  $\mathcal{A}$ . A complex is denoted as

$$F^\bullet : \dots \rightarrow F^{n-1} \xrightarrow{d_F^{n-1}} F^n \xrightarrow{d_F^n} F^{n+1} \rightarrow \dots$$

A morphism in  $\mathcal{K}(\mathcal{A})$  is a homotopy equivalence class of chain maps. A trivial complex is a split exact sequences of projective modules. Truncations of a

complex  $F^\bullet$  are defined as follows:

$$\tau_{\leq n} F^\bullet : \cdots \rightarrow F^{n-2} \xrightarrow{d_F^{n-2}} F^{n-1} \xrightarrow{d_F^{n-1}} F^n \rightarrow 0 \rightarrow 0 \rightarrow \cdots,$$

$$\tau_{\geq n} F^\bullet : \cdots \rightarrow 0 \rightarrow 0 \rightarrow F^n \xrightarrow{d_F^n} F^{n+1} \xrightarrow{d_F^{n+1}} F^{n+2} \rightarrow \cdots$$

An  $R$ -dual  $F_\bullet^*$  of a complex  $F^\bullet$  is the cocomplex such as  $F_n^* = (F_n)^*$ ,  $d_n^{F^*} = (d_F^{n-1})^*$  where  $*$  means  $\text{Hom}_R(\_, R)$ .

The projective stabilization  $\underline{\text{mod}} R$  is defined as follows.

- Each object of  $\underline{\text{mod}} R$  is an object of  $\text{mod } R$ .
- For  $A, B \in \text{mod } R$ , a set of morphisms from  $A$  to  $B$  is  $\underline{\text{Hom}}_R(A, B) = \text{Hom}_R(A, B) / \mathcal{P}(A, B)$  where  $\mathcal{P}(A, B) := \{f \in \text{Hom}_R(A, B) \mid f \text{ factors through some projective module}\}$ . Each element is denoted as  $\underline{f} = f \text{ mod } \mathcal{P}(A, B)$ . A morphism  $f : A \rightarrow B$  in  $\text{mod } R$  is called a *stable isomorphism* if  $\underline{f}$  is an isomorphism in  $\underline{\text{mod}} R$  and we write  $A \stackrel{st}{\cong} B$ .

For an  $R$ -module  $M$ , define a transpose  $\text{Tr } M$  of  $M$  to be  $\text{Cok } \delta^*$  where  $P \xrightarrow{\delta} Q \rightarrow M \rightarrow 0$  is a projective presentation of  $M$ . The transpose of  $M$  is uniquely determined as an object of  $\underline{\text{mod}} R$ . If  $\underline{f} \in \underline{\text{Hom}}_R(M, N)$ , then  $f$  induces a map  $\text{Tr } N \rightarrow \text{Tr } M$ , which represents a morphism  $\text{Tr } \underline{f} \in \underline{\text{Hom}}_R(\text{Tr } N, \text{Tr } M)$ .

A kernel of projective cover of  $M$  is called the first syzygy module of  $M$  and denoted as  $\Omega_R^1(M)$ . The first syzygy module of  $M$  is uniquely determined as an object of  $\underline{\text{mod}} R$ . Inductively, we define  $\Omega_R^1(M) = \Omega_R^1(\Omega_R^{n-1}(M))$ . If  $\underline{f} \in \underline{\text{Hom}}_R(M, N)$ , then  $f$  induces a map  $\Omega_R^n(M) \rightarrow \Omega_R^n(N)$ , which represents a morphism  $\underline{\Omega_R^n(f)} \in \underline{\text{Hom}}_R(\Omega_R^n(M), \Omega_R^n(N))$ .

**Lemma 2.1** *On the commutative diagram with exact rows in  $\text{mod } R$*

$$\begin{array}{ccccccccc} 0 & \rightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \rightarrow & 0 \\ & & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \\ 0 & \rightarrow & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \rightarrow & 0, \end{array}$$

if  $\beta$  and  $\gamma$  are stable isomorphisms, so is  $\alpha$ .

**proof.** We show that  $\alpha$  is a stable isomorphism in the following case:

- 1)  $\gamma$  is an isomorphism and  $\beta$  is a stable isomorphism.

2)  $\beta$  and  $\gamma$  are stable isomorphisms and  $\gamma$  is an epimorphism.

3)  $\beta$  and  $\gamma$  are stable isomorphisms.

1) Adding a projective cover  $\rho_A : P_A \rightarrow A$  to the given diagram, we get the following:

$$\begin{array}{ccccccc}
& & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \\
& & Q & = & Q & & \\
& & \downarrow v & & \downarrow w & & \\
0 & \rightarrow & A \oplus P_A & \xrightarrow{\begin{pmatrix} f \\ 0 \end{pmatrix}} & B \oplus P_A & \xrightarrow{\begin{pmatrix} g \\ 0 \end{pmatrix}} & C \rightarrow 0 \\
& & \downarrow (\alpha \ \rho_A) & & \downarrow (\beta \ f' \circ \rho_A) & & \downarrow \cong \\
0 & \rightarrow & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' \rightarrow 0 \\
& & \downarrow & & \downarrow & & \\
& & 0 & & 0 & & 
\end{array}$$

Since  $(\beta \ f' \circ \rho_A)$  is an epimorphism and a stable isomorphism at the same time, it is a split epimorphism and  $Q$  is projective. In other words,  $w$  is a split monomorphism. Hence  $v$  is also a split monomorphism and  $(\alpha \ \rho_A)$  a split epimorphism with a projective kernel. In particular,  $\alpha$  is a stable isomorphism.

2) Since  $\gamma$  is a split epimorphism with a projective kernel, there exists  $\gamma' : C' \rightarrow C$  such that  $\gamma \circ \gamma' = \text{id}_{C'}$ . On the diagram

$$\begin{array}{ccccccc}
0 & \rightarrow & A & \rightarrow & B'' & \rightarrow & C' \rightarrow 0 \\
& & \parallel & & \downarrow \beta' & & \downarrow \gamma' \\
0 & \rightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C \rightarrow 0,
\end{array}$$

$\beta'$  is a stable isomorphism because  $\text{Ext}_{\mathbb{R}}^1(\gamma', A) : \text{Ext}_{\mathbb{R}}^1(C, A) \rightarrow \text{Ext}_{\mathbb{R}}^1(C', A)$  is an isomorphism. Connection of the above and the given diagrams yields another diagram:

$$\begin{array}{ccccccc}
0 & \rightarrow & A & \rightarrow & B'' & \rightarrow & C' \rightarrow 0 \\
& & \downarrow \alpha & & \downarrow \beta \circ \beta' & & \parallel \\
0 & \rightarrow & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' \rightarrow 0
\end{array}$$

Since  $\beta \circ \beta'$  is a stable isomorphism, we can apply 1).

3) Adding a projective cover  $\rho_{B'} : P_{B'} \rightarrow B'$ , we get

$$\begin{array}{ccccccc}
0 & \rightarrow & A & \xrightarrow{\begin{pmatrix} f \\ 0 \end{pmatrix}} & B \oplus P_{B'} & \xrightarrow{\begin{pmatrix} g \\ 0 \end{pmatrix}} & C \oplus P_{B'} \rightarrow 0 \\
& & \downarrow \alpha & & \downarrow (\beta \ \rho_{B'}) & & \downarrow (\gamma \ g' \circ \rho_B) \\
0 & \rightarrow & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' \rightarrow 0.
\end{array}$$

Since  $\beta$  is a stable isomorphism and  $(\gamma \circ g' \circ \rho_B)$  is an epimorphic stable isomorphism, we can apply (2) to get the conclusion. (q.e.d.)

Let  $\mathcal{L}$  be a full subcategory of  $\mathcal{K}(\text{mod } R)$  defined as

$$\mathcal{L} = \{F^\bullet \in \mathcal{K}(\text{proj } R) \mid H^i(F^\bullet) = 0 \ (i < 0), \quad H_j(F^\bullet) = 0 \ (j \geq 0)\}.$$

**Lemma 2.2** ( [5] **Proposition 2.3, Proposition 2.4** )

1) For  $A \in \underline{\text{mod}} R$ , there exists  $F_A^\bullet \in \mathcal{L}$  that satisfies

$$H^0(\tau_{\leq 0} F_A^\bullet) \stackrel{st}{\cong} A.$$

Such an  $F_A^\bullet$  is uniquely determined by  $A$  up to isomorphisms. We fix the notation  $F_A^\bullet$  and call this a standard resolution of  $A$ .

2) For  $\underline{f} \in \underline{\text{Hom}}_R(A, B)$ , there exists  $f^\bullet \in \text{Hom}_{\mathcal{K}(\text{mod } R)}(F_A^\bullet, F_B^\bullet)$  that satisfies

$$\underline{H^0}(\tau_{\leq 0} f^\bullet) = \underline{f}.$$

Such an  $f^\bullet$  is uniquely determined by  $\underline{f}$  up to isomorphisms, so we use the notation  $f^\bullet$  to describe a chain map with this property for given  $\underline{f}$ .

**Theorem 2.3** ( [5] **Theorem 2.6** ) The mapping  $A \mapsto F_A^\bullet$  gives a functor from  $\underline{\text{mod}} R$  to  $\mathcal{K}(\text{mod } R)$ , and this gives a category equivalence between  $\underline{\text{mod}} R$  and  $\mathcal{L}$ .

For  $f \in \text{Hom}_R(A, B)$ , there exists a triangle

$$C(f)^{\bullet-1} \xrightarrow{n^\bullet} F_A^\bullet \xrightarrow{f^\bullet} F_B^\bullet \xrightarrow{c^\bullet} C(f)^\bullet. \quad (2.1)$$

In general,  $C^\bullet$  does not belong to  $\mathcal{L}$  any more but it satisfies the following:

$$H^i(C^\bullet) = 0 \ (i < -1), \quad H_j(C^\bullet) = 0 \ (j > -1).$$

**Definition and Lemma 2.4** ([5], **Definition and Lemma 3.1**) As objects of  $\underline{\text{mod}} R$ ,  $\underline{\text{Ker}} \underline{f} := H^{-1}(\tau_{\leq -1} C(f)^\bullet)$  and  $\underline{\text{Cok}} \underline{f} := H^0(\tau_{\leq 0} C^\bullet)$  are uniquely determined by  $\underline{f}$ , up to isomorphisms. We call these the pseudo-kernel and the pseudo-cokernel of  $\underline{f}$ . And we have

$$\underline{\text{Cok}} \underline{f} = \text{Tr } \underline{\text{Ker}} \text{ Tr } \underline{f}.$$

**Notations.** The pseudo-kernel or the pseudo-cokernel is determined as a stable equivalence class of modules, not as an isomorphic class of  $\text{mod } R$ . Although, for the simplicity, a module  $M$  with  $M \cong^{st} \underline{\text{Ker}} f$  or  $M \cong^{st} \underline{\text{Cok}} f$  is denoted by  $\underline{\text{Ker}} f$  or  $\underline{\text{Cok}} f$  respectively.

Chain maps  $n^\bullet$  and  $c^\bullet$  in (2.1) induce  $c_f : \underline{\text{Cok}} f \rightarrow A$  and  $n_f : B \rightarrow \underline{\text{Ker}} f$ . These maps have the properties  $\underline{f} \circ n_f = \underline{0}$  and  $c_f \circ \underline{f} = \underline{0}$ . The following lemma shows why  $\underline{\text{Ker}} f$  and  $\underline{\text{Cok}} f$  are called the pseudo-kernel and the pseudo-cokernel respectively.

**Lemma 2.5** ([5] **Lemma 3.3, Lemma 3.5**) *Let  $f : A \rightarrow B$  be a homomorphism of  $R$ -modules.*

- 1) *If  $\underline{x} \in \underline{\text{Hom}}_R(X, A)$  satisfies  $\underline{f} \circ \underline{x} = \underline{0}$ , there exists  $\underline{h}_x \in \underline{\text{Hom}}_R(X, \underline{\text{Ker}} f)$  such that  $\underline{x} = \underline{n}_f \circ \underline{h}_x$ .*
- 2) *If  $\underline{y} \in \underline{\text{Hom}}_R(B, Y)$  satisfies  $\underline{y} \circ \underline{f} = \underline{0}$ , there exists  $\underline{e}_y \in \underline{\text{Hom}}_R(\underline{\text{Cok}} f, Y)$  such that  $\underline{y} = \underline{e}_y \circ \underline{c}_f$ .*

From (2.1), we have an exact sequence

$$0 \rightarrow \underline{\text{Ker}} f \rightarrow A \oplus P \xrightarrow{(f \ p)} B \rightarrow 0$$

with some projective module  $P$ . This characterizes the pseudo-kernel.

**Lemma 2.6** *For a given  $f \in \text{Hom}_R(A, B)$ , suppose both  $A \oplus P \xrightarrow{(f \ p)} B$  and  $A \oplus P' \xrightarrow{(f \ p')} B$  are epimorphisms with projective modules  $P$  and  $P'$ . Then there are stable isomorphisms  $\lambda : A \oplus P \rightarrow A \oplus P'$  and  $\kappa : \text{Ker}(f \ p) \rightarrow \text{Ker}(f \ p')$  that make the following diagram commutative:*

$$\begin{array}{ccccccc} 0 & \rightarrow & \text{Ker}(f \ p) & \rightarrow & A \oplus P & \xrightarrow{(f \ p)} & B \rightarrow 0 \\ & & \downarrow \kappa & & \downarrow \lambda & & \parallel \\ 0 & \rightarrow & \text{Ker}(f \ p') & \rightarrow & A \oplus P' & \xrightarrow{(f \ p')} & B \rightarrow 0 \end{array}$$

**proof.** Set  $\pi : B \rightarrow \text{Cok } f$  as a canonical map. Both  $\pi \circ p$  and  $\pi \circ p'$  are projective covers of  $\text{Cok } f$ , hence there exists  $l : P \rightarrow P'$  such that  $\pi \circ p = \pi \circ p' \circ l$ . Then  $\text{Im}(p' \circ l - p) \subset \text{Im } f$ , so we get  $h : P \rightarrow \text{Im } f$  as  $h$  coincides with  $p' \circ l - p$ . Via  $A \rightarrow \text{Im } f$  which is surjective,  $h$  can be lifted to a map  $j : P \rightarrow A$ . This shows the equation  $f \circ j + p' \circ l = p$ . The map  $\lambda = \begin{pmatrix} 1 & j \\ 0 & l \end{pmatrix} : A \oplus P \rightarrow A \oplus P'$  yields the desired diagram. Obviously  $\lambda$

is a stable isomorphism, which implies that  $\kappa$  is a stable isomorphism from Lemma 2.1. (q.e.d.)

**Lemma 2.7 ([5] Lemma 3.6)**

1) *There is an exact sequence*

$$0 \rightarrow \text{Ker } f \rightarrow \underline{\text{Ker}} f \rightarrow \Omega_R^1(\text{Cok } f) \rightarrow 0.$$

2) *There is an exact sequence*

$$0 \rightarrow L \rightarrow \underline{\text{Cok}} f \rightarrow \text{Cok } f \rightarrow 0$$

*such that  $\Omega_R^1(L)$  is the surjective image of  $\text{Ker } f$ .*

**Lemma 2.8** *The following holds for  $f \in \text{Hom}_R(A, B)$ .*

1)  *$\underline{\text{Ker}} f$  is projective if and only if  $f^\bullet$  can be taken as  $f^i$  are isomorphisms for  $i \leq -1$ .*

2) *If  $\underline{\text{Ker}} f$  is projective, then  $\Omega_R^1(f)$  is a stable isomorphism.*

**proof.** 1) The "if" part is obvious. First notice that  $\underline{\text{Ker}} f$  is projective if and only if we can choose  $C(f)^\bullet$  such that  $C(f)^i = 0$  ( $i \leq -2$ ) as an element of  $\mathcal{K}(\text{mod } R)$ . There is an exact sequence

$$0 \rightarrow C(f)^{\bullet-1} \rightarrow E^\bullet \xrightarrow{f'^\bullet} F^\bullet \rightarrow 0$$

where  $E^\bullet \cong F_A^\bullet$ ,  $F^\bullet \cong F_B^\bullet$  in  $\mathcal{K}(\text{mod } R)$  and via these isomorphisms,  $f^\bullet$  is isomorphic to  $f'^\bullet$ . Easily we see that  $C(f)^i = 0$  if and only if  $f'^i$  is an isomorphism in  $\text{mod } R$ .

2) The triangle

$$C(f)^{\bullet-1} \xrightarrow{n^\bullet} F_A^\bullet \xrightarrow{f^\bullet} F_B^\bullet \xrightarrow{c^\bullet} C(f)^\bullet$$

induces an exact sequence of complexes

$$0 \rightarrow F_A^\bullet \rightarrow \tilde{F}_B^\bullet \rightarrow C(f)^\bullet \rightarrow 0$$

which again induces

$$0 \rightarrow \tau_{\leq -1} F_A^\bullet \rightarrow \tau_{\leq -1} \tilde{F}_B^\bullet \rightarrow \tau_{\leq -1} C(f)^\bullet \rightarrow 0$$

where  $\tilde{F}_B^\bullet$  is a direct sum of  $F_B^\bullet$  and a trivial complex. Hence the exact sequence of homology groups is

$$0 \rightarrow \Omega_R^1(A) \xrightarrow{(\Omega_R^1(f))^\star} \Omega_R^1(B) \oplus P \rightarrow \underline{\text{Ker}} f \rightarrow 0$$

with projective modules  $P$ . From the assumption,  $(\Omega_R^1(f))^\star$  is a split monomorphism, in particular,  $\Omega_R^1(f)$  is a stable isomorphism. (q.e.d.)

**Corollary 2.9** *Suppose  $R$  is local. If a morphism  $f \in \text{End}_R(A)$  satisfies that  $\underline{\text{Ker}} f$  is projective, then  $f$  is a stable isomorphism.*

**proof.** We may assume  $C(f)^i = 0$  ( $i \leq -2$ ). Therefore the complex  $C(f)^\bullet$  has no cohomology except for  $H_{-1}(C(f)^\bullet)$ . The triangle

$$C(f)^\bullet \rightarrow F_{A^\bullet} \rightarrow F_{A^\bullet} \rightarrow C(f)_{\bullet-1}^\bullet$$

induces an exact sequence

$$0 \rightarrow H_{-1}(C(f)^\bullet) \rightarrow H_{-1}(F_{A^\bullet}) \rightarrow H_{-1}(F_{A^\bullet}) \rightarrow 0.$$

Since  $R$  is local, a surjective endomorphism on a finite module is always an automorphism. Thus we get  $H_{-1}(C(f)^\bullet) = 0$ . It follows that  $C(f)^\bullet$  is an exact sequence of projective modules, equivalently,  $f$  is a stable isomorphism. (q.e.d.)

### 3 Representation by monomorphisms and perfect exact sequences

**Definition 3.1** *A morphism  $f : A \rightarrow B$  in  $\text{mod } R$  is said to be represented by monomorphisms (rbm for short) if some monomorphism  $f' : A' \rightarrow B'$  in  $\text{mod } R$  is projective-stably equivalent to  $f$ , that is, there exist stable isomorphisms  $\alpha : A \rightarrow A'$  and  $\beta : B \rightarrow B'$  such that  $\underline{\beta \circ f} = \underline{f' \circ \alpha}$ .*

Each morphism is not always rbm.

**Example 3.2** *Let  $R$  be a ring of dimension  $n \geq 3$ ,  $N$  an  $R$ -module with  $\text{pd } N = n$ , and  $\varphi_N : N \rightarrow N^{**}$  the natural map. Then any map  $N \oplus P \rightarrow N^{**} \oplus Q$  of the form  $\begin{pmatrix} \varphi_N & \star \\ \star & \star \end{pmatrix}$  with projective modules  $P$  and  $Q$ , is never be monomorphic. If otherwise,  $N \oplus P$  is a submodule of a projective module; this is a contradiction because  $N$  has a maximal projective dimension.*



It was Auslander and Bridger who first defined and studied "represented by monomorphisms" property.

**Theorem 3.3 (Auslander-Bridger)** *The following are equivalent for a morphism  $f : A \rightarrow B$  in  $\text{mod } R$ .*

- 1) *There exists a monomorphism  $f' : A \rightarrow B \oplus P$  with a projective module  $P$  such that  $f = s \circ f'$  via some split epimorphism  $s : B \oplus P \rightarrow B$ .*
- 2) *There exists a monomorphism  $f' : A \rightarrow B \oplus P$  with a projective module  $P$  such that  $f = s \circ f'$  via some split epimorphism  $s : B \oplus P \rightarrow B$ , and  $f'^*$  is an epimorphism.*
- 3)  *$\underline{\text{Hom}}_R(B, I) \rightarrow \underline{\text{Hom}}_R(A, I)$  is surjective if  $I$  is an injective module.*

Auslander and Bridger's original definition of "represented by monomorphisms" condition is 1) of Theorem 3.3. Seemingly this is different from our definition. But we show that two conditions are equivalent.

**Lemma 3.4** *For a morphism  $f : A \rightarrow B$  in  $\text{mod } R$ ,  $f$  is rbm if and only if there exists a monomorphism  $f' : A \rightarrow B \oplus P$  with a projective module  $P$  such that  $f = s \circ f'$  via some split epimorphism  $s : B \oplus P \rightarrow B$ .*

**proof.** The "if" part is clear. We shall show "only if" part. Suppose there exists a monomorphism  $f' : A' \rightarrow B'$ , stable isomorphisms  $\alpha : A \rightarrow A'$  and  $\beta : B \rightarrow B'$  such that  $\underline{\beta \circ f} = \underline{f' \circ \alpha}$ . We first take projective covers  $\rho_A : P_A \rightarrow A$  and  $\rho_B : P_B \rightarrow B$  such that the induced map  $f_P : P_A \rightarrow P_B$  by  $f$  is a monomorphism. Since  $\alpha$  is a stable isomorphism, there exists a morphism  $\alpha' : A' \rightarrow A$  such that  $\underline{\alpha \circ \alpha'} = \underline{\text{id}_{A'}}$  and  $\underline{\alpha' \circ \alpha} = \underline{\text{id}_A}$ .

From the last equation there exists a morphism  $s_A : A \rightarrow P_A$  such that  $\alpha' \circ \alpha + \rho_A \circ s_A = \text{id}_A$ , equivalently  $(\alpha' \rho_A) \circ \begin{pmatrix} \alpha \\ s_A \end{pmatrix} = \text{id}_A$ . In particular,  $(\alpha' \rho_A) : A' \oplus P_A \rightarrow A$  is a split epimorphism and  $\begin{pmatrix} \alpha \\ s_A \end{pmatrix} : A \rightarrow A' \oplus P_A$  is a split monomorphism. Similarly we get morphisms  $\beta' : B' \rightarrow B$ ,  $s_B : B \rightarrow P_B$  and  $s_{B'} : B' \rightarrow P_{B'}$  such that  $(\beta' \rho_B) \circ \begin{pmatrix} \beta \\ s_B \end{pmatrix} = \text{id}_B$  and  $(\beta \rho_{B'}) \circ \begin{pmatrix} \beta' \\ s_{B'} \end{pmatrix} = \text{id}_{B'}$ .

Given equation  $\underline{\beta \circ f} = \underline{f' \circ \alpha}$  induces  $\underline{\beta' \circ (\beta \circ f) \circ \alpha'} = \underline{\beta' \circ (f' \circ \alpha) \circ \alpha'}$ , that is,  $\underline{f \circ \alpha'} = \underline{\beta' \circ f'}$ . Hence there exists a homomorphism  $t : A' \rightarrow P_B$  such that  $f \circ \alpha' - \beta' \circ f' = \rho_B \circ t$ .

Now we get a commutative diagram

$$\begin{array}{ccc} A' \oplus P_A & \xrightarrow{\begin{pmatrix} f' & 0 \\ t & f_P \end{pmatrix}} & B' \oplus P_B \\ \downarrow (\alpha' \rho_A) & & \downarrow (\beta' \rho_B) \\ A & \xrightarrow{f} & B. \end{array}$$

Since the composite of maps  $\begin{pmatrix} \beta' & 0 \\ s_{B'} & 0 \\ 0 & \text{id}_{P_B} \end{pmatrix} : B' \oplus P_B \rightarrow B \oplus P_{B'} \oplus P_B$  and  $\begin{pmatrix} \beta & \rho_{B'} & 0 \\ 0 & 0 & \text{id}_{P_B} \end{pmatrix} : B \oplus P_{B'} \oplus P_B \rightarrow B' \oplus P_B$  is equal to  $\text{id}_{B' \oplus P_B}$ ,  $(\beta' \ \rho_B) \circ \begin{pmatrix} f' & 0 \\ t & f_P \end{pmatrix} = (\beta' \ \rho_B) \circ \text{id}_{B' \oplus P_B} \circ \begin{pmatrix} f' & 0 \\ t & f_P \end{pmatrix} = (\beta' \ \rho_B) \circ \begin{pmatrix} \beta & \rho_{B'} & 0 \\ 0 & 0 & \text{id}_{P_B} \end{pmatrix} \circ \begin{pmatrix} \beta' & 0 \\ s_{B'} & 0 \\ 0 & \text{id}_{P_B} \end{pmatrix} \circ \begin{pmatrix} f' & 0 \\ t & f_P \end{pmatrix}$ , and the following diagram commutes:

$$\begin{array}{ccc} A' \oplus P_A & \xrightarrow{f''} & B \oplus P_{B'} \oplus P_B \\ \downarrow (\alpha' \ \rho_A) & & \downarrow \rho \\ A & \xrightarrow{f} & B \end{array} .$$

where

$$f'' = \begin{pmatrix} \beta' & 0 \\ s_{B'} & 0 \\ 0 & \text{id}_{P_B} \end{pmatrix} \circ \begin{pmatrix} f' & 0 \\ t & f_P \end{pmatrix},$$

and

$$\rho = (\beta' \ \rho_B) \circ \begin{pmatrix} \beta & \rho_{B'} & 0 \\ 0 & 0 & \text{id}_{P_B} \end{pmatrix}.$$

It is easy to see that  $f''$  is a monomorphism and  $\rho$  is a split epimorphism.

Finally putting  $f''' = f'' \circ \begin{pmatrix} \alpha \\ s_A \end{pmatrix}$  which is a monomorphism, we have

$$f = f \circ \text{id}_A = f \circ (\alpha' \ \rho_A) \circ \begin{pmatrix} \alpha \\ s_A \end{pmatrix} = \rho \circ f'' \circ \begin{pmatrix} \alpha \\ s_A \end{pmatrix} = \rho \circ f'''.$$

(q.e.d.)

The most remarkable point in Auslander-Bridger's Theorem is that being rfm is equivalent to being represented by "perfect monomorphisms" whose  $R$ -dual is an epimorphism.

**Definition 3.5** *An exact sequence  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$  of  $R$ -modules is called a perfect exact sequence or to be perfectly exact if its  $R$ -dual  $0 \rightarrow \text{Hom}_R(C, R) \rightarrow \text{Hom}_R(B, R) \rightarrow \text{Hom}_R(A, R) \rightarrow 0$  is also exact.*

**Proposition 3.6** ([5] **Lemma 2.7**) *The following are equivalent for an exact sequence*

$$\theta : 0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0.$$

- 1)  $\theta$  is perfectly exact.
- 2)  $0 \rightarrow F_A^\bullet \xrightarrow{f^\bullet} F_B^\bullet \xrightarrow{g^\bullet} F_C^\bullet \rightarrow 0$  is exact.
- 3)  $F_C^{\bullet-1} \rightarrow F_A^\bullet \xrightarrow{f^\bullet} F_B^\bullet \xrightarrow{g^\bullet} F_C^\bullet$  is a distinguished triangle in  $\mathcal{K}(\text{mod } R)$ .
- 4)  $F_A^\bullet \cong C(g)^{\bullet-1}$  in  $\mathcal{K}(\text{proj } R)$ .

If these conditions are satisfied, we have the following.

- 5)  $C \cong \underline{\text{Cok}} f$ .
- 6)  $F_C^\bullet \cong C(f)^\bullet$  in  $\mathcal{K}(\text{proj } R)$ .

**proof.**

In [5] Lemma 2.7, we see the equivalence between 1) and 2).

The implication 3)  $\Rightarrow$  2) is obvious.

For the rest of the proof, consider the following diagram:

$$\begin{array}{ccccccccc}
0 & \rightarrow & F_A^\bullet & \rightarrow & Cyl(f)^\bullet & \rightarrow & C(f)^\bullet & \rightarrow & 0 \\
& & \parallel & & \downarrow \beta'^\bullet & & \downarrow \gamma'^\bullet & & \\
0 & \rightarrow & F_A^\bullet & \xrightarrow{f^\bullet} & F_B^\bullet & \xrightarrow{g^\bullet} & F_C^\bullet & \rightarrow & 0 \\
& & \downarrow \alpha^\bullet & & \downarrow \beta''^\bullet & & \downarrow \gamma''^\bullet & & \\
0 & \rightarrow & C(g)^{\bullet-1} & \rightarrow & \widetilde{F}_B^\bullet & \rightarrow & Cyl(g)^\bullet & \rightarrow & 0
\end{array}$$

The top-row and the bottom-row are exact. Chain maps  $\beta'^\bullet$ ,  $\beta''^\bullet$  and  $\gamma''^\bullet$  are isomorphisms up to homotopy.

2)  $\Rightarrow$  3),4),5) and 6). If the middle row is also exact, then  $C(\gamma')^\bullet \cong C(\beta')^\bullet$ , which are trivial complexes, hence  $\gamma'^\bullet$  is an isomorphism. Now  $\beta''^\bullet \circ \beta'^\bullet$  and  $\gamma''^\bullet \circ \gamma'^\bullet$  are isomorphisms,  $C(\alpha)^\bullet = 0$  follows from the exact sequence

$$0 \rightarrow C(\alpha)^\bullet \rightarrow C(\beta'' \circ \beta')^\bullet \rightarrow C(\gamma'' \circ \gamma')^\bullet \rightarrow 0.$$

4)  $\Rightarrow$  3). On the above diagram  $\alpha^i = \text{id}$  for  $i \leq 0$ , so  $F_A^\bullet \cong C(g)^{\bullet-1}$  implies that  $\alpha^\bullet$  is an isomorphism. Therefore  $\gamma''^\bullet \circ \gamma'^\bullet$  is also an isomorphism. (q.e.d.)

If  $R$  is local, all the conditions above are equivalent. We shall give the proof later at the end of this section.

**Lemma 3.7** *Let the sequence*

$$\theta : 0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$$

*be exact. If  $R$  is local, the conditions 1) - 4) in Proposition 3.6 are equivalent to the conditions 5) and 6).*

5)  $C \cong \underline{\text{Cok}} f$ .

6)  $F_C^\bullet \cong C(f)^\bullet$  in  $\mathcal{K}(\text{proj } R)$ .

For a morphism  $f : A \rightarrow B$ ,  $A \oplus P_B \xrightarrow{(f \ \rho_B)} B$  is an epimorphism with a projective cover  $\rho_B : P_B \rightarrow B$ . Thus each morphism is represented by epimorphisms. And the choice of the representing epimorphism is unique up to direct sum of projective modules, as we have seen in Lemma 2.6.

Unlikely, we already know an example of a morphism that is not rbm. And moreover, even if a given map is represented by a monomorphism, there would be another representing monomorphism. We see it in Example 3.10.

However, uniqueness theorem is obtained in this way. Due to Theorem 3.3, a morphism is rbm if and only if it is represented by a perfect monomorphism. And if this is the case, the representing perfect monomorphism is uniquely determined up to stable isomorphisms. These are the statements in Theorem 3.9, before which, we need some preparations.

For given exact sequence of modules  $A \xrightarrow{f} B \xrightarrow{g} C$ , we have a diagram of triangles

$$\begin{array}{ccccccc} F_A^\bullet & \xrightarrow{f^\bullet} & F_B^\bullet & \rightarrow & C(f)^\bullet & \rightarrow & F_A^\bullet \\ \downarrow \alpha^\bullet & & \parallel & & \downarrow \gamma^\bullet & & \downarrow \alpha^{\bullet+1} \\ C(g)^{\bullet-1} & \rightarrow & F_B^\bullet & \xrightarrow{g^\bullet} & F_C^\bullet & \rightarrow & C(g)^\bullet \end{array} \quad (3.2)$$

which induces a diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \rightarrow & H^{-1}(C(f)^\bullet) & \rightarrow & A & \xrightarrow{\binom{f}{\alpha}} & B \oplus F_A^1 & \xrightarrow{(c_f \ \pi)} & \underline{\text{Cok}} f & \rightarrow & 0 \\ & & & & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \\ 0 & \rightarrow & \underline{\text{Ker}} g & \rightarrow & B \oplus P_C & \xrightarrow{(g \ \rho_C)} & C & \rightarrow & 0 \end{array} \quad (3.3)$$

We observe some facts below.

**Lemma 3.8** *With the notations above, the following holds.*

1)  $\beta$  is a stable isomorphism.

2)  $C(\alpha)^{\bullet+1} \cong C(\gamma)^{\bullet}$ .

3)  $\alpha$  is the composite of natural maps  $A \rightarrow \text{Im } f = \text{Ker } g$  and  $\text{Ker } g \rightarrow \underline{\text{Ker}} g$ . So if  $f$  is injective and  $g$  is surjective, then from Lemma 2.7,  $\alpha$  is a stable isomorphism,  $\tau_{\leq -1} C(\alpha)^{\bullet} = 0$ , and  $\tau_{\leq -2} C(\gamma)^{\bullet} = 0$ .

4) If  $H^{-1}(C(f)^{\bullet}) = 0$ , then the upper row of (3.3) is the short exact sequence

$$\theta_f : 0 \rightarrow A \xrightarrow{\begin{pmatrix} f \\ \epsilon \end{pmatrix}} B \oplus F_A^1 \xrightarrow{(c_f \ \pi)} \underline{\text{Cok}} f \rightarrow 0$$

which is a perfect exact sequence.

**Theorem 3.9** Let  $f : A \rightarrow B$  be a morphism in  $\text{mod } R$ . Then  $f$  is rbm if and only if  $H^{-1}(C(f)^{\bullet})$  vanishes. If this is the case, we have the following:

1) We have a perfect exact sequence

$$\theta_f : 0 \rightarrow A \xrightarrow{\begin{pmatrix} f \\ \epsilon \end{pmatrix}} B \oplus F_A^1 \xrightarrow{(c_f \ \pi)} \underline{\text{Cok}} f \rightarrow 0.$$

2) For any exact sequence of the form

$$\sigma : 0 \rightarrow A \xrightarrow{\begin{pmatrix} f \\ q \end{pmatrix}} B \oplus P' \xrightarrow{(g \ p)} C \rightarrow 0$$

with some projective module  $P'$ , there is a commutative diagram

$$\begin{array}{ccccccc} \theta_f : 0 & \rightarrow & A & \xrightarrow{\begin{pmatrix} f \\ \epsilon \end{pmatrix}} & B \oplus F_A^1 & \xrightarrow{(c_f \ \pi)} & \underline{\text{Cok}} f & \rightarrow & 0 \\ & & \downarrow \tilde{\alpha} & & \downarrow \tilde{\beta} & & \downarrow \tilde{\gamma} & & \\ \sigma : 0 & \rightarrow & A & \xrightarrow{\begin{pmatrix} f \\ q \end{pmatrix}} & B \oplus P' & \xrightarrow{(g \ p)} & C & \rightarrow & 0 \end{array}$$

where  $\tilde{\alpha}$  and  $\tilde{\beta}$  are stable isomorphisms.

3) There is an exact sequence with some projective module  $Q$  and  $Q'$

$$0 \rightarrow Q' \rightarrow \underline{\text{Cok}} f \oplus Q \xrightarrow{(\tilde{\gamma} \ \star)} C \rightarrow 0.$$

4) If  $\sigma$  is also perfectly exact, then  $\sigma$  is isomorphic to  $\theta_f$  up to direct sum of trivial complexes.

**proof.** Suppose that  $f$  is rbm; there is an exact sequence

$$\sigma : 0 \rightarrow A \xrightarrow{\begin{pmatrix} f \\ q \end{pmatrix}} B \oplus P' \xrightarrow{(g \ p)} C \rightarrow 0.$$

The maps  $\tilde{f} = \begin{pmatrix} f \\ q \end{pmatrix}$  and  $\tilde{g} = (g \ p)$  produce the same diagram as (3.2) because we may consider  $\tilde{f}^\bullet = f^\bullet$  and  $\tilde{g}^\bullet = g^\bullet$ . Apply Lemma 3.8 3) to this sequence, and we get  $\tau_{\leq -2}C(\gamma)^\bullet = 0$  as for  $\gamma^\bullet : C(f)^\bullet \rightarrow F_C^\bullet$ . From the long exact sequence of homology groups  $H^{-2}(C(\gamma)^\bullet) \rightarrow H^{-1}(C(f)^\bullet) \rightarrow H^{-1}(F_C^\bullet)$ , we get  $H^{-1}(C(f)^\bullet) = 0$ . Conversely, suppose that  $H^{-1}(C(f)^\bullet) = 0$ . Then Lemma 3.8 4) shows that  $\theta_f$  is perfectly exact. Now it remains to prove 2) - 4) in the case  $H^{-1}(C(f)^\bullet) = 0$ .

2) Applying the argument of Lemma 3.8 to the sequence  $\sigma$ , we get a similar diagram as (3.3) :

$$\begin{array}{ccccccc} 0 & \rightarrow & A & \xrightarrow{\begin{pmatrix} f \\ q \\ \epsilon \end{pmatrix}} & B \oplus P' \oplus F_A^1 & \rightarrow & \underline{\text{Cok}}\begin{pmatrix} f \\ q \end{pmatrix} \rightarrow 0 \\ & & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma \\ 0 & \rightarrow & \underline{\text{Ker}}(g \ p) & \rightarrow & B \oplus P' \oplus P_C & \xrightarrow{(g \ p \ \rho_C)} & C \rightarrow 0. \end{array}$$

The upper row is a direct sum of  $\theta_f$  and a trivial complex, and the lower row is that of  $\sigma$  and a trivial complex. Hence we get a desired diagram.

$$\begin{array}{ccccccc} 0 & \rightarrow & A & \xrightarrow{\begin{pmatrix} f \\ \epsilon \end{pmatrix}} & B \oplus F_A^1 & \xrightarrow{(c_f \ \pi)} & \underline{\text{Cok}}f \rightarrow 0 \\ & & \downarrow \tilde{\alpha} & & \downarrow \tilde{\beta} & & \downarrow \tilde{\gamma} \\ 0 & \rightarrow & A & \xrightarrow{\begin{pmatrix} f \\ q \end{pmatrix}} & B \oplus P & \xrightarrow{(g \ p)} & C \rightarrow 0. \end{array}$$

Notice that  $\tilde{\beta}$  is a stable isomorphism. From Lemma 3.8 3),  $\tilde{\alpha}$  is also a stable isomorphism.

3) Consider the exact sequence of complex

$$0 \rightarrow C(\gamma)^{\bullet-1} \rightarrow \widetilde{C}(f)^\bullet \xrightarrow{\gamma} F_C^\bullet \rightarrow 0,$$

where  $\widetilde{C}(f)^\bullet \cong C(f)^\bullet$  in  $\mathcal{K}(proj R)$ . Applying the truncation  $\tau_{\leq 0}$ , we get

$$0 \rightarrow (\tau_{\leq -1}C(\gamma))^{\bullet-1} \rightarrow \tau_{\leq 0}\widetilde{C}(f)^\bullet \rightarrow \tau_{\leq 0}F_C^\bullet \rightarrow 0,$$

which induces an exact sequence of homology  $0 \rightarrow Q' \rightarrow \underline{\text{Cok}}f \oplus Q \rightarrow C \rightarrow 0$  with projective modules  $Q' = C(\gamma)^{-1}$  and  $Q$  from Lemma 3.8 3).

4) Suppose  $\sigma$  is perfect. From Proposition 3.6,  $F_C^{\bullet-1} \rightarrow F_A^{\bullet} \xrightarrow{f^{\bullet}} F_B^{\bullet} \xrightarrow{g^{\bullet}}$   $F_C^{\bullet}$  is a distinguished triangle, and  $F_C^{\bullet} \cong C(f)^{\bullet}$ , hence the induced sequence  $\sigma$  is isomorphic to  $\theta_f$ . (q.e.d.)

**Example 3.10** Let  $k$  be a field and  $R = k[[X, Y, Z]]/(X^2 - YZ)$ . Let  $M$  be an  $R$ -module defined as  $M = R/(XY, Y^2, YZ)$ . The minimal Cohen-Macaulay approximation  $\theta_f : 0 \rightarrow Y_k \xrightarrow{f} X_k \rightarrow k \rightarrow 0$  of  $k$  is perfectly exact since  $\text{Ext}_R^1(k, R) = 0$ . On the other hand, the minimal Cohen-Macaulay approximation (see [2] for definition) of  $M$ ,  $\sigma : 0 \rightarrow Y_M \xrightarrow{g} X_M \rightarrow M \rightarrow 0$  is not perfect since  $\text{Ext}_R^1(M, R) \neq 0$ . The map  $g$  is decomposed as  $Y_M \cong Y_k \oplus R$ ,  $X_M \cong X_k \oplus R$  and  $g = \begin{pmatrix} f & q \\ 0 & Y \end{pmatrix}$ . We easily check the statement of Theorem 3.9;  $k \cong \underline{\text{Cok}} f$  and there is an exact sequence

$$0 \rightarrow R \rightarrow k \oplus R \rightarrow M \rightarrow 0$$

which clearly does not split.

**proof of Lemma 3.7.** 5)  $\Leftrightarrow$  6). Since  $f$  is injective, we get  $H^{-1}(C(f)^{\bullet}) = 0$  from Theorem 3.9. In other words,  $C(f)^{\bullet} \cong F_{\underline{\text{Cok}} f}^{\bullet}$ . In this situation, 5) and 6) are clearly equivalent.

6)  $\Rightarrow$  4). Since the assumption implies  $H^{-1}(C(f)^{\bullet}) = 0$ , we get an exact sequence  $0 \rightarrow Q' \rightarrow C \oplus Q \xrightarrow{(\gamma^{\bullet}, \ast)} C \rightarrow 0$  with projective modules  $Q$  and  $Q'$ , applying Theorem 3.9 3). Since  $R$  is local, we can apply Corollary 2.9, which shows that  $\gamma$  is a stable isomorphism, equivalently  $\gamma^{\bullet}$  is an isomorphism, hence  $\alpha^{\bullet}$  is also an isomorphism. (q.e.d.)

An exact sequence  $\theta : 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$  is perfectly exact if  $\text{Ext}_R^1(C, R) = 0$ . But as we see in the next example, the vanishing of  $\text{Ext}_R^1(C, R)$  is not the sufficient condition for  $\theta$  to be perfectly exact. @

**Example 3.11** Let  $R$  be  $R = k[[X, Y]]/(XY)$  and let  $f$  be a map defined via projective resolutions as follows:

$$\begin{array}{ccccccc} \dots & \rightarrow & R^2 & \xrightarrow{\begin{pmatrix} X & Y \end{pmatrix}} & R & \rightarrow & k \\ & & \downarrow \begin{pmatrix} X & Y \end{pmatrix} & & \downarrow \begin{pmatrix} X \\ Y \end{pmatrix} & & \downarrow f \\ 0 & \rightarrow & R & \xrightarrow{\begin{pmatrix} X \\ Y \end{pmatrix}} & R^2 & \rightarrow & Y^k \end{array}$$

Then the sequence

$$0 \rightarrow X \rightarrow k \oplus R^2 \xrightarrow{(f \ \rho_{Y^k})} Y^k \rightarrow 0$$

with a projective cover  $\rho_{Y^k}$  of  $Y^k$  is a perfect exact sequence since  $\text{Ext}_R^1(k, R) \cong \text{Ext}_R^1(Y^k, R)$ . But  $\text{Ext}_R^1(Y^k, R) \neq 0$ .

Notice that the dual of a perfect exact sequence is not always perfect as we see in the next example.

**Example 3.12** Let  $(R, \mathfrak{m}, k)$  be three-dimensional Gorenstein local ring. Set  $L = \text{Tr } k$ ,  $T_L = \Omega_R^3(\text{Tr})\Omega_R^3(\text{Tr})L$  and  $\varphi_L : L \rightarrow T_L$  to be a natural map. Since  $H^i(F_L^\bullet) = 0$  ( $i \leq 1$ ) and  $F_{T_L}^\bullet$  is exact,  $H^{-1}(C(\varphi_L)^\bullet) = 0$ . Putting  $N = \underline{\text{Cok}}\varphi_L$ , we have a perfectly exact sequence

$$\theta_{\varphi_L} : 0 \rightarrow L \xrightarrow{\begin{smallmatrix} (\varphi_L) \\ * \end{smallmatrix}} T_L \oplus P \rightarrow N \rightarrow 0$$

with some projective module  $P$ . It is easy to see that  $(N)^*$  is free. Dualizing  $\theta^*$ , we get an exact sequence

$$0 \rightarrow L^{**} \rightarrow T_L^{**} \rightarrow (N)^{**} \rightarrow \text{Ext}_R^1(L^*, R) \rightarrow 0.$$

But  $\text{Ext}_R^1(L^*, R) \cong \text{Ext}_R^3(\text{Tr } L, R) = \text{Ext}_R^3(k, R) \neq 0$ .

**Remark 3.13** Let  $\theta : 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$  be a perfect exact sequence. Then  $\theta^*$  is also perfectly exact if and only if the induced map  $H^2(F_A^\bullet) \rightarrow H^2(F_B^\bullet)$  is a monomorphism.

## 4 Representation by monomorphisms and torsionless modules.

In the previous section, we see that a given map  $f$  is represented by monomorphisms if and only if  $H^{-1}(C(f)^\bullet) = 0$ . If this is the case,  $\underline{\text{Ker}}f = \text{Cok } d_{C(f)}^{-2}$  is the first syzygy of  $\underline{\text{Cok}}f = \text{Cok } d_{C(f)}^{-1}$ . So it is natural to ask the converse: Is a given map  $f$  represented by monomorphisms if  $\underline{\text{Ker}}f$  is a first syzygy? This section deals with the problem. As a conclusion, the answer is yes if the total ring of fractions  $Q(R)$  of  $R$  is Gorenstein. Notice that if  $Q(R)$  is Gorenstein, then  $Q(R)$  is Artinian as we see in Lemma 4.3. What is more, if  $Q(R)$  is Gorenstein, instead of a pseudo-kernel, we can use a (usual) kernel to describe rbm condition. We begin with seeing equivalent conditions for a module to be a first syzygy.

**Definition 4.1** An  $R$ -module  $M$  is said to be torsionless <sup>1</sup> if the natural map  $\phi : M \rightarrow M^{**}$  is a monomorphism.

<sup>1</sup>In [1], Auslander and Bridger use the term "1-torsion free" for "torsionless". Usually a module  $M$  is called torsion-free if the natural map  $M \rightarrow M \otimes Q(R)$  is injective.



The next theorem is well known. We use the proof in [1] and [4].

**Lemma 4.2** *The following are equivalent for an  $R$ -module  $M$ .*

- 1)  $M$  is torsionless.
- 2)  $\text{Ext}_R^1(\text{Tr } M, R) = 0$
- 3)  $M$  is a first syzygy; there exists a monomorphism from  $M$  to a projective module.

proof. Let  $\phi : M \rightarrow M^{**}$  be the natural map. The well known formula  $\text{Ker } \phi \cong \text{Ext}_R^1(\text{Tr } M, R)$  shows the equivalence between 1) and 2).

2)  $\Rightarrow$  3). We may assume that  $M$  is a submodule of a free module  $R^l$ . Let  $\dots P^{-2} \xrightarrow{d^{-2}} P^{-1} \xrightarrow{d^{-1}} P^0 \rightarrow \text{Tr } M$  be a free resolution of  $\text{Tr } M$ . Then 2) says  $(P^0)^* \xrightarrow{(d^{-1})^*} (P^{-1})^* \xrightarrow{(d^{-2})^*} (P^{-2})^*$  is exact. By definition,  $M \cong \text{Cok } (d^{-1})^*$  which is isomorphic to  $\Omega_R^1(\text{Cok } d^{-2*})$ .

3)  $\Rightarrow$  1). We may assume that  $M$  is a submodule of a free module  $R^l$ .

Let  $\begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_l \end{pmatrix} : M \rightarrow R^l$  be a monomorphisms. If  $m \in M$  is not zero,  $f(m)$

is not zero, so there exists some  $i$  such that  $0 \neq f_i(m) = \phi(m)(f_i)$ , which implies  $\phi(m) \neq 0$ . Thus  $\text{Ker } \phi = (0)$ . (q.e.d.)

**Lemma 4.3** *If  $Q(R)$  is Cohen-Macaulay, then  $Q(R)$  is of dimension zero.*

**proof.** Suppose there exists a non-minimal prime ideal  $\mathfrak{q}$ . Then  $\mathfrak{q} \notin \text{Ass } Q(R)$ , since  $Q(R)$  has no embedded prime. This implies  $\mathfrak{q} \not\subseteq \bigcup_{\mathfrak{p} \in \text{Ass } Q(R)} \mathfrak{p}$ ,

hence  $\mathfrak{q}$  contains a non-zero-divisor which is a unit. (q.e.d.)

**Lemma 4.4** *Let  $R$  be a Noetherian ring and  $f : A \rightarrow B$  be a morphism in  $\text{mod } R$ . Suppose  $Q(R)$  is Gorenstein. If  $\underline{\text{Ker}} f$  is projective, then  $f$  is rbm.*

**proof.** The assumption says  $\tau_{\leq -2} C(f)^\bullet = 0$ . From Theorem 3.9,  $f$  is rbm if and only if  $H^{-1}(C(f)^\bullet) = 0$ , which means that  $d_{C(f)}^{-1}$  is injective. So we have only to show  $\text{Ker } d_{C(f)}^{-1} = (\text{Cok } (d_{C(f)}^{-1})^*)^* = 0$ . A triangle  $F_A^\bullet \xrightarrow{f^\bullet} F_B^\bullet \rightarrow C(f)^\bullet \rightarrow F_A^{\bullet+1}$  induces an exact sequence

$$0 \rightarrow H_{-1}(C(f)^\bullet) \rightarrow H_{-1}(F_B^\bullet) \rightarrow H_{-1}(F_A^\bullet) \rightarrow 0.$$

Note that  $\text{Cok}(d_{C(f)}^{-1})^* \cong \text{H}_{-1}(C(f)_\bullet^*)$  and  $\text{H}_{-1}(F_B)_\bullet^* = \text{Ext}_R^1(B, R)$ . Since  $Q(R)$  is Gorenstein of dimension zero,  $\text{Ext}_R^1(B, R) \otimes Q(R) = 0$ . So if  $\mathfrak{p}$  is any associated prime ideal of  $R$ ,  $\text{Ext}_R^1(B, R)_\mathfrak{p} = 0$ . Hence  $\text{Cok}(d_{C(f)}^{-1})_\mathfrak{p}^* = 0$ , which implies  $(\text{Cok}(d_{C(f)}^{-1})^*)^* = 0$ . (q.e.d.)

To solve our problem, the special kind of maps is a key. For  $M \in \text{mod } R$ , consider a module  $J^2M = \text{Tr } \Omega_R^1 \text{Tr } \Omega_R^1 M$ . Since  $\text{Tr } J^2M$  is a first syzygy, we have  $\text{Ext}_R^1(J^2M, R) = 0$ , which means  $\text{H}_{-1}(F_{J^2M})_\bullet^* = 0$  and  $\tau_{\geq -2} F_{J^2M})_\bullet^*$  is a projective resolution of  $\text{Tr } \Omega_R^1 M = \text{Cok}(d_{F_{J^2M}}^{-2})^* = \text{Cok}(d_{F_M}^{-2})^*$ . The identity map on  $\text{Tr } \Omega_R^1 M$  induces a chain map  $(F_M)_\bullet^* \rightarrow (F_{J^2M})_\bullet^*$  and  $\psi_M)_\bullet^* : F_{J^2M})_\bullet^* \rightarrow F_M)_\bullet^*$  subsequently. The maps  $\psi_M^i$  are identity maps for  $i \leq -1$ , in other words,  $\tau_{\leq -2} C(\psi_M)_\bullet^* = 0$  and  $\underline{\text{Ker}} \psi_M$  is projective. Thus we can apply the argument in Lemma 4.4 for  $f = \psi_M$ ;  $\text{H}^{-1}(C(\psi_M)_\bullet^*) \cong (\text{H}_{-1} C(\psi_M)_\bullet^*)^*$ . Since  $\text{H}_{-1}(F_{J^2M})_\bullet^* = 0$ , we have  $\text{H}_{-1}(C(\psi_M)_\bullet^*) \cong \text{H}_{-1}((F_M)_\bullet^*) \cong \text{Ext}_R^1(M, R)$ . Therefore we have  $\text{H}^{-1}(C(\psi_M)_\bullet^*) \cong (\text{Ext}_R^1(M, R))^*$ . Now we get a result as follows:

**Corollary 4.5** *The map  $\psi_M : J^2M \rightarrow M$  is rbm if and only if an  $R$ -module  $M$  has  $\text{Ext}_R^1(M, R)^* = 0$ .*

For given morphism of  $R$ -modules  $f : A \rightarrow B$ , adding a projective cover of  $B$  to  $f$ , we get an exact sequence

$$0 \rightarrow \underline{\text{Ker}} f \xrightarrow{\binom{n_f}{*}} A \oplus P_B \xrightarrow{(f \ \rho_B)} B \rightarrow 0.$$

Due to Theorem 3.9, we have a perfect exact sequence  $\theta_{n_f}$ , because  $n_f$  is rbm:

$$\begin{array}{ccccccc} \theta_{n_f} : & 0 & \rightarrow & \underline{\text{Ker}} f & \xrightarrow{\binom{n_f}{\diamond}} & A \oplus F_{\underline{\text{Ker}} f}^1 & \rightarrow & \underline{\text{Cok}} n_f & \rightarrow & 0 \\ & & & \downarrow \cong & & \downarrow \cong & & \downarrow \omega_f & & \\ & 0 & \rightarrow & \underline{\text{Ker}} f & \xrightarrow{\binom{n_f}{*}} & A \oplus P_B & \xrightarrow{(f \ \rho_B)} & B & \rightarrow & 0 \end{array}$$

**Lemma 4.6** *With notation as above, suppose  $(\text{Ext}_R^1(\underline{\text{Cok}} f, R))^* = 0$ . Then the following conditions are equivalent.*

- 1)  $f$  is rbm.
- 2)  $\underline{\text{Ker}} f$  is torsionless and  $\omega_f : \underline{\text{Cok}} n_f \rightarrow B$  is rbm.

**proof.** On the diagram of triangles

$$\begin{array}{ccccccc}
F_{\underline{\text{Ker}}f}^\bullet & \xrightarrow{n_f^\bullet} & F_A^\bullet & \rightarrow & C(n_f)^\bullet & \rightarrow & F_{\underline{\text{Ker}}f}^{\bullet+1} \\
\downarrow \chi_f^\bullet & & \parallel & & \downarrow \omega_f^\bullet & & \downarrow \chi_f^{\bullet+1} \\
C(f)^{\bullet-1} & \rightarrow & F_A^\bullet & \xrightarrow{f^\bullet} & F_B^\bullet & \rightarrow & C(f)^\bullet,
\end{array}$$

we observe  $C(\chi_f)^{\bullet+1} \cong C(\omega_f)^\bullet$ . We have  $H^{-1}(C(n_f)^\bullet) = 0$  because  $n_f$  is rbm. There is an exact sequence

$$H^0(F_{\underline{\text{Ker}}f}^\bullet) \rightarrow H^{-1}(C(f)^\bullet) \rightarrow H^{-1}(C(\omega_f)^\bullet).$$

2)  $\Rightarrow$  1). Since  $\underline{\text{Ker}}f$  is torsionless,  $H^0(F_{\underline{\text{Ker}}f}^\bullet) = \text{Ext}_R^1(\text{Tr } \underline{\text{Ker}}f, R) = 0$ . And  $H^{-1}(C(\omega_f)^\bullet) = 0$  because  $\omega_f$  is rbm. From the above exact sequence, we have  $H^{-1}(C(f)^\bullet) = 0$ .

1)  $\Rightarrow$  2). From the assumption,  $H^{-1}(C(f)^\bullet) = 0$  which implies  $\underline{\text{Ker}}f \stackrel{st}{\cong} \Omega_R^1(\underline{\text{Cok}}f)$ . We show that  $H^{-1}(C(\omega_f)^\bullet) = H^0(C(\chi_f)^\bullet)$  vanishes. The equation  $\underline{\text{Ker}}f \stackrel{st}{\cong} \Omega_R^1(\underline{\text{Cok}}f)$  implies  $F_{\underline{\text{Ker}}f}^{\bullet+1} \cong F_{J^2(\underline{\text{Cok}}f)}^\bullet$ . On the other hand,  $C(f)^\bullet \cong F_{\underline{\text{Cok}}f}^\bullet$ . Via these isomorphisms,  $\chi_f^{\bullet+1}$  is regarded as  $\psi_{\underline{\text{Cok}}f}^\bullet$ . Hence from the proof of Corollary 4.5,  $H^0(C(\chi_f)^\bullet) \cong H^{-1}(C(\psi_{\underline{\text{Cok}}f}^\bullet)^\bullet) \cong (\text{Ext}_R^1(\underline{\text{Cok}}f, R))^* = 0$ . (q.e.d.)

**Proposition 4.7** *Suppose  $Q(R)$  is Gorenstein. A morphism  $f$  of  $R$ -modules is rbm if and only if  $\underline{\text{Ker}}f$  is torsionless.*

**proof.** In the previous section, we already have "only if" part. Apply Theorem 3.9 to  $n_f$  which is rbm, Theorem 3.9 3) says that  $\underline{\text{Ker}}\omega_f$  is projective. Therefore  $\omega_f$  is rbm from Lemma 4.4. Since  $Q(R)$  is Gorenstein, we can use Lemma 4.6, which completes the proof. (q.e.d.)

Now we go to the next stage; we are to state rbm condition in terms of normal kernel.

**Lemma 4.8** *Suppose  $Q(R)$  is Gorenstein. Let the sequence of  $R$ -modules  $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$  be exact. If  $A$  and  $C$  are torsionless, then so is  $B$ .*

**proof.** From the assumption,  $A \cong \underline{\text{Ker}}g$  is torsionless. Due to Proposition 4.7,  $g$  is rbm; there exists an exact sequence

$$\theta_g : 0 \rightarrow B \xrightarrow{\binom{g}{q}} C \oplus Q \rightarrow \underline{\text{Cok}}g \rightarrow 0$$

with a projective module  $Q$  and a map  $q : B \rightarrow Q$ . Since  $C$  is a submodule of some projective module, so is  $B$ . (q.e.d.)

**Corollary 4.9** *Suppose  $Q(R)$  is Gorenstein. For a given morphism  $f$ ,  $\text{Ker } f$  is torsionless if and only if  $\underline{\text{Ker}} f$  is torsionless.*

**proof.** From Lemma 2.7, there is an exact sequence  $0 \rightarrow \text{Ker } f \rightarrow \underline{\text{Ker}} f \rightarrow \Omega_R^1(\text{Cok } f) \rightarrow 0$ . So the "if" part is obvious, and the "only if" part comes from Lemma 4.8. (q.e.d.)

**Theorem 4.10** *Suppose  $Q(R)$  is Gorenstein. The following are equivalent for a morphism  $f : A \rightarrow B$  in  $\text{mod } R$ .*

- 1)  $f$  is *rbm*.
- 2)  $\text{Ker } f$  is torsionless.
- 3)  $\underline{\text{Ker}} f$  is torsionless.
- 4)  $H^{-1}(C(f)^\bullet) = 0$ .
- 5)  $\Omega_R^1(\underline{\text{Cok}} f) \stackrel{st}{\cong} \underline{\text{Ker}} f$ .
- 6) There exists  $f'$  such that  $f' \stackrel{st}{\cong} f$  and  $\text{Ker } f'$  is torsionless.
- 7) For any  $f'$  with  $f' \stackrel{st}{\cong} f$ ,  $\text{Ker } f'$  is torsionless.

**proof.** Implications 5)  $\Rightarrow$  3), 7)  $\Rightarrow$  2) and 7)  $\Rightarrow$  6) are obvious. We already showed 1)  $\Leftrightarrow$  4) in Theorem 3.9, 1)  $\Leftrightarrow$  3) in Proposition 4.7, and 3)  $\Leftrightarrow$  2) in Corollary 4.9. Implications 3)  $\Rightarrow$  7) and 6)  $\Rightarrow$  3) are obtained from "if" and "only if" part of Corollary 4.9 respectively.

4)  $\Rightarrow$  5). It is clear since  $\text{Cok } d_{C(f)}^0 = \underline{\text{Ker}} f$  and  $\text{Cok } d_{C(f)}^{-1} = \underline{\text{Cok}} f$ . (q.e.d.)

The statement of Theorem 4.10 does not hold for ring  $R$  with  $Q(R)$  non-Gorenstein.

**Corollary 4.11** *The following are equivalent for a Noetherian ring  $R$ .*

- 1)  $Q(R)$  is Gorenstein.
- 2) Every morphism with torsionless kernel is *rbm*.
- 3)  $\text{Ext}_R^1(M, R)^* = 0$  for each  $M \in \text{mod } R$ .

**proof.**

1)  $\Rightarrow$  2). It comes directly from Theorem 4.10.

For  $M \in \text{mod } R$ ,  $\text{Ext}_R^1(M, R)^* = 0$  means  $\text{Ext}_{R_{\mathfrak{p}}}^1(M_{\mathfrak{p}}, R_{\mathfrak{p}}) = 0$  for every  $\mathfrak{p} \in \text{Ass } R$ . Hence 3) says that  $R_{\mathfrak{p}}$  is a zero-dimensional Gorenstein for each  $\mathfrak{p} \in \text{Ass } R$ , which is equivalent to 1) from Lemma 4.3.

2)  $\Rightarrow$  3). With no assumption,  $\text{Ker } \psi_M$  is torsionless. Because  $\underline{\text{Ker}}\psi_M$  is projective and  $\text{Ker } \psi_M$  is a submodule of  $\underline{\text{Ker}}\psi_M$  from Lemma 2.7 1). So if 2) holds,  $\psi_M$  is rbm and  $\text{Ext}_R^1(M, R)^* = 0$  from Corollary 4.5. (q.e.d.)

**Example 4.12** In the case  $R = k[[X, Y, Z]]/(XY, X^2)$  with any field  $k$ , consider the map  $\psi_k : k \rightarrow J^2k$ . We know  $\underline{\text{Ker}}\psi_k$  is projective. We shall show that  $\psi_k$  is not rbm; that is,  $H^{-1}(C(\psi_k)^\bullet)$  does not vanish.

As we have seen,  $C(\psi_k)^i = 0$  ( $i \leq -2$ ),  $H^{-1}(C(\psi_k)^\bullet) = (\text{Ext}_R^1(k, R))^*$ . From a free resolution of  $k$

$$R^3 \begin{pmatrix} Y & X & 0 \\ 0 & 0 & X \end{pmatrix} R^2 \begin{pmatrix} X & Y \end{pmatrix} R \rightarrow k \rightarrow 0,$$

we get a free resolution of  $\text{Ext}_R^1(k, R)$

$$R^4 \begin{pmatrix} X & Y & 0 & 0 \\ 0 & 0 & X & Y \end{pmatrix} R^2 \rightarrow \text{Ext}_R^1(k, R) \rightarrow 0.$$

We easily see that  $(\text{Ext}_R^1(k, R))^* \cong H^{-1}(C(\psi_k)^\bullet)$  does not vanish.

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